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THE THEORY OF SEMIGROUPS WITH WEAK SINGULARITY AND ITS APPLICATIONS TO PARTIAL DIFFDRENTIAL EQUATIONS

Dedicated to Professor Daisuke FUJIWARA on his 50th birthday

By

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Abstract. This paper provides a careful and accessible exposition of the theory of analytic semigroups which may have weak singularity at the origin, with emphasis on its applications to a class of *subelliptic* initial-boundary value problems for linear parabolic differential equations of second order.

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§ 0. Introduction and results

Let Ω be a bounded domain of Euclidean space R^n, with C^∞ boundary Γ; its closure $\bar{\Omega}$ is an n-dimensional, compact C^∞ manifold with boundary Γ. Let

$$A=\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x_i}+c(x)$$

be a second-order *elliptic* differential operator with real C^∞ coefficients on $\bar{\Omega}=\Omega\cup\Gamma$ such that:

1) $a^{ij}(x)=a^{ji}(x), \quad x\in\Omega,\ 1\leq i,\ j\leq n.$

2) $\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j\geq c_0|\xi|^2, \quad x\in\Omega,\ \xi\in R^n,$

with a constant $c_0>0$.

We consider the following *oblique derivative problem*: Given functions f and φ defined in Ω and on Γ respectively, find a function u in Ω such that

$$(*)\qquad \begin{cases} (A-\lambda)u=f & \text{in } \Omega, \\ Bu\equiv a\dfrac{\partial u}{\partial \nu}+\alpha u+bu|_\Gamma=\varphi & \text{on } \Gamma. \end{cases}$$

Here:

1° λ is a complex number.

2° a and b are real-valued C^∞ functions on Γ.

3° α is a real C^∞ vector field on Γ.

4° $\partial/\partial\nu$ is the conormal derivative associated with the matrix (a^{ij}):

$$\frac{\partial}{\partial \nu}=\frac{1}{\left(\sum_{i,j=1}^{n} a^{ij}n_i n_j\right)^{1/2}}\sum_{i,j=1}^{n} a^{ij}n_j\frac{\partial}{\partial x_i},$$

$n=(n_1, \cdots, n_n)$ being the unit exterior normal to Γ.

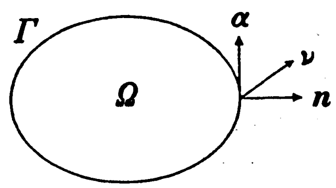

Figure 1

First we consider the problem of existence and uniqueness of solutions of problem (*) in the framework of Sobolev spaces when $|\lambda|$ tends to $+\infty$.

Our starting point is the following result, which is proved by Taira ([4], Théorème 11):

THEOREM 1. *Let* $s \geqq 2$. *The following two assertions are equivalent*:

(i) *For every* $-\pi<\theta<\pi$, *there exists a constant* $R(\theta)>0$ *depending on* θ *such that, for all* $\lambda=r^2e^{i\theta}$ *satisfying* $r\geqq 0$ *and* $|\lambda|=r^2\geqq R(\theta)$, *problem* (*) *admits a unique solution* u *in* $H^{s-1+\delta}(\Omega)$ $(0<\delta\leqq 1)$ *for any* $f\in H^{s-2}(\Omega)$ *and* $\varphi\in H^{s-3/2}(\Gamma)$, *and that*

$$(0.1)\qquad \|u\|^2_{H^{s-1+\delta}(\Omega)}+|\lambda|^{s-1+\delta}\|u\|^2_{L^2(\Omega)} \leqq C(\theta)(\|f\|^2_{H^{s-2}(\Omega)}+|\lambda|^{s-2}\|f\|^2_{L^2(\Omega)}+|\varphi|^2_{H^{s-3/2}(\Gamma)}+|\lambda|^{s-3/2}|\varphi|^2_{L^2(\Gamma)}),$$

where $C(\theta)>0$ *is a constant depending only on* θ.

(ii) *Hypothesis* $(H)_\delta$ *is satisfied* (*cf. Figure* 2):

$(H)_\delta$ *The vector field* α *is* non zero *on* $\Gamma_0=\{x'\in\Gamma;\ a(x')=0\}$ *and, along the integral curve* $x(t, x_0')$ *of* α *passing* $x_0'\in\Gamma_0$ *at* $t=0$, *the function*: $t\mapsto a(x(t, x_0'))$ *has zeros of* even *order* $\leqq 2k$, *and* $\delta=1/(1+2k)$.

Here $H^s(\Omega)$ (resp. $H^s(\Gamma)$) denotes the Sobolev space on Ω (resp. Γ) of order s.

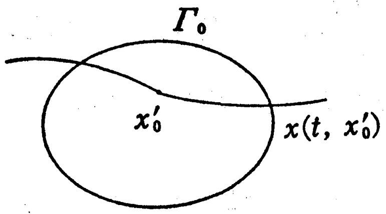

Figure 2

REMARKS 2. 1) The constant δ may take only the discrete values:

$$1,\ 1/3,\ \cdots,\ 1/(2k+1),\ \cdots.$$

2) Hypothesis $(H)_1$ $(\delta=1)$ is satisfied if and only if $a(x)\neq 0$ on Γ. In other words, problem (*) is *elliptic* (or coercive) if and only if $a(x)\neq 0$ on Γ. If $0<\delta<1$, problem (*) is said to be *subelliptic*.

We associate with problem (*) a closed linear operator $\mathfrak{A}$ from $L^2(\Omega)$ into itself as follows:

(a) The domain of definition $\mathcal{D}(\mathfrak{A})$ of $\mathfrak{A}$ is the space

$$\mathcal{D}(\mathfrak{A})=\left\{u\in L^2(\Omega);\ Au\in L^2(\Omega),\ Bu\equiv a\frac{\partial u}{\partial\nu}+\alpha u+bu|_\Gamma=0\right\}.$$

(b) $\mathfrak{A}u=Au,\ u\in\mathcal{D}(\mathfrak{A})$.

Here the function Au is taken in the sense of distributions and the boundary condition Bu can be defined as a distribution on Γ (cf. [5], Theorem 5.6.5 and

Proposition 8.3.2).

Using Theorem 1 with $s=2$, we can prove the following:

THEOREM 3. *Assume that hypothesis* $(H)_\delta$ *is satisfied. Then:*

(i) *For every* $\varepsilon>0$, *there exists a constant* $r(\varepsilon)>0$ *such that the resolvent set of* $\mathfrak{A}$ *contains the set* $\Sigma_\varepsilon=\{\lambda=r^2e^{i\theta};\ r\geqq r(\varepsilon),\ -\pi+\varepsilon\leqq\theta\leqq\pi-\varepsilon\}$, *and that the resolvent* $(\mathfrak{A}-\lambda I)^{-1}$ *satisfies the estimate*

$$\|(\mathfrak{A}-\lambda I)^{-1}\|\leqq\frac{c(\varepsilon)}{|\lambda|^{(1+\delta)/2}},\qquad \lambda\in\Sigma_\varepsilon, \tag{0.2}$$

where $c(\varepsilon)>0$ *is a constant depending only on* ε.

(ii) *The operator* $\mathfrak{A}$ *generates a semigroup* $U(z)$ *on* $L^2(\Omega)$ *which is* analytic *in the sector* $\Delta_\varepsilon=\{z=t+is;\ z\neq 0,\ |\arg z|<\pi/2-\varepsilon\}$ *for any* $0<\varepsilon<\pi/2$, *and enjoys the following properties:*

(a) *The operators* $\mathfrak{A}U(z)$ *and* $(dU/dz)(z)$ *are bounded operators on* $L^2(\Omega)$ *for each* $z\in\Delta_\varepsilon$, *and satisfy the relation*

$$\frac{dU}{dz}(z)=\mathfrak{A}U(z),\qquad z\in\Delta_\varepsilon.$$

(b) *For each* $0<\varepsilon<\pi/2$, *there exist constants* $\tilde{M}_0(\varepsilon)>0$, $\tilde{M}_1(\varepsilon)>0$ *and* $\mu_\varepsilon>0$ *such that*

$$\|U(z)\|\leqq\frac{\tilde{M}_0(\varepsilon)}{|z|^{(1-\delta)/2}}e^{\mu_\varepsilon\cdot\mathrm{Re}\,z},\qquad z\in\Delta_\varepsilon. \tag{0.3}$$

$$\|\mathfrak{A}U(z)\|\leqq\frac{\tilde{M}_1(\varepsilon)}{|z|^{(3-\delta)/2}}e^{\mu_\varepsilon\cdot\mathrm{Re}\,z},\qquad z\in\Delta_\varepsilon.$$

(c) *For each* $u_0\in\mathcal{D}(\mathfrak{A})$, *we have*

$$U(z)u_0\longrightarrow u_0\quad \text{in } L^2(\Omega)$$

as $z\to 0$, $z\in\Delta_\varepsilon$ $(0<\varepsilon<\omega/2)$.

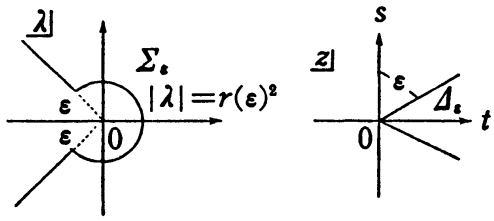

Figure 3

Now, as an application of Theorem 3, we consider the following *initial-boundary value problem*: Given functions f and u_0 defined in $\Omega\times[0, T]$ and in Ω respectively, find a function u in $\Omega\times[0, T]$ such that

$$(**)\qquad \begin{cases} \left(\dfrac{\partial}{\partial t}-A\right)u(x, t)=f(x, t) & \text{in } \Omega\times(0, T], \\ Bu(x, t)=0 & \text{on } \Gamma\times(0, T], \\ u(x, 0)=u_0(x) & \text{in } \Omega. \end{cases}$$

By using the operator $\mathfrak{A}$, one can formulate problem (**) in terms of the *Cauchy problem* in the space $L^2(\Omega)$:

$$(**)'\qquad \begin{cases} \dfrac{du}{dt}=\mathfrak{A}u(t)+f(t), & 0<t\leqq T, \\ u|_{t=0}=u_0. \end{cases}$$

Here $u(t)=u(\cdot, t)$ and $f(t)=f(\cdot, t)$ are functions defined on the interval $[0, T]$, taking values in the space $L^2(\Omega)$.

First we consider the elliptic case $\delta=1$. The next result is well known (cf. Friedman [1], Part 2, Theorem 9.1; Tanabe [6], Theorems 3.8.2 and 3.3.4):

THEOREM 4. *Assume that*

$$a(x)\neq 0 \quad \text{on } \Gamma.$$

Let $f(t)=f(\cdot, t)$ be a Hölder continuous function with exponent γ $(0<\gamma\leqq 1)$:

$$\|f(\cdot, t)-f(\cdot, s)\|_{L^2(\Omega)}\leqq C|t-s|^{\gamma}, \qquad t, s\in[0, T].$$

Then, for every function u_0 of $L^2(\Omega)$, the function

$$(0.4)\qquad u(t)=U(t)u_0+\int_0^t U(t-s)f(s)ds$$

belongs to $C([0, T]; L^2(\Omega))\cap C^1((0, T]; L^2(\Omega))$, and is a unique solution of problem (**)'.

Here $C([0, T]; L^2(\Omega))$ denotes the space of continuous functions on $[0, T]$ taking values in $L^2(\Omega)$, and $C^1((0, T]; L^2(\Omega))$ denotes the space of continuously differentiable functions on $(0, T]$ taking values in $L^2(\Omega)$, respectively.

By Theorem 3, we can define the fractional power $(-\mathfrak{A})^{-\alpha}$ for $1-\theta<\alpha<1$ by the formula

$$(-\mathfrak{A})^{-\alpha}=-\frac{\sin\alpha\pi}{\pi}\int_0^\infty s^{-\alpha}(\mathfrak{A}-sI)^{-1}ds,$$

and

$$(-\mathfrak{A})^{\alpha}=\text{the inverse of } (-\mathfrak{A})^{-\alpha}.$$

Then we can extend Theorem 4 to the subelliptic case $0<\delta<1$ as follows:

THEOREM 5. *Assume that hypothesis* $(H)_\delta$ *is satisfied with* $0<\delta<1$. *Let* $f(t)=f(\cdot, t)$ *be a Hölder continuous function with exponent* γ *satisfying* $(1-\delta)/2<\gamma\leqq 1$. *Then, for every function* u_0 *of* $\mathcal{D}((-\mathfrak{A})^\eta)$ *with* $(1-\delta)/2<\eta<(1+\delta)/2$, *the function* $u(t)$, *defined by formula* (0.4), *belongs to* $C([0, T]; L^2(\Omega))\cap C^1((0, T]; L^2(\Omega))$, *and is a unique solution of problem* $(**)'$.

The rest of this paper is organized as follows. Section 1 through Section 8 provide a careful and accessible exposition of the theory of analytic semigroups with weak singularity. Our presentation of semigroup theory follows the book of Kreĭn [2] and also part of Pazy's [3] fairly closely. Section 1 describes the basic definitions and facts about the abstract Cauchy problem for a densely defined, closed linear operator A in a Banach space E:

$$(P) \qquad \begin{cases} \dfrac{dx}{dt}=Ax(t), & 0<t\leqq T, \\ x(0)=x_0. \end{cases}$$

We study the semigroup $U(t)$ associated with the well-posed Cauchy problem. In Section 2 we prove a representation formula for solutions $x(t)$ of the Cauchy problem (P) in terms of the Laplace transform. This formula allows us to construct explicitly solutions of problem (P) in Section 4. Section 3 gives a sufficient condition for the uniqueness of solutions of problem (P) in terms of growth conditions on the resolvent $R(\lambda)=(A-\lambda I)^{-1}$. In Section 5 we consider the semigroup $U(t)$ under the condition that the operator A satisfies such a condition as (0.2). We prove that the semigroup $U(t)$ can be extended to an analytic semigroup $U(z)$ in some sector containing the positive real axis, but may have such a *weak singularity* as (0.3) at $z=0$ according to the decay order of the resolvent $R(\lambda)$ (Theorem 5.3). It is in this point that our semigroup $U(t)$ is different from the usual analytic semigroups. In Section 6 we study the fractional powers $(-A)^\alpha$. Section 7 is devoted to the characterization of admissible initial data x_0 for problem (P) in terms of the domains of $(-A)^\alpha$ (Theorem 7.1). Section 8 is devoted to the non-homogeneous Cauchy problem:

$$(NP) \qquad \begin{cases} \dfrac{dx}{dt}=Ax(t)+f(t), & 0<t\leqq T. \\ x(0)=x_0. \end{cases}$$

We prove an existence and uniqueness theorem for problem (NP) under Hölder continuous conditions on the non-homogeneous term $f(t)$ (Theorem 8.2). Our results (Theorems 7.1 and 8.2) are a generalization of the well known results for the usual analytic semigroups. In Section 9, as an application of the arguments developed in the previous sections, we study the oblique derivative problem, and prove Theorems 3 and 5.

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§ 1. The abstract Cauchy problem

Let E be a Banach space over the real or complex number field, and let $A: E \to E$ be a densely defined, closed linear operator with domain $\mathcal{D}(A)$.

A function $x(t)$, defined on the interval $I=[0, T]$ with values in E, is called a *solution* of the equation

$$(*) \qquad \frac{dx}{dt}=Ax(t) \qquad \text{on } [0, T],$$

if it satisfies the following three conditions:

$$\begin{cases} (1) & x(t)\in C([0, T]; E)\cap C^1((0, T]; E). \\ (2) & x(t)\in \mathcal{D}(A) \qquad \text{for all } 0<t\leqq T. \\ (3) & \dfrac{dx}{dt}=Ax(t) \qquad \text{for all } 0<t\leqq T. \end{cases}$$

Here $C([0, T]; E)$ denotes the space of continuous functions on $[0, T]$ taking values in E, and $C^1((0, T]; E)$ denotes the space of continuously differentiable functions on $(0, T]$ taking values in E, respectively.

We shall consider the following *Cauchy problem*: Given an element $x_0\in E$, find a solution $x(t)$ of equation $(*)$ satisfying the initial condition $x(0)=x_0$, that is,

$$(P) \qquad \begin{cases} \dfrac{dx}{dt}=Ax(t), & 0<t\leqq T. \\ x(0)=x_0. \end{cases}$$

We say that the Cauchy problem (P) is *well posed* on $[0, T]$ if we have:

(I) For any $x_0\in\mathcal{D}(A)$, there exists a unique solution $x(t)$ of problem (P).

(II) For each $t\in(0, T]$, the solution $x(t)$ depends continuously on x_0, that is,

$$x_0 \longrightarrow 0 \text{ in } E \Longrightarrow x(t) \longrightarrow 0 \text{ in } E.$$

The Cauchy problem is said to be well posed on $[0, \infty)$ if it is well posed on $[0, T]$ for every $T>0$.

First we have the following (cf. Kreĭn [2], Chapter I, Theorem 1.1):

THEOREM 1.1. *If the Cauchy problem (P) is well posed on* $[0, \infty)$, *then the operators* $U(t): E \to E$, *defined by*

$$x(t)=U(t)x_0, \qquad x_0 \in \mathcal{D}(A),$$

can be extended uniquely to bounded linear operators on E which form a strongly continuous semigroup, that is,

(i) $U(t+s)=U(t)\cdot U(s)$, $t, s>0$.

(ii) *The function* $U(t)x$ *is continuous on* $(0, \infty)$, *for each* $x \in E$.

REMARK 1.2. We do not know whether the limit $\lim_{t\downarrow 0} U(t)x$ exists or not if $x \in E$, while $\lim_{t\downarrow 0} U(t)x_0=x_0$ if $x_0 \in \mathcal{D}(A)$. Further, we do not know whether the function $U(t)x$, $t>0$, is strongly differentiable or not, and whether it belongs to $\mathcal{D}(A)$ or not.

PROOF OF THEOREM 1.1. 1) We remark that condition (II) implies that the operators $U(t)$ are continuous on $\mathcal{D}(A)$ and hence can be extended uniquely to bounded linear operators on E, since the domain $\mathcal{D}(A)$ is dense in E.

First we show the semigroup property:

$$U(t+s)=U(t)\cdot U(s), \qquad t, s>0. \tag{1.1}$$

If we let

$$w(t)=x(t+s)=U(t+s)x_0, \qquad x_0 \in \mathcal{D}(A),$$

then we have

$$\begin{cases} \dfrac{dw}{dt}(t)=\dfrac{d}{dt}(x(t+s))=\dfrac{dx}{dt}(t+s)=Ax(t+s)=Aw(t), \\ w(0)=x(s)=U(s)x_0. \end{cases}$$

Thus, it follows from the uniqueness of solutions of problem (P) that

$$U(t+s)x_0=w(t)=U(t)(U(s)x_0), \qquad x_0 \in \mathcal{D}(A).$$

This proves property (1.1), since the operators $U(t)$ are bounded on E and since the domain $\mathcal{D}(A)$ is dense in E.

2) Next we show that:

$$\sup_{\delta \le t \le 1/\delta} \|U(t)\| < +\infty \qquad \text{for every } \delta>0.$$

Assume to the contrary that there exist a constant $\delta'>0$ and a sequence $\{t_n\} \subset [\delta', 1/\delta']$ such that

$$\|U(t_n)\| \longrightarrow +\infty .$$

Then, by the resonance theorem (cf. Appendix C, Theorem C), it follows that

$$\|U(t_n)x_0\| \longrightarrow +\infty \qquad \text{for some } x_0 \in E .$$

For simplicity, we suppose that

$$\|U(t_n)x_0\| \geqq n .$$

Further, by passing to a subsequence we may suppose that the sequence $\{t_n\}$ converges to some $\gamma \in [\delta', 1/\delta']$: $t_n \to \gamma$.

Since the operators $U(t)$ are bounded on E, it follows that

$$(0, \gamma) = \bigcup_{M=1}^{\infty} \{0<t<\gamma ; \|U(t)x_0\| \leqq M\}.$$

Here we remark that the function $t \mapsto \|U(t)x_0\|$ is Lebesgue measurable, for it is the limit of continuous functions. Thus there exists an integer $M' \geqq 1$ such that

$$\mu(\{0<t<\gamma ; \|U(t)x_0\| \leqq M'\}) > \rho \qquad \text{for some } \rho > 0 ,$$

where μ is the Lebesgue measure on $\boldsymbol{R}$.

We let

$$F_{M'} = \{0<t<\gamma ; \|U(t)x_0\| \leqq M'\}.$$

Then we have

$$\lim_{\varepsilon \downarrow 0} \mu(F_{M'} \cap (0, \gamma-\varepsilon)) = \mu(F_{M'}) > \rho .$$

Since $t_n \to \gamma$, this implies that for all sufficiently large n

$$\mu(F_{M'} \cap (0, t_n)) > \rho .$$

If we let

$$\mathcal{E}_n = -F_{M'} \cap (0, t_n) + t_n ,$$

then we obtain that the set $\mathcal{E}_n$ is Lebesgue measurable and satisfies

$$\mu(\mathcal{E}_n) = \mu(F_{M'} \cap (0, t_n)) > \rho \qquad \text{for all sufficiently large } n.$$

Furthermore, we have for all $\tau \in F_{M'} \cap (0, t_n)$

$$n \leqq \|U(t_n)x_0\| = \|U(t_n-\tau)U(\tau)x_0\| \leqq \|U(t_n-\tau)\| M' .$$

This proves that

$$\|U(\sigma)\| \geqq \frac{n}{M'} \qquad \text{for all } \sigma \in \mathcal{E}_n . \tag{1.2}$$

Now we let

$$\mathcal{E} = \bigcap_{n=0}^{\infty} \bigcup_{\nu=n}^{\infty} \mathcal{E}_\nu .$$

Then the set $\mathcal{E}$ is not empty, for we have

$$\mu(\mathcal{E})=\lim_{n\to\infty}\mu\Big(\bigcup_{\nu=n}^{\infty}\mathcal{E}_\nu\Big)\geqq\rho\,.$$

But, we find that a point σ belongs to the set $\mathcal{E}$ if and only if it belongs to infinitely many $\mathcal{E}_\nu$'s. Thus, in view of assertion (1.2), it follows that

$$\|U(\sigma)\|=+\infty \qquad \text{for all } \sigma\in\mathcal{E}\,.$$

This is a contradiction, since $\|U(\sigma)\|$ is finite for every $\sigma\in(0,\infty)$.

3) Now let x be an arbitrary element of E, and choose a sequence $\{x_j\}\subset\mathcal{D}(A)$ such that

$$x_j\longrightarrow x \qquad \text{in } E\,.$$

Then we have for $\delta\leqq t\leqq 1/\delta$

$$\|U(t)x_j-U(t)x\|\leqq\sup_{[\delta,1/\delta]}\|U(t)\|\cdot\|x_j-x\|\longrightarrow 0 \qquad \text{as } j\to\infty\,.$$

Since this convergence is uniform in $t\in[\delta,1/\delta]$, and since the functions $U(t)x_j$ are continuous, it follows that the function $U(t)x$ is continuous on $[\delta,1/\delta]$, for every $\delta>0$. This proves that the function $U(t)x$ is continuous on $(0,\infty)$, for each $x\in E$.

The proof of Theorem 1.1 is now complete.

Assume that the operator A has a resolvent $R(\lambda_0)=(A-\lambda_0 I)^{-1}$ for some $\lambda_0\in\boldsymbol{C}$. Then, for each $x\in E$, the function $U(t)(R(\lambda_0)x)$ is a solution of problem (P), since $R(\lambda_0)x\in\mathcal{D}(A)$. Hence we have for all $T>0$

$$\sup_{0\leq t\leq T}\|U(t)(R(\lambda_0)x)\|<+\infty\,, \qquad x\in E\,.$$

By the resonance theorem (cf. Theorem C), this implies that

$$\sup_{0\leq t\leq T}\|U(t)R(\lambda_0)\|<+\infty\,.$$

Further we have the following:

PROPOSITION 1.3. *Assume that the operator A has a resolvent $R(\lambda_0)=(A-\lambda_0 I)^{-1}$ for some $\lambda_0\in\boldsymbol{C}$. If the Cauchy problem (P) is well-posed on $[0,\infty)$, then we have for all $t>0$*

$$R(\lambda_0)U(t)=U(t)R(\lambda_0) \qquad \text{on } E. \tag{1.3}$$

$$AU(t)=U(t)A \qquad \text{on } \mathcal{D}(A)\,. \tag{1.4}$$

PROOF. i) Let x_0 be an arbitrary element of $\mathcal{D}(A)$. Since the function $R(\lambda_0)(U(t)x_0)$ is a solution of problem (P) with initial condition $R(\lambda_0)x_0$, it follows that

$$R(\lambda_0)(U(t)x_0)=U(t)(R(\lambda_0)x_0).$$

This proves formula (1.3), since the operators $U(t)R(\lambda_0)$ and $R(\lambda_0)U(t)$ are both bounded on E and since the domain $\mathcal{D}(A)$ is dense in E.

ii) By formula (1.3), we have for any $x_0\in\mathcal{D}(A)$

$$\begin{aligned} AU(t)x_0&=AU(t)R(\lambda_0)(A-\lambda_0 I)x_0\\ &=AR(\lambda_0)U(t)(A-\lambda_0 I)x_0\\ &=(A-\lambda_0 I)R(\lambda_0)U(t)(A-\lambda_0 I)x_0+\lambda_0 R(\lambda)U(t)(A-\lambda_0 I)x_0\\ &=U(t)(A-\lambda_0 I)x_0+\lambda_0 U(t)R(\lambda_0)(A-\lambda_0 I)x_0\\ &=U(t)(A-\lambda_0 I)x_0+\lambda_0 U(t)x_0\\ &=U(t)Ax_0. \end{aligned}$$

This proves formula (1.4).

§ 2. Representation of solutions

The next theorem gives a representation formula for the solutions of problem (P) in terms of the Laplace transform (cf. Kreĭn [2], Chapter I, Theorem 3.2):

THEOREM 2.1. *Assume that the operator A has a resolvent $R(\lambda)=(A-\lambda I)^{-1}$ for any $\lambda\in\boldsymbol{C}$ satisfying $\operatorname{Re}\lambda\geqq\alpha$, and that there exists a constant $M>0$ such that*

$$\|R(\lambda)\|\leqq M(1+|\lambda|),\qquad \operatorname{Re}\lambda\geqq\alpha. \tag{2.1}$$

Then every solution $x(t)$ of problem (P) can be written as

$$\begin{aligned} x(t)&=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}R(\lambda)x_0\,d\lambda\\ &=-\frac{1}{2\pi i}\lim_{N\to+\infty}\int_{\alpha-iN}^{\alpha+iN}e^{\lambda t}R(\lambda)x_0\,d\lambda,\qquad 0<t<T. \end{aligned}\tag{2.2}$$

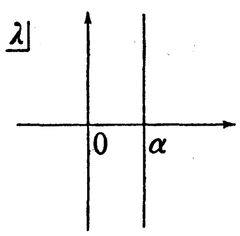

Figure 4

PROOF. Since $x(t)\in C^1((0,T];E)$, by integration by parts, we have for any

$\varepsilon>0$

$$\int_\varepsilon^T e^{-\lambda s}x(s)ds=\left[-\frac{1}{\lambda}e^{-\lambda s}x(s)\right]_\varepsilon^T+\frac{1}{\lambda}\int_\varepsilon^T e^{-\lambda s}x'(s)ds$$
$$=\frac{1}{\lambda}\Big(e^{-\lambda\varepsilon}x(\varepsilon)-e^{-\lambda T}x(T)\Big)+\frac{1}{\lambda}\int_\varepsilon^T e^{-\lambda s}Ax(s)ds\,.$$

But, since $x(t)\in C([0, T]; E)$, it follows that as $\varepsilon\downarrow 0$

$$\int_\varepsilon^T e^{-\lambda s}x(s)ds \longrightarrow \int_0^T e^{-\lambda s}x(s)ds \qquad \text{in } E\,.$$

Hence we find that the improper integral $\int_0^T e^{-\lambda s}Ax(s)ds$ exists and satisfies

$$\int_0^T e^{-\lambda s}Ax(s)ds=\lambda\int_0^T e^{-\lambda s}x(s)ds+e^{-\lambda T}x(t)-x(0).$$

On the other hand, by the closedness of A, it follows that

$$\begin{cases} \int_0^T e^{-\lambda s}x(s)ds\in\mathcal{D}(A)\,, \\ A\Big(\int_0^T e^{-\lambda s}x(s)ds\Big)=\int_0^T e^{-\lambda s}Ax(s)ds\,. \end{cases}$$

Thus we have

$$(A-\lambda I)\Big(\int_0^T e^{-\lambda s}x(s)\,ds\Big)=e^{-\lambda T}x(T)-x(0)\,,$$

and hence

$$\int_0^T e^{-\lambda s}x(s)ds=R(\lambda)(e^{-\lambda T}x(T)-x(0))\,. \tag{2.3}$$

If we let

$$X(s)=\begin{cases} x(s) & \text{if } 0\leqq s\leqq T\,, \\ 0 & \text{if } s>T\,, \end{cases}$$

then formula (2.3) can be written as

$$\int_0^\infty e^{-\lambda s}X(s)ds=R(\lambda)(e^{-\lambda T}X(T)-X(0))\,. \tag{2.3'}$$

Since $X(s)\in C^1((0, T); E)$, using the Laplace inversion formula (cf. Appendix B, Theorem B), we obtain that for $0<t<T$

$$X(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}R(\lambda)(e^{-\lambda T}X(T)-X(0))d\lambda$$
$$=\frac{1}{2\pi i}\lim_{N\to+\infty}\int_{\alpha-iN}^{\alpha+iN}e^{\lambda t}R(\lambda)(e^{-\lambda T}X(T)-X(0))d\lambda\,.$$

Hence we have for $0<t<T$

$$x(t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{\lambda t}R(\lambda)x(0)d\lambda$$
$$+\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{-\lambda(T-t)}R(\lambda)x(T)d\lambda .$$

We will show that the second term on the right-hand side vanishes:

$$\int_{a-i\infty}^{a+i\infty}e^{-\lambda(T-t)}R(\lambda)x(T)d\lambda=0, \qquad 0<t<T,$$

which proves formula (2.2).

a) First we remark that by inequality (2.1)

(2.1′) $\frac{\|R(\lambda)\|}{|\lambda|}$ is bounded for all sufficiently large $|\lambda|$.

b) Since we have for any $x\in\mathscr{D}(A)$

$$x=R(\lambda)(A-\lambda I)x,$$

if follows from assertion (2.1′) that

$$\left\|\frac{R(\lambda)x}{\lambda}\right\|\leqq\frac{1}{|\lambda|^2}\|x\|+\frac{1}{|\lambda|}\frac{\|R(\lambda)\|}{|\lambda|}\|Ax\|\longrightarrow 0 \qquad \text{as } |\lambda|\to\infty .$$

This proves that

$$\left\|\frac{R(\lambda)y}{\lambda}\right\|\longrightarrow 0 \qquad \text{for any } y\in E,$$

since the domain $\mathscr{D}(A)$ is dense in E. Hence we have for any $x\in\mathscr{D}(A)$

$$\|R(\lambda)x\|=\left\|-\frac{1}{\lambda}x+\frac{R(\lambda)}{\lambda}Ax\right\| \tag{2.4}$$
$$\leqq\frac{1}{|\lambda|}\|x\|+\left\|\frac{R(\lambda)}{\lambda}Ax\right\|\longrightarrow 0 \qquad \text{as } |\lambda|\to\infty .$$

c) Since $x(T)\in\mathscr{D}(A)$, it follows from assertion (2.4) that

$$\|R(\lambda)x(T)\|\longrightarrow 0 \qquad \text{as } |\lambda|\to\infty .$$

Hence, using Cauchy's theorem and Jordan's lemma (cf. Lemma A), we obtain that

$$\int_{a-i\infty}^{a+i\infty}e^{-\lambda(T-t)}R(\lambda)x(T)d\lambda=0, \qquad 0<t<T .$$

The proof of Theorem 2.1 is complete.

§ 3. Uniqueness of solutions

The next theorem gives a criterion for the uniqueness of solutions of problem (P) (cf. Kreĭn [2], Chapter I, Theorem 3.1; Pazy [3], Chapter 4, Theorem

1.2):

THEOREM 3.1. *Assume that the operator* A *has a resolvent* $R(\lambda)=(A-\lambda I)^{-1}$ *for all sufficiently large* $\lambda>0$, *and that*

$$\limsup_{\lambda\to+\infty}\frac{\log\|R(\lambda)\|}{\lambda}\leqq 0. \tag{3.1}$$

Then the Cauchy problem (P) *is uniquely solvable on the interval* $[0, T]$, *for every* $T>0$.

PROOF. 1) We remark that $x(t)$ is a solution of problem (P) if and only if $e^{\lambda t}x(t)$ $(\lambda\in\boldsymbol{C})$ is a solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dt}=(A+\lambda I)y(t), & 0<t\leqq T, \\ y(0)=x_0. \end{cases}$$

Thus one may translate A by a constant multiple of the identity, and assume that the resolvent $R(\lambda)$ exists for all $\lambda>0$.

The proof of the theorem is based on the following lemma (cf. Pazy [3], Chapter 4, Lemma 1.1):

LEMMA 3.2. *Let* $u:[0, T]\to E$ *be a continuous function. If there exists a constant* $M>0$ *such that*

$$\left\|\int_0^T e^{ns}u(s)ds\right\|\leqq M \qquad \textit{for all integer } n\geqq 1. \tag{3.2}$$

then it follows that $u(t)\equiv 0$ *on* $[0, T]$.

PROOF. Take an arbitrary element e^* of the dual space E^*, and let

$$\varphi(t)=\langle u(t), e^*\rangle.$$

Then the function φ is continuous on $[0, T]$ and, in view of inequality (3.2), it follows that for all integer $n\geqq 1$

$$\begin{aligned}\left|\int_0^T e^{ns}\varphi(s)ds\right| &= \left|\left\langle e^*, \int_0^T e^{ns}u(s)ds\right\rangle\right| \\ &\leqq \|e^*\|M.\end{aligned} \tag{3.3}$$

We shall show that $\varphi(s)\equiv 0$ on $[0, T]$; then the lemma follows, since $e^*\in E^*$ is arbitrary.

Now we consider the function

$$1-\exp\{-e^{n\tau}\}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}e^{k\tau n}.$$

The series converges uniformly in τ on bounded intervals of $\boldsymbol{R}$. Thus we have by inequality (3.3)

$$\begin{aligned}&\left|\int_0^T\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}e^{kn(t-T+s)}\varphi(s)ds\right|\\&=\left|\sum_{k=1}^{\infty}\int_0^T\frac{(-1)^{k-1}}{k!}e^{kn(t-T+s)}\varphi(s)ds\right|\\&\leqq\sum_{k=1}^{\infty}\frac{1}{k!}e^{kn(t-T)}\left|\int_0^T e^{kns}\varphi(s)ds\right|\\&=(\exp\{e^{n(t-T)}\}-1)M\|e^*\|.\end{aligned}$$

This proves that

$$\lim_{n\to\infty}\int_0^T\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}e^{kn(t-T+s)}\varphi(s)ds=0,\qquad 0\leqq t<T, \tag{3.4}$$

since the function $\exp\{e^{n(t-T)}\}-1$ tends to zero as $n\to\infty$, for every $0\leqq t<T$.

On the other hand, it follows from an application of Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty}\int_0^T(1-\exp\{-e^{n(t-T+s)}\})\varphi(s)ds=\int_{T-t}^T\varphi(s)ds,\qquad 0<t<T. \tag{3.5}$$

Therefore, combining formulas (3.4) and (3.5), we obtain that

$$\int_{T-t}^T\varphi(s)ds=0\qquad\text{for every } 0<t<T.$$

This implies that $\varphi(s)\equiv 0$ on $[0, T]$, since φ is continuous.

The lemma is proved.

2) End of Proof of Theorem 3.1. Let $x(t)$ be a solution of problem (P) with $x(0)=0$. Then we have

$$\frac{d}{dt}(R(\lambda)x(t))=R(\lambda)\frac{dx}{dt}=R(\lambda)Ax(t)=\lambda R(\lambda)x(t)+x(t),$$

so that

$$R(\lambda)x(t)=\int_0^t e^{\lambda(t-s)}x(s)ds, \tag{3.6}$$

since $x(0)=0$.

Now, condition (3.1) implies that, for each $\varepsilon>0$, there exists a constant $M_\varepsilon>0$ such that

$$\|R(\lambda)\|\leqq M_\varepsilon e^{\varepsilon\lambda}\qquad\text{for all } \lambda>0.$$

Thus it follows that

$$\lim_{\lambda\to+\infty} e^{-\sigma\lambda}\|R(\lambda)\|=0 \qquad \text{for each } \sigma>0.$$

Hence we have by formula (3.6)

$$\lim_{\lambda\to+\infty}\int_0^t e^{\lambda(t-\sigma-s)}x(s)ds=\lim_{\lambda\to+\infty} e^{-\lambda\sigma}R(\lambda)x(t)=0. \tag{3.7}$$

On the other hand, it follows from an application of Lebesgue's dominated convergence theorem that

$$\lim_{\lambda\to+\infty}\int_{t-\sigma}^t e^{\lambda(t-\sigma-s)}x(s)ds=\lim_{\lambda\to+\infty}\int_0^\sigma e^{-\lambda\tau}x(\tau+t-\sigma)d\tau=0. \tag{3.8}$$

Combining formulas (3.7) and (3.8), we obtain that

$$\begin{aligned}\lim_{\lambda\to+\infty}\int_0^{t-\sigma} e^{\lambda\tau}x(t-\sigma-\tau)d\tau&=\lim_{\lambda\to+\infty}\int_0^{t-\sigma} e^{\lambda(t-\sigma-s)}x(s)ds\\&=\lim_{\lambda\to+\infty}\int_0^t e^{\lambda(t-\sigma-s)}x(s)ds\\&\quad-\lim_{\lambda\to+\infty}\int_{t-\sigma}^t e^{\lambda(t-\sigma-s)}x(s)ds\\&=0.\end{aligned}$$

Therefore, applying Lemma 3.2 to the function $x(t-\sigma-\cdot)$, we find that

$$x(s)\equiv 0 \qquad \text{on } [0, t-\sigma].$$

This proves that

$$x(s)\equiv 0 \qquad \text{on } [0, T], \quad \text{for every } T>0,$$

since t and σ are arbitrary.

The proof of Theorem 3.1 is now complete.

§ 4. Existence of solutions

The next theorem plays an essential role in the construction of solutions of the Cauchy problem (P) (cf. Kreĭn [2], Chapter I, Theorem 1.5):

THEOREM 4.1. *Assume that the operator A has a resolvent $R(\lambda)=(A-\lambda I)^{-1}$ for any $\lambda\in \boldsymbol{C}$ satisfying* $\operatorname{Re}\lambda\geqq\alpha$, *and that there exist constants $0<\beta\leqq 1$ and $M>0$ such that*

$$\|R(\lambda)\|\leqq\frac{M}{(1+|\operatorname{Im}\lambda|)^\beta}, \qquad \operatorname{Re}\lambda\geqq\alpha. \tag{4.1}$$

Then the Cauchy problem (P) has a solution $x(t)\in C^1([0, \infty); E)$ for any initial condition $x_0\in\mathcal{D}(A^2)$.

Furthermore, the solution $x(t)$ *is given by the following formula*:

$$x(t)=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}\frac{R(\lambda)((A-\lambda_0 I)^2 x_0)}{(\lambda-\lambda_0)^2}d\lambda \tag{4.2}$$

$$=-\frac{1}{2\pi i}\lim_{N\to+\infty}\int_{\alpha-iN}^{\alpha+iN}e^{\lambda t}\frac{R(\lambda)((A-\lambda_0 I)^2 x_0)}{(\lambda-\lambda_0)^2}d\lambda, \qquad t\geqq 0.$$

Here λ_0 *is a complex number satisfying* $\mathrm{Re}\,\lambda_0>\alpha$.

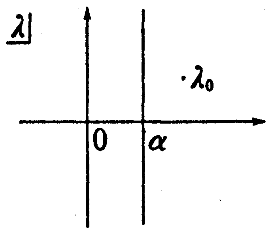

Figure 5

PROOF. By Theorem 2.1, we know that a solution $x(t)$ of problem (P), if it exists, is given by formula (2.2):

$$x(t)=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}R(\lambda)x_0 d\lambda, \qquad t>0. \tag{2.2}$$

i) First we show that:

If $x_0\in\mathcal{D}(A^2)$, *then formula* (2.2) *can be written in the form* (4.2).

Since every $x_0\in\mathcal{D}(A^2)$ can be written as

$$x_0=R(\lambda_0)^2 z_0 \qquad \text{with } z_0=(A-\lambda_0 I)^2 x_0,$$

it follows from the resolvent equation that

$$\begin{aligned}R(\lambda)x_0&=R(\lambda)R(\lambda_0)(R(\lambda_0)z_0)\\&=\frac{R(\lambda)-R(\lambda_0)}{\lambda-\lambda_0}R(\lambda_0)z_0\\&=-\frac{R(\lambda_0)^2 z_0}{\lambda-\lambda_0}+\frac{1}{\lambda-\lambda_0}\left(\frac{R(\lambda)-R(\lambda_0)}{\lambda-\lambda_0}\right)z_0\\&=\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}-\frac{R(\lambda_0)^2 z_0}{\lambda-\lambda_0}-\frac{R(\lambda_0)z_0}{(\lambda-\lambda_0)^2}.\end{aligned}$$

Thus we have for any $t>0$

$$\begin{aligned}x(t)&=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}R(\lambda)x_0 d\lambda\\&=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}d\lambda\end{aligned}$$

$$+\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t}\frac{R(\lambda_0)^2 z_0}{\lambda-\lambda_0}d\lambda$$

$$+\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t}\frac{R(\lambda_0)z_0}{(\lambda-\lambda_0)^2}d\lambda .$$

But, the second and third terms on the right-hand side vanish. In fact, since we have for $k=1, 2$

$$\frac{1}{|\lambda-\lambda_0|^k} \longrightarrow 0 \qquad \text{as } |\lambda|\to\infty ,$$

applying Cauchy's theorem and Jordan's lemma (cf. Appendix A, Remark after Lemma A), we obtain that for any $t>0$

$$\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t}\frac{1}{(\lambda-\lambda_0)^k}d\lambda = \lim_{r\to+\infty}\int_{C_r} e^{\lambda t}\frac{1}{(\lambda-\lambda_0)^k}d\lambda$$
$$=0, \qquad k=1, 2,$$

where the path C_r is a semicircle of radius r shown in the following figure:

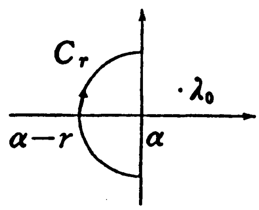

Figure 6

Therefore, we have the following formula for $x(t)$:

(4.2′) $$x(t)=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t}\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}d\lambda , \qquad t>0 ,$$

with

(4.2″) $$z_0=(A-\lambda_0 I)^2 x_0 .$$

ii) Next we show that:

The function $x(t)$, defined by formulas (4.2′) and (4.2″), belongs to $C^1([0, \infty); E)$, and satisfies

$$\begin{cases} x(t)\in\mathcal{D}(A), & t>0 , \\ Ax(t)=x'(t). \end{cases}$$

By inequality (4.1), we have for all $\operatorname{Re}\lambda=\alpha$

$$\left\| e^{\lambda t} \frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2} \right\| \leqq e^{\alpha t} \frac{\|R(\lambda)\|}{|\lambda-\lambda_0|^2} \|z_0\|$$

$$\leqq e^{\alpha t} \frac{M}{|\lambda-\lambda_0|^2 (1+\text{Im}\,\lambda|)^{\beta}} \|z_0\|$$

$$\leqq M_0 e^{\alpha t} \|z_0\| \frac{1}{(1+|\text{Im}\,\lambda|)^{\beta+2}},$$

where $M_0>0$ is a constant depending on M and λ_0. Thus, one can differentiate formula (4.2′) under the integral sign to obtain that

$$x'(t) = -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \lambda e^{\lambda t} \frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2} d\lambda, \qquad t \geqq 0.$$

Since $0<\beta\leqq 1$, this proves that

$$x(t) \in C^1([0, \infty); E).$$

Furthermore, if $t>0$, we have

$$x'(t) = -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \lambda e^{\lambda t} \frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2} d\lambda$$

$$= -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t} (AR(\lambda)z_0 - z_0) \frac{1}{(\lambda-\lambda_0)^2} d\lambda$$

$$= -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t} \frac{AR(\lambda)z_0}{(\lambda-\lambda_0)^2} d\lambda$$

$$+ \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t} \frac{z_0}{(\lambda-\lambda_0)^2} d\lambda.$$

But, applying once again Cauchy's theorem and Jordan's lemma, we obtain that

$$\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t} \frac{z_0}{(\lambda-\lambda_0)^2} d\lambda = 0.$$

Hence we have the following formula for $x'(t)$:

$$x'(t) = -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t} \frac{AR(\lambda)z_0}{(\lambda-\lambda_0)^2} d\lambda, \qquad t>0.$$

By the closedness of A, this implies that for any $t>0$

$$x(t) = -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t} \frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2} d\lambda \in \mathcal{D}(A),$$

and

$$Ax(t) = -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t} \frac{AR(\lambda)z_0}{(\lambda-\lambda_0)^2} d\lambda = x'(t).$$

iii) Finally we show that:

$$x(0)=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}d\lambda=x_0 .$$

Let D_r be a semicircle of radius r shown in the following figure:

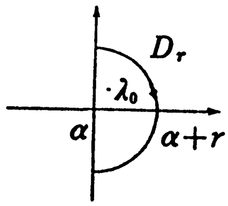

Figure 7

Then, by inequality (4.1), it follows that

$$\left\|\int_{D_r}\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}d\lambda\right\|\leq M_0'\|z_0\|\int_{-\pi/2}^{\pi/2}\frac{d\theta}{r^2}$$

$$=\frac{M_0'\pi\|z_0\|}{r^2}\longrightarrow 0 \qquad \text{as } r\rightarrow+\infty .$$

Thus, using the residue theorem, we obtain that

$$\begin{aligned} x(0)&=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}d\lambda \\ &=-\frac{1}{2\pi i}\lim_{r\to+\infty}\left\{\int_{\alpha-ir}^{\alpha+ir}\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}d\lambda+\int_{D_r}\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}d\lambda\right\} \\ &=\mathrm{Res}\left[\frac{R(\lambda)z_0}{(\lambda-\lambda_0)^2}\right]_{\lambda=\lambda_0} \\ &=\frac{d}{d\lambda}(R(\lambda)z_0)|_{\lambda=\lambda_0} \\ &=R(\lambda_0)^2z_0 \\ &=x_0 . \end{aligned}$$

The proof of Theorem 4.1 is now complete.

REMARK 4.2.. Furthermore, we can prove that:

If $x_0\in\mathcal{D}(A^k)$ *for some* $k\in N$, *then* $x(t)\in C^{k-1}([0, \infty); E)$.
This implies that if the initial datum x_0 is "smooth", so is the solution $x(t)$.

Combining Theorem 3.1 and Theorem 4.1, we can obtain the following existence and uniqueness theorem for problem (P) (cf. Kreĭn [2], Chapter I, Theorem 3.3):

THEOREM 4.3. *Assume that the operator* A *satisfies condition* (4.1). *Then* *the Cauchy problem (P) has a unique solution* $x(t)\in C([0, \infty); E)\cap C^{\infty}((0, \infty); E)$ *for any initial condition* $x_0\in\mathcal{D}(A)$.

PROOF. 1) The uniqueness for problem (P) is an immediate consequence of Theorem 3.1, since we have by inequality (4.1)

$$\limsup_{\sigma\to\infty}\frac{\log\|R(\sigma)\|}{\sigma}\leqq\lim_{\sigma\to\infty}\frac{\log M}{\sigma}=0.$$

2) Next we show that:

The Cauchy problem (P) has a solution

$$x(t)\in C([0, \infty); E)\cap C^{\infty}((0, \infty); E) \qquad \textit{for any } x_0\in\mathcal{D}(A).$$

(a) We let

$$x_1(t)=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}\frac{R(\lambda)((A-\lambda_0 I)x_0)}{(\lambda-\lambda_0)^2}d\lambda, \qquad x_0\in\mathcal{D}(A).$$

We remark that

$$(A-\lambda_0 I)x_0=(A-\lambda_0 I)^2(R(\lambda_0)x_0), \qquad R(\lambda_0)x_0\in\mathcal{D}(A^2).$$

Thus, arguing as in the proof of Theorem 4.1, we obtain that the function $x_1(t)$ belongs to $C^1([0, \infty); E)$, and is a solution of the Cauchy problem (P) with initial condition

$$x_1(0)=R(\lambda_0)x_0. \tag{4.3}$$

Here it is worth pointing out that the function $x_1(t)$ may be *formally* written as

$$x_1(t)=e^{At}R(\lambda_0)x_0.$$

Now we let

$$x(t)=(A-\lambda_0 I)x_1(t) \quad (=(A-\lambda_0 I)e^{At}R(\lambda_0)x_0). \tag{4.4}$$

Then we have for all $t\geqq 0$

$$\begin{aligned} x(t)&=Ax_1(t)-\lambda_0 x_1(t) \\ &=x_1'(t)-\lambda_0 x_1(t). \end{aligned} \tag{4.5}$$

This proves that

$$x(t)\in C([0, \infty); E),$$

since $x_1(t)\in C^1([0, \infty); E)$.

(b) Next we show that

$$x(t)\in C^{\infty}((0, \infty); E). \tag{4.6}$$

In view of formula (4.5), it suffices to show that

$$x_1(t)\in C^{\infty}((0, \infty); E). \tag{4.7}$$

To do so, we make use of the representation formula (2.2) for the solution $x_1(t)$:

$$x_1(t)=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}R(\lambda)x_1(0)d\lambda$$

$$=-\frac{1}{2\pi i}\lim_{N\to+\infty}\int_{\alpha-iN}^{\alpha+iN}e^{\lambda t}R(\lambda)x_1(0)d\lambda\,, \qquad x_1(0)=R(\lambda_0)x_0\,.$$

By integration by parts, it follows that

$$x_1(t)=-\frac{1}{2\pi i}\lim_{N\to+\infty}\left\{\left[\frac{e^{\lambda t}}{t}R(\lambda)x_1(0)\right]_{\alpha-iN}^{\alpha+iN}-\int_{\alpha-iN}^{\alpha+iN}\frac{e^{\lambda t}}{t}R(\lambda)^2x_1(0)d\lambda\right\}.$$

But, the first term on the right-hand side vanishes, for we have as $N\to+\infty$

$$\left\|\left[\frac{e^{\lambda t}}{t}R(\lambda)x_1(0)\right]_{\alpha-iN}^{\alpha+iN}\right\|\leqq\frac{e^{\lambda t}}{t}\frac{2M}{(1+N)^\beta}\|x_1(0)\|\longrightarrow 0\,.$$

Therefore, we have the following formula for $x_1(t)$:

$$x_1(t)=\frac{1}{2\pi i}\frac{(-1)^2}{t}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}R(\lambda)^2x_1(0)d\lambda\,, \qquad t>0\,.$$

Repeating this process, we have after n-steps

$$x_1(t)=\frac{1}{2\pi i}\frac{(-1)^n(n-1)!}{t^{n-1}}\int_{\alpha-i\infty}^{\alpha+i\infty}e^{\lambda t}R(\lambda)^nx_1(0)d\lambda\,, \qquad t>0\,. \tag{4.8}$$

Now let k be an arbitrary positive integer, and take a positive integer n such that

$$n>\frac{k+1}{\beta}\,.$$

Then one can differentiate formula (4.8) k-times with respect to t to obtain that

$$x_1^{(k)}(t)=\frac{(-1)^n(n-1)!}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{d^k}{dt^k}\left(\frac{e^{\lambda t}}{t^{n-1}}\right)R(\lambda)^nx_1(0)d\lambda \tag{4.9}$$

$$=\frac{(-1)^n(n-1)!}{2\pi i}\lim_{N\to+\infty}\int_{\alpha-iN}^{\alpha+iN}\frac{d^k}{dt^k}\left(\frac{e^{\lambda t}}{t^{n-1}}\right)R(\lambda)^nx_1(0)d\lambda\,.$$

In fact, by inequality (4.1), we have for all $\operatorname{Re}\lambda=\alpha$

$$\|\lambda^ke^{\lambda t}R(\lambda)^nx_1(0)\|\leqq M'e^{\alpha t}\|x_1(0)\|\frac{1}{(1+|\operatorname{Im}\lambda|)^{n\beta-k}}\,,$$

so that the integral in formula (4.9) converges absolutely for $t>0$, since $n\beta-k>1$.

Formula (4.9) proves assertion (4.7) and hence (4.6), since k is arbitrary.

(c) We show that for any $t>0$

$$\begin{cases} x(t)\in\mathcal{D}(A)\,,\\ Ax(t)=x'(t)\,.\end{cases}$$

Since we have for any $t>0$

$$(4.10)\qquad \begin{cases} x_1(t)\in\mathcal{D}(A), \\ Ax_1(t)=x_1'(t), \end{cases}$$

in view of assertion (4.7), it follows that

$$\lim_{h\to 0}\frac{x_1(t+h)-x_1(t)}{h}=x_1'(t),$$

and also

$$\lim_{h\to 0}A\left(\frac{x_1(t+h)-x_1(t)}{h}\right)=\lim_{h\to 0}\frac{x_1'(t+h)-x_1'(t)}{h}=x_1''(t).$$

By the closedness of A, this implies that for any $t>0$

$$(4.11)\qquad \begin{cases} x_1'(t)\in\mathcal{D}(A), \\ Ax_1'(t)=x_1''(t). \end{cases}$$

Therefore, we have by assertions (4.5), (4.10) and (4.11)

$$x(t)=x_1'(t)-\lambda_0 x_1(t)\in\mathcal{D}(A), \qquad t>0,$$

and

$$\begin{aligned} Ax(t)&=Ax_1'(t)-\lambda_0 Ax_1(t) \\ &=x_1''(t)-\lambda_0 x_1'(t) \\ &=x'(t). \end{aligned}$$

(d) Finally we have by formulas (4.4) and (4.3)

$$x(0)=(A-\lambda_0 I)x_1(0)=(A-\lambda_0 I)R(\lambda_0)x_0=x_0.$$

Summing up, we have proved that the function $x(t)$, defined by formula (4.4), belongs to $C([0,\infty);E)\cap C^\infty((0,\infty);E)$, and is a (unique) solution of problem (P).

The proof of Theorem 4.3 is now complete.

COROLLARY 4.4. *Assume that the operator A satisfies condition* (4.1). *Then the Cauchy problem* (P) *is well posed on* $[0,\infty)$.

PROOF. It remains to verify that:

(4.12) For each $t>0$, the solution $x(t)$, given by formula (4.4), depends continuously on the initial datum $x_0\in\mathcal{D}(A)$.

By Theorem 2.1, it follows that the solution $x(t)$ can also be written in the form (2.2):

$$(2.2)\qquad x(t)=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t}R(\lambda)x_0 d\lambda\,, \qquad t>0\,.$$

Thus, arguing as in the proof of Theorem 4.3, we obtain the following formulas (cf. formulas (4.8) and (4.9)):

$$(4.13)\qquad x(t)=\frac{1}{2\pi i}\frac{(-1)^n(n-1)!}{t^{n-1}}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t}R(\lambda)^n x_0 d\lambda\,, \qquad t>0\,,$$

$$(4.14)\qquad x^{(k)}(t)=\frac{(-1)^n(n-1)!}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{d^k}{dt^k}\Big(\frac{e^{\lambda t}}{t^{n-1}}\Big)R(\lambda)^n x_0 d\lambda\,, \qquad t>0\,.$$

Hence, by formula (4.13), we have for an integer $n>1/\beta$

$$(4.15)\qquad \|x(t)\|\leqq M'\frac{e^{\alpha t}}{t^{n-1}}\|x_0\|\,, \qquad t>0\,.$$

This proves assertion (4.12).

§ 5. The semigroup $U(t)$

Assume that the operator A satisfies condition (4.1):

$$(4.1)\qquad \|R(\lambda)\|\leqq\frac{M}{(1+|\mathrm{Im}\,\lambda|)^{\beta}}\,, \qquad \mathrm{Re}\,\lambda\geqq\alpha\,.$$

Then, by Theorems 4.3 and 2.1, we can define a linear operator

$$U(t)\colon E\longrightarrow E\,, \qquad t>0\,,$$

by the formula (cf. formula (2.2)):

$$(5.1)\qquad U(t)x_0=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{\lambda t}R(\lambda)x_0 d\lambda$$

$$=-\frac{1}{2\pi i}\lim_{N\to+\infty}\int_{\alpha-iN}^{\alpha+iN} e^{\lambda t}R(\lambda)x_0 d\lambda\,, \qquad x_0\in\mathcal{D}(A)\,.$$

We remark that

$$\lim_{t\downarrow 0} U(t)x_0=x_0\,.$$

Further, in view of Corollary 4.4 and Theorem 1.1, it follows that the operators $U(t)$ can be extended uniquely to bounded linear operators on E which form a strongly continuous semigroup, and satisfy the estimate

$$(5.2)\qquad \|U(t)\|\leqq M'\frac{e^{\alpha t}}{t^{n-1}}\,, \qquad n>1/\beta\,.$$

In fact, by inequality (4.15), we have

$$\|U(t)x_0\|\leqq M'\frac{e^{\alpha t}}{t^{n-1}}\|x_0\|\,, \qquad x_0\in\mathcal{D}(A)\,.$$

This proves estimate (5.2), since the domain $\mathcal{D}(A)$ is dense in E.

This section is devoted to the study of the semigroup $U(t)$. First we have the following:

THEOREM 5.1. *Assume that the operator A satisfies condition* (4.1). *Then the semigroup $U(t)$, defined by formula* (5.1), *is differentiable infinitely many times for $t>0$, that is, for every $x \in E$, the function $t \mapsto U(t)x$ is differentiable infinitely many times for $t>0$.*

PROOF. Theorem 4.3 tells us that

$$U(t)x_0 \in C([0, \infty); E) \cap C^{\infty}((0, \infty); E) \qquad \text{if } x_0 \in \mathcal{D}(A).$$

Hence we have for any integer $k \geqq 1$

$$U^{(k-1)}(t)x_0 = \int_{t_0}^{t} U^{(k)}(s)x_0 ds + U^{(k-1)}(t_0)x_0, \qquad 0<t_0<t. \tag{5.3}$$

But, in view of formula (4.14), we find that the derivative $U^{(k)}(t)$ also satisfies an estimate of the form (5.2). Thus one can pass to the limit in formula (5.3) to obtain that formula (5.3) remains valid for all x_0 of E. This proves the k-times differentiability of $U(t)$ for $t>0$. Thus the semigroup $U(t)$ is differentiable infinitely many times for $t>0$, since k is arbitrary.

REMARK 5.2. The infinitely many times differentiability of $U(t)$ for $t>0$ already follows from the differentiability of $U(t)$. More precisely, we have for any integer $k \geqq 1$

$$U^{(k)}(t) = \left(AU\left(\frac{t}{k}\right)\right)^k = \left(U'\left(\frac{t}{k}\right)\right)^k, \qquad t>0. \tag{5.4}$$

PROOF. If, for every $x \in E$, the function $t \mapsto U(t)x$ is differentiable for $t>0$, then it follows that

$$\begin{cases} U(t)x \in \mathcal{D}(A), & t>0; \\ AU(t)x = U'(t)x, \end{cases}$$

that is,

$$U'(t) = AU(t), \qquad t>0.$$

But, since $AU(t)$ is a closed linear operator defined on all of E, applying the closed graph theorem, we find that the operator $U'(t)=AU(t)$ is bounded for all $t>0$. Hence, by formula (1.4), we have for $0<s<t$

$$U'(t) = AU(t) = AU(t-s) \cdot U(s) = U(t-s) \cdot AU(s).$$

Differentiating this formula with respect to t, we obtain that

$$U^{(2)}(t)=U'(t-s)\cdot AU(s)=AU(t-s)\cdot AU(s).$$

Taking $s=t/2$ yields formula (5.4) for $k=2$:

$$U^{(2)}(t)=\left(AU\left(\frac{t}{2}\right)\right)^2=\left(U'\left(\frac{t}{2}\right)\right)^2, \qquad t>0.$$

Repeating this process, we have formula (5.4) for general $k\in \boldsymbol{N}$.

Now assume that the operator A satisfies a stronger condition than condition (4.1):

1) The resolvent set of A contains the region $\Sigma_\omega=\{\lambda\in \boldsymbol{C}\,;\,\lambda\neq 0,\ |\arg\lambda|<\pi/2+\omega\}$.
2) For each small $\varepsilon>0$, there exist constants $0<\theta\leq 1$ and $M_\varepsilon>0$ such that the resolvent $R(\lambda)=(A-\lambda I)^{-1}$ satisfies the estimate

(5.5) $$\|R(\lambda)\|\leq \frac{M_\varepsilon}{|\lambda|^\theta},\ \lambda\in\Sigma_\omega^\varepsilon=\{\lambda\in \boldsymbol{C}\,;\,\lambda\neq 0,\ |\arg\lambda|\leq\pi/2+\omega-\varepsilon\}.$$

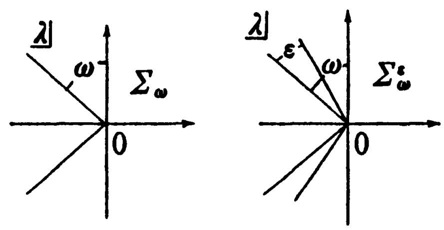

Figure 8

Then we let

(5.6) $$U(t)=-\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda)d\lambda.$$

Here Γ is a path in the set $\Sigma_\omega^\varepsilon$ consisting of the following three curves:

$$\Gamma^{(1)}=\{re^{-i(\pi/2+\omega-\varepsilon)}\,;\,1\leq r<\infty\},$$

$$\Gamma^{(2)}=\{e^{i\eta}\,;\,-(\pi/2+\omega-\varepsilon)\leq\eta\leq\pi/2+\omega-\varepsilon\},$$

$$\Gamma^{(3)}=\{re^{i(\pi/2+\omega-\varepsilon)}\,;\,1\leq r<\infty\}.$$

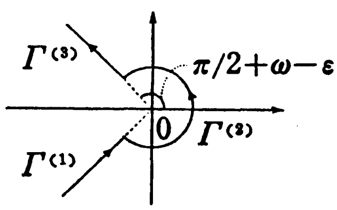

Figure 9

It is easy to see that the integral

$$U(t)=-\frac{1}{2\pi i}\sum_{k=1}^{3}\int_{\Gamma(k)}e^{\lambda t}R(\lambda)d\lambda$$

converges in the uniform operator topology for $t>0$, and thus defines a bounded linear operator on E. By Cauchy's theorem and Jordan's lemma, we find that the line of integration in formula (5.1) may be deformed into the path of integration Γ in formula (5.6); hence the two definitions (5.1) and (5.6) of $U(t)$ coincide (cf. the argument in the proof of formula (4.2′)).

The next theorem states that the semigroup $U(t)$ can be extended to an analytic semigroup in some sector containing the positive real axis, but may be unbounded at the origin:

THEOREM 5.3. *Assume that the operator A satisfies condition* (5.5). *Then the semigroup $U(t)$, defined by formula* (5.6), *can be extended to a semigroup $U(z)$ which is* analytic *in the sector $\Delta_\omega=\{z\in \boldsymbol{C}\,;\, z\neq 0,\ |\arg z|<\omega\}$, and enjoys the following properties:*

(a) *The operators $AU(z)$ and $(dU/dz)(z)$ are bounded operators on E for each $z\in\Delta_\omega$, and satisfy the relation*

$$\frac{dU}{dz}(z)=AU(z), \qquad z\in\Delta_\omega. \tag{5.7}$$

(b) *For each $0<\varepsilon<\omega/2$, there exist constants $\tilde{M}_0(\varepsilon)>0$ and $\tilde{M}_1(\varepsilon)>0$ such that*

$$\|U(z)\|\leqq\frac{\tilde{M}_0(\varepsilon)}{|z|^{1-\theta}}, \qquad z\in\Delta_\omega^{2\varepsilon}, \tag{5.8}$$

$$\|AU(z)\|\leqq\frac{\tilde{M}_1(\varepsilon)}{|z|^{2-\theta}}, \qquad z\in\Delta_\omega^{2\varepsilon}, \tag{5.9}$$

where

$$\Delta_\omega^{2\varepsilon}=\{z\in \boldsymbol{C}\,;\, z\neq 0,\ |\arg z|\leqq\omega-2\varepsilon\}.$$

(c) *For each $x_0\in\mathcal{D}(A)$, we have*

$$U(z)x_0 \longrightarrow x_0 \quad in\ E$$

as $z\to 0$, $z\in\Delta_\omega^{2\varepsilon}$ $(0<\varepsilon<\omega/2)$.

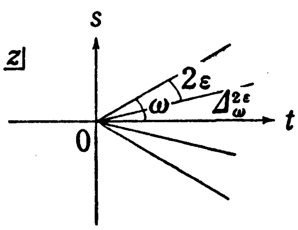

Figure 10

PROOF. (i) The analyticity of $U(z)$: If $\lambda\in\Gamma^{(3)}$ and $z\in\Delta_{\omega}^{2\varepsilon}$, that is, if

$$\begin{cases}\lambda=|\lambda|e^{i\eta}, & \eta=\pi/2+\omega-\varepsilon,\\ z=|z|e^{i\varphi}, & |\varphi|\leqq\omega-2\varepsilon,\end{cases}$$

then we have

$$\lambda z=|\lambda||z|e^{i(\eta+\varphi)},$$

with

$$\pi/2+\varepsilon\leqq\eta+\varphi\leqq\pi/2+2\omega-3\varepsilon<3\pi/2-3\varepsilon.$$

Note that

$$\cos(\eta+\varphi)\leqq\cos(\pi/2+\varepsilon)=-\sin\varepsilon.$$

Hence it follows that

$$|e^{\lambda z}|\leqq e^{-|\lambda||z|\sin\varepsilon}, \qquad \lambda\in\Gamma^{(3)},\ z\in\Delta_{\omega}^{2\varepsilon}. \tag{5.10}$$

Similarly, we have

$$|e^{\lambda z}|\leqq e^{-|\lambda||z|\sin\varepsilon}, \qquad \lambda\in\Gamma^{(1)},\ z\in\Delta_{\omega}^{2\varepsilon}. \tag{5.11}$$

Now, for each small $\varepsilon>0$, we let

$$\begin{aligned}K_{\omega}^{\varepsilon}&=\Delta_{\omega}^{2\varepsilon}\cap\{z\in\boldsymbol{C};\ |z|\geqq\varepsilon\}\\ &=\{z\in\boldsymbol{C};\ |z|\geqq\varepsilon,\ |\arg z|\leqq\omega-2\varepsilon\}.\end{aligned}$$

Then, combining estimates (5.5), (5.10) and (5.11), we obtain that

$$\|e^{\lambda z}R(\lambda)\|\leqq\frac{M_{\varepsilon}}{|\lambda|^{\theta}}e^{-\varepsilon\sin\varepsilon\cdot|\lambda|}, \qquad \lambda\in\Gamma^{(1)}\cup\Gamma^{(3)},\ z\in K_{\omega}^{\varepsilon}. \tag{5.12}$$

On the other hand, we have

$$\|e^{\lambda z}R(\lambda)\|\leqq M_{\varepsilon}e^{|z|}, \qquad \lambda\in\Gamma^{(2)},\ z\in K_{\omega}^{\varepsilon}. \tag{5.13}$$

Therefore, we find that the integral

$$U(z)=-\frac{1}{2\pi i}\int_{\Gamma}e^{\lambda z}R(\lambda)d\lambda=-\frac{1}{2\pi i}\sum_{k=1}^{3}\int_{\Gamma^{(k)}}e^{\lambda z}R(\lambda)d\lambda \tag{5.6'}$$

converges in the uniform operator topology, uniformly in $z\in K_{\omega}^{\varepsilon}$, for every $\varepsilon>0$. This proves that the operator $U(z)$ is analytic in the domain $\Delta_{\varepsilon}=\bigcup_{\varepsilon>0}K_{\omega}^{\varepsilon}$.

By the analyticity of $U(z)$, it follows that the operators $U(z)$ also enjoy the semigroup property:

$$U(z+w)=U(z)\cdot U(w), \qquad z,\ w\in\Delta_{\omega}.$$

(ii) We prove that the operators $U(z)$ enjoy properties (a) and (b).

(b) First, using Cauchy's theorem, we obtain that

$$U(z)=-\frac{1}{2\pi i}\int_{\Gamma}e^{\lambda z}R(\lambda)d\lambda$$
$$=-\frac{1}{2\pi i}\int_{\Gamma_{|z|}}e^{\lambda z}R(\lambda)d\lambda,$$

where $\Gamma_{|z|}$ is a path consisting of the following three curves:

$$\Gamma_{|z|}^{(1)}=\left\{re^{-i(\pi/2+\omega-\varepsilon)};\ \frac{1}{|z|}\leqq r<\infty\right\},$$
$$\Gamma_{|z|}^{(2)}=\left\{\frac{1}{|z|}e^{i\eta};\ -(\pi/2+\omega-\varepsilon)\leqq\eta\leqq\pi/2+\omega-\varepsilon\right\},$$
$$\Gamma_{|z|}^{(3)}=\left\{re^{i(\pi/2+\omega-\varepsilon)};\ \frac{1}{|z|}\leqq r<\infty\right\}.$$

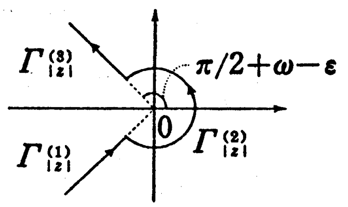

Figure 11

But, by estimates (5.5), (5.10) and (5.11), it follows that

$$\|e^{\lambda z}R(\lambda)\|\leqq\frac{M_\varepsilon}{|\lambda|^\theta}e^{-|\lambda||z|\sin\varepsilon},\qquad \lambda\in\Gamma_{|z|}^{(1)}\cup\Gamma_{|z|}^{(3)},\ z\in\Delta_\omega^{2\varepsilon}.$$

Hence we have for $k=1, 3$

$$\int_{\Gamma_{|z|}^{(k)}}\|e^{\lambda z}R(\lambda)\|\,|d\lambda|\leqq M_\varepsilon\int_{\frac{1}{|z|}}^{\infty}e^{-\rho|z|\sin\varepsilon}\rho^{-\theta}d\rho$$
$$=M_\varepsilon\int_1^\infty e^{-\sin\varepsilon\cdot s}s^{-\theta}ds\cdot|z|^{\theta-1}.$$

We have also for $k=2$

$$\int_{\Gamma_{|z|}^{(2)}}\|e^{\lambda z}R(\lambda)\|\,|d\lambda|\leqq M_\varepsilon\int_{-(\pi/2+\omega-\varepsilon)}^{\pi/2+\omega-\varepsilon}e\,d\eta\cdot|z|^{\theta-1}$$
$$=2eM_\varepsilon(\pi/2+\omega-\varepsilon)|z|^{\theta-1}$$
$$\leqq2\pi eM_\varepsilon|z|^{\theta-1}.$$

Summing up, we obtain the following estimate:

$$\|U(z)\|\leqq\frac{1}{2\pi}\sum_{k=1}^{3}\int_{\Gamma_{|z|}^{(k)}}\|e^{\lambda z}R(\lambda)\|\,|d\lambda|$$

$$\leqq \frac{1}{2\pi}\Big(2M_\varepsilon\int_1^\infty s^{-\theta}e^{-\sin\varepsilon\cdot s}ds+2\pi e M_\varepsilon\Big)|z|^{\theta-1}$$

$$=\frac{M_\varepsilon}{\pi}\Big(\int_1^\infty s^{-\theta}e^{-\sin\varepsilon\cdot s}ds+\pi e\Big)|z|^{\theta-1}.$$

This proves estimate (5.8), with

$$\tilde{M}_0(\varepsilon)=\frac{M_\varepsilon}{\pi}\Big(\int_1^\infty s^{-\theta}e^{-\sin\varepsilon\cdot s}ds+\pi e\Big).$$

To prove estimate (5.9), note that

$$AR(\lambda)=(A-\lambda I+\lambda I)R(\lambda)=I+\lambda R(\lambda),$$

so that

$$\|AR(\lambda)\|\leqq 1+M_\varepsilon|\lambda|^{1-\theta}, \qquad \lambda\in\Sigma_\omega^\varepsilon.$$

Hence, arguing as in the proof of estimate (5.8), we obtain that

$$\Big\|\int_\Gamma e^{\lambda z}AR(\lambda)d\lambda\Big\|\leqq 2\int_{1/|z|}^\infty e^{-\rho|z|\sin\varepsilon}(1+M_\varepsilon\rho^{1-\theta})d\rho \tag{5.14}$$

$$+\int_{-(\pi/2+\omega-\varepsilon)}^{\pi/2+\omega-\varepsilon}(1+M_\varepsilon|z|^{\theta-1})e\frac{d\eta}{|z|}$$

$$\leqq 2\Big(\int_1^\infty e^{-\sin\varepsilon\cdot s}ds+\pi e\Big)|z|^{-1}$$

$$+2M_\varepsilon\Big(\int_1^\infty s^{1-\theta}e^{-\sin\varepsilon\cdot s}ds+\pi e\Big)|z|^{\theta-2}.$$

This proves that the integral $\int_\Gamma e^{\lambda z}AR(\lambda)d\lambda$ is convergent for every $z\in\Delta_\omega^{2\varepsilon}$. By the closedness of A, this implies that for any $z\in\Delta_\omega^{2\varepsilon}$

$$U(z)\in\mathcal{D}(A),$$

and

$$AU(z)=-\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}AR(\lambda)d\lambda. \tag{5.15}$$

Therefore, estimate (5.9) follows from estimate (5.14), with

$$\tilde{M}_1(\varepsilon)=\frac{1}{\pi}\Big(\int_1^\infty e^{-\sin\varepsilon\cdot s}ds+\pi e\Big)+\frac{M_\varepsilon}{\pi}\Big(\int_1^\infty s^{1-\theta}e^{-\sin\varepsilon\cdot s}ds+\pi e\Big).$$

We remark that formula (5.15) remains valid for all $z\in\Delta_\omega$, since $\Delta_\omega=\bigcup_{\varepsilon>0}\Delta_\omega^{2\varepsilon}$.

(a) By estimates (5.12) and (5.13), one can differentiate formula (5.6′) under the integral sign to obtain that

$$\frac{dU}{dz}(z)=-\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}\lambda R(\lambda)d\lambda, \qquad z\in\Delta_\omega. \tag{5.16}$$

On the other hand, it follows from formula (5.15) that

$$
\begin{aligned}
(5.17)\qquad AU(z) &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z} AR(\lambda)d\lambda \\
&= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}(I+\lambda R(\lambda))d\lambda \\
&= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}\lambda R(\lambda)d\lambda\,, \qquad z\in\Delta_\omega\,,
\end{aligned}
$$

for we have by Cauchy's theorem

$$\int_\Gamma e^{\lambda z}d\lambda=0\,.$$

Therefore, formula (5.7) follows immediately from formulas (5.16) and (5.17).

(c) Let x_0 be an arbitrary element of $\mathcal{D}(A)$. By the residue theorem, it follows that

$$x_0=\frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda z}}{\lambda}x_0 d\lambda\,,$$

so that

$$
\begin{aligned}
U(z)x_0-x_0 &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}\Big(R(\lambda)+\frac{1}{\lambda}\Big)x_0 d\lambda \\
&= -\frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda z}}{\lambda}R(\lambda)Ax_0 d\lambda\,.
\end{aligned}
$$

Here we remark that

$$\frac{1}{|\lambda|}\|R(\lambda)\|\leqq \frac{M_\varepsilon}{|\lambda|^{1+\theta}}\,,\qquad \lambda\in\Gamma\,.$$

$$|e^{\lambda z}|\leqq 2e^{-|\lambda||z|\sin\varepsilon}+e^{|z|}\,,\qquad z\in\Delta_\omega^{2\varepsilon},\ \lambda\in\Gamma\,.$$

Thus it follows from an application of Lebesgue's dominated convergence theorem that as $z\to 0$, $z\in\Delta_\omega^{2\varepsilon}$

$$U(z)x_0-x_0\longrightarrow -\frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda}R(\lambda)Ax_0 d\lambda\,.$$

But we have

$$\int_\Gamma \frac{1}{\lambda}R(\lambda)Ax_0 d\lambda=0\,.$$

In fact, by Cauchy's theorem, it follows that

$$
\begin{aligned}
\int_\Gamma \frac{1}{\lambda}R(\lambda)Ax_0 d\lambda &= \lim_{r\to+\infty}\int_{\Gamma\cap\{|\lambda|\leq r\}}\frac{1}{\lambda}R(\lambda)Ax_0 d\lambda \\
&= \lim_{r\to+\infty}\int_{C_r}\frac{1}{\lambda}R(\lambda)Ax_0 d\lambda \\
&= 0\,,
\end{aligned}
$$

where C_r is a closed path shown in the following figure:

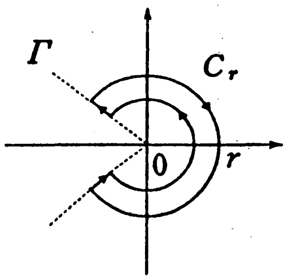

Figure 12

Summing up, we have proved that

$$U(z)x_0 \longrightarrow x_0 \quad \text{as } z\to 0,\ z\in\Delta_\omega^{2\varepsilon},$$

for each $z_0\in\mathcal{D}(A)$.

REMARK 5.4. Assume that the operator A satisfies a stronger condition than condition (5.5):

$$\|R(\lambda)\|\leqq\frac{M_\varepsilon}{(|\lambda|+1)^\theta}, \qquad \lambda\in\Sigma_\omega^\varepsilon. \tag{5.5'}$$

Then we have the following estimates:

$$\|U(z)\|\leqq\frac{\tilde{M}_0(\varepsilon)}{|z|^{1-\theta}}e^{-a\cdot\mathrm{Re}\,z}, \qquad z\in\Delta_\omega^{2\varepsilon}, \tag{5.8'}$$

$$\|AU(z)\|\leqq\frac{\tilde{M}_1(\varepsilon)}{|z|^{2-\theta}}e^{-a\cdot\mathrm{Re}\,z}, \qquad z\in\Delta_\omega^{2\varepsilon}, \tag{5.9'}$$

with some constant $a>0$.

PROOF. Take a real number a such that

$$0<a<\frac{1}{M_\varepsilon}.$$

Then we have by estimate (5.5')

$$a\|(A-\lambda I)^{-1}\|\leqq\frac{aM_\varepsilon}{(|\lambda|+1)^\theta}\leqq aM_\varepsilon<1, \qquad \lambda\in\Sigma_\omega^\varepsilon.$$

Hence it follows that the operator $(A+aI)-\lambda I$ has the inverse

$$((A+aI)-\lambda I)^{-1}=(I+a(A-\lambda I)^{-1})^{-1}(A-\lambda I)^{-1},$$

and

$$\|((A+aI)-\lambda I)^{-1}\|\leqq\|(I+a(A-\lambda I)^{-1}\|\cdot\|(A-\lambda I)^{-1}\|$$

$$\leqq\frac{M_\varepsilon}{(|\lambda|+1)^\theta}\frac{1}{1-a\|(A-\lambda I)^{-1}\|}$$

$$\leq \frac{M_\varepsilon}{(|\lambda|+1)^\theta} \frac{1}{1-aM_\varepsilon}$$

$$\leq \left(\frac{M_\varepsilon}{1-aM_\varepsilon}\right) \frac{1}{|\lambda|^\theta}.$$

This proves that the operator $A+aI$ satisfies condition (5.5), so that estimates (5.8) and (5.9) remain valid for the operator $A+aI$:

(5.18) $$\|V(z)\| \leq \frac{\tilde{M}_0'(\varepsilon)}{|z|^{1-\theta}}, \qquad z \in \Delta_\omega^{2\varepsilon},$$

(5.19) $$\|(A+aI)V(z)\| \leq \frac{\tilde{M}_1'(\varepsilon)}{|z|^{2-\theta}}, \qquad z \in \Delta_\omega^{2\varepsilon},$$

where

$$V(z) = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}(A+aI-\lambda I)^{-1} d\lambda.$$

But, we have by Cauchy's theorem

(5.20) $$V(z) = -\frac{1}{2\pi i}\int_{\Gamma+a} e^{\lambda z}(A+aI-\lambda I)^{-1} d\lambda$$

$$= -\frac{1}{2\pi i}\int_\Gamma e^{\mu z} e^{az}(A-\mu I)^{-1} d\mu$$

$$= e^{az} U(z).$$

In view of formula (5.20), the desired estimates (5.8′) and (5.9′) follow from estimates (5.18) and (5.19).

§ 6. The fractional powers $(-A)^\alpha$

Assume that the operator A satisfies a stronger condition than condition (5.5):

1) The resolvent set of A contains the following region Σ:

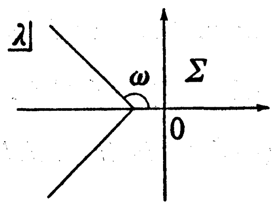

Figure 13

2) There exist constants $0 < \theta \leq 1$ and $M > 0$ such that the resolvent $R(\lambda) = (A-\lambda I)^{-1}$ satisfies the estimate

$$\|R(\lambda)\| \leq \frac{M}{(1+|\lambda|)^{\theta}}, \qquad \lambda \in \Sigma. \tag{6.1}$$

If $\alpha > 1-\theta$, we define the fractional power $(-A)^{-\alpha}$ of $-A$ by the following formula:

$$(-A)^{-\alpha} = -\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{-\alpha}R(\lambda)d\lambda. \tag{6.2}$$

Here the path Γ runs in the set Σ from $b+\infty e^{-i\omega}$ to $b+\infty e^{i\omega}$, avoiding the positive real axis and the origin (cf. Figure 14), and for the function $(-\lambda)^{-\alpha} = e^{-\alpha \log(-\lambda)}$, we choose the branch whose argument lies between $-\alpha\pi$ and $\alpha\pi$; it is analytic in the region obtained by omitting the positive real axis.

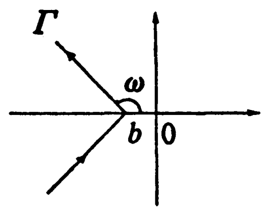

Figure 14

It is easy to see that the integral (6.2) converges in the uniform operator topology for $\alpha+\theta>1$, and thus defines a bounded linear operator on E. In fact, it suffices to note the following:

$$\begin{cases} |(-\lambda)^{-\alpha}| = |e^{-\alpha \log(-\lambda)}| = e^{-\alpha \log|\lambda|} = |\lambda|^{-\alpha}, \\ \|R(\lambda)\| \leq \dfrac{M}{(1+|\lambda|)^{\theta}}. \end{cases}$$

Some basic properties of $(-A)^{-\alpha}$ are summarized in the following:

PROPOSITION 6.1. (i) *We have for all* $\alpha, \beta > 1-\theta$

$$(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-(\alpha+\beta)}.$$

(ii) *If* α *is a positive integer* n, *then we have*

$$(-A)^{-\alpha} = ((-A)^{-1})^{n}.$$

(iii) *The fractional power* $(-A)^{-\alpha}$ *is invertible for all* $\alpha > 1-\theta$.

PROOF. (i) By Cauchy's theorem, one may suppose that

$$(-A)^{-\beta} = -\frac{1}{2\pi i}\int_{\Gamma'}(-\mu)^{-\beta}R(\mu)d\mu, \tag{6.2'}$$

where Γ' is a path obtained from Γ by translating each point of Γ to the right by a fixed small positive distance.

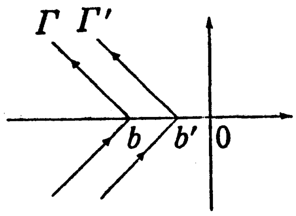

Figure 15

Then we have by Fubini's theorem

$$(-A)^{-\alpha}(-A)^{-\beta}=\frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'}(-\lambda)^{-\alpha}(-\mu)^{-\beta}R(\lambda)R(\mu)d\lambda d\mu$$

$$=\frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'}(-\lambda)^{-\alpha}(-\mu)^{-\beta}\frac{R(\lambda)-R(\mu)}{\lambda-\mu}d\lambda d\mu$$

$$=\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{-\alpha}R(\lambda)\left[\frac{1}{2\pi i}\int_{\Gamma'}\frac{(-\mu)^{-\beta}}{\lambda-\mu}d\mu\right]d\lambda$$

$$-\frac{1}{2\pi i}\int_{\Gamma'}(-\mu)^{-\beta}R(\mu)\left[\frac{1}{2\pi i}\int_{\Gamma}\frac{(-\lambda)^{-\alpha}}{\lambda-\mu}d\lambda\right]d\mu .$$

We calculate each term on the right-hand side.

a) We let

$$f(\mu)=\frac{(-\mu)^{-\beta}}{\lambda-\mu} .$$

Then, applying the residue theorem, we obtain that (cf. Figure 16)

$$\int_0^R f(b'+re^{i\omega})e^{i\omega}dr+\int_{\omega}^{2\pi-\omega}f(b'+Re^{i\eta})Rie^{i\eta}d\eta$$

$$+\int_R^0 f(b'+re^{i(2\pi-\omega)})e^{i(2\pi-\omega)}dr$$

$$=2\pi i\,\mathrm{Res}\,[f(\mu)]_{\mu=\lambda}$$

$$=-2\pi i(-\lambda)^{-\beta} .$$

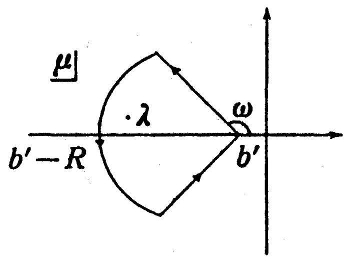

Figure 16

But we have as $R\to+\infty$

$$\int_0^R f(b'+re^{i\omega})e^{i\omega}dr \longrightarrow \int_0^{+\infty} f(b'+re^{i\omega})e^{i\omega}dr,$$

$$\int_R^0 f(b'+re^{i(2\pi-\omega)})e^{i(2\pi-\omega)}dr \longrightarrow \int_{+\infty}^0 f(b'+re^{i(2\pi-\omega)})e^{i(2\pi-\omega)}dr,$$

and

$$\left|\int_\omega^{2\pi-\omega} f(b'+Re^{i\eta})Rie^{i\eta}d\eta\right| \leqq \frac{1}{R^\beta}\int_\omega^{2\pi-\omega}\frac{d\eta}{\left|\frac{\lambda-b'}{R}-e^{i\eta}\right|} \longrightarrow 0.$$

Therefore, we find that

$$\frac{1}{2\pi i}\int_{\Gamma'}\frac{(-\mu)^{-\beta}}{\lambda-\mu}d\mu=-(-\lambda)^{-\beta}.$$

b) Similarly, since Γ lies to the left of Γ', we find that

$$\frac{1}{2\pi i}\int_\Gamma\frac{(-\lambda)^{-\alpha}}{\lambda-\mu}d\lambda=0.$$

Summing up, we obtain that

$$\begin{aligned}(-A)^{-\alpha}(-A)^{-\beta}&=-\frac{1}{2\pi i}\int_\Gamma(-\lambda)^{-(\alpha+\beta)}R(\lambda)d\lambda\\&=(-A)^{-(\alpha+\beta)}.\end{aligned}$$

(ii) Since we have by inequality (6.1)

$$\lim_{r\to+\infty}\int_{-\omega}^{\omega}(-re^{i\eta})^{-n}R(re^{i\eta})ire^{i\eta}d\eta=0 \qquad \text{for any integer } n\geqq 1,$$

it follows that

$$\begin{aligned}(-A)^{-n}&=-\frac{1}{2\pi i}\int_\Gamma(-\lambda)^{-n}R(\lambda)d\lambda\\&=\frac{1}{2\pi i}\lim_{r\to+\infty}\int_{C_r}(-\lambda)^{-n}R(\lambda)d\lambda,\end{aligned}$$

where C_r is a closed path shown in the following figure:

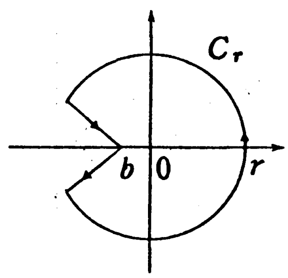

Figure 17

Thus, by the residue theorem, we obtain that

$$
\begin{aligned}
(-A)^{-n} &= \operatorname{Res}\,[(-\lambda)^{-n}R(\lambda)]_{\lambda=0} \\
&= \frac{(-1)^n}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}((A-\lambda I)^{-1})|_{\lambda=0} \\
&= (-1)^n (A^{-1})^n \\
&= ((-A)^{-1})^n .
\end{aligned}
$$

(iii) Since the operator $(-A)^{-1}$ is injective, it follows that $(-A)^{-n}=((-A)^{-1})^n$ is injective for all integer $n \geqq 1$. Assume that $(-A)^{-\alpha}x=0$. Then, taking an integer n such that $n-\alpha>1-\theta$, we obtain that

$$(-A)^{-n}x=(-A)^{-(n-\alpha)}((-A)^{-\alpha}x)=0\,,$$

so that

$$x=0\,.$$

This proves part (iii).

The proof of Proposition 6.1 is complete.

If $1-\theta<\alpha<1$, we have the following useful expression for the fractional power $(-A)^{-\alpha}$:

THEOREM 6.2. *We have for* $1-\theta<\alpha<1$

$$(-A)^{-\alpha}=-\frac{\sin\alpha\pi}{\pi}\int_0^\infty s^{-\alpha}R(s)ds\,. \tag{6.3}$$

PROOF. By Cauchy's theorem, one may deform the path Γ in formula (6.2) into the upper and lower sides of the positive real axis. But, we have

$$(-\lambda)^{-\alpha}=e^{-\alpha\log(-\lambda)}=\begin{cases} |\lambda|^{-\alpha}e^{\alpha\pi i} & \text{if } \operatorname{Im}\lambda>0\,, \\ |\lambda|^{-\alpha}e^{-\alpha\pi i} & \text{if } \operatorname{Im}\lambda<0\,. \end{cases}$$

Hence it follows that

$$
\begin{aligned}
(-A)^{-\alpha} &= -\frac{1}{2\pi i}\int_0^\infty s^{-\alpha}e^{\alpha\pi i}R(s)ds-\frac{1}{2\pi i}\int_\infty^0 s^{-\alpha}e^{-\alpha\pi i}R(s)ds \\
&= -\frac{1}{\pi}\frac{e^{\alpha\pi i}-e^{-\alpha\pi i}}{2i}\int_0^\infty s^{-\alpha}R(s)ds \\
&= -\frac{\sin\alpha\pi}{\pi}\int_0^\infty s^{-\alpha}R(s)ds\,.
\end{aligned}
$$

COROLLARY 6.3. *We have for* $1-\theta<\alpha<1$

$$\|(-A)^{-\alpha}\| \leqq M\frac{\Gamma(\alpha+\theta-1)}{\Gamma(\alpha)\Gamma(\theta)}\,. \tag{6.4}$$

PROOF. By formula (6.3), it follows that

$$\|(-A)^{-\alpha}\| \leqq \frac{\sin \alpha\pi}{\pi}\int_0^\infty s^{-\alpha}\|R(s)\|ds$$
$$\leqq M\frac{\sin \alpha\pi}{\pi}\int_0^\infty s^{-\alpha}(1+s)^{-\theta}ds\,.$$

But we have

$$\int_0^\infty s^{-\alpha}(1+s)^{-\theta}ds=\int_0^1\left(\frac{\sigma}{1-\sigma}\right)^{-\alpha}(1-\sigma)^{\theta}\frac{d\sigma}{(1-\sigma)^2}$$
$$=\int_0^1 \sigma^{-\alpha}(1-\sigma)^{\alpha+\theta-2}d\sigma$$
$$=B(1-\alpha,\ \alpha+\theta-1)$$
$$=\frac{\Gamma(1-\alpha)\Gamma(\alpha+\theta-1)}{\Gamma(\theta)},$$

and also

$$\frac{\sin \alpha\pi}{\pi}=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\,.$$

Summing up, we obtain that

$$\|(-A)^{-\alpha}\|\leqq M\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)\Gamma(\alpha+\theta-1)}{\Gamma(\theta)}=M\frac{\Gamma(\alpha+\theta-1)}{\Gamma(\alpha)\Gamma(\theta)}\,.$$

REMARK 6.4. Estimate (6.1) with $\lambda=0$ tells us that estimate (6.4) remains valid for $\alpha=1$: $\|(-A)^{-1}\|\leqq M$.

In view of part (iii) of Proposition 6.1, we can define the fractional power $(-A)^{\alpha}$ for $\alpha>1-\theta$ as follows:

$$(-A)^{\alpha}=\text{the inverse of } (-A)^{-\alpha}, \qquad \alpha>1-\theta\,.$$

The next theorem states that the domain $\mathcal{D}((-A)^{\alpha})$ of $(-A)^{\alpha}$ is bigger than the domain $\mathcal{D}(A)$ of A when $1-\theta<\alpha<\theta$.

THEOREM 6.5. *We have for any* $1-\theta<\alpha<\theta$

$$\mathcal{D}(A)\subset\mathcal{D}((-A)^{\alpha})\,.$$

PROOF. Let x be an arbitrary element of $\mathcal{D}(A)$. Then there exists a unique element $y\in E$ such that

$$x=(-A)^{-1}y\,.$$

But, if $1-\theta<\alpha<\theta$, one can define the fractional powers $(-A)^{-\alpha}$ and $(-A)^{-(1-\alpha)}$, and write $(-A)^{-1}$ as follows:

$$(-A)^{-1}=(-A)^{-\alpha}(-A)^{-(1-\alpha)}.$$

Hence we have

$$x=(-A)^{-1}y=(-A)^{-\alpha}((-A)^{-(1-\alpha)}y).$$

This proves that

$$x\in\mathcal{D}((-A)^{\alpha}).$$

§ 7. Homogenous solutions

Assume that the operator A satisfies condition (6.1):

$$\|R(\lambda)\|\leqq\frac{M}{(1+|\lambda|)^{\theta}},\qquad \lambda\in\Sigma. \tag{6.1}$$

In this section we characterize admissible initial data x_0 for the Cauchy problem (P) in terms of the domains of the fractional powers $(-A)^{\alpha}$.

The next theorem states that problem (P) has a unique solution $x(t)$ for any initial condition $x_0\in\mathcal{D}((-A)^{\eta})$ with $1-\theta<\eta<\theta$.

THEOREM 7.1. *Assume that the operator A satisfies condition* (6.1) *with* $0<\theta\leqq 1$. *Then, for every $x_0\in\mathcal{D}((-A)^{\eta})$ with $1-\theta<\eta<\theta$, the function*

$$x(t)=U(t)x_0=-\frac{1}{2\pi i}\int_{\Gamma}e^{\lambda t}R(\lambda)x_0 d\lambda,\qquad t>0, \tag{7.1}$$

belongs to $C([0,\infty);E)\cap C^{\infty}((0,\infty);E)$, *and is a unique solution of the Cauchy problem*:

$$(P)\qquad \begin{cases} \dfrac{dx}{dt}=Ax(t), & t>0,\\ x(0)=x_0. \end{cases}$$

REMARK 7.2. By Theorem 6.5, it follows that the domain $\mathcal{D}((-A)^{\eta})$ is bigger than the domain $\mathcal{D}(A)$ when $1-\theta<\eta<\theta$. Hence Theorem 7.1 is a generalization of Theorem 4.3.

PROOF OF THEOREM 7.1. 1) First we show that:

$$\|U(t)(-A)^{-\alpha}\|\leqq\frac{M}{\pi}\frac{\Gamma(1-\alpha)\Gamma(\alpha+\theta-1)}{\Gamma(\theta)},\qquad 1-\theta<\alpha<1. \tag{7.2}$$

By formulas (5.6) and (6.2′), it follows that

$$\begin{aligned} U(t)(-A)^{-\alpha}&=(-A)^{-\alpha}U(t)\\ &=\frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'}e^{\lambda t}(-\mu)^{-\alpha}R(\lambda)R(\mu)d\lambda d\mu\\ &=\frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'}e^{\lambda t}(-\mu)^{-\alpha}\frac{R(\lambda)-R(\mu)}{\lambda-\mu}d\lambda d\mu \end{aligned}$$

$$=\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}R(\lambda)\left[\frac{1}{2\pi i}\int_{\Gamma'}\frac{(-\mu)^{-\alpha}}{\lambda-\mu}d\mu\right]d\lambda$$

$$-\frac{1}{2\pi i}\int_{\Gamma'}(-\mu)^{-\alpha}R(\mu)\left[\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{\lambda t}}{\lambda-\mu}d\lambda\right]d\mu$$

$$=-\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{-\alpha}e^{\lambda t}R(\lambda)d\lambda .$$

In fact, it suffices to note the following (cf. the proof of Proposition 6.1):

$$\frac{1}{2\pi i}\int_{\Gamma'}\frac{(-\mu)^{-\alpha}}{\lambda-\mu}d\mu=-(-\lambda)^{-\alpha},$$

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{\lambda t}}{\lambda-\mu}d\lambda=0 .$$

Therefore we obtain that

$$\|U(t)(-A)^{-\alpha}\|\leqq\frac{M}{2\pi}\int_{\Gamma}e^{bt}\frac{d|\lambda|}{|\lambda|^{\alpha}(1+|\lambda|)^{\theta}}$$

$$\leqq\frac{M}{\pi}\int_0^{\infty}s^{-\alpha}(1+s)^{-\theta}ds$$

$$=\frac{M}{\pi}\frac{\Gamma(1-\alpha)\Gamma(\alpha+\theta-1)}{\Gamma(\theta)} .$$

2) Next we show that:

(7.3) $$\lim_{t\downarrow 0} U(t)(-A)^{-\alpha}x=(-A)^{-\alpha}x \qquad \text{for each } x\in E.$$

Let x_0 be an arbitrary element of $\mathcal{D}(A)$. Then we have

$$U(t)x_0 \longrightarrow x_0 \qquad \text{as } t\downarrow 0,$$

so that

(7.4) $$U(t)((-A)^{-\alpha}x_0)=(-A)^{-\alpha}(U(t)x_0)\longrightarrow(-A)^{-\alpha}x_0 \qquad \text{as } t\downarrow 0 .$$

Now take an arbitrary element x of E. For each $\varepsilon>0$, one can find an element $x_0\in\mathcal{D}(A)$ such that

$$\|x-x_0\|<\varepsilon .$$

Then we have by inequality (7.2)

$$\|U(t)(-A)^{-\alpha}x-(-A)^{-\alpha}x\|$$

$$\leqq\|U(t)(-A)^{-\alpha}(x-x_0)\|+\|U(t)(-A)^{-\alpha}x_0-(-A)^{-\alpha}x_0\|+\|(-A)^{-\alpha}(x_0-x)\|$$

$$\leqq\left(\frac{M}{\pi}\frac{\Gamma(1-\alpha)\Gamma(\alpha+\theta-1)}{\Gamma(\theta)}+\|(-A)^{-\alpha}\|\right)\varepsilon+\|U(t)(-A)^{-\alpha}x_0-(-A)^{-\alpha}x_0\| .$$

Hence, by assertion (7.4), it follows that

$$\limsup_{t\downarrow 0} \|U(t)(-A)^{-\alpha}x-(-A)^{-\alpha}x\| \leqq \Big(\frac{M}{\pi}\frac{\Gamma(1-\alpha)\Gamma(\alpha+\theta-1)}{\Gamma(\theta)}+\|(-A)^{-\alpha}\|\Big)\varepsilon\,.$$

This proves assertion (7.3), since $\varepsilon>0$ is arbitrary.

3) Now let x_0 be an arbitrary element of $\mathcal{D}((-A)^\eta)$ with $1-\theta<\eta<\theta$. Then the function $x(t)=U(t)x_0$ can be written as

$$x(t)=U(t)(-A)^{-\eta}((-A)^\eta x_0)\,.$$

Thus, by assertion (7.3), we find that

$$x(t)\in C([0,\infty);E)\,,$$

and also

$$x(0)=(-A)^{-\eta}((-A)^\eta x_0)=x_0\,.$$

On the other hand, we know from Theorem 5.3 that

$$\begin{cases} x(t)\in C^\infty((0,\infty);E)\,, \\ x'(t)=Ax(t)\,, \qquad t>0\,. \end{cases}$$

Summing up, we have proved that the function $x(t)=U(t)x_0$, defined by formula (7.1), is a (unique) solution of problem (P).

REMARK 7.3. In the case $\theta=1$, Theorem 7.1 remains valid for every $x_0\in E$ (i.e., $\eta=0$). In fact, it suffices to note the following:

1) $\lim_{t\downarrow 0} U(t)x_0=x_0$ for each $x_0\in\mathcal{D}(A)$.

2) We have by estimate (5.8′) with $\theta=1$ (Remark 5.4)

$$\sup_{0<t\leqq 1}\|U(t)\|<\infty\,.$$

§ 8. Non-homogeneous solutions

Let $f:[0,T]\to E$ be a continuous function. Now we consider the following non-homogeneous Cauchy problem:

$$(NP)\qquad \begin{cases} \dfrac{dx}{dt}=Ax(t)+f(t)\,, \qquad 0<t\leqq T\,. \\ x(0)=x_0\,. \end{cases}$$

The next theorem gives an explicit formula for the solutions of problem (NP) (cf. Krein [2], Chapter I, Theorem 6.1):

THEOREM 8.1. *Assume that the Cauchy problem* (P) *is well posed on* $[0,\infty)$, *and that the operator* A *has a resolvent* $R(\lambda_0)=(A-\lambda_0 I)^{-1}$ *for some* $\lambda_0\in\boldsymbol{C}$. *Then* *a solution $x(t)$ of problem (NP), if it exists, is given by the following formula:*

$$x(t)=U(t)x_0+\int_0^t U(t-s)f(s)ds\,. \tag{8.1}$$

Here $U(t)$ is the strongly continuous semigroup on E constructed in Theorem 1.1.

PROOF. First, applying the operators $U(t-s)$, $0<s<t$, to the equation

$$x'(s)=Ax(s)+f(s)\,, \qquad 0<s\leqq T\,,$$

we obtain that

$$U(t-s)x'(s)=U(t-s)Ax(s)+U(t-s)f(s)\,, \qquad 0<s<t\,. \tag{8.2}$$

On the other hand, since $\sup_{[\delta,1/\delta]}\|U(t)\|<\infty$ for every $\delta>0$, it follows that

$$\begin{aligned}\frac{d}{ds}(U(t-s)x(s)) &= \lim_{\sigma\to 0}\left(\frac{U(t-s-\sigma)x(s+\sigma)-U(t-s)x(s)}{\sigma}\right)\\ &= \lim_{\sigma\to 0}\left\{U(t-s-\sigma)\left(\frac{x(s+\sigma)-x(s)}{\sigma}\right)\right.\\ &\qquad \left.-U(t-s-\sigma)\left(\frac{U(\sigma)-I}{\sigma}\right)x(s)\right\}\\ &= U(t-s)x'(s)-U(t-s)Ax(s)\,, \qquad 0<s<t\,.\end{aligned} \tag{8.3}$$

Hence it follows from formulas (8.2) and (8.3) that

$$\frac{d}{ds}(U(t-s)x(s))=U(t-s)f(s)\,, \qquad 0<s<t\,.$$

Integrating this equation from 0 to $t-h$ ($h>0$) with respect to s, we obtain that

$$\int_0^{t-h} U(t-s)f(s)ds=[U(t-s)x(s)]_0^{t-h}=U(h)x(t-h)-U(t)x_0\,.$$

But we have as $h\downarrow 0$

$$\begin{aligned}U(h)x(t-h) &= U(h)R(\lambda_0)[Ax(t-h)-\lambda_0 x(t-h)]\\ &\longrightarrow R(\lambda_0)(Ax(t)-\lambda_0 x(t))=x(t)\,.\end{aligned}$$

In fact, it suffices to note the following:

1) $\sup_{0<h\leq 1}\|U(h)R(\lambda_0)\|<\infty$.
2) $\lim_{h\downarrow 0} U(h)R(\lambda_0)x=R(\lambda_0)x$ for each $x\in E$.
3) $x(t)\in C([0, T]; E)$.
4) $Ax(t)=x'(t)\in C((0, T]; E)$.

Therefore, we find that the improper integral $\int_0^t U(t-s)f(s)ds$ exists, and satisfies

$$\int_0^t U(t-s)f(s)ds = x(t) - U(t)x_0 .$$

This proves formula (8.1).

Now assume that the operator A satisfies condition (6.1):

$$\|R(\lambda)\| \leqq \frac{M}{(1+|\lambda|)^{\theta}}, \qquad \lambda \in \Sigma . \tag{6.1}$$

The next theorem states that the function $x(t)$, defined by formula (8.1), is a solution of problem (NP) (cf. Kreĭn [2], Chapter I, Theorem 6.9):

THEOREM 8.2. *Assume that the operator A satisfies condition* (6.1) *with* $0 < \theta \leqq 1$. *Let* $f: [0, T] \to E$ *be a Hölder continuous function with exponent* γ *satisfying* $1-\theta < \gamma \leqq 1$:

$$\|f(t)-f(s)\| \leqq C|t-s|^{\gamma}, \qquad t, s \in [0, T] . \tag{8.4}$$

Then, for every $x_0 \in \mathcal{D}((-A)^{\eta})$ *with* $1-\theta < \eta < \theta$, *the function*

$$x(t) = U(t)x_0 + \int_0^t U(t-s)f(s)ds \tag{8.1}$$

belongs to $C([0, T]; E) \cap C^1((0, T]; E)$, *and is a unique solution of problem* (NP). *Here* $U(t)$ *is the semigroup on* E *defined by formula* (5.6):

$$U(t) = -\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t} R(\lambda) d\lambda , \qquad t > 0 . \tag{5.6}$$

PROOF. Theorem 7.1 tells us that the function $U(t)x_0$ belongs to $C([0, \infty); E) \cap C^{\infty}((0, \infty); E)$, and is a (unique) solution of problem (P).

Thus it suffices to consider the function

$$y(t) = \int_0^t U(t-s)f(s)ds . \tag{8.1'}$$

We remark that the function $y(t)$ is well defined for all $0 \leqq t \leqq T$. In fact, we have by estimate (5.8') (Remark 5.4)

$$\|U(t)\| \leqq M_T t^{\theta-1}, \qquad 0 < t \leqq T , \tag{8.5}$$

so that

$$\begin{aligned} \|y(t)\| &\leqq \int_0^t \|U(t-s)\| \cdot \|f(s)\| ds \\ &\leqq \left(\max_{0 \leq s \leq T} \|f(s)\|\right) \int_0^t \|U(\tau)\| d\tau \end{aligned}$$

$$\leqq M_T\Big(\max_{0\leq s\leq T}\|f(s)\|\Big)\frac{t^\theta}{\theta}, \qquad 0\leqq t\leqq T.$$

In particular, we have

$$y(0)=0.$$

(a) The continuity of $y(t)$: We have for $0\leqq t\leqq T$, $0<t+h<T$

$$\begin{aligned} y(t+h)-y(t) &= \int_0^{t+h} U(t+h-s)f(s)ds-\int_0^t U(t-s)f(s)ds \\ &= \int_0^{t+h} U(\tau)f(t+h-\tau)d\tau-\int_0^t U(\tau)f(t-\tau)d\tau \\ &= \int_0^t U(\tau)(f(t+h-\tau)-f(t-\tau))d\tau \\ &\quad +\int_t^{t+h} U(\tau)f(t+h-\tau)d\tau. \end{aligned}$$

Hence it follows from inequalities (8.4) and (8.5) that

$$\begin{aligned} \|y(t+h)-y(t)\| &\leqq \Big(\max_{0\leq\sigma\leq T}\|f(\sigma+h)-f(\sigma)\|\Big)\int_0^T \|U(\tau)\|d\tau \\ &\quad +\Big(\max_{0\leq\sigma\leq T}\|f(\sigma)\|\Big)\left|\int_t^{t+h}\|U(\tau)\|d\tau\right| \\ &\leqq \frac{M_T}{\theta}\Big[C|h|^\gamma T^\theta+\Big(\max_{0\leq\sigma\leq T}\|f(\sigma)\|\Big)|(t+h)^\theta-t^\theta|\Big]. \end{aligned}$$

Therefore, we obtain that

$$\|y(t+h)-y(t)\| \longrightarrow 0 \qquad \text{as } h\to 0.$$

This proves that

$$y(t)\in C([0,T];E).$$

(b) The differentiability of $y(t)$: For each small $\delta>0$, we let

$$y_\delta(t)=\begin{cases} 0 & \text{if } 0\leqq t\leqq\delta, \\ \int_0^{t-\delta} U(t-s)f(s)ds & \text{if } \delta<t\leqq T. \end{cases}$$

Then, in view of estimate (8.5), it follows that

$$\|y_\delta(t)-y(t)\|\leqq\begin{cases} K_1\delta^\theta & \text{if } 0\leqq t\leqq\delta, \\ K_2(T^\theta-(T-\delta)^\theta) & \text{if } \delta<t\leqq T. \end{cases}$$

Hence we find that the the function $y_\delta(t)$ converges uniformly to the function $y(t)$ as $\delta\downarrow 0$. Furthermore, we have for $\delta<t\leqq T$

$$(8.6)\qquad y_\delta'(t)=U(\delta)f(t-\delta)+\int_0^{t-\delta}\frac{d}{dt}(U(t-s)f(s))ds$$
$$=U(\delta)f(t-\delta)+\int_0^{t-\delta}AU(t-s)f(s)ds$$
$$=U(\delta)(f(t-\delta)-f(t))+U(t)f(t)+\int_0^{t-\delta}AU(t-s)(f(s)-f(t))ds$$
$$+\Big((U(\delta)-U(t))f(t)+\int_0^{t-\delta}AU(t-s)f(t)ds\Big).$$

But the last term on the right-hand side vanishes:

$$(8.7)\qquad (U(\delta)-U(t)f(t)+\int_0^{t-\delta}AU(t-s)f(t)ds=0.$$

In fact, since the inverse $(-A)^{-1}$ exists and $AU(t)=U(t)A$ on $\mathcal{D}(A)$, it follows that

$$(8.8)\qquad \int_0^{t-\delta}U(t-s)f(t)ds=\int_0^{t-\delta}U(t-s)(-A)(-A)^{-1}f(t)ds$$
$$=\int_0^{t-\delta}\frac{d}{ds}(U(t-s)((-A)^{-1}f(t)))ds$$
$$=[U(t-s)((-A)^{-1}f(t))]_{s=0}^{s=t-\delta}$$
$$=U(\delta)(-A)^{-1}f(t)-U(t)(-A)^{-1}f(t).$$

Hence, by the closedness of A, we find that

$$U(\delta)f(t)-U(t)f(t)=-A\Big(\int_0^{t-\delta}U(t-s)f(t)ds\Big)$$
$$=-\int_0^{t-\delta}AU(t-s)f(t)ds.$$

This proves equation (8.7).

Therefore, combining formulas (8.6) and (8.7), we obtain that

$$(8.9)\qquad y_\delta'(t)=U(\delta)(f(t-\delta)-f(t))+U(t)f(t)$$
$$+\int_0^{t-\delta}AU(t-s)(f(s)-f(t))ds,\qquad \delta<t\leqq T.$$

Now we estimate each term on the right-hand side of (8.9):

1° We obtain from inequalities (8.4) and (8.5) that

$$\|U(\delta)(f(t-\delta)-f(t))\|\leqq CM_T\delta^{(\gamma+\theta)-1},\qquad \delta<t\leqq T.$$

2° By estimate (5.9′) (Remark 5.4), it follows that

$$\|AU(t)\| = \|U'(t)\| \leqq M_T' t^{\theta-2}, \qquad 0<t\leqq T.$$

Hence we have for $0<\delta'<\delta$

$$\left\|\int_{t-\delta}^{t-\delta'} AU(t-s)(f(s)-f(t))ds\right\| \leqq CM_T'\int_{t-\delta}^{t-\delta'} |t-s|^{\gamma+\theta-2}ds$$

$$= CM_T'\int_{\delta'}^{\delta} \tau^{\gamma+\theta-2}d\tau$$

$$= \frac{CM_T'}{\gamma+\theta-1}[\delta^{\gamma+\theta-1}-\delta'^{\gamma+\theta-1}].$$

Since $\gamma+\theta>1$, this proves that the improper integral

$$\int_0^t AU(t-s)(f(s)-f(t))ds = \lim_{\delta\downarrow 0}\int_0^{t-\delta} AU(t-s)(f(s)-f(t))ds \tag{8.10}$$

exists, and the convergence is uniform in $t\in[\varepsilon, T]$, for every $\varepsilon>0$.

Summing up, we find that

$$y_\delta'(t) \longrightarrow U(t)f(t)+\int_0^t AU(t-s)(f(s)-f(t))ds \qquad \text{as } \delta\downarrow 0$$

uniformly in $t\in[\varepsilon, T]$, for every $\varepsilon>0$. Thus one can let $\delta\downarrow 0$ in the formula

$$y_\delta(t)=\int_\varepsilon^t y_\delta'(\tau)d\tau+y_\delta(\varepsilon), \qquad 0<\delta<\varepsilon,$$

to obtain that

$$y(t)=\int_\varepsilon^t\left[U(\tau)f(\tau)+\int_0^\tau AU(\tau-s)(f(s)-f(\tau))ds\right]d\tau+y(\varepsilon), \quad 0<\varepsilon\leqq t\leqq T.$$

Since ε is arbitrary, this proves that

$$y(t)\in C^1((0, T]; E),$$

and

$$y'(t)=U(t)f(t)+\int_0^t AU(t-s)(f(s)-f(t))ds. \tag{8.11}$$

(c) Finally we show that for any $0<t\leqq T$

$$\begin{cases} y(t)\in \mathcal{D}(A), \\ Ay(t)=y'(t)-f(t). \end{cases} \tag{8.12}$$

First, one can let $\delta\downarrow 0$ in formula (8.8) to obtain that

$$\int_0^t U(t-s)f(t)ds=(-A)^{-1}f(t)-U(t)(-A)^{-1}f(t)$$

$$=(-A)^{-1}(f(t)-U(t)f(t)).$$

Hence it follows that for any $0<t\leqq T$

(8.13a) $$\int_0^t U(t-s)f(t)ds \in \mathcal{D}(A),$$

and

(8.13b) $$A\left(\int_0^t U(t-s)f(t)ds\right) = U(t)f(t) - f(t).$$

On the other hand, since we have for $\delta < t \leqq T$

$$\int_0^{t-\delta} U(t-s)(f(s)-f(t))ds \in \mathcal{D}(A),$$

in view of the closedness of A, it follows from assertion (8.10) that for any $0 < t \leqq T$

(8.14a) $$\int_0^t U(t-s)(f(s)-f(t))ds \in \mathcal{D}(A),$$

and

(8.14b) $$A\left(\int_0^t U(t-s)(f(s)-f(t))ds\right) = \int_0^t AU(t-s)(f(s)-f(t))ds.$$

Therefore, by assertions (8.13), (8.14) and (8.11), we have for any $0 < t \leqq T$

$$\begin{aligned} y(t) &= \int_0^t U(t-s)f(s)ds \\ &= \int_0^t U(t-s)f(t)ds + \int_0^t U(t-s)(f(s)-f(t))ds \in \mathcal{D}(A), \end{aligned}$$

and

$$\begin{aligned} Ay(t) &= A\left(\int_0^t U(t-s)f(t)ds\right) + A\left(\int_0^t U(t-s)(f(s)-f(t))ds\right) \\ &= (U(t)f(t)-f(t)) + (y'(t)-U(t)f(t)) \\ &= y'(t) - f(t). \end{aligned}$$

This proves assertion (8.12).

Summing up, we find that the function $y(t)$, defined by formula (8.1′), belongs to $C([0, T]; E) \cap C^1((0, T]; E)$, and is a solution of problem (NP) with initial condition $y(0)=0$.

Now the proof of Theorem 8.2 is complete.

REMARK 8.3. In the case $\theta=1$, Theorem 8.2 remains valid for every $x_0 \in E$ (i.e., $\eta=0$). This is an immediate consequence of Remark 7.3.

§ 9. Proof of Theorems 3 and 5

PROOF OF THEOREM 3. (i) We find from the proof of Théorème 11 of Taira [4] (cf. [5], Section 8.4) that the constants $R(\theta)$ and $C(\theta)$ in Theorem 1 depend continuously on $\theta\in(-\pi, \pi)$, so that they may be chosen uniformly in $\theta\in[-\pi+\varepsilon, \pi+\varepsilon]$, for every $\varepsilon>0$. This proves the existence of the constants $r(\varepsilon)$ and $c(\varepsilon)$. Estimate (0.2) is an immediate consequence of estimate (0.1) with $s=2$ and $\varphi=0$.

(ii) By part (i), one may assume that, for $\mu_\varepsilon>0$ large enough, the operator $\mathfrak{A}-\mu_\varepsilon I$ satisfies condition (6.1) with $\theta=(1+\delta)/2$:

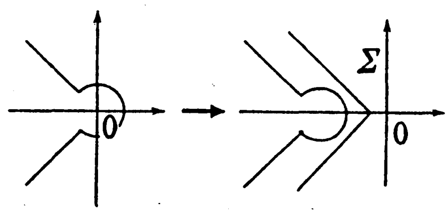

Figure 18

Thus we can apply Theorem 5.3 (and Remark 5.4) to the operator $\mathfrak{A}-\mu_\varepsilon I$ to obtain part (ii).

PROOF OF THEOREM 5. Theorem 5 follows immediately from Theorem 3 and Theorem 8.2. (We remark that Theorem 5 includes Theorem 4 as the particular case $\delta=1$, as is seen from Remark 8.3.)

Appendix A Jordan's lemma

LEMMA A (Jordan). *If $f(z)$ is a continuous function on the half plane $\{z\in\boldsymbol{C}; \operatorname{Re} z\geqq 0\}$, and if $f(re^{i\theta})\to 0$ as $r\to+\infty$, uniformly in $\theta\in[-\pi/2, \pi/2]$, then we have for any $m>0$*

$$\lim_{r\to+\infty}\int_{C_r} e^{-mz}f(z)dz=0,$$

where

$$C_r=\{z=re^{i\theta};\ -\pi/2\leqq\theta\leqq\pi/2\}.$$

REMARK. If $f(z)$ is a continuous function on the half plane $\{z\in\boldsymbol{C}; \operatorname{Re} z\leqq 0\}$, and if $f(re^{i\theta})\to 0$ as $r\to+\infty$, uniformly in $\theta\in[\pi/2, 3\pi/2]$, then we have for any $m>0$

$$\lim_{r\to+\infty}\int_{D_r} e^{m\zeta}f(\zeta)d\zeta=0,$$

where

$$D_r=\{\zeta=re^{i\theta};\ \pi/2\leqq\theta\leqq 3\pi/2\}.$$

Appendix B The Laplace transform

Let E be a Banach space, and let $u(t)$: $\boldsymbol{R}\to E$ be a function satisfying the following three conditions:

(a) $u(t)=0$ for all $t<0$.

(b) There exist constants $C>0$ and $\beta\in\boldsymbol{R}$ such that

$$\|u(t)\|\leqq Ce^{\beta t}, \qquad t\geqq 0 .$$

(c) u is Riemannian integrable on every bounded interval $[0, T]$, $T>0$.

Then we let

$$U(\lambda)=\int_0^{\infty}e^{-\lambda t}u(t)dt, \qquad \mathrm{Re}\,\lambda>\beta .$$

It is easy to verify that $U(\lambda)$ is a holomorphic function of λ in the half plane $\{\lambda\in\boldsymbol{C}; \mathrm{Re}\,\lambda>\beta\}$. The function $U(\lambda)$ is called the *Laplace transform* of $u(t)$.

The most fundamental result is the following:

THEOREM B (The Laplace inversion formula). *Let* $u: \boldsymbol{R}\to E$ *be a function which satisfies the following conditions*:

(a) $u(t)=0$ *for all* $t<0$.

(b) *There exist constants* $C>0$ *and* $\beta\in\boldsymbol{R}$ *such that*

$$\|u(t)\|\leqq Ce^{\beta t}, \qquad t\geqq 0 .$$

(c′) *u is of* bounded variation *on any bounded interval* $[0, T]$, $T>0$.

Then we have for $\xi>\beta$

$$\lim_{A\to+\infty}\frac{1}{2\pi i}\int_{\xi-iA}^{\xi+iA}e^{\lambda t}U(\lambda)d\lambda=\frac{1}{2}[u(t+0)+u(t-0)] \qquad \text{in } E.$$

The convergence is uniform in t on any bounded interval of continuity of u.

Appendix C The resonance theorem

Let X, Y be normed linear spaces over the same scalar field and let $L(X, Y)$ be the space of bounded (continuous) linear operators on X into Y.

Then we have the following:

THEOREM C (The resonance theorem). *Let H be a subset of* $L(X, Y)$. *If X is a Banach space, then the boundedness of* $\{\|Tx\|; T\in H\}$ *at each* $x\in X$ *implies the boundedness of* $\{\|T\|; T\in H\}$.

The material in Appendices B and C is standard and can be found in textbooks on functional analysis such as Yosida [7].

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