TSUKUBA J. MATH.
Vol. 13 No. 2 (1989), 483–493

PARACOMPACTNESS AND CLOSED SUBSETS

By

F. G. LUPIAÑEZ* and E. OUTERELO*

Abstract. In this paper we introduce some covering properties related with paracompactness, which allow us to characterize paracompact, compact, pseudocompact, lightly compact, collectionwise normal with respect to paracompact and closed sets, and countably compact spaces.

AMS Subject Classification (1980; 1985 Revision): 54B05, 54D10, 54D15, 54D18, 54D30.

Key words and phrases: paracompact space, compact space, countably compact space, pseudocompact space, lightly compact space, α-paracompact subset, σ-paracompact subset, collectionwise normal w. r. t. poracompact and closed sets space.

Introduction.

The idea of the research of this paper is based in two theorems obtained by J. Abdelhay in 1948, and a theorem obtained by J. M. Boyte in 1973.

I) "The following conditions on a T_2 space are equivalent:
  1) Given an open covering $\mathcal{U}$ of X and given $a\in X$, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is locally finite at a.
  2) X is regular" (See J. Abdelhay, "Teorema" 1 in [1]; this theorem has been rediscovered by J. Chew in 1972 [5] and J. M. Boyte in 1973 [4]).

II) "If X is T_2, then X is normal, if and only if, each open covering $\mathcal{U}$ of X has, for each $U\in\mathcal{U}$ and each closed set $F\subset U$ an open refinement which is locally finite in F, i.e.: in every point of F" (See J. Abdelhay, "Teorema" 2 in [1]).

III) "If X is T_2, then X is normal, if and only if, each open covering $\mathcal{U}$ of X has, for each $U\in\mathcal{U}$ and each closed set $F\subset U$ an open refinement which

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(*) Partially supported by "Comisión Asesora de Investigación Científica y Técnica".
Received May 2, 1988. Revised September 30, 1988.

is locally finite with respect to F" (Def.: A collection of sets $\mathcal{G}$ is locally finite with respect to a set S if there exists an open set $V(S)$ containing S such that $V(S)\cap G\neq\emptyset$ for only finitely many members $G\in\mathcal{G}$. See J. M. Boyte, Theorem 8 in [4]).

In [1], Abdelhay defined the concept of paracompact in E. Analogously, using the hypothesis of theorems I and III, we define some new covering properties: A-paracompact in E, A-paracompact with respect to E, and paracompact with respect to E.

In Theorems 1.3, 1.4, 1.5 and 1.6 of Section 1 we characterize all these properties for an arbitrary closed subset.

In Section 2 of this paper we study these properties for certain subsets. We give important characterizations of lightly compact spaces (Theorem 2.2) and countably compact spaces (Proposition 2.9) using these methods.

The following definitions are used in this paper:

1) A subset E of a topological space X is said to be α-paracompact (σ-paracompact) in X if every covering of E by open subsets of X has a refinement by open subsets of X, locally finite in X (σ-locally finite in X) which covers E. (See [2]).
2) A subset E of a topological space X is said to be normal (regular) in X if E and each closed (one-point) subset of X, non intersecting E, have disjoint open neighbourhoods. (See [10]).

1. On paracompactness.

We introduce various covering properties related with paracompactness.

DEFINITION 1. Let X be a topological space and let E be a subset of X.

1) We will say that X is A-paracompact in E if every open covering $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ which is locally finite in E (i.e.: in every point of E).
2) We will say that X is paracompact in E if every open covering $\mathcal{U}$ of X such that $E\subset U$ for some $U\in\mathcal{U}$ has an open refinement $\mathcal{V}$ which is locally finite in E. (J. Abdelhay).
3) We will say that X is A-paracompact with respect to E if every open covering $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ which is locally finite with respect to E.
4) We will say that X is paracompact with respect to E if every open covering

$\mathcal{U}$ of X such that $E\subset U$ for some $U\in\mathcal{U}$ has an open refinement $\mathcal{V}$ which is locally finite with respect to E (see [4]).

REMARK 1.1.
1) If X is A-paracompact in (with respect to) E, then X is A-paracompact in (with respect to) F for every $F\subset E$.
2) We have:
   a) X A-paracompact in $E \Rightarrow X$ paracompact in E (The converse is false: see 2.6 and 2.7).
   b) X A-paracompact with respect to $E \Rightarrow X$ A-paracompact in E (The converse is false: see 1.3 and 1.4).
   c) X paracompact with respect to $E \Rightarrow X$ paracompact in E (This implication is an equivalence if X is a regular and T_0 space: see 1.7).
   d) X A-paracompact with respect to $E \Rightarrow X$ paracompact with respect to E (The converse is false: see 1.4 and 1.6).
3) A topological space X is paracompact, if and only if, is A-paracompact in X.
   A topological space X is compact, if and only if, is A-paracompact with respect to X.

LEMMA 1.2. *Let X be a regular space, and E a closed subset of X, if X is A-paracompact in E, then E is a normal subset in X.*

PROOF. Let F a closed subset of X non intersecting E, for every $x\in F$ there is an open neighbourhood V^x of x in X such that $\overline{V}^x\cap E=\emptyset$. Then, the open covering $\mathcal{U}=\{V^x \mid x\in F\}\cup\{X\setminus F\}$ of X has an open refinement $\mathcal{V}$ which is locally finite in E.

Clearly, $st(F, \mathcal{V})$ and $X\setminus\overline{st(F, \mathcal{V})}$ are disjoint open neighbourhoods of F and E, respectively.

THEOREM 1.3. *Let X be a regular space and E a closed subset of X. The following properties are equivalent:*
a) *X is A-paracompact in E.*
b) *E is α-paracompact in X.*

PROOF. Let E be α-paracompact in X, then for every open covering $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}'$ which covers E and is locally finite in X, and there exists an open subset W such that $E\subset W\subset\overline{W}\subset \bigcup_{V\in\mathcal{V}'} V$ (since E is a normal subset in X, [2, Theorem 6]).

Thus $\mathcal{V}=\mathcal{V}'\cup\{U\setminus\overline{W} \mid U\in\mathcal{U},\ U\setminus\overline{W}\neq\emptyset\}$ is an open refinement of $\mathcal{U}$ which is locally finite in E.

For the only if part, let $\mathcal{U}'$ be a covering of E by open subsets of X, then $\mathcal{U}=\mathcal{U}'\cup\{X\setminus E\}$ is an open covering of X. By the hypothesis, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ which is locally finite in E, then for each $x\in E$ there exists an open neighbourhood W^x such that $W^x\subset\bigcup_{V\in\mathcal{V}'}V$, where $\mathcal{V}'=\{V\in\mathcal{V}\mid V\cap E\neq\emptyset\}$, and W^x meets only finitely many members of $\mathcal{V}'$.

Hence $E\subset\bigcup_{x\in E}W^x=W$ which is open in X, and $\mathcal{V}'$ is an open refinement of $\mathcal{U}'$ which is locally finite in W.

Finally, since E is a normal subset of X (by Lemma 1.2), there exists an open subset G such that $E\subset G\subset\overline{G}\subset W$, and $\mathcal{V}''=\{V\cap G\mid V\in\mathcal{V}\}$ is a refinement of $\mathcal{U}'$ by open subsets of X, locally finite in X, which covers E.

THEOREM 1.4. *Let X be a regular space and E a closed subset of X. The following properties are equivalent:*

a) *X is A-paracompact with respect to E.*

b) *E is compact,*

PROOF. Let E be a compact subset of X, then for every open covering $\mathcal{U}$ of X there exists a finite subcovering $\{U_1, \cdots, U_r\}$ which covers E. Since E is a normal subset of X, there is an open subset V such that $E\subset V\subset\overline{V}\subset U_1\cup\cdots\cup U_r$.

Then $\mathcal{V}=\{U_i\}_{i=1,\dots,r}\cup\{U\setminus\overline{V}\mid U\in\mathcal{U}, U\setminus\overline{V}\neq\emptyset\}$ is an open refinement of $\mathcal{U}$, locally finite with respect to E.

For the only if part let $\mathcal{U}'$ be a covering of E by open subsets of X, then $\mathcal{U}=\mathcal{U}'\cup\{X\setminus E\}$ is an open covering of X. By the hypothesis, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$, which is locally finite with respect to E; then E meets only finitely many members of $\mathcal{V}$ and $\{V\mid V\in\mathcal{V}, V\cap E\neq\emptyset\}$ is a finite open refinement of $\mathcal{U}'$ which covers E.

THEOREM 1.5. *Let X be a T_1 space and E a closed subset of X. The following properties are equivalent:*

a) *X is paracompact in E, and E is a regular subset in X.*

b) *E is a normal subset in X,*

PROOF. Let F a closed subset of X such that $F\cap E=\emptyset$. Since E is regular in X, for every $y\in F$ there is an open neighbourhood V^y of y in X such that $\overline{V}^y\cap E=\emptyset$. Then $\mathcal{U}=\{V^y\mid y\in F\}\cup\{X\setminus F\}$ is an open covering of X and $E\subset X\setminus F\in\mathcal{U}$, thus there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ which is locally finite in E. Then $st(F, \mathcal{V})$ and $X\setminus\overline{st(F, \mathcal{V})}$ are disjoint open neighbourhoods of

F and E, respectively.

Conversely, if E is a normal subset in a T_1 space X then E is regular in X. Let $\mathcal{U}$ be an open covering of X such that $E \subset U$ for some $U \in \mathcal{U}$. Then there is an open subset G such that $E \subset G \subset \bar{G} \subset U$ and $\mathcal{V} = \{U\} \cup \{V \setminus \bar{G} \mid V \in \mathcal{U}, V \setminus \bar{G} \neq \emptyset\}$ is an open refinement of $\mathcal{U}$ which is locally finite in E,

THEOREM 1.6. *Let X be a T_1 space and E a closed subset of X. The following properties are equivalent:*

a) *X is paracompact with respect to E, and E is a regular subset in X.*

b) *E is a normal subset in X.*

PROOF. The implication a)$\Rightarrow$b) follows from Remark 1.1.2) and Theorem 1.5. The proof of the converse implication is analogous to the last theorem.

COROLLARY 1.7. *Let X be a regular and T_0 space and E a closed subset of X. The following properties are equivalent:*

a) *X is paracompact in E.*

b) *X is paracompact with respect to E.*

c) *E is a normal subset in X.*

2. On some covering properties.

The properties of the last Section allow us to give characterizations of some covering properties.

PROPOSITION 2.1. *Let X be a T_2 space. The following properties are equivalent:*

a) *X is regular,*

b) *X is A-paracompact in each point of X.*

c) *X is A-paracompact in each compact subset of X.*

d) *X is A-paracompact with respect to each compact subset of X.*

e) *X is paracompact in each compact subset of X.*

f) *X is paracompact with respect to each compact subset of X.*

g) *X is A-paracompact in each α-paracompact subset of X.*

h) *X is paracompact in each α-paracompact subset of X.*

i) *X is paracompact with respect to each α-paracompact subset of X.*

PROOF. It follows from theorems of the last Section. The equivalence a)$\Leftrightarrow$b) is the "Teorema" 1 in [1].

THEOREM 2.2. *Let X be a T_2 space. The following properties are equivalent:*
a) *X is lightly compact.*
b) *Every α-paracompact subset of X is compact.*

PROOF. Let E be an α-paracompact subset of X, then E is closed by [2, Corollary 4]. Let $\mathcal{U}$ be a covering of E by open subsets of X, then there is an open refinement $\mathcal{V}$ of $\mathcal{U}$, locally finite in X, which covers E. Then $\mathcal{V}'=\mathcal{V}\cup\{X\setminus E\}$ is a locally finite open covering of X, thus $\mathcal{V}'$ is finite by the hypothesis, hence E is compact.

Conversely, if X is not lightly compact, there is a locally finite open covering $\mathcal{U}=\{U_j\}_{j\in J}$ of X which is infinite. For every $j\in J$, let $x_j\in U_j$. Then $E=\{x_j|j\in J\}$ is α-paracompact in X by [7, Proposition 1.1.2)] and, by the hypothesis, E is compact.

Since $\mathcal{U}$ is locally finite and J is infinite, it follows that E is infinite. Let S be an infinite sequence in E. Clearly, there is a cluster point x of S in E, and x is an accumulation point of E. Hence, each open neighbourhood of x meets infinitely many members of $\mathcal{U}$, and this is a contradiction. (In this implication, the hypothesis "X is a T_2 space" is not necessary. See [9, "Observación" VIII. 1. 160. 2)]).

COROLLARY 2.3. *Let X be a T_{3a} space. The following properties are equivalent:*
a) *X is pseudocompact.*
b) *All α-paracompact subset of X is compact.*

PROOF. It follows from [3, Theorem 3] and the last theorem.

PROPOSITION 2.4. *Let X be a T_3 space, then the following properties are equivalent:*
a) *X is lightly compact.*
b) *X is A-paracompact with respect to each α-paracompact subset of X.*

PROOF. It follows from Theorem 2.2, Theorem 1.4 and [2, Corollary 4]. For the implication b)$\Rightarrow$a), we use only that X is a T_2 space.

COROLLARY 2.5. *Let X be a T_{3a} space, then the following properties are equivalent:*
a) *X is pseudocompact.*
b) *X is A-paracompact with respect to each α-paracompact subset of X.*

PROPOSITION 2.6. *Let X be a T_2 space. The following properties are equivalent:*

a) *X is normal.*

b) *X is paracompact in each closed subset of X.*

c) *X is paracompact in each regular subset of X.*

d) *X is paracompact with respect to each closed subset of X.*

e) *X is paracompact with respect to each regular subset of X.*

PROOF. If follows from theorems of the last Section. The equivalence a)⇔b) is the "Teorema" 2 in [1] and a)⇔d) is the Theorem 8 in [4].

PROPOSITION 2.7. *Let X be a topological space, the following properties are equivalent:*

a) *X is paracompact.*

b) *X is A-paracompact in each closed subset of X.*

c) *X is A-paracompact in each regular subset of X.*

d) *X is A-paracompact in each normal and closed subset of X.*

PROPOSITION 2.8. *Let X be a T_2 space. The following properties are equivalent:*

a) *X is compact.*

b) *X is A-paracompact with respect to each closed subset of X.*

c) *X is A-paracompact with respect to each regular subset of X.*

d) *X is A-paracompact with respect to each normal subset of X.*

PROPOSITION 2.9. *Let X be a T_1 space. The following properties are equivalent:*

a) *X is countably compact.*

b) *All paracompact and closed subset of X is compact.*

PROOF. Let X be a countably compact space, then, if E is a paracompact and closed subset of X, we have that E is paracompact and countably compact. Thus E is compact.

If X is not countably compact, then there exists a discrete countable closed subset E of X which is infinite. Thus E is paracompact and is not compact.

COROLLARY 2.10. *Let X be a T_3 space. The following properties are equivalent:*

a) *X is countably compact.*

b) *X is A-paracompact with respect to each paracompact and closed subset of X.*

PROPOSITION 2.11. *Let X be a regular space. The following properties are equivalent:*

a) *X is collectionwise normal w.r.t. paracompact and closed sets, and all paracompact and closed subset of X is normal in X.*

b) *All paracompact and closed subset of X is α-paracompact in X.*

PROOF. Let X be a collectionwise normal w.r.t. paracompact and closed set and regular space, such that all paracompact and closed subset of X is normal in X, and let E be a paracompact and closed subset of X; if E is not α-paracompact in X, then E is not α-collectionwise normal in X [8, Proposition 1.13], and this is a contradiction.

Conversely, let X such that all paracompact and closed subset is α-paracompact in X, then all paracompact and closed subset is normal in X. If $\{D_j\}_{j\in J}$ is a discrete family of paracompact and closed subsets of X, then $E=\bigcup_{j\in J} D_j$ is paracompact and closed in X, thus E is α-paracompact in X, and E is α-collectionwise normal in X [2, Theorem 8], and this proves the result.

REMARK 2.12. In the last proposition, the hypothesis "all paracompact and closed subset of X is normal X" can not be omitted, because there exists a collectionwise normal w.r.t. paracompact and closed sets, and T_{3a} space such that has a paracompact and closed subset that is not normal in X.

EXAMPLE. Let $T=[0, \Omega]\times[0, \omega]\setminus\{(\Omega, \omega)\}$ be the Tychonoff plank, T is a T_{3a} space and has the paracompact and closed subset $\{\Omega\}\times[0, \omega)$ that is not normal in T.

1. Let P be a paracompact and closed subset in T, then:
   a) There is $a\in[0, \Omega)$ such that $P\cap([a, \Omega)\times\{\omega\})=\emptyset$ (because P is paracompact and closed in T).
   b) There is $n_0\in[0, \omega]$ such that $P\cap([a, \Omega)\times[n_0, \omega])=\emptyset$ (if, for each $n\in[0, \omega]$ there is $p_n\geqq n$ and $x_{p_n}\in[a, \Omega)$ such that $(x_{p_n}, p_n)\in P$, then $c=\sup\{x_{p_n}\mid n\in[0, \omega]\}\in[a, \Omega)$, and $(c, \omega)\in P$, but this is a contradiction).

   Thus each paracompact and closed subset in T is the union of a discrete and closed subset of $\{\Omega\}\times[n_0, \omega)$ (for some $n_0\in[0, \omega)$), and a compact subset of $[0, \Omega]\times[0, \omega]\setminus((a, \Omega]\times[n_0, \omega])$ (for some $n_0\in[0, \omega)$ and $a\in[0, \Omega)$).
2. Let a discrete family of paracompact and closed subsets of X, $\{P_j\}_{j\in J}$, then for each $j\in J$ there is a discrete set $D_j\subset\{\Omega\}\times[0, \omega)$ and a compact set K_j pairwise disjoints, such that $P_j=D_j\cup K_j$.

Since $\bigcup_{j\in J} P_j=P$ is paracompact and closed in T, there is a discrete set D, and a compact set K parwise disjoints, such that $P=D\cup K$. Then K is the union of finitely many members of $\{K_j\}_{j\in J}$. Since T is a regular space, it follows the result.

COROLLARY 2.13. *Let X be a regular and normal space. The following properties are equivalent:*

a) *X is A-paracompact in each paracompact and closed subset of X.*

b) *X is collectionwise normal w.r.t. paracompact and closed sets.*

c) *All paracompact and closed subset of X is α-paracompact in X.*

PROPOSITION 2.14. *Let X be a T_3 space. The following properties are equivalent:*

a) *X is paracompact in each paracompact and closed subset of X.*

b) *X is paracompact with respect to each paracompact and closed subset of X.*

c) *All paracompact and closed subset of X is normal in X.*

REMARK. The results 2.11, 2.12 and 2.14 cause the next definition.

DEFINITION 2. Let X be a T_1 space. We say that X is P-normal if all paracompact and closed subset of X is normal in X.

REMARK 2.15. We have that: T_4 implies P-normal, and P-normal implies T_3. From Remark 2.12 it follows that $T_{3a} \not\Rightarrow P$-normal. Also, from Proposition 2.9 it follows that all countably compact and T_3 space is P-normal, this proves that P-normal $\not\Rightarrow T_4$ (let $[0, \Omega]\times[0, \Omega)$).

PROPOSITION 2.16. *Let X be a T_3 space. The following properties are equivalent:*

a) *X is countably compact.*

b) *All σ-paracompact and closed subset of X is compact.*

PROOF. Analogously to proof of Proposition 2.9 (a σ-paracompact subset of a regular space is paracompact).

COROLLARY 2.17. *Let X be a T_3 space. The following properties are equivalent:*

a) *X is countably compact.*

b) *X is A-paracompact with respect to each σ-paracompact and closed subset of X.*

PROPOSITION 2.18. *Let X be a regular space. The following properties are equivalent:*

a) *All σ-paracompact and closed subset is α-paracompact in X.*

b) *All σ-paracompact and closed subset is normal in X.*

PROOF. It follows from Theorem 1.3 in [7].

COROLLARY 2.19. *Let X be a T_3 space. The following properties are equivalent:*

a) *X is A-paracompact in each σ-paracompact and closed subset of X.*

b) *X is paracompact in each σ-paracompact and closed subset of X.*

c) *X is paracompvct with respect to each σ-paracompact and closed subset of X.*

COROLLARY 2.20. *If X is T_4 then X is A-paracompact in each σ-paracompact and closed subset.*

OPEN PROBLEMS.

1) Characterization of completely regular space using these methods.
2) Characterization of P-normal spaces.
3) Characterization of those T_1 spaces such that all σ-paracompact and closed subset is normal in them.

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Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad Complutense de Madrid
28040 Madrid
SPAIN