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ON THE EXISTENCE OF A STRAIGHT LINE

By

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§ 1. Introduction.

Let M be a connected, complete, non-compact, oriented and finitely connected Riemannian 2-manifold. The total curvature of such an M is defined to be an improper integral of the Gaussian curvature G with respect to the volume element of M and expressed as $C(M)=\int_{M} G d_{M}$. The influence of total curvature of such an M have been investigated by many people. The pioneering work on total curvature was done by Cohn-Vossen in [1], which stated that if M admits total curvature, then $C(M)\leqq 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. He also proved in [2] that if a Riemannian plane M (i.e. M is a complete Riemannian manifold homeomorphic to $\boldsymbol{R}^{2}$) admits total curvature and if there exists a straight line on M, then $C(M)\leqq 0$. It is known that this is generalized as follows. (Confer section 4 in [4].); Let M have only one end. If such an M admits total curvature and if M contains a straight line, then $C(M)\leqq 2\pi(\chi(M)-1)$.

It is natural to consider whether the converse of the fact mentioned above is true or not. In this paper, we shall prove the following theorem.

THEOREM. *Let M be a connected, complete, non-compact, oriented and finitely connected Riemannian 2-manifold having one end. If M admits total curvature which is smaller than $2\pi(\chi(M)-1)$, then M contains a straight line.*

In the case where $C(M)=2\pi(\chi(M)-1)$, it is not always that M contains a straight line. In section 4, we shall show an example of a C^{2}-surface M whose total curvature is equal to 0 and on which there are no straight lines. Finally we shall note that if M has more than one end, then it is obvious that M contains a straight line.

§ 2. Preliminaries.

This section is devoted to introduce some definitions and the properties used throughout this paper.

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From completeness and non-compactness of M, through every point on M there is at least a *ray* $\gamma: [0, \infty)\rightarrow M$, where it is a unit speed geodesic satisfying $d(\gamma(t_1), \gamma(t_2))=|t_1-t_2|$ for all $t_1, t_2\geqq 0$, and d is the distance function induced from the Riemannian metric on M. A unit speed geodesic $\gamma: \boldsymbol{R}\rightarrow M$ is called a *straight line* if $d(\gamma(t_1), \gamma(t_2))=|t_1-t_2|$ for all $t_1, t_2\in\boldsymbol{R}$. From now on, geodesics are assumed to be unit speed unless otherwise mentioned. By definition, M is said to be *finitely connected* if it is homeomorphic to be a compact 2-manifold (without boundary) with finitely many point removed. The number of these points removed is equal to the number of ends on M.

For a point p on M let M_p and S_p be the tangent space to M at p and the unit circle of M_p centered at the origin. S_p is equipped with the natural measure which is induced from the Riemannian metric on M. Let $A(p)$ be the set of all unit vectors tangent to rays emanating from p. Then the following lemma is known. (Confer section 4 in [4].)

LEMMA 1. *Let M be a connected, complete, non-compact, oriented and finitely connected Riemannian 2-manifold having one end. If M admits total curvature and if $D\subset M$ is a domain bounded by two rays emanating from a point $p\in\partial D$ such that any ray starting from p dose not intersect D and if $M\setminus D$ is homeomorphic to a closed half-plane, then*

$$C(D)=2\pi(\chi(M)-1)+\sphericalangle(u, v),$$

where $u, v\in A(p)$ are tangent to the rays lying in the boundary of D.

§3. Proof of Theorem.

First we consider the case that $\int_M G_- d_M>-\infty$, where $G_-=\min(G, 0)$. We put $\varepsilon=\{2\pi(\chi(M)-1)-C(M)\}/2>0$. Then there exists a compact set $K\subset M$ such that

$$\int_K G_- d_M<\int_M G_- d_M+\varepsilon \qquad \text{and}$$

$$M\setminus K \text{ is homeomorphic to } S^1\times[0, \infty),$$

where S^1 denotes a unit circle. For an arbitrarily point p on $M\setminus K$, we shall show that there exists a ray emanating from p which intersects with the interior of K.

Now, we suppose that such a ray dose not exists. Let Ω denote the set of all elements $(u, v)\in A(p)\times A(p)$. Note that Ω is not empty from the non-emptiness of $A(p)$ and is closed on $S_p\times S_p$ from the closedness of $A(p)$. Then there exists the element (u, v) of Ω satisfying

$$\sphericalangle(u, v) \leqq \sphericalangle(u', v') \qquad \text{for all } (u', v') \in \Omega ,$$

where the angle is measured with respect to the domain containing K. It should be noted that if $u=v$, then the angle is understood as $\sphericalangle(u, v)=2\pi$. Let E be a component containing K and bounded by $\gamma_u([0, \infty))$ and $\gamma_v([0, \infty))$, where γ_u is a ray with initial vector $\gamma_u'(0)=u$. From Lemma 1, we have

$$C(E)=2\pi(\chi(M)-1)+\sphericalangle(u, v)>2\pi(\chi(M)-1) .$$

On the other hand, we have

$$\int_K G_+ d_M \leqq \int_E G_+ d_M \leqq \int_M G_+ d_M \qquad \text{and}$$

$$\int_E G_- d_M \leqq \int_K G_- d_M < \int_M G_- d_M + \varepsilon ,$$

where $G_+ = \max(G, 0)$ and last inequality is due to the construction of K. Hence

$$C(E) < C(M) + \varepsilon < 2\pi(\chi(M)-1) .$$

This is a contradiction. Therefore there exists a ray emanating from p which intersects with the interior of K.

Let $\{p_j\}$ be the sequence of points on $M \setminus K$ such that $\{d(p_j, K)\}$ is a monotone divergent sequence. As is shown above, for each j there exists a ray γ_j emanating from p_j which intersects with the interior of K. Since K is compact there exists a subsequence $\{\gamma_k\}$ of $\{\gamma_j\}$ such that γ_k converges to a straight line as k tends to infinity.

Next we consider the case that $\int_M G_- d_M = -\infty$. Since M admits total curvature, $\int_M G_+ d_M < \infty$. We can choose the positive number ε satisfying $\varepsilon > \int_M G_+ d_M$. Then there exists a compact set $K \subset M$ such that

$$\int_K G_- d_M < 2\pi(\chi(M)-1) - \varepsilon \qquad \text{and}$$

$$M \setminus K \text{ is homeomorphic to } S^1 \times [0, \infty) .$$

In the sequel similarly as the privious case we can prove the existence of a straight line passing through K. Thus the proof of Theorem is complete.

§ 4. Example.

We shall construct a C^2-surface M in E^3 whose total curvature is equal to 0 and on which there are no straight lines. The construction is carried out as follows. Consider the C^2-function $f : (-\infty, 1] \to [0, \infty)$ defined by

$$f(x)=x^4-(x^2/2)+1 \qquad \text{for } x\leqq 0,$$

$$f(x)=(1-x^2)^{1/2} \qquad \text{for } 0\leqq x\leqq 1.$$

Then M is defined as a surface of revolusion around the x-axis whose generating line is the graph of f in the (xz)-plane. It is easy to see that $C(M)=0$. Next we shall see that there are no straight lines on M. Let $K=\{(x, y, z)\in M \mid x\geqq -1/2\}$. Since the boundary of K is a closed geodesic, it is obviously that there are no straight lines passing through any point on K. Furthermore there are no straight lines on $M\setminus K$. In fact, suppose that there exists a straight line α on $M\setminus K$. Then α divides M into two components $M_1\supset K$ and M_2. Now, it has already been proved by Cohn-Vossen in [2] that $C(M_1)\leqq 0$ and $C(M_2)\leqq 0$. In particular, $C(M_2)<0$ because the Gaussian curvature is negative on $M\setminus K$. Hence $C(M)=C(M_1)+C(M_2)<0$. This is a contradiction.

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