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NATURAL TRANSFORMATIONS OF VECTOR FIELDS ON MANIFOLDS TO VECTOR FIELDS ON TANGENT BUNDLES

By

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There are well known classical examples of vector fields on the tangent bundle TM which can be constructed from a vector field on the base manifold M, namely the vertical lift and the complete lift. Furthermore, if we consider the tangent bundle over an affine manifold (M, ∇), we can define the horizontal lift of a vector field on (M, ∇) to TM. As we shall see in Section 1, the classical constructions are examples of "natural transformations of the second order".

We have two goals in this paper. The first is to describe explicitly *all* second order natural transformations of vector fields on manifolds into vector fields on their respective tangent bundles. The second is to describe explicitly *all pointwise* second order natural transformations of vector fields on manifolds with symmetric affine connections to vector fields on their respective tangent bundles.

As we have done in previous papers [1], [2], [7], we shall use for our purposes the concepts and methods developed by D. Krupka [3]-[5]. This leads to a system of partial differential equations to solve, and to the problem of geometric interpretation of all solutions. Our main results are formulated in Theorems 2.4 and 3.3.

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1. Classical lifts of vector fields to tangent bundles

In this paper we shall adopt the Einstein summation convention, unless otherwise stated. Also, we assume all manifolds and geometrical objects to be of class C^∞.

Let $(U; x^1, x^2, \cdots, x^n)$ and $(\bar{U}; \bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$ be two systems of local coordinates in a smooth manifold M of dimension n such that the domain $U \cap \bar{U}$ is not

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empty. The coordinate vector fields $E_i=\partial/\partial x^i$ and $\bar{E}_i=\partial/\partial\bar{x}^i$ $(i=1, 2, \cdots, n)$ are related by the transformation formulas

$$E_i=A_i^a\bar{E}_a \quad \text{or} \quad \bar{E}_i=B_i^aE_a \qquad (i=1, 2, \cdots, n), \tag{1.1}$$

where $[A_i^k]=[\partial\bar{x}^k/\partial x^i]$ and $[B_i^k]=[\partial x^k/\partial\bar{x}^i]$ are the (mutually inverse) Jacobi matrices. If $y=y^kE_k=\bar{y}^k\bar{E}_k$ is a tangent vector field on $U\cap\bar{U}$ then we get by (1.1)

$$\bar{y}^k=A_a^ky^a \qquad (k=1, 2, \cdots, n). \tag{1.2}$$

Further, differentiating both sides of (1.2) with respect to $\bar{x}^i$, we get

$$\bar{y}^k{}_{,i}=A_{ab}^kB_i^by^a+A_a^kB_i^by^a{}_{,b}, \tag{1.3}$$

where $\bar{y}^k{}_{,i}=\partial\bar{y}^k/\partial\bar{x}^i$, $y^k{}_{,i}=\partial y^k/\partial x^i$ and $A_{ij}^k=\partial^2\bar{x}^k/\partial x^i\partial x^j$ $(i, j, k=1, 2, \cdots, n)$.

Now let TM be the tangent bundle over M with the natural projection p. Let $(p^{-1}U; x^1, x^2, \cdots, x^n, u^1, u^2, \cdots, u^n)$ and $(p^{-1}\bar{U}; \bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n, \bar{u}^1, \bar{u}^2, \cdots, \bar{u}^n)$ be two systems of local coordinates in TM induced from $(U; x^1, x^2, \cdots, x^n)$ and $(\bar{U}; \bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$, respectively. Here $u=u^kE_k=\bar{u}^k\bar{E}_k$ is a tangent vector field on $U\cap\bar{U}$. Then the transformation law on $p^{-1}(U\cap\bar{U})$ is given by

$$\begin{cases} \bar{x}^k=\bar{x}^k(x^1, x^2, \cdots, x^n), \\ \bar{u}^k=A_a^ku^a, \qquad (k=1, 2, \cdots, n). \end{cases} \tag{1.4}$$

Let $X_i=\partial/\partial x^i$, $X_{i*}=\partial/\partial u^i$ $(i=1, 2, \cdots, n)$ denote the coordinate vector fields on the tangent bundle TM, put $\bar{X}_i=\partial/\partial\bar{x}^i$, $\bar{X}_{i*}=\partial/\partial\bar{u}^i$ accordingly. Then the two bases $\{X_1, X_2, \cdots, X_n, X_{1*}, X_{2*}, \cdots, X_{n*}\}$ and $\{\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_n, \bar{X}_{1*}, \bar{X}_{2*}, \cdots, \bar{X}_{n*}\}$ are related to each other by

$$\begin{cases} X_i=A_i^a\bar{X}_a+A_{ib}^aB_c^b\bar{u}^c\bar{X}_{a*}, \\ X_{i*}=A_i^a\bar{X}_{a*}, \qquad (i=1, 2, \cdots, n). \end{cases} \tag{1.5}$$

Let now $Y=Y^kX_k+Y^{k*}X_{k*}=\bar{Y}^k\bar{X}_k+\bar{Y}^{k*}\bar{X}_{k*}$ be a vector field on $p^{-1}(U\cap\bar{U})$. Then, using (1.5), we obtain the following transformation formulas for the components of Y:

$$\begin{cases} \bar{Y}^k=A_a^kY^a, \\ \bar{Y}^{k*}=A_{ab}^ku^aY^b+A_a^kY^{a*}, \quad (k=1, 2, \cdots, n). \end{cases} \tag{1.6}$$

(In Section 2 we shall show that the formulas (1.2), (1.3) and (1.6) define actions of the second order differential group L_n^2 on some manifolds.)

Next, we shall recall briefly the definitions of the vertical and the complete lift of vector fields on M to TM. The *vertical lift* of a vector field y on M is a vector field y^v such that $y^v(df)=y(f)\circ p$ for all functions f on M. Here we consider the 1-form df on M as the function on TM defined by $(df)(x, u)=u(f)$. If $y=y^k E_k$ with respect to a system of local coordinates $(U; x^1, x^2, \cdots, x^n)$ in M, then we obtain

$$y^v=y^k X_{k*}. \tag{1.7}$$

Obviously, the vertical lift of a *single vector* $y\in TM$ is also well-defined. The *complete lift* of a vector field y on M is a vector field y^c such that $y^c(df)=d(y(f))$ for all functions f on M. In terms of local coordinates we obtain

$$y^c=y^k X_k+y^k{}_{,a}u^a X_{k*}. \tag{1.8}$$

We note that the vertical and the complete lift of vector fields depend only on a differential structure of the base manifold.

When an affine connection is defined on the base manifold, we can define another lift of vector fields on M to TM. Let ∇ be an affine connection on M. Then the tangent space of TM at any point $(x, u)\in TM$ splits into the horizontal and the vertical subspace with respect to ∇:

$$(TM)_{(x,u)}=H_{(x,u)}\oplus V_{(x,u)}.$$

For any vector field y on M, there exists a unique vector field y^h on TM such that $y^h(x, u)\in H_{(x,u)}$ and $p_* y^h(x, u)=y(u)$ for any point $(x, u)\in TM$. We call y^h the *horizontal lift* of y. In terms of the local coordinates, y^h is expressed as

$$y^h=y^k X_k-\Gamma^k_{ab}u^a y^b X_{k*}, \tag{1.9}$$

where Γ^k_{ij} $(i, j, k=1, 2, \cdots, n)$ are the local components of ∇.

REMARK A. The local components Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$ $(i, j, k=1, 2, \cdots, n)$ of ∇ given by $\nabla_{E_i}E_j=\Gamma^a_{ij}E_a$ and $\nabla_{\bar{E}_i}\bar{E}_j=\bar{\Gamma}^a_{ij}\bar{E}_a$ are related by

$$\bar{\Gamma}^k_{ij}=A^k_a(B^b_i B^c_j \Gamma^a_{bc}+B^a_{ij}) \quad (i, j, k=1, 2, \cdots, n), \tag{1.10}$$

where we put $B^k_{ij}=\partial^2 x^k/\partial\bar{x}^i\partial\bar{x}^j$ $(i, j, k=1, 2, \cdots, n)$. We shall also use the formula (1.10) in order to define actions of the second order differential group L^2_n on some manifolds.

REMARK B. The components Y^A of any of the classical lifted vector field y^v, y^c and y^h above depend only on the components u^k of the tangent vector $u\in TM$, on the components y^k of the original vector field y, on their first derivatives $y^k{}_{,i}$, and on the components Γ^k_{ij}. Because, firstly, all these quantities are subjected to the transformation formulas (1.2)-(1.4) and (1.6) depending on the second order jets of the coordinate transformations, and, secondly, the corresponding constructions are geometrically invariant, we say that the classical constructions $y \to y^v$, y^c, y^h are "second order natural transformations". We shall make this concept more precise in the next section.

REMARK C. We know already that each "classical lift" can be expressed locally in the form

$$Y^A = Y^A(u^k, y^k, y^k{}_{,i}) \quad \text{or} \quad Y^A = Y^A(u^k, y^k, \Gamma^k_{ij}).$$

Yet, it will be more precise to say that our second order natural transformations are of the form

$$(u^k, y^k, y^k{}_{,i}) \longmapsto (u^k, Y^A(u^k, y^k, y^k{}_{,i})) \tag{1.11}$$

or

$$(u^k, y^k, \Gamma^k_{ij}) \longmapsto (u^k, Y^A(u^k, y^k, \Gamma^k_{ij})). \tag{1.12}$$

2. Natural transformations.

Let us now recall the general theory of natural transformations due to D. Krupka. We refer to [3] and [4] for more details.

Let L^r_n be the *r-th order differential group* of the n-dimensional Euclidean space $\boldsymbol{R}^n$, that is, the Lie group of all r-jets of local diffeomorphisms of $\boldsymbol{R}^n$ with source and target at the origin $o \in \boldsymbol{R}^n$, where r is any nonnegative integer. Let P and Q be smooth manifolds on which the group L^r_n acts to the left. An *r-th order differential invariant* $f: P \to Q$ is an L^r_n-equivariant map of the left L^r_n-space P to the left L^r_n-space Q, i.e., a map satisfying $f(j^r_0\alpha \cdot p) = j^r_0\alpha \cdot f(p)$ for all $j^r_0\alpha \in L^r_n$ and all $p \in P$. Here the dot $\cdot$ denotes the action of L^r_n on P (or on Q, respectively).

Further, let F^rM denote the bundle of all frames of r-th order over M, which carries a natural structure of a principal L^r_n-bundle $F^rM(M, L^r_n, \pi^r_n)$. We get a natural functor from the category D_n of smooth n-manifolds and injective immersions into the category of principal L^r_n-bundles and L^r_n-bundle morphisms. Here, for any morphism $\varphi: M_1 \to M_2$ of D_n the corresponding morphism $F^r\varphi: F^rM_1 \to F^rM_2$ is given in a familiar way (see [6]).

Finally, for a left L^r_n-space P, let F^r_PM denote the fibre bundle with fibre P, associated to the principal L^r_n-bundle F^rM. We obtain a natural functor F^r_P from the category D_n into the category of fibre bundles and their morphisms. Here, for any morphism $\varphi: M_1 \to M_2$ of D_n the corresponding morphism $F^r_P\varphi: F^r_PM_1 \to F^r_PM_2$ is given by

$$F^r_P\varphi([y, p]) = [F^r\varphi(y), p]$$

for any $[y, p]\in F^r_P M_1$ ($[y, p]$ is the equivalence class of a pair $(y, p)\in F^r M_1\times P$ with respect to the equivalence relation defined by the right action $(y, p)\cdot j^r_0\alpha=(y\cdot j^r_0\alpha, j^r_0\alpha^{-1}\cdot p)$ of L^r_n on $F^r M_1\times P$).

For each manifold M and each differential invariant $f: P\to Q$ we can define a morphism $f_M: F^r_P M\to F^r_Q M$ over the identity map $id: M\to M$ by

$$f_M([y, p])=[y, f(p)]$$

for all $[y, p]\in F^r_P M$. This morphism f_M is called the *realization of a differential invariant* f on the manifold M.

An *r-th order natural transformation* T of the functor F^r_P into the functor F^r_Q is a collection of bundle morphisms $T_M: F^r_P M\to F^r_Q M$ over the identity map $id: M\to M$ $(M\in D_n)$ such that the diagram

$$\begin{array}{ccc} F^r_P M_1 & \xrightarrow{T_{M_1}} & F^r_Q M_1 \\ \downarrow{\scriptstyle F^r_P\varphi} & & \downarrow{\scriptstyle F^r_Q\varphi} \\ F^r_P M_2 & \xrightarrow{T_{M_2}} & F^r_Q M_2 \end{array}$$

is commutative for every morphism $\varphi: M_1\to M_2$ of D_n.

The following theorem due to Krupka [3] says that the problem to find all r-th order natural transformations of F^r_P to F^r_Q is equivalent to the problem of finding all r-th order differential invariants f from P to Q.

THEOREM A. *Let $f: P\to Q$ be an r-th order differential invariant. Then the correspondence $T_f: M\to f_M$, where M is an object of D_n, is a natural transformation of the functor F^r_P to the functor F^r_Q. Moreover, the correspondence $f\to T_f$ is a bijection between the set of all r-th order differential invariants from P to Q and the set of all r-th order natural transformations of F^r_P to F^r_Q.*

In order to apply the method by Krupka to our problem, we shall restrict ourselves to the second order differential invariants. We define functions A^k_i, A^k_{ij} $(1\leqq i\leqq j\leqq n,\ 1\leqq k\leqq n)$ on L^2_n by

$$A^k_i(j^2_0\alpha)=D_i\alpha^k(o)\,,\qquad A^k_{ij}(j^2_0\alpha)=D_iD_j\alpha^k(o)$$

for any local diffeomorphism $\alpha=(\alpha^1, \alpha^2, \cdots, \alpha^n)$ with $\alpha(o)=o\in R^n$, where D_i denotes the partial derivative with respect to the i-th variable in $\boldsymbol{R}^n$. The *system of the canonical* (*global*) *coordinates* on L^2_n is the system of functions $\{B^k_i, B^k_{ij}\}$ $(1\leqq i\leqq j\leqq n,\ 1\leqq k\leqq n)$ on L^2_n given by

$$B^k_i(j^2_0\alpha)=A^k_i(j^2_0\alpha^{-1})\,,\quad B^k_{ij}(j^2_0\alpha)=A^k_{ij}(j^2_0\alpha^{-1})\,,\quad (1\leqq i\leqq j\leqq n,\ 1\leqq k\leqq n)\,.$$

Obviously

$$B_a^k(j_0^2\alpha)A_i^a(j_0^2\alpha)=\delta_i^k \qquad (i,\ k=1,\ 2,\ \cdots,\ n)$$

for all $j_0^2\alpha\in L_n^2$. Thus A_i^k $(i,\ k=1,\ 2,\ \cdots,\ n)$ are well-determined functions of the canonical coordinates B_i^k $(i,\ k=1,\ 2,\ \cdots,\ n)$.

Let us consider the product $P=\boldsymbol{R}^n\times T_n^1\boldsymbol{R}^n$ of $\boldsymbol{R}^n$ and the set $T_n^1\boldsymbol{R}^n$ of all 1-jets with source at $o\in\boldsymbol{R}^n$ and target in $\boldsymbol{R}^n$. We denote by $\{u^k,\ y^k,\ y^k{}_{,i}\}$ $(i,\ k=1,\ 2,\ \cdots,\ n)$ the canonical coordinates on P. We define an action of L_n^2 on P by the formulas

$$\begin{cases} u^k(j_0^2\alpha\cdot u)=A_a^k(j_0^2\alpha)u^a(u)\,, \\ y^k(j_0^2\alpha\cdot y)=A_a^k(j_0^2\alpha)y^a(y)\,, \\ y^k{}_{,i}(j_0^2\alpha\cdot y)=A_{ab}^k(j_0^2\alpha)B_i^b(j_0^2\alpha)y^a(y)+A_a^k(j_0^2\alpha)B_i^b(j_0^2\alpha)y^a{}_{,b}\,, \quad (i,\ k=1,2,\cdots,n) \end{cases}$$

for all $j_0^2\alpha\in L_n^2$, $u\in\boldsymbol{R}^n$ and $y\in T_n^1\boldsymbol{R}^n$. Or briefly, we can write

$$\begin{cases} \bar{u}^k=A_a^ku^a\,, \qquad \bar{y}^k=A_a^ky^a\,, \\ \bar{y}^k{}_{,i}=A_{ab}^kB_i^by^a+A_a^kB_i^by^a{}_{,b}\,, \qquad (i,\ k=1,\ 2,\ \cdots,\ n)\,, \end{cases} \tag{2.1}$$

which is formally the same as (1.2) and (1.3).

Put $Q=\boldsymbol{R}^n\oplus\boldsymbol{R}^{2n}$ and denote by $\{v^k,\ Y^A\}$ $(k=1,\ 2,\ \cdots,\ n,\ A=1,\ 2,\ \cdots,\ 2n)$ the system of canonical coordinates on Q. This can be also written in the form $\{v^k,\ Y^k,\ Y^{k*}\}$ $(k=1,\ 2,\ \cdots,\ n)$, where k^* stands for $n+k$ $(k=1,\ 2,\ \cdots,\ n)$. The space Q has a structure of a left L_n^2-space defined by

$$\begin{cases} \bar{v}^k=A_a^kv^a\,, \qquad \bar{Y}^k=A_a^kY^a\,, \\ \bar{Y}^{k*}=A_{ab}^kv^aY^b+A_a^kY^{a*}\,, \qquad (k=1,\ 2,\ \cdots,\ n)\,, \end{cases} \tag{2.2}$$

which coincides with (1.2) and (1.6).

We see easily that the corresponding associated L_n^2-bundles F_P^2M and F_Q^2M over a manifold M always have canonical bundle projections

$$F_P^2M \longrightarrow TM\,, \qquad F_Q^2M \longrightarrow TM\,. \tag{2.3}$$

Now *we define the problem to find all second order natural transformation of vector fields on a manifold to vector fields on its tangent bundle as the problem to find all those natural transformations of* F_P^2 *to* F_Q^2 *which, for each* M *and via the projections* (2.3), *induce the identity map* $id: TM\to TM$. Hence, by Theorem A, this reduces to the problem to find all second order differential invariants $f:(u^k,\ y^k,\ y^k{}_{,i})\mapsto(v^k,\ Y^A(u^k,\ y^k,\ y^k{}_{,i}))$ from P to Q such that $v^k=u^k$ $(k=1,2,\cdots,n)$ (cf. (1.11)).

In the sequel, we shall use the method of differential equations proposed by

Krupka (see [5]). For the computational purposes, we shall extend the symbols A^k_{ij} and B^k_{ij} also for the case $i \geqq j$ by putting $A^k_{ij}=A^k_{ji}$ and $B^k_{ij}=B^k_{ji}$. Then the range of all indices denoted by small letters will be $\{1, 2, \cdots, n\}$, independently of the other indices. We note that, with respect to this notation, we have to use the following conventions (cf. [4]):

$$\frac{\partial B^k_{ij}}{\partial B^p_{qr}}=-\frac{\partial A^k_{ij}}{\partial B^p_{qr}}=\frac{1}{2}\delta^k_p(\delta^q_i\delta^r_j+\delta^q_j\delta^r_i) \quad (i, j, k, p, q, r=1, 2, \cdots, n).$$

PROPOSITION 2.1. *For any differential invariant $f: P \to Q$ as above, the functions $Y^A(u^k, y^k, y^k{}_{,i})$ satisfy the following system of differential equations:*

$$y^q\frac{\partial Y^k}{\partial f^p}+y^q{}_{,a}\frac{\partial Y^k}{\partial y^p{}_{,a}}-y^a{}_{,p}\frac{\partial Y^k}{\partial y^a{}_{,q}}+u^q\frac{\partial Y^k}{\partial u^p}=Y^q\delta^k_p, \tag{2.4}$$

$$y^q\frac{\partial Y^{k*}}{\partial y^p}+y^q{}_{,a}\frac{\partial Y^{k*}}{\partial y^p{}_{,a}}-y^a{}_{,p}\frac{\partial Y^{k*}}{\partial y^a{}_{,q}}+u^q\frac{\partial Y^{k*}}{\partial u^p}=Y^{q*}\delta^k_p, \tag{2.5}$$

$$y^q\frac{\partial Y^k}{\partial y^p{}_{,r}}+y^r\frac{\partial Y^k}{\partial y^p{}_{,q}}=0, \tag{2.6}$$

$$y^q\frac{\partial Y^{k*}}{\partial y^p{}_{,r}}+y^r\frac{\partial Y^{k*}}{\partial y^p{}_{,q}}=(u^rY^q+u^qY^r)\delta^k_p \tag{2.7}$$

for all $k, p, q, r \in \{1, 2, \cdots, n\}$.

PROOF. The fundamental vector fields on P relative to the action (2.1) are given by

$$\begin{aligned}\xi^q_p&=\frac{\partial \bar{y}^a}{\partial B^q_p}(e)\frac{\partial}{\partial y^a}+\frac{\partial \bar{y}^a{}_{,b}}{\partial B^p_q}(e)\frac{\partial}{\partial y^a{}_{,b}}+\frac{\partial \bar{u}^a}{\partial B^p_q}(e)\frac{\partial}{\partial u^a}\\&=-y^q\frac{\partial}{\partial y^p}-y^q{}_{,a}\frac{\partial}{\partial y^p{}_{,a}}+y^a{}_{,p}\frac{\partial}{\partial y^a{}_{,q}}-u^q\frac{\partial}{\partial u^p},\\ \xi^{qr}_p&=\frac{\partial \bar{y}^a{}_{,b}}{\partial B^p_{qr}}(e)\frac{\partial}{\partial y^a{}_{,b}}=-\frac{1}{2}\left(y^q\frac{\partial}{\partial y^p{}_{,r}}+y^r\frac{\partial}{\partial y^p{}_{,q}}\right),\\ &\qquad (p, q, r=1, 2, \cdots, n),\end{aligned}$$

where e denotes the identity element $j^2_0(id)$ of L^2_n. The *corresponding* fundamental vector fields on Q relative to the action (2.2) are given by

$$\Xi^q_p=\frac{\partial \bar{Y}^A}{\partial B^p_q}(e)\frac{\partial}{\partial Y^A}+\frac{\partial \bar{v}^a}{\partial B^p_q}(e)\frac{\partial}{\partial v^a}=-Y^q\frac{\partial}{\partial Y^p}-Y^{q*}\frac{\partial}{\partial Y^{p*}}-v^q\frac{\partial}{\partial v^p},$$

$$\Xi^{qr}_p=\frac{\partial \bar{Y}^A}{\partial B^p_{qr}}(e)\frac{\partial}{\partial Y^A}=-\frac{1}{2}(u^qY^r+u^rY^q)\frac{\partial}{\partial Y^{p*}}. \quad (p, q, r=1, 2, \cdots, n).$$

Since any L^2_n-equivariant map f from P to Q satisfies

$$f_*(\xi_p^q)=\Xi_p^q \quad \text{and} \quad f_*(\xi_p^{qr})=\Xi_p^{qr}, \quad (p, q, r=1, 2, \cdots, n),$$

we get

$$\xi_p^q(Y^A \circ f)=\Xi_p^q(Y^A) \quad \text{and} \quad \xi_p^{qr}(Y^A \circ f)=\Xi_p^{qr}(Y^A),$$

$$(p, q, r=1, 2, \cdots, n, \ A=1, 2, \cdots, 2n).$$

We have to use also the condition $v^k \circ f=u^k$ $(k=1, 2, \cdots, n)$. Hence we get the system (2.4)-(2.7). *q.e.d.*

The following Lemma will be used for solving the system (2.4)-(2.7) and also the system (3.2)-(3.5) in Section 3.

LEMMA 2.2. *The complete solution of the system of partial differential equations*

$$y^q \frac{\partial Y^k}{\partial y^p}+u^q \frac{\partial Y^k}{\partial u^p}=Y^q \delta_p^k \quad (k, p, q=1, 2, \cdots, n; \ n \geqq 2), \tag{2.8}$$

where $Y^1, Y^2, \cdots, Y^n$ *are functions of* $y^1, y^2, \cdots, y^n, u^1, u^2, \cdots, u^n$ *only, is given by*

$$Y^k=\alpha y^k+\beta u^k \qquad (k=1, 2, \cdots, n),$$

where α *and* β *are arbitrary constants.*

PROOF. (We *do not* use the Einstein summation convention in this proof.) Let us introduce new independent variables in (2.8) by putting

$$z^p=\frac{y^p}{u^p}, \qquad v^p=u^p, \quad (p=1, 2, \cdots, n).$$

Then (2.8) takes of the form

$$\frac{v^q}{v^p}(z^q-z^p)\frac{\partial Y^k}{\partial z^p}+v^q \frac{\partial Y^k}{\partial v^p}=Y^q \delta_p^k. \tag{2.9}$$

Putting $p=q$ in (2.9) we get

$$v^p \frac{\partial Y^k}{\partial v^p}=Y^p \delta_p^k, \tag{2.10}$$

which can be rewritten (at generic points where $v^1 v^2 \cdots v^n \neq 0$) in the form

$$v^k \frac{\partial Y^k}{\partial v^k}=Y^k, \qquad \frac{\partial Y^k}{\partial v^p}=0 \quad (p \neq k). \tag{2.11}$$

We can integrate (2.11) easily, and we obtain at all generic points and hence at any point by continuiy

$$Y^k=\gamma^k v^k \qquad (k=1, 2, \cdots, n), \tag{2.12}$$

where γ^k $(k=1, 2, \cdots, n)$ depend only on z^p $(p=1, 2, \cdots, n)$.

Now, substituting (2.12) into (2.9) we get

(2.13) $$\frac{v^q}{v^p}(z^q-z^p)v^k\frac{\partial\gamma^k}{\partial z^p}+\gamma^k v^q\delta_p^k=\gamma^q v^q\delta_p^k .$$

Putting $k=q$ in (2.13) we obtain

$$\frac{v^k}{v^p}(z^k-z^p)v^k\frac{\partial\gamma^k}{\partial z^p}=0,$$

which implies that $\partial\gamma^k/\partial z^p=0$ holds for $p\neq k$ at all generic points, and hence at any point. Thus γ^k $(k=1, 2, \cdots, n)$ depend only on z^k, $\gamma^k=\gamma^k(z^k)$. Substituting into (2.13) we get, for $p=k$,

$$v^q(z^q-z^k)\frac{\partial\gamma^k}{\partial z^k}+\gamma^k v^q=\gamma^q v^q$$

and, at any generic point,

(2.14) $$(z^q-z^k)\frac{\partial\gamma^k}{\partial z^k}=\gamma^q-\gamma^k .$$

Here $\partial\gamma^k/\partial z^k$ is a function of z^k only, say, $\partial\gamma^k/\partial z^k=f(z^k)$. Then (2.14) gives

$$\gamma^q=[\gamma^k-z^k f(z^k)]+z^q f(z^k) .$$

Because γ^q is a function of z^q only, we see that $f(z^k)$ and $\gamma^k-z^k f(z^k)$ are constants. Let us denote these constants by α^k and β^k, respectively. Then $\gamma^q=\alpha^k z^q+\beta^k$ for each q and each k $(k, q=1, 2, \cdots, n)$. Hence, there exist constants α and β such that

$$\gamma^q=\alpha z^q+\beta \qquad \text{for } q=1, 2, \cdots, n .$$

Substituting into (2.12) we get

(2.15) $$Y^k=\alpha y^k+\beta u^k \quad (k=1, 2, \cdots, n) .$$

On the other hand, we see that (2.15) gives a solution of (2.8) for α and β arbitrary.

THEOREM 2.3. *All differential invariants* $f:(u^k, y^k, y^k{}_{,i})\mapsto(v^k, Y^A)$ *from* $P=\boldsymbol{R}^n\times T_n^1\boldsymbol{R}^n$ *into* $Q=\boldsymbol{R}^n\oplus\boldsymbol{R}^{2n}$ $(n\geqq 2)$ *such that* $v^k=u^k$ $(k=1, 2, \cdots, n)$ *are given, in the canonical coordinates, by*

$$Y^k=\alpha y^k, \quad Y^{k*}=\alpha y^k{}_{,a}u^a+\beta y^k+\gamma^k \qquad (k=1, 2, \cdots, n),$$

where α, β *and* γ *are constants.*

PROOF. (We *do not* use the Einstein summation convention in this proof.) To obtain our differential invariants we solve the system (2.4)-(2.7).

Putting $q=r$ in (2.6) we get

$$y^q \frac{\partial Y^k}{\partial y^p{}_{,q}} = 0 \qquad (k, p, q = 1, 2, \cdots, n),$$

which implies that $\partial Y^k / \partial y^p{}_{,q} = 0$ at each generic point where $y^q \neq 0$ and hence, by the continuity, at any point. Thus Y^k do not depend on $y^p{}_{,q}$ for $p, q = 1, 2, \cdots, n$. Hence (2.4) reduces to (2.8). By Lemma 2.2, the solution of (2.4) and (2.6) is given by

$$Y^k = \alpha y^k + \beta u^k \qquad (k = 1, 2, \cdots, n), \tag{2.16}$$

where α and β are constants.

Substituting (2.16) into (2.7) and putting $q = r$ in the obtained equations we get

$$y^q \frac{\partial Y^{k*}}{\partial y^p{}_{,q}} = u^q(\alpha y^q + \beta u^q)\delta^k_p \qquad (k, p, q = 1, 2, \cdots, n),$$

which can be rewritten in the form

$$y^q \frac{\partial Y^{k*}}{\partial y^k{}_{,q}} = u^q(\alpha y^q + \beta u^q), \qquad \frac{\partial Y^{k*}}{\partial y^p{}_{,q}} = 0, \qquad (k \neq p). \tag{2.17}$$

We can integrate (2.17) with respect to $y^p{}_{,q}$ and obtain (at all generic points and hence at any point)

$$Y^{k*} = \alpha \sum y^k{}_{,a} u^a + \beta \sum y^k{}_{,a} \frac{(u^a)^2}{y^a} + \gamma^k \quad (k = 1, 2, \cdots, n), \tag{2.18}$$

where γ^k $(k = 1, 2, \cdots, n)$ do not depend on $y^p{}_{,q}$ $(p, q = 1, 2, \cdots, n)$. Now, substitute (2.16) and (2.18) into (2.7) for $p = k$. We get

$$\beta\left[\frac{y^q}{y^r}(u^r)^2 + \frac{y^r}{y^q}(u^q)^2\right] = 2\beta u^r u^q \qquad (q, r = 1, 2, \cdots, n),$$

which is possible if and only if

$$\beta = 0. \tag{2.19}$$

Thus we obtain

$$Y^{k*} = \alpha \sum y^k{}_{,a} u^a + \gamma^k \qquad (k = 1, 2, \cdots, n), \tag{2.20}$$

where γ^k $(k = 1, 2, \cdots, n)$ depend on y^p and u^p $(p = 1, 2, \cdots, n)$.

Next, substituting (2.20) into (2.5) we get

$$y^q \frac{\partial \gamma^k}{\partial y^p} + u^q \frac{\partial \gamma^k}{\partial u^p} = \gamma^q \delta^k_p,$$

which is the same as (2.8). Thus, by Lemma 2.2 once again the solution is given by

$$\gamma^k = \beta' y^k + \gamma u^k. \tag{2.21}$$

where β' and γ are constant.

Finally, by (2.19) and (2.21), (2.18) takes of the form

$$Y^{k*}=\alpha\sum y^{k}{}_{,a}u^{a}+\beta' y^{k}+\gamma u^{k} \qquad (k=1, 2, \cdots, n). \tag{2.22}$$

Thus, the solution of the system (2.4)-(2.7) is given by (2.16) with $\beta=0$ and (2.22). This is our assertion.

Let us introduce the *canonical vertical vector field* V on TM by the formula

$$V_{(x,u)}=\text{vertical lift of } u \text{ at } (x, u).$$

This means, in a system of local coordinates on $U\subset M$,

$$V_{(x,u)}=u^{k}X_{k*}. \tag{2.23}$$

Summarizing now Theorem 2.3 and formulas (1.7), (1.8), (2.23), we obtain our first main Theorem in the following form:

THEOREM 2.4. *Any vector field on the tangent bundle TM which comes from a second order natural transformation of a vector fields on a manifold M ($\dim M\geqq 2$) is a linear combination (with constant coefficients) of the vertical lift and complete lift of the given vector field on M and of the canonical vector field on TM.*

3. Pointwise natural transformations.

In this Section we shall find all "pointwise" second order natural transformations of vector fields on a manifold with a symmetric affine connection to vector fields on its tangent bundle. Here "pointwise" means that any *value* of the transformed vector field on TM is determined by the corresponding *value* of the given vector field on M (and by the given connection). As we recalled in Section 1, the horizontal lift is a classical example of such a transformation. Our main Theorem in this section is Theorem 3.3.

Let us consider the vector space $V=\boldsymbol{R}^{n}\oplus\boldsymbol{R}^{n}\oplus[\boldsymbol{R}^{n}\otimes(\boldsymbol{R}^{n*}\odot\boldsymbol{R}^{n*})]$, where $\boldsymbol{R}^{n*}$ is the dual space to $\boldsymbol{R}^{n}$ and $\oplus$, $\otimes$, $\odot$ denote the direct sum, the tensor product, the symmetric product, respectively. We denote by $\{u^{k}, y^{k}, \Gamma^{k}_{ij}\}$ ($1\leqq i\leqq j\leqq n$, $1\leqq k\leqq n$) the canonical coordinates on V. We define an action of L^{2}_{n} on V by the formulas

$$\left\{\begin{array}{l}\bar{u}^{k}=A^{k}_{a}u^{a}, \quad \bar{y}^{k}=A^{k}_{a}y^{a}, \\ \bar{\Gamma}^{k}_{ij}=A^{k}_{a}(B^{b}_{i}B^{c}_{j}\Gamma^{a}_{bc}+B^{a}_{ij}), \qquad (1\leqq i\leqq j\leqq n,\ 1\leqq k\leqq n),\end{array}\right. \tag{3.1}$$

which is formally the same as (1.2) and (1.9).

Further, consider the vector space Q in Section 2 with a natural structure of a left L_n^2-space defined by (2.2).

By a similar observation to in Section 2, our problem reduces to fined all second order differential invariants $f:(u^k, y^k, \Gamma_{ij}^k)\mapsto(v^k, Y^A(u^k, y^k, \Gamma_{ij}^k))$ from V to Q such that $v^k=u^k$ $(k=1, 2, \cdots, n)$ (cf. (1.12)).

In the sequel, we also extend the symbol Γ_{ij}^k for the case $i\geqq j$ by putting $\Gamma_{ij}^k=\Gamma_{ji}^k$. With respect to this notation, we have to use the following conventions:

$$\frac{\partial \Gamma_{ij}^k}{\partial \Gamma_{qr}^p}=\frac{1}{2}\delta_p^k(\delta_i^q\delta_j^r+\delta_j^q\delta_i^r) \qquad (i, j, k, p, q, r=1, 2, \cdots, n).$$

PROPOSITION 3.1. *For any differential invariant $f: V\to Q$ as above, the functions $Y^A(u^k, y^k, \Gamma_{ij}^k)$ satisfy the following system of differential equations:*

$$y^q\frac{\partial Y^k}{\partial y^p}+(\delta_p^a\Gamma_{bc}^q-\delta_b^q\Gamma_{pc}^a-\delta_c^q\Gamma_{bp}^a)\frac{\partial Y^k}{\partial \Gamma_{bc}^a}+u^q\frac{\partial Y^k}{\partial u^p}=Y^q\delta_p^k, \tag{3.2}$$

$$y^q\frac{\partial Y^{k*}}{\partial y^p}+(\delta_p^a\Gamma_{bc}^q-\delta_b^q\Gamma_{pc}^a-\delta_c^q\Gamma_{bp}^a)\frac{\partial Y^{k*}}{\partial \Gamma_{bc}^a}+u^q\frac{\partial Y^{k*}}{\partial u^p}=Y^{q*}\delta_p^k, \tag{3.3}$$

$$\frac{\partial Y^k}{\partial \Gamma_{qr}^p}=0, \tag{3.4}$$

$$\frac{\partial Y^{k*}}{\partial \Gamma_{qr}^p}=-\frac{1}{2}(u^qY^r+u^rY^q)\delta_p^k \tag{3.5}$$

for all $k, p, q, r\in\{1, 2, \cdots, n\}$.

PROOF. The fundamental vector fields on V relative to the action (3.1) are

$$\eta_p^q=-y^q\frac{\partial}{\partial y^p}+(-\delta_p^a\Gamma_{bc}^q+\delta_b^q\Gamma_{pc}^a+\delta_c^q\Gamma_{bp}^a)\frac{\partial}{\partial \Gamma_{bc}^a}-u^q\frac{\partial}{\partial u^p},$$

$$\eta_p^{qr}=\frac{\partial}{\partial \Gamma_{qr}^p}, \qquad (p, q, r=1, 2, \cdots, n).$$

Since any L_n^2-equivariant map f from V to Q satisfies

$$f_*(\eta_p^q)=\Xi_p^q \quad\text{and}\quad f_*(\eta_p^{qr})=\Xi_p^{qr} \qquad (p, q, r=1, 2, \cdots, n),$$

we get the system (3.2)-(3.5). *q.e.d.*

THEOREM 3.2. *All differential invariants $f:(u^k, y^k, \Gamma_{ij}^k)\mapsto(v^k, Y^A)$ from $V=\boldsymbol{R}^n\oplus\boldsymbol{R}^n\oplus[\boldsymbol{R}^n\otimes(\boldsymbol{R}^{n*}\odot\boldsymbol{R}^{n*})]$ into $Q=\boldsymbol{R}^n\oplus\boldsymbol{R}^{2n}$ $(n\geqq 2)$ such that $v^k=u^k$ $(k=1, 2, \cdots, n)$ are given, in the canonical coordinates, by*

$$Y^k=\alpha y^k+\beta u^k,$$

$$Y^{k*}=-\alpha\Gamma^{k}_{ab}u^{a}y^{b}-\beta\Gamma^{k}_{ab}u^{a}u^{b}+\varphi y^{k}+\psi u^{k},$$

where α, β, φ *and* ψ *are constants.*

PROOF. We shall solve the system (3.2)-(3.5). According to (3.4), the system (3.2) reduces to (2.8). Thus, by Lemma 2.2, the solution of (3.2) and (3.4) is given by

$$Y^{k}=\alpha y^{k}+\beta u^{k}\qquad (k=1,2,\cdots,n), \tag{3.6}$$

where α and β are constants.

Substituting (3.6) into (3.5) we get

$$\frac{\partial Y^{k*}}{\partial\Gamma^{p}_{qr}}=-\frac{1}{2}\delta^{k}_{p}(\delta^{q}_{a}\delta^{r}_{b}+\delta^{q}_{b}\delta^{r}_{a})(\alpha u^{a}y^{b}+\beta u^{a}u^{b}),$$

from which we get by integration with respect to Γ^{p}_{qr}

$$Y^{k*}=-\alpha\Gamma^{k}_{ab}u^{a}y^{b}-\beta\Gamma^{k}_{ab}u^{a}u^{b}+\gamma^{k}, \tag{3.7}$$

where γ^{k} $(k=1,2,\cdots,n)$ do not depend on Γ^{p}_{qr} $(p,q,r=1,2,\cdots,n)$. Substituting (3.7) into (3.3) we obtain, after some calculations,

$$y^{q}\frac{\partial\gamma^{k}}{\partial y^{p}}+u^{q}\frac{\partial\gamma^{k}}{\partial u^{p}}=\gamma^{q}\delta^{k}_{p}.$$

Thus, by Lemma 2.2 once again, we obtain

$$\gamma^{k}=\varphi y^{k}+\psi u^{k}\qquad (k=1,2,\cdots,n), \tag{3.8}$$

where φ and ψ are constants.

Summarizing above, the solution of the system (3.2)-(3.5) is given by (3.6) and (3.7) with (3.8). This is our assertion.

According to Theorem A, we have, from (1.7), (1.9) and Theorem 3.2, our second main Theorem in the following form.

THEOREM 3.3. *Let* M ($\dim M\geqq 2$) *be a manifold with a symmetric affine connection* ∇ *and denote by* TM *its tangent bundle. Then a vector field* Y *on* TM *comes from a pointwise second order natural transformation of the connection* ∇ *and a vector field* y *on* M *if and only if, at each point* $(x,u)\in TM$, $Y(x,u)$ *is a linear combination with constant coefficients of the vertical and horizontal lifts of* y *and* u *with respect to* ∇, *that is,*

$$Y(x,u)=\alpha y^{h}(x,u)+\beta u^{h}(x,u)+\varphi y^{v}(x,u)+\psi u^{v}(x,u),$$

where α, β, φ *and* ψ *are constants.*

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Added in proof. I. Kolář investigates natural operators transforming every vector field on a manifold M into a vector field on FM, where F is any natural bundle corresponding to a product preserving functor. His method differs from ours.

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