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APPROXIMATIVE SHAPE III
—FIXED POINT THEOREMS—

By

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§ 0. Introduction.

This paper is a continuation of [**45–46**]. We introduced approximative shape in [**45**], and discussed approximative shape properties of spaces and generalized ANRs in [**46**]. In this paper we shall discuss approximative shape properties of maps and fixed point theorems.

The Lefschetz-Hopf fixed point theorem is a well-known fixed point theorem formulated in homological or cohomological terms. It was first discovered by Lefschetz for compact manifolds and then extended by him to manifolds with boundary. Hopf gave a completly different and simple proof for finite polyhedra and then Lefschetz extended it to compact metric ANRs (see Lefschetz [**33**]). It was extended to compact metric AANR_Ms by Granas [**22**], to compact metric AANR_Cs by Clapp [**9**] and to metric AANR_Cs by Powers [**40**].

Borsuk [**3, 5**] introduced nearly extendable sets, in notation NE-sets, and nearly extendable maps, in notation NE-maps, between compact metric spaces. He [**4, 6**] showed the Lefschetz-Hopf fixed point theorem for NE-maps and Gauthier [**19–21**] extended it to NE-maps between compact spaces.

Borsuk and Ulam [**7**] introduced symmetric products. This notion was generalized as G-product where G is a subgroup of all permutations of coordinates. Maxwell [**36**] showed a fixed point theorem for maps into G-products of finite polyhedra. The Maxwell fixed point theorem contains the Lefschetz-Hopf fixed point theorem as a special case. The Maxwell fixed point theorem is extended to maps into G-products of compact metric ANRs by Masih [**35**] and to maps into G-products of compact metric AANR_Ns by Vora [**42**].

In this paper we investigate the following topics. In § **1** we introduce NE-sets and NE-maps between arbitrary spaces. We show that the notions of ap-approximative movability and NE-sets are equivalent. We show that the notions of AANR_C and NE-sets are equivalent for compact metric spaces, but not for compact spaces. This gives a negative answer to a question of Gauthier [**20**]. In § **2** we show that products, suspensions and cones preserve NE-maps. In § **3**

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we investigate approximative shape properties of hyperspaces. In § **4** we show that G-products induce shape functors. In § **5** we introduce Maxwell homomorphisms for shapings and investigate their properties. In § **6** we show the Maxwell fixed point theorem for NE-maps between compact spaces. It contains the Lefschetz-Hopf fixed point theorem for NE-maps between compact spaces. Our proofs depend only on the Maxwell and the Lefschetz-Hopf fixed point theorems for finite polyhedra.

We show the fixed point property of cones and hyperspaces of approximatively movable compact spaces. These give partial answers to questions raised by Rogers [**41**] and Nadler [**39**].

We assume that the reader is familiar with the theory of ANRs and with shape theory. Borsuk [**1**] and Hu [**23**] are standard textboogs for the theory of ANRs. Borsuk [**2**] and Mardešić and Segal [**34**] are standard textbooks for shape theory. For undefined notations and terminology see Hu [**23**] and Mardešić and Segal [**34**], which is quoted by MS [**34**]. We use the same notations and terminology as in [**45-46**]. We quote results in [**45-46**] as follows; for example (I. 3.3) and (II. 5.5) denote theorem (3.3) in [**45**] and theorem (5.5) in [**46**], respectively.

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§ 1. Nearly extendable maps.

The notions of nearly extendable maps and nearly extendable sets were introduced by Borsuk [**3-6**] for compact metric spaces and then by Gauthier [**19-21**] for compact spaces. Dugundji [**11**] introduced the notion of Borsuk presentations. In this section we show that resolutions and approximative resolutions are better than Borsuk presentations. We discuss these notions for arbitrary maps and spaces, and we study their properties.

Let $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ and $(\mathcal{Y}, \mathcal{V})=\{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ be approximative inverse systems in **TOP**. Let $\boldsymbol{f}=\{f, f_b : b\in B\} : (\mathcal{X}, \mathcal{U})\rightarrow(\mathcal{Y}, \mathcal{V})$ be an approximative system map in **TOP**.

We say that $\boldsymbol{f}$ is nearly extendable provided that it satisfies the following condition:

(NE) For each $b\in B$ there exists $a_0>f(b)$ with the property; for each $b'>b$ there exists a map $h : X_{a_0}\rightarrow Y_{b'}$ such that $(f_b p_{a_0,f(b)}, q_{b',b}h)<st\mathcal{V}_b$.

(1.1) LEMMA. *Let* $\boldsymbol{f}, \boldsymbol{f}'$: $(\mathcal{X}, \mathcal{U})\rightarrow(\mathcal{Y}, \mathcal{V})$ *be approximative system maps and*

$\boldsymbol{f} \equiv : \boldsymbol{f}'$. *If* $\boldsymbol{f}$ *is nearly extendable, then so is* $\boldsymbol{f}'$.

PROOF. We put $\boldsymbol{f}'=\{f', f'_b : b\in B\}$. Without loss of generality we may assume that $\boldsymbol{f}=:\boldsymbol{f}'$ and show that $\boldsymbol{f}'$ satisfies (NE). Take any $b\in B$ and then by (AI3) there exists $b_1>b$ such that $q_{b_1,b}^{-1}\mathcal{V}_b > st\mathcal{V}_{b_1}$. By the assumption there exists $a_1>f(b_1)$ satisfying (NE) for $\boldsymbol{f}$ and b_1. Since $\boldsymbol{f}$ satisfies (AM2) and $\boldsymbol{f}=:\boldsymbol{f}'$, there exists $a_2>f(b), f'(b), a_1$ such that

(1) $$(f_b p_{a_2,f(b)}, q_{b_1,b} f_{b_1} p_{a_2,f(b_1)}) < \mathcal{V}_b \quad \text{and}$$

(2) $$(f_b p_{a_2,f(b)}, f'_b p_{a_2,f'(b)}) < \mathcal{V}_b.$$

Take any $b_2>b$ and then there exists $b_3>b_1, b_2$. By the choice of a_1 there exists a map $k: X_{a_1}\to Y_{b_3}$ such that

(3) $$(f_{b_1} p_{a_1,f(b_1)}, q_{b_3,b_1} k) < st\mathcal{V}_{b_1}.$$

By (3) and the choice of b_1

(4) $$(q_{b_1,b} f_{b_1} p_{a_2,f(b_1)}, q_{b_3,b} k p_{a_2,a_1}) < \mathcal{V}_b.$$

From (1), (2) and (4)

(5) $$(f'_b p_{a_2,f'(b)}, q_{b_3,b} k p_{a_2,a_1}) < st\mathcal{V}_b.$$

(5) means that a_2 and the map $q_{b_3,b_2} k p_{a_2,a_1} : X_{a_2}\to Y_{b_2}$ satisfy (NE) for $\boldsymbol{f}'$ and b. Hence $\boldsymbol{f}'$ is nearly extendable. ■

Thus by (1.1) we say that $[\boldsymbol{f}]$ is nearly extendable provided that $\boldsymbol{f}$ is nearly extendable. Let $\boldsymbol{g}=\{g, g_c : c\in C\} : (\mathcal{Y}, \mathcal{V})\to(\mathcal{X}, \mathcal{W})=\{(Z_c, \mathcal{W}_c), r_{c',c}, C\}$ be an approximative system map.

(1.2) LEMMA. *If one of* $[\boldsymbol{f}]$ *and* $[\boldsymbol{g}]$ *is nearly extendable, then so is* $[\boldsymbol{g}][\boldsymbol{f}]$.

PROOF. Let $u: C\to C$ be a 1-refinement function of $(\mathcal{X}, \mathcal{W})$. Then $[\boldsymbol{g}][\boldsymbol{f}] =[\boldsymbol{r}(u)(\boldsymbol{gf})]$. First we assume that $[\boldsymbol{g}]$ is nearly extendable and show that $\boldsymbol{r}(u)(\boldsymbol{gf})$ is nearly extendable. Take any $c\in C$. By the assumption there exists $b_0>gu(c)$ satisfying (NE) for $\boldsymbol{g}$ and $u(c)$. By (AM2) there exists $a_0>fgu(c), f(b_0)$ such that

(1) $$(f_{gu(c)} p_{a_0,fgu(c)}, q_{b_0,gu(c)} f_{b_0} p_{a_0,f(b_0)}) < \mathcal{V}_{gu(c)}.$$

Take any $c_1>c$. By the choice of b_0 there exists a map $k: Y_{b_0}\to Z_{u(c_1)}$ such that $(g_{u(c)} q_{b_0,gu(c)}, r_{u(c_1),u(c)} k) < st\mathcal{W}_{u(c)}$. Since u is a 1-refinement function,

(2) $$(r_{u(c),c} g_{u(c)} q_{b_0,gu(c)} f_{b_0} p_{a_0,f(b_0)}, r_{u(c_1),c} k f_{b_0} p_{a_0,f(b_0)}) < \mathcal{W}_c.$$

By (1)

(3) $$(r_{u(c),c}g_{u(c)}f_{gu(c)}p_{a_0,fgu(c)},\ r_{u(c),c}g_{u(c)}q_{b_0,gu(c)}f_{b_0}p_{a_0,f(b_0)})<\mathcal{W}_c.$$

By (2) and (3) $(r_{u(c),c}g_{u(c)}f_{gu(c)}p_{a_0,fgu(c)},\ r_{u(c_1),c}kf_{b_0}p_{a_0,f(b_0)})<st\mathcal{W}_c$. This means that a_0 and the map $r_{u(c_1),c_1}kf_{b_0}p_{a_0,f(b_0)}: X_{a_0}\to Z_{c_1}$ satisfy the required condition. Then $\boldsymbol{r}(u)(\boldsymbol{gf})$ is nearly extendable and hence so is $[\boldsymbol{g}][\boldsymbol{f}]$.

Next we assume that $[\boldsymbol{f}]$ is nearly extendable and show that $[\boldsymbol{g}][\boldsymbol{f}]$ is nearly extendable. Take any $c\in C$. By the assumption there exists $a_0>fgu(c)$ satisfying (NE) for $\boldsymbol{f}$ and $gu(c)$. Take any $c_1>c$. By (AM2) there exists $b_0>gu(c_1),\ gu(c)$ such that

(4) $$(g_{u(c)}q_{b_0,gu(c)},\ r_{u(c_1),u(c)}g_{u(c_1)}q_{b_0,gu(c_1)})<\mathcal{W}_{u(c)}.$$

By the choice of a_0 there exists a map $k: X_{a_0}\to Y_{b_0}$ such that $(f_{gu(c)}p_{a_0,fgu(c)},\ q_{b_0,gu(c)}k)<st\mathcal{V}_{gu(c)}$. Since u is a 1-refinement function, by (AM1) and (2.2)

(5) $$(r_{u(c),c}g_{u(c)}f_{gu(c)}p_{a_0,fgu(c)},\ r_{u(c),c}g_{u(c)}q_{b_0,gu(c)}k)<\mathcal{W}_c.$$

By (4) $(r_{u(c),c}g_{u(c)}q_{b_0,gu(c)}k,\ r_{u(c_1),c}g_{u(c_1)}q_{b_0,gu(c_1)}k)<\mathcal{W}_c$. Then by this and (5) $(r_{u(c),c}g_{u(c)}f_{gu(c)}p_{a_0,fgu(c)},\ r_{u(c_1),c}g_{u(c_1)}q_{b_0,gu(c_1)}k)<st\mathcal{W}_c$. This means that a_0 and the map $r_{u(c_1),c_1}g_{u(c_1)}q_{b_0,gu(c_1)}k: X_{a_0}\to Z_{c_1}$ satisfy the required condition. Then $\boldsymbol{r}(u)(\boldsymbol{gf})$ is nearly extendable and hence so is $[\boldsymbol{g}][\boldsymbol{f}]$. ■

(1.3) COROLLARY. *Let $(\mathcal{X}, \mathcal{U})$ be an approximative inverse system. Then the following statements are equivalent:*

(i) $1_{(\mathcal{X},\mathcal{U})}: (\mathcal{X}, \mathcal{U})\to(\mathcal{X}, \mathcal{U})$ *is nearly extendable.*

(ii) *Any approximative system map $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ is nearly extendable for each approximative inverse system $(\mathcal{Y}, \mathcal{V})$.*

(iii) *Any approximative system map $\boldsymbol{f}: (\mathcal{Y}, \mathcal{V})\to(\mathcal{X}, \mathcal{U})$ is nearly extendable for each approximative inverse system $(\mathcal{Y}, \mathcal{V})$.* ■

Let $f: X\to Y$ be a map. Let $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$, $\boldsymbol{p}': X\to(\mathcal{X}, \mathcal{U})'$ and $\boldsymbol{q}: Y\to(\mathcal{Y}, \mathcal{V})$, $\boldsymbol{q}': Y\to(\mathcal{Y}, \mathcal{V})'$ be approximative **AP**-resolutions. Let $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{f}': (\mathcal{X}, \mathcal{U})'\to(\mathcal{Y}, \mathcal{V})'$ be approximative resolutions of f with respect to $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{p}'$, $\boldsymbol{q}'$ respectively.

(1.4) LEMMA. *If $\boldsymbol{f}$ is nearly extendable, then so is $\boldsymbol{f}'$.*

(1.4) follows from (I. 5.1), (iv) of (I. 5.3) and (1.1). ■

Let $\boldsymbol{p}=\{p_a: a\in A\}: X\to\mathcal{X}=\{X_a, p_{a',a}, A\}$ and $\boldsymbol{q}=\{q_b: b\in B\}: Y\to\mathcal{Y}=\{Y_b, q_{b',b}, B\}$ be **AP**-resolutions. Let $\boldsymbol{f}=\{f, f_b: b\in B\}: \mathcal{X}\to\mathcal{Y}$ be a system map and $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ an **AP**-resolution of f. We say that $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is nearly extendable provided that it satisfies the following condition:

$(NE)_1$ For each $b\in B$ and for each $\mathcal{V}\in Cov(Y_b)$ there exists $a_0>f(b)$ with the property; for each $b'>b$ there exists a map $k: X_{a_0}\to Y_{b'}$ such that $(f_b p_{a_0, f(b)}, q_{b', b}k)<\mathcal{V}$.

(1.5) LEMMA. *Let* $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ *and* $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$ *be* **AP**-*resolutions of* f. *If* $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ *satisfies* (NE)$_1$, *then so does* $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$.

PROOF. Put $\boldsymbol{r}=\{r_c: c\in C\}: X\to\mathcal{R}=\{R_c, r_{c', c}, C\}$, $\boldsymbol{s}=\{s_d: d\in D\}: Y\to\mathcal{S}=\{S^d, s_{d', d}, D\}$ and $\boldsymbol{g}=\{g, g_d: d\in D\}$. Take any $d\in D$ and any $\mathcal{W}\in Cov(S_d)$. Take $\mathcal{W}_1\in Cov(S_d)$ such that $st\mathcal{W}_1<\mathcal{W}$. There exist $\mathcal{W}_2, \mathcal{W}_3\in Cov(S_d)$ such that $\mathcal{W}_2$ satisfies (R2) for $\boldsymbol{r}$ and $\mathcal{W}_1$, and $\mathcal{W}_3$ satisfies (R2) for $\boldsymbol{q}$ and $\mathcal{W}_1$. Take $\mathcal{W}_4\in Cov(S_d)$ such that $st\mathcal{W}_4<\mathcal{W}_2\wedge\mathcal{W}_3$. By (R1) there exist $b\in B$ and a map $h: Y_b\to S_d$ such that

$$(1)\qquad (hq_b, s_d)<\mathcal{W}_4.$$

By the assumption there exists $a_0>f(b)$ satisfying (NE)$_1$ for $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$, b and $h^{-1}\mathcal{W}_1$. By (R1) there exist $c_0>g(d)$ and a map $i: R_{c_0}\to X_{a_0}$ such that $(ir_{c_0}, p_{a_0})<(hf_b p_{a_0, f(b)})^{-1}\mathcal{W}_4$. Thus $(hf_b p_{a_0, f(b)}ir_{c_0}, hf_b p_{f(b)})<\mathcal{W}_4$. Since $f_b p_{f(b)}=q_b f$ by (RM2),

$$(2)\qquad (hf_b p_{a_0, f(b)}ir_{c_0}, hq_b f)<\mathcal{W}_4.$$

By (1) $(hq_b f, s_d f)<\mathcal{W}_4$. Since $s_d f=g_d r_{g(d)}=g_d r_{c_0, g(d)}r_{c_0}$ by (RM2),

$$(3)\qquad (hq_b f, g_d r_{c_0, g(d)}r_{c_0})<\mathcal{W}_4.$$

By (2) and (3) $(hf_b p_{a_0, f(b)}ir_{c_0}, g_d r_{c_0, g(d)}r_{c_0})<st\mathcal{W}_4<\mathcal{W}_2$. By the choice of $\mathcal{W}_2$ there exists $c_1>c_0$ such that

$$(4)\qquad (hf_b p_{a_0, f(b)}ir_{c_1, c_0}, g_d r_{c_1, g(d)})<\mathcal{W}_1.$$

We now show that c_1 is the required index. Take and $d_1>d$. By (R1) there exist $b_1>b$ and a map $j: Y_{b_1}\to S_{d_1}$ such that $(jq_{b_1}, s_{d_1})<s_{d_1, d}^{-1}\mathcal{W}_4$. Thus $(s_{d_1, d}jq_{b_1}, s_d)<\mathcal{W}_4$ and then by (1) $(hq_{b_1, b}q_{b_1}, s_{d_1, d}jq_{b_1})<st\mathcal{W}_4<\mathcal{W}_3$. By the choice of $\mathcal{W}_3$ there exists $b_2>b_1$ such that

$$(5)\qquad (hq_{b_2, b}, s_{d_1, d}jq_{b_2, b_1})<\mathcal{W}_1.$$

By the choice of a_0 there exists a map $k: X_{a_0}\to Y_{b_2}$ such that $(f_b p_{a_0, f(b)}, q_{b_2, b}k)<h^{-1}\mathcal{W}_1$. Thus

$$(6)\qquad (hf_b p_{a_0, f(b)}ir_{c_1, c_0}, hq_{b_2, b}kir_{c_1, c_0})<\mathcal{W}_1.$$

By (5) $(hq_{b_2, b}kir_{c_1, c_0}, s_{d_1, d}jq_{b_2, b_1}kir_{c_1, c_0})<\mathcal{W}_1$. From this, (4) and (6) $(g_d r_{c_1, g(d)}, s_{d_1, d}jq_{b_2, b_1}kir_{c_1, c_0})<st\mathcal{W}_1<\mathcal{W}$. This means that c_1 and the map $jq_{b_2, b_1}kir_{c_1, c_0}: R_{c_1}\to S_{d_1}$ satisfy (NE)$_1$ for $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$. Hence $(\boldsymbol{g}, \boldsymbol{r}, \boldsymbol{s})$ is nearly extendable. ■

By (I. 4.9) there exist approximative **ANR**-resolutions $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V})$ and an approximative resolution $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$ such that $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is an **ANR**-resolution of f.

(1.6) LEMMA. *$\boldsymbol{f}$ satisfies* (NE) *iff $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ satisfies* $(\mathrm{NE})_1$.

In the same way as in (II. 1.6) we can easily show (1.6). Thus from (1.4)-(1.6) we have the following:

(1.7) THEOREM. *Let $f: X \to Y$ be a map. Then the following statements are equivalent*:

(i) *Any/some approximative* **AP**-*resolution of f is nearly extendable.*

(ii) *Any/some* **AP**-*resolution of f is nearly extendable.* ■

Thus we say that a map $f: X \to Y$ is a nearly extendable map, in notation NE-map, provided that it satisfies one of the conditions in (1.7). A space X is a nearly extendable set, in notation NE-set, provided that $1_X: X \to X$ is an NE-map.

(1.8) COROLLARY. *Let $f: X \to Y$ and $g: Y \to Z$ be maps. If one of maps f and g is an* NE-*map, then so is gf.* ■

(1.9) COROLLARY. *Let X be a space. Then the following statements are equivalent*:

(i) *X is an* NE-*set.*

(ii) *Any map $f: X \to Y$ is an* NE-*map for any space Y.*

(iii) *Any map $f: Y \to X$ is an* NE-*map for any space Y.* ■

(1.10) LEMMA. *A map $f: X \to Y$ is nearly extendable iff so is $CT(f): CT(X) \to CT(Y)$.*

(1.10) follows from (I. 6.8), (I. 6.10), (1.1) and (1.2). ■

(1.11) LEMMA. *Let $(\mathcal{X}, \mathcal{U})$ be an approximative inverse system. Then $(\mathcal{X}, \mathcal{U})$ is AM iff $1_{(\mathcal{X}, \mathcal{U})}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{X}, \mathcal{U})$ is nearly extendable.*

PROOF. First we assume that $(\mathcal{X}, \mathcal{U})$ is AM. Take any $a \in A$. By the assumption there exists $a_0 > a$ satisfying (AM) for a. Take any $a' > a$. By the choice of a_0 there exists a map $r: X_{a_0} \to X_{a'}$ such that $(p_{a_0 a}, p_{a' a} r) < \mathcal{U}_a$. Thus $(1_{X_a} p_{a_0 a}, p_{a' a} r) < st\mathcal{U}_a$ and hence $1_{(\mathcal{X}, \mathcal{U})}$ is nearly extendable.

Next we assume that $1_{(\mathcal{X}, \mathcal{U})}$ is nearly extendable. Take any $a \in A$. Then there exists $a_1 > a$ such that $p_{a_1 a}^{-1} \mathcal{U}_a > st\mathcal{U}_{a_1}$. Since $1_{(\mathcal{X}, \mathcal{U})}$ is nearly extendable, there exists $a_2 > a_1$ satisfying (NE) for $1_{(\mathcal{X},\mathcal{U})}$ and a_1. Take any $a' > a$ and take any $a_3 > a', a_1$. By the choice of a_2 there exists a map $r : X_{a_2} \to X_{a_3}$ such that $(1_{X_{a_1}} p_{a_2, a_1}, p_{a_3, a_1} r) < st\mathcal{U}_{a_1}$. Thus $(p_{a_2, a}, p_{a_3, a} r) < \mathcal{U}_a$. This means that a_2 and the map $p_{a_3, a'} r : X_{a_2} \to X_{a'}$ satisfying (AM) for $(\mathcal{X}, \mathcal{U})$ and a. Hence $(\mathcal{X}, \mathcal{U})$ is AM. ■

(1.11) implies the following:

(1.12) THEOREM. *A space X is an* NE-*set iff X is* AM. ■

We now consider Borsuk's approach to NE-maps. Let X and Y be paracompact M-spaces and let $f : X \to Y$ be a map. By (I. 3.17) there exist closed embeddings $h_X : X \to M_X$ and $h_Y : Y \to M_Y$ into AR(**PM**)s M_X and M_Y. We say that f is nearly extendable with respect to h_X and h_Y provided that there exists a map $F : M_X \to M_Y$ satisfying the following two conditions:

(NE1) $F | h_X(X) = h_Y f h_X^{-1}$.

(NE2) For each $\mathcal{V} \in \mathcal{C}ov(M_Y)$ there exists a neighborhood U_0 of $h_X(X)$ in M_X with the following property; for each neighborhood V of $h_Y(Y)$ in M_Y there exists a map $g : U_0 \to V$ such that $(F | U_0, jg) < \mathcal{V}$. Here $j : V \to M_Y$ is the inclusion map.

We say that F realizes the NE-property of f with respect to h_X and h_Y.

(1.13) LEMMA. *If F realizes the* NE-*property of f with respect to h_X and h_Y and $F' : M_X \to M_Y$ satisfies* (NE1), *then F' also realizes the* NE-*property of f with respect to h_X and h_Y.*

It is not difficult to show (1.13). Thus the choice of maps $F : M_Y \to M_Y$ satisfying (NE1) is immaterial. Let $h'_X : X \to M'_X$ and $h'_Y : Y \to M'_Y$ be closed embeddings into AR(**PM**)s M'_X and M'_Y.

(1.14) LEMMA. *If f is nearly extendable with respect to h_X and h_Y, then f is nearly extendable with respect to h'_X and h'_Y.*

By a straightforward argument we can show (1.14). Thus we may say that f is a nearly extendable map, in notation NE-map, in the sense of Borsuk provided that f is nearly extendable with respect to any/some h_X and h_Y.

(1.15) THEOREM. *Let X and Y be paracompact M-spaces and let $f : X \to Y$ be a map. Then f is nearly extendable in our sense iff f is nearly extendable in the sense of Borsuk.*

PROOF. By (I. 3.17) there exist AR(**PM**)s M_X and M_Y which contain X and

Y as closed subsets, respectively. Then there exists a map $F: M_X \to M_Y$ such that $F|X=f$. By using F and (I. 4.10) we have ANR(**PM**)-resolutions $\boldsymbol{p}=\{p_a : a\in A\} : X\to \mathcal{A}\mathcal{V}(X, M_X)=\{U_a, p_{a',a}, A\}$, $\boldsymbol{q}=\{q_b : b\in B\} : Y\to \mathcal{A}\mathcal{V}(Y, M_Y)=\{V_b, q_{b',b}, B\}$ and a resolution $\boldsymbol{f}=\{f, f_b : b\in B\} : \mathcal{A}\mathcal{V}(X, M_X)\to \mathcal{A}\mathcal{V}(Y, M_Y)$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. By (ii) of (I. 3.17) we may assume that all U_a and V_b are ANR(**PM**)-open neighberhoods of $h_X(X)$ and $h_Y(Y)$ in M_X and M_Y, respectively.

First we assume that f is nearly extendable in the sense of Borsuk and show that $\boldsymbol{f}$ satisfies $(\mathrm{NE})_1$. Take any $b\in B$ and any $\mathcal{V}'\in \mathcal{C}ov(V_b)$. By (ii) of (I. 3.17) there exists $b_1>b$ such that $V_{b_1}\subset \overline{V}_{b_1}\subset V_b$. Put $\mathcal{V}=\{V : V\in \mathcal{V}'\}\cup \{M_Y-\overline{V}_{b_1}\}\in \mathcal{C}ov(M_Y)$. There exists a neigborhood U of X in M_X satisfying (NE2) for $\mathcal{V}$. By (ii) of (I. 3.17) there exists $a_0>f(b)$ such that $U_{a_0}\subset U_{f(b)}\cap U$. We show that a_0 is the required index. Take any $b'>b$ and then there exists $b_2>b', b_1$ such that $V_{b_2}\subset V_{b_1}\cap V_{b'}$. By the choice of U there exists a map $g: U\to V_{b_2}$ such that

$$(1) \qquad (F|U, jg)<\mathcal{V}.$$

Here $j: V_{b_2}\to M_Y$ is the inclusion map.

For each $x\in U_{a_0}$ by (1) there exists $K\in \mathcal{V}$ such that $F(x)$, $g(x)\in K$. However $g(x)\in V_{b_2}\subset \overline{V}_{b_1}$ and $F(x)\in V_b$ because $U_{a_0}\subset U_{f(b)}\subset F^{-1}(V_b)$. Thus $K\in \mathcal{V}'$ by the definition of $\mathcal{V}$. Since $F(x)=f_b p_{a_0,f(b)}(x)$ and $g(x)=(q_{b_2,b} g|U_{a_0})(x)$, this means that $(f_b p_{a_0,f(b)}, q_{b',b} r)<\mathcal{V}'$, where $r=q_{b_2,b'} g|U_{a_0}: U_{a_0}\to V_{b'}$. Thus $\boldsymbol{f}$ satisfies $(\mathrm{NE})_1$ and hence it is nearly extendable in our sense.

Next we assume that $\boldsymbol{f}$ is nearly extendable in our sense and show that f is nearly extendable in the sense of Borsuk. Take any $\mathcal{V}\in \mathcal{C}ov(M_Y)$. Since M_Y is an ANR(**PM**)-neigborhood of Y in M_Y, $V_b=M_Y$ for some $b\in B$. By the assumption there exists $a_0>f(b)$ satisfying $(\mathrm{NE})_1$ for b and $\mathcal{V}$. Take any neigborhood V of Y in M_Y and then by (ii) of (I. 3.17) there exists $b_1>b$ such that $V_{b_1}\subset V$. By the choice of a_0 there exists a map $g: U_{a_0}\to V_{b_1}$ such that $(f_b p_{a_0,f(b)}, q_{b_1,b} g)<\mathcal{V}$. Hence $(F|U_{a_0}, jg')<\mathcal{V}$ where $g'=kg: U_{a_0}\to V$, and $k: V_{b_1}\to V$, $j: V\to M_Y$ are the inclusion maps. Hence f is nearly extendable in the sense of Borsuk. ■

The following gives an answer to a question in Gauthier [20].

(1.15) THEOREM. *The notions of* AANR_C(**PM**) *and* NE-*sets are equivalent for compact metric spaces, but are not equivalent for compact spaces.*

(1.15) follows from (II. 2.5), (II. 5.10), (II. 6.12) and (1.12). ■

Let $\boldsymbol{p}: X\to \mathcal{X}$ be a resolution and $f: X\to Y$ a map. We say that f is approximatively extendable with respect to $\boldsymbol{p}$ provided that it satisfies the following:

(AE) For any $\mathcal{V}\in Cov(Y)$ there exist $a\in A$ and a map $g_a: X_a\to Y$ such that $(f, g_a p_a)<\mathcal{V}$.

(1.16) LEMMA. *Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be* **AP**-*resolutions of X. If f is approximatively extendable with respect to $\boldsymbol{p}$, then so is f with respect to $\boldsymbol{p}'$.*

We easily show (1.16). Thus we may say that f is an approximatively extendable map, in notation AE-map, provided that f is approximatively extendable with respect to any/some **AP**-resolution of X.

(1.17) LEMMA. *Let $f: X\to Y$ and $g: Y\to Z$ be maps. If one of f and g is an* AE-*map, then so is gf.* ■

(1.18) LEMMA. *Let $f: X\to Y$ be a map. If f is an* AE-*map, then $CT(f): CT(X)\to CT(Y)$ is an* AE-*map.* ■

(1.19) PROPOSITION. *If a map $f: X\to Y$ is an* AE-*map, then f is an* NE-*map.* ■

(1.20) THEOREM. *A space X is* IAM *iff $1_X: X\to X$ is an* AE-*map.*

PROOF. Let $\boldsymbol{p}: X\to \mathcal{X}$ be an **AP**-resolution. We assume that X is IAM. Then $\boldsymbol{p}$ is IAM and then $\boldsymbol{p}$ satisfies (C) by (II. 2.14). Thus for each $\mathcal{U}\in Cov(X)$ there exist $a\in A$ and a map $r: X_a\to X$ such that $(rp_a, 1_X)<\mathcal{U}$. This means that $1_X: X\to X$ is an AE-map.

Next, we assume that 1_X is an AE-map. Then for each $\mathcal{U}\in Cov(X)$ there exist $a\in A$ and a map $r: X_a\to X$ such that $(rp_a, 1_X)<\mathcal{U}$. Thus $\boldsymbol{p}$ satisfies (C) and then $\boldsymbol{p}$ is IAM by (II. 2.14). Hence X is IAM. ■

(1.21) THEOREM. *Let X be a space. Then the following statements are equivalent:*

(i) *X is an* AP.

(ii) *$1_X: X\to X$ is an* AE-*map.*

(iii) *Any map $f: X\to Y$ is an* AE-*map for any space Y.*

(iv) *Any map $f: Y\to X$ is an* AE-*map for any space Y.*

This follows from (II. 2.17), (1.17) and (1.20). ■

§ 2. Products, suspensions and cones.

In this section we shall show that product, suspension and cone preserve AE- and NE-maps between compact spaces. Some results are proved by Borsuk [3-6] for maps between compact metric spaces.

In this section all spaces are compact spaces. Let C be a non-empty set. Let X_c and Y_c be non-empty spaces and $f_c: X_c \to Y_c$ maps for $c \in C$. Let $X = \pi\{X_c : c \in C\}$ and $Y = \pi\{Y_c : c \in C\}$ be product spaces. Let $f = \pi\{f_c : c \in C\} : X \to Y$ be the product of the maps f_c. Let $\boldsymbol{p}^c = \{p^c_a : a \in A(c)\} : X_c \to (\mathcal{X}, \mathcal{U})^c = \{(X^c_a, \mathcal{U}^c_a), p^c_{a',a}, A(c)\}$ and $\boldsymbol{q}^c = \{q^c_b : b \in B(c)\} : Y_c \to (\mathcal{Y}, \mathcal{V})^c = \{(Y^c_b, \mathcal{V}^c_b), q^c_{b',b}, B(c)\}$ be approximative finite polyhedral resolutions of X_c and Y_c for each $c \in C$, respectively.

We may assume that $A(c) \cap A(c') = \emptyset$ and $B(c) \cap B(c') = \emptyset$ for $c, c' \in C$ with $c \neq c'$. Let $A = \cup\{A(c) : c \in C\}$ and $B = \cup\{B(c) : c \in C\}$ and $M = \{m : m$ is a non empty finite subset of $C\}$. Take any $m = \{c_1, c_2, \cdots, c_k\} \in M$. We say that a function $g : m \to A$ is a choice function provided that $g(c_i) \in A(c_i)$ for $c_i \in m$. Let $F(A) = \{g : m \in M$ and $g : m \to A$ is a choice function$\}$. Let $g : m \to A$ and $g' : m \to A$ be choice functions. We say that $g' > g$ provided that $m' \supset m$ and $g'(c) > g(c)$ in $A(c)$ for $c \in m$. Since all $A(c)$ are cofinite and directed, $(F(A), >)$ forms a cofinite directed set. Similarly we may define a cofinite directed set $(F(B), >)$.

For each choice function $g : m \to A$ we define a space X_g, a covering $\mathcal{U}_g \in \mathcal{C}ov(X_g)$ and a map $p_g : X \to X_g$ as follows: To simplify notations sometimes *ai* and *ci* denote a_i and c_i, respectively. $X_g = X_{g(c1)} \times X_{g(c2)} \times \cdots \times X_{g(ck)}$, $\mathcal{U}_g = \mathcal{U}^{c1}_{g(c1)} \times \cdots \times \mathcal{U}^{ck}_{g(ck)} = \{U_1 \times \cdots \times U_k : U_i \in \mathcal{U}^{ci}_{g(ci)}$ for $i = 1, 2, \cdots, k\}$ and $p_g((x_c)) = (p^{c1}_{g(c1)}(x_{c1}), \cdots, p^{ck}_{g(ck)}(x_{ck}))$ for each $(x_c) \in X$. For $g' > g$ we define a map $p_{g',g} : X_{g'} \to X_g$ as follows: $p_{g',g}(x_{c1}, x_{c2}, \cdots, x_{ck}, x_{ck+1}, \cdots, x_{ck'}) = (p^{c1}_{g'(c1), g(c1)}(x_{c1}), \cdots, p^{ck}_{g'(ck), g(ck)}(x_{ck}))$ for each $(x_{c1}, \cdots, x_{ck'}) \in X_{g'}$. Here $g' : m' = \{c_1, c_2, \cdots, c_k, c_{k+1}, \cdots, c_{k'}\} \to A$.

It is not difficult by (I. 3.13) and (I. 7.1) to show that $(\mathcal{X}, \mathcal{U}) = \{(X_g, \mathcal{U}_g), p_{g',g}, F(A)\}$ forms an approximative finite polyhedral inverse system and $\boldsymbol{p} = \{p_g : g \in F(A)\} : X \to (\mathcal{X}, \mathcal{U})$ is an approximative finite polyhedral resolution of X. In the same way we can construct an approximative finite polyhedral resolution $\boldsymbol{q} = \{q_h : h \in F(B)\} : Y \to (\mathcal{Y}, \mathcal{V}) = \{(Y_h, \mathcal{V}_h), q_{h',h}, F(B)\}$ of Y.

Let $\boldsymbol{f}^c = \{f^c_\ast, f^c_b : b \in B(c)\} : (\mathcal{X}, \mathcal{U})^c \to (\mathcal{Y}, \mathcal{V})^c$ be an approximative resolution of f_c with respect to $\boldsymbol{p}^c$ and $\boldsymbol{q}^c$ for each $c \in C$. We define a function $f : F(B) \to F(A)$ and maps $f_h : X_{f(h)} \to Y_h$ for $h \in F(B)$ as follows: Take any choice function $h : m \to B$. We define a choice function $f(h) : m \to A$ by $f(h)(c) = f^c(h(c))$ for $c \in m$. We define a map $f_h : X_{f(h)} \to Y_h$ by $f_h(x_1, x_2, \cdots, x_k) = (f^{c1}_{h(c1)}(x_1), \cdots, f^{ck}_{h(ck)}(x_k))$ for $(x_1, \cdots, x_k) \in X_{f(h)}$. It is not difficult to show that $\boldsymbol{f} = \{f, f_h : h \in$

$F(B)\}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ forms an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$.

(2.1) THEOREM. (i) *f is an* AE-*map iff all f_c are* AE-*maps.*

(ii) *f is an* NE-*map iff all f_c are* NE-*maps.*

PROOF. We show (ii). In a similar way we can show (i). First we assume that all f_c are NE-maps and show that f is an NE-map. Take any $h \in F(B)$ and put $h: m=\{c_1, c_2, \cdots, c_k\} \to B$. There exists $a_i \in A(c_i)$ with $a_i > f^{ci}h(c_i)$ satisfying (NE) for $\boldsymbol{f}_{ci}$ and $h(c_i)$. We define a choice function $g: m \to A$ by $g(c_i) = a_i$ for $i=1, 2, \cdots, k$, and then $g > f(h)$. We show that g is the required map. Take any $h' \in F(B)$ with $h' > h$ and put $h': m'=\{c_1, c_2, \cdots, c_k, c_{k+1}, \cdots, c_{k'}\} \to B$. By the choice of a_i there exist maps $r_i: X^{ci}_{ai} \to Y^{ci}_{h'(ci)}$ such that

$$(1) \qquad (q^{ci}_{h'(ci), h(ci)} r_i, f^{ci}_{h(ci)} p^{bi}_{ai}, f^{ci}_{h(ci)}) < st\mathcal{V}^{ci}_{h(ci)} \qquad \text{for} \quad i=1, 2, \cdots, k.$$

We define a map $r: X_g \to Y_{h'}$ by $r=j\,\pi\{r_i: i=1, 2, \cdots k\}$. Here $j: Y^{c1}_{h'(c1)} \times Y^{c2}_{h'(c2)} \times \cdots \times Y^{ck}_{h'(ck)} \to Y_{h'}$ is a natural inclusion map. Then by (1) we have that $(q_{h',h} r, f_h p_{g, f(h)}) < st\mathcal{V}_h$. Hence $\boldsymbol{f}$ satisfies (NE) and then f is an NE-map.

Next we assume that f is an NE-map and show that all f_c are NE-maps. Take any $c_0 \in C$ and any $b \in B(c_0)$. We define a choice function $h_0: \{c_0\} \to B$ by $h_0(c_0)=b$. By the assumption there exists $g > f(h_0)$ satisfying (NE) for $\boldsymbol{f}$ and h_0. Put $g(c_0)=a_0$ and then $a_0 > f_{c0}(b)$. We show that a_0 is the required index. Take any $b' \in B(c_0)$ with $b' > b$. We define a choice function $h': \{c_0\} \to B$ by $h'(c_0)=b'$. Since $h' > h$, by the choice of g there exists a map $r: X_g \to X_{h'} = Y^{c0}_{b'}$ such that

$$(2) \qquad (f_{h_0} p_{g, f(h_0)}, q_{h', h} r) < st\mathcal{V}_{h_0}.$$

We define a map $r': X^{c0}_{a0} \to Y^{c0}_{a0}$ by $r'=rj$. Here $j: X^{c0}_{a0} \to X_g$ is the inclusion map. Thus by (2) $(f^{c0}_b p^{c0}_{a0, f_{c0}(b)}, q^{c0}_{b', b} r') < st\mathcal{V}_b$. Then f_{c0} is an NE-map and hence all f_c are NE-maps. ■

By a straightforward argument we can show the following:

(2.2) LEMMA. *We assume that C is finite. Then X satisfies the condition M iff all X_c satisfy the condition M.* ■

(2.3) COROLLARY. (i) *X is an* AP *iff all X_c are* APs.

(ii) *X is* AM *iff all X_c are* AM.

(iii) *X is an* AANR$_C$ *for* **COM** *iff all X_c are* AANR$_C$ *for* **COM**.

(iv) *X is an* AAR *for* **COM** *iff all X_c are* AAR *for* **COM**.

(v) *When C is finite, X is an* AANR$_N$ *for* **COM** *iff all X_c are* AANR$_N$ *for* **COM**.

PROOF. (i) and (iii) follow from (II. 5.10), (1.21) and (2.1). (ii) follows from (1.12) and (2.1). (iv) follows from (i), (II. 5.12) and the following fact: X has trivial shape iff all X_c have trivial shape. (v) follows from (II. 5.11), (i) and (2.2). ■

Let I be the unit interval. Cone(X) denotes the quotient space $X\times I/X\times\{1\}$ and $t_X: X\times I\to$Cone(X) the quotient map. Then any map $f: X\to Y$ induces the unique map Cone(f): Cone$(f)\to$Cone(Y) satisfying Cone$(f)t_X=t_Y(f\times 1_I)$. Let $\mathcal{Z}\in\mathcal{C}ov(X\times I)$ and put $\mathcal{Z}^*=\{t_X(st(X\times\{1\},\mathcal{Z}))\}\cup\{t_X(Z): Z\in\mathcal{Z}$ and $Z\cap(X\times\{1\})=\emptyset\}$. Then $\mathcal{Z}^*$ forms a covering of Cone(X).

Let $\mathcal{J}=\{I_i, r_{j,i}, N\}$ be an inverse sequence such that $I_i=I$ and $r_{j,i}=1_I$ for all $j\geqq i$. Let $r_i=1_I: I\to I_i$ for all i. Then $\boldsymbol{r}=\{r_i: i\in N\}: I\to\mathcal{J}$ forms an **POL**-resolution. By (I. 3.8) there are coverings $\mathcal{W}_i\in\mathcal{C}ov(I_i)$ such that $\boldsymbol{r}: I\to(\mathcal{J},\mathcal{W})=\{(I_i,\mathcal{W}_i), r_{j,i}, N\}$ forms an approximative **POL**-resolution. Let $\boldsymbol{p}=\{p_a: a\in A\}: X\to(\mathcal{X},\mathcal{U})=\{(X_a,\mathcal{U}_a), p_{a',a}, A\}$ be an approximative finite polyhedral resolution of a space X. Let $A\times N$ be the directed product set. We define $X_{(a,i)}$, $p_{(a,i)}: X\times I\to X_{(a,i)}$ and $p_{(a',i'),(a,i)}: X_{(a',i')}\to X_{(a,i)}$ for $(a',i')>(a,i)$ as follows: $X_{(a,i)}=X_a\times I_i$, $p_{(a,i)}=p_a\times r_i$ and $p_{(a',i'),(a,i)}=p_{a',a}\times r_{i',i}$. It is easy to show that $\boldsymbol{p}\times\boldsymbol{r}=\{p_{(a,i)}:(a,i)\in A\times N\}: X\times I\to(\mathcal{X},\mathcal{U})\times(\mathcal{J},\mathcal{W})=\{(X_{(a,i)},\mathcal{U}_a\times\mathcal{W}_i), p_{(a',i'),(a,i)}, A\times N\}$ forms an approximative polyhedral resolution. Similarly we have an approximative polyhedral resolution $\boldsymbol{q}\times\boldsymbol{r}: Y\times I\to(\mathcal{Y},\mathcal{V})\times(\mathcal{J},\mathcal{W})$ for an approximative polyhedral resolution $\boldsymbol{q}=\{q_b: b\in B\}: Y\to(\mathcal{Y},\mathcal{V})=\{(Y_b\ '\mathcal{V}_b), q_{b',b}, B\}$ of a space Y.

Let $\boldsymbol{f}=\{f, f_b: b\in B\}: (\mathcal{X},\mathcal{U})\to(\mathcal{Y},\mathcal{V})$ be an approximative resolution of a map $f: X\to Y$ with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. We put $\boldsymbol{f}\times 1_I=\{f\times 1_N, f_b\times 1_I: (b,i)\in B\times N\}$. Then $\boldsymbol{f}\times 1_I: (\mathcal{X},\mathcal{U})\times(\mathcal{J},\mathcal{W})\to(\mathcal{Y},\mathcal{V})\times(\mathcal{J},\mathcal{W})$ forms an approximative resolution of $f\times 1_I$ with respect to $\boldsymbol{p}\times\boldsymbol{r}$ and $\boldsymbol{q}\times\boldsymbol{r}$.

Maps $p_{(a,i)}: X\times I\to X_a\times I_i$ and $p_{(a',i'),(a,i)}: X_{a'}\times I_{i'}\to X_a\times I_i$ induce maps $p'_{(a,i)}$: Cone$(X)\to X_a\times I_i/X_a\times\{1\}=X'_{(a,i)}$ $(=$Cone$(X_a))$ and $p'_{(a',i'),(a,i)}: X'_{(a',i')}\to X'_{(a,i)}$. Moreover $(\mathcal{X},\mathcal{U})'=\{(X'_{(a,i)},(\mathcal{U}_a\times\mathcal{W}_i)^*), p'_{(a',i'),(a,i)}, A\times N\}$ forms an approximative inverse system. It is not difficult to show that $\boldsymbol{p}'=\{p'_{(a,i)}:(a,i)\in A\times N\}$: Cone$(X)\to(\mathcal{X},\mathcal{U})'$ is an approximative polyhedral resolution. Similarly $\boldsymbol{q}'=\{q'_{(b,i)}: (b,i)\in B\times N\}$: Cone$(Y)\to(\mathcal{Y},\mathcal{V})'=\{(Y'_{(b,i)},(\mathcal{V}_b\times\mathcal{W}_i)^*), q'_{(b,i)}, B\times N\}$ forms an approximative polyhedral resolution. Maps $f_b\times 1_I: X_{(f(b),i)}=X_{f(b)}\times I_i\to Y_{(b,i)}=Y_b\times I_i$ induce maps $f'_{(b,i)}: X'_{(f(b),i)}\to Y'_{(b,i)}$. It is not difficult to show that $\boldsymbol{f}'=\{f\times 1_N, f'_{(b,i)}, B\times N\}: (\mathcal{X},\mathcal{U})'\to(\mathcal{Y},\mathcal{V})'$ forms an approximative resolution of Cone(f) with respect to $\boldsymbol{p}'$ and $\boldsymbol{q}'$.

(2.4) THEOREM. (i) *f is an* AE-*map iff* Cone(f): Cone$(X)\to$Cone(Y) *is an* AE-*map.*

(ii) *f is an* NE-*map iff* Cone(f) *is an* NE-*map.*

PROOF. We show (ii). In a similar way we can show (i). First we assume that f is an NE-map. We show that f' satisfies (NE). Take any $(b, i)\in B\times N$. Since f satisfies (NE), there exists $a>f(b)$ satisfying (NE) for f and b. We show that (a, i) is the required index. To do so take any $(b', i')>(b, i)$. By the choice of a there exists a map $r: X_a\to Y_{b'}$ such that

$$(1)\qquad (f_b p_{a,f(b)}, q_{b',b}r)<st\mathcal{V}_b.$$

The map $r\times 1_I: X_a\times I\to Y_{b'}\times I$ induces a map $r': X'_{(a,i)}\to Y'_{(b',i')}$. By (1) $(f'_{(b,i)}p'_{(a,i),(f(b),i)}, q'_{(b',i'),(b,i)}r')<st((\mathcal{V}_b\times\mathcal{W}_i)^*)$. This means that f' satisfies (NE) and hence Cone(f) is an NE-map.

Next we assume that Cone(f) is an NE-map. Then f' satisfies (NE). We show that f satisfies (NE). Take any $b\in B$. Put $\mathcal{W}=\{[0, 2/6), (1/6, 3/6), (2/6, 4/6), (3/6, 5/6), (4/6, 1]\}\in Cov(I)$. By (R1) there exists $i\in N$ such that $\mathcal{W}>r_i^{-1}\mathcal{W}_i=\mathcal{W}_i$. There exists $(a, j)>(f(b), i)$ satisfying (NE) for f' and (b, i). We show that $a>f(b)$ is the required index. To do so take any $b'>b$. By the choice of (a, j) there exists a map $k: X'_{(a,j)}\to Y'_{(b',i)}$ such that $(f'_{(b,i)}p'_{(a,j),(f(b),i)}, q'_{(b',i),(b,i)}k)<st((\mathcal{V}_b\times\mathcal{W}_i)^*)$. By the choice of i $(f'_{(b,i)}p'_{(a,j),(f(b),i)}t_{X_a}, q'_{(b',i),(b,i)}kt_{X_a})<st((\mathcal{V}_b\times\mathcal{W})^*)$. Since $f'_{(b,i)}p'_{(a,j),(f(b),i)}t_{X_a}=t_{Y_b}(f_b p_{a,f(b)}\times 1_I)$, for each $x\in X_a$ there exists $V\in\mathcal{V}_b$ such that

$$(2)\qquad t_{Y_b}(f_b p_{a,f(b)}(x), 0), q'_{(b',i),(b,i)}kt_{X_a}(x, 0)\in t_{Y_b}(st(V, \mathcal{V}_b)\times[0, 4/6)).$$

Thus $kt_{X_a}(x, 0)\in t_{Y_{b'}}(Y_{b'}\times[0, 4/6))$. Since $t_{Y_{b'}}|Y_{b'}\times[0, 4/6): Y_{b'}\times[0, 4/6)\to t_{Y_{b'}}(Y_{b'}\times[0, 4/6))$ is a homeomorphism, we can define a map $k': X_a\to Y_{b'}$ by $k'(x)=ut_{Y_{b'}}^{-1}kt_{X_a}(x, 0)$ for $x\in X_a$. Here $u: Y_{b'}\times I\to Y_{b'}$ is the projection. Then (2) means that $f_b p_{a,f(b)}(x), q_{b',b}k'(x)\in st(V, \mathcal{V}_b)$. Hence $(f_b p_{a,f(b)}, q_{b',b}k')<st\mathcal{V}_b$ and f is an NE-map. ■

(2.5) COROLLARY. (i) *X is an* AP *iff* Cone(X) *is an* AP.

(ii) *X is* AM *iff* Cone(X) *is* AM.

(iii) *X is an* AANR$_C$ *for* **COM** *iff* Cone(X) *is an* AAR *for* **COM**.

PROOF. (i) follows from (1.21) and (2.4). (ii) follows from (1.12) and (2.4). (iii) follows from (II.5.10), (II.5.12) and the following fact: Cone(X) is contractible and hence has the trivial shape. ■

$S(X)$ denotes the suspension of X. A map $f: X\to Y$ induces a map $S(f): S(X)\to S(Y)$.

(2.6) THEOREM. (i) *f is an* AE-*map iff* $S(f)$ *is an* AE-*map.*

(ii) *f is an* NE-*map iff* $S(f)$ *is an* NE-*map.*

(2.7) COROLLARY. (i) X *is an* AP *iff* $S(X)$ *is an* AP.
(ii) X *is* AM *iff* $S(X)$ *is* AM.

(2.8) LEMMA. *If* X *satisfies the condition* M, *then so does* $S(X)$.

(2.9) COROLLARY. (i) X *is an* AANR_C *for* **COM** *iff so is* $S(X)$.
(ii) *If* X *is an* AANR_N *for* **COM**, *then so is* $S(X)$.

In a way similar to the one used in (2.4) and (2.5) we can show (2.6), (2.7) and (i) of (2.9). By a straightforward argument as used in (2.2) we can show (2.8). (ii) of (2.9) follows from (II. 5.11) and (i) of (2.9). ■

Let X^c be a connected component of X and X_a^c a connected component of X_a with $p_a(X^c)\subset X_a^c$ for $a\in A$. Put $\mathcal{U}_a^c=\{U\cap X_a^c: U\in\mathcal{U}_a\}\in Cov(X_a^c)$. Let $p_a^c: X^c\to X_a^c$ and $p_{a',a}^c: X_{a'}^c\to X_a^c$ be induced maps by p_a and $p_{a',a}$ for $a'>a$. Then it is easy to show that $\boldsymbol{p}^c=\{p_a^c: a\in A\}: X^c\to(\mathcal{X}, \mathcal{U})^c=\{(X_a^c, \mathcal{U}_a^c), p_{a'a}^c, A\}$ forms an approximative finite polyhedral resolution.

$f: X\to Y$ induces a map $f^c=f|X^c: X^c\to Y$. For each $b\in B$ $f_b: X_{f(b)}\to Y_b$ induces a map $f_b^c=f_b|X_{f(b)}^c: X_{f(b)}^c\to Y_b$. Since $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ is an appoximative resolution of f with $\boldsymbol{p}$ and $\boldsymbol{q}$, $\boldsymbol{f}^c=\{f, f_b^c: b\in B\}: (\mathcal{X}, \mathcal{U})^c\to(\mathcal{Y}, \mathcal{V})$ is an approximative resolution of f^c with respect to $\boldsymbol{p}^c$ and $\boldsymbol{q}$.

(2.10) THEOREM. (i) $f: X\to Y$ *is an* AE-*map iff* $f^c: X^c\to Y$ *is an* AE-*map for each connected component* X^c *of* X.
(ii) f *is an* NE-*map iff all* f^c *are* NE-*maps.*

PROOF. We show (ii). In a similar way we can show (i). First we assume that f is an NE-map. Take any connected component X^c of X and let $j^c: X^c\to X$ be the inclusion map. Since $f^c=fj^c$, by (1.8) f^c is an NE-map.

Next we assume that all f^c are NE-maps and show that f is an NE-map. Take any $b\in B$ and any connected component X^c of X. Since $\boldsymbol{f}^c: (\mathcal{X}, \mathcal{U})^c\to(\mathcal{Y}, \mathcal{V})$ satisfies (NE), there exists $a(c)>f(b)$ satisfying (NE) for $\boldsymbol{f}^c$ and b. Since all X_a are finite polyhedra, all connected components are open and closed. Then $X_{a(c)}^c$ is open and closed. Put $U'(X^c)=p_{a(c)}^{-1}(X_{a(c)}^c)$ and then $U'(X^c)$ is an open and closed neighborhood of X^c in X. Put $\mathcal{U}'=\{U'(X^c): X^c$ is a connected component of $X\}\in Cov(X)$. Since X is compact, there exist finitely many connected components $X^{c1}, X^{c2}, \cdots, X^{cn}$ such that $X=\cup\{U'(X^{ci}): i=1, 2, \cdots, n\}$. Put $U_i=U'(X^{ci})-\cup\{U'(X^{cj}): j=1, 2, \cdots, i-1\}$ for each i and then

(1) all U_i are open and closed, $U_i \cap U_j = \emptyset$ for $i \neq j$, and
$\mathcal{V} = \{U_i : i = 1, 2, \cdots, n\} \in \mathcal{C}ov(X)$.

By (1) there exists a canonical map $\boldsymbol{p}: X \to N(\mathcal{V})$ such that

(2) $p^{-1}(U) = U$ for $U \in \mathcal{V}$ and $N(\mathcal{V})$ is 0-dimensional.

Here $N(\mathcal{V})$ denotes the nerve of $\mathcal{V}$. Since $N(\mathcal{V})$ is 0-dimensional by (1), then by (R1) there exist $a > a(c1), \cdots, a(cn)$ and a map $h: X_a \to N(\mathcal{V})$ such that

(3) $$p = hp_a.$$

We define $K_i = \cup\{T : T$ is a connected component of X_a and $T \cap p_a(U_i) \neq \emptyset\}$ for each i. Then all K_i are open and closed in X_a such that

(4) $p_a(X) \subset \cup\{K_i : i = 1, 2, \cdots, n\}$ and $p_a(X) \cap K_i \neq \emptyset$ for each i.

CLAIM 1. $K_i \cap K_j = \emptyset$ for $i \neq j$.

We assume $K_i \cap K_j \neq \emptyset$ for some $i \neq j$. Then there exists a connected component T of X_a satisfying $T \cap p_a(U_i) \neq \emptyset$ and $T \cap p_a(U_j) \neq \emptyset$. Since $N(\mathcal{V})$ is 0-dimensional, $h(T) \in hp_a(U_i)$ and $h(T) \in hp_a(U_j)$. Since $hp_a(U_i) = p_a(U_i) = \{U_i\}$ and $hp_a(U_j) = \{U_j\}$ by (3), $h(T) = \{U_i\} = \{U_j\}$ and hence $i = j$. This is a contradiction. Hence we have Claim 1.

CLAIM 2. $p_{a, a(ci)}(K_i) \subset X^{ci}_{a(ci)}$ for $1 \leqq i \leqq n$.

Take any $y \in K_i$ and then there exists a connected component T such that $y \in T$ and $T \cap p_a(U_i) \neq \emptyset$. Thus there exists $x \in U_i$ with $p_a(x) \in T$. Since $U_i \subset U'(X^{ci}) = p^{-1}_{a(ci)}(X^{ci}_{a(ci)})$, $p_{a(ci)}(x) \in X^{ci}_{a(ci)}$. Since T is connected, $p_{a, a(ci)}(y)$, $p_{a(ci)}(x) \in p_{a, a(ci)}(T) \subset X^{ci}_{a(ci)}$. Thus $p_{a, a(ci)}(y) \in X^{ci}_{a(ci)}$ and hence we have Claim 2.

Put $K_0 = X_a - \cup\}K_i : 0 \leqq i \leqq n\}$ and then K_0 is open and closed. Let $\mathcal{K} = \{K_i : 0 \leqq i \leqq n\} \in \mathcal{C}ov(X_a)$ and then $st(p_a(X), \mathcal{K}) = \cup\{K_i : 1 \leqq i \leqq n\}$. By (B4) there exists $a_1 > a$ such that $p_{a_1, a}(X_{a_1}) \subset st(p_a(X), \mathcal{K})$. Put $L_i = p^{-1}_{a_1, a}(K_i)$ for $1 \leqq i \leqq n$ and then by Claim 1

(5) All L_i are open and closed in X_{a_1} and $X_{a_1} = \cup\{L_i : i = 1, 2, \cdots, n\}$
and $L_i \cap L_j = \emptyset$ for $i \neq j$.

We show that a_1 is the required index. Take any $b' > b$. By the choice of $a(ci)$ there exists a map $g_{ci}: X^{ci}_{a(ci)} \to Y_{b'}$ such that $(q_{b', b}g_{ci}, f^{ci}_b p^{ci}_{a(ci), f(b)}) < st\mathcal{V}_b$, i.e.,

(6) $$(q_{b', b}g_{ci}, f_b p_{a(ci), f(b)} | X^{ci}_{a(ci)}) < st\mathcal{V}_b.$$

By Claim 2

(7) $$p_{a_1, a(ci)}(L_i) \subset X^{ci}_{a(ci)} \quad \text{for } 1 \leqq i \leqq n.$$

Now we define a map $g: X_{a_1} \to Y_{b'}$ as follows: For each $y \in L_i$ $g(y) = g_{ci} p_{a_1, a(ci)}(y)$. By (5) and (7) g is well defined and continuous. By (6) $(q_{b', b} g, f_b p_{a_1, f(b)}) < st \mathcal{V}_b$. This means that $\boldsymbol{f}$ satisfies (NE) and hence f is an NE-map. ■

For any connected component X^c of X there exists a connected component Y^d of Y such that $f(X^c) \subset Y^d$. f induces a map $f^{c,d}: X^c \to Y^d$.

(2.11) COROLLARY. (i) *If all* $f^{c,d}: X^c \to Y^d$ *are* AE-*maps, then* f *is an* AE-*map.*

(ii) *If all* $f^{c,d}: X^c \to Y^d$ *are* NE-*maps, then* f *is an* NE-*map.*

PROOF. We show (ii). In the same way we can show (i). Since $f^{c,d}: X^c \to Y^d$ are NE-maps, $f^c = j^d f^{c,d}: X^c \to Y$ are NE-maps by (1.8). Here $j^d: Y^d \to Y$ is the inclusion map. By (2.10) f is an NE-map. ■

(2.12) COROLLARY. (i) *If all connected components of* X *are* APs, *then so is* X.

(ii) *If all connected components of* X *are* AM, *then so is* X. ■

In general the converse assertions of (2.11) and (2.12) do not hold, because we have the following example.

(2.13) EXAMPLE. Let X be a non-movable compact connected metric space. For example we can choose for X the 2-adic solenoid. Let $\mathcal{X} = \{X_i, p_{j,i}, N\}$ be an inverse sequence of finite complexes such that $\lim \mathcal{X} = X$. Put $Y_i = v\ \{X_k : k = 1, 2, \cdots, i-1\}$, that is, disjoint sum of $X_1, \cdots, X_{i-1}$ for each $i \in N$. We define $r_{i+1,i}: Y_{i+1} \to Y_i$ for each $i \in N$ as follows: $r_{i+1,i}(y) = y$ for $y \in Y_i$ and $r_{i+1,i}(y) = p_{i+1,i}(y)$ for $y \in X_{i+1}$. Put $r_{j,i} = r_{j,j-1} \cdots r_{i+1,i}$ for $j \geqq i$. Thus $\mathcal{Y} = \{Y_i, r_{j,i}, N\}$ forms an inverse sequence of finite polyhedra. Put $Y = \min \mathcal{Y}$ and then Y is a compact metric space. It is easy to show that Y is an approximative polyhedron and approximatively movable. Then $1_Y: Y \to Y$ is an AE-map and an NE-map. X is a connected component of Y. We assume that $1_X: X \to X$ is an NE-map. Then X is AM and hence X is movable. This is a contradiction. Hence 1_X is not an NE-map and X is not approximatively movable. We see in a similar way that 1_X is not an AE-map and X is not an AP. ■

§ 3. Hyperspaces.

In this section we discuss approximative properties of hyperspaces.

In this section all spaces are compact spaces. Let X be a space. We denote by 2^X the set of all non-empty closed subsets of X, by $C(X)$ the set of all non-empty connected closed subsets of X, and by $X(n)$, n is a positive integer, the set of all non-empty subsets of X consisting of at most n points. $C(X)$ and $X(n)$ are subsets of 2^X.

For open subsets $U_1, U_2, \cdots, U_k$ of X we put $\langle U_1, U_2, \cdots, U_k\rangle=\{K\in 2^X : K\subset\bigcup_{i=1}^k U_i$ and $K\cap U_i\neq\emptyset$ for each $0\leqq i\leqq k\}$. Then $\{\langle U_1, U_2, \cdots, U_k\rangle : U_1, U_2, \cdots, U_k$ are open subsets of X and $k=1, 2, \cdots\}$ forms a base of a topology of 2^X. This topology of 2^X is called the finite topology or the Vietoris topology. 2^X denotes the topological space with the Vietoris topology. We consider $C(X)$ and $X(n)$ as subspaces of 2^X. These spaces are called hyperspaces of X. $X(n)$ is the n-th symmetric product of X (see Borsuk-Ulam [7] and Jaworowski [**24**]). Concerning hyperspaces see Kuratowski [**32**], Michael [**37**] and Nadler [**39**].

Let $\mathcal{U}\in Cov(X)$ and put $\langle\mathcal{U}\rangle=\{\langle U_1, U_2, \cdots, U_k\rangle : U_1, \cdots, U_k\in\mathcal{U}$ and $k=1, 2, \cdots\}$. Then $\langle\mathcal{U}\rangle$ forms an open covering of 2^X and $\{\langle\mathcal{U}\rangle : \mathcal{U}\in Cov(X)\}$ forms a uniformity of 2^X by Morita [38]. By the uniqueness of uniformities on compact spaces we have the following:

(3.1) LEMMA. *The uniformities* $\{\langle\mathcal{U}\rangle : \mathcal{U}\in Cov(X)\}$ *and* $Cov(2^X)$ *are equivalent, that is, for each* $\mathcal{W}\in Cov(2^X)$ *there exists* $\mathcal{U}\in Cov(X)$ *such that* $\mathcal{W}>\langle\mathcal{U}\rangle$.

Let Y be a space and $f: X\to Y$ a map. Then f induces a map $f^*: 2^X\to 2^Y$ as follows: $f^*(K)=f(K)$ for each $K\in 2^X$. Clearly f^* induces maps $f^*: C(X)\to C(Y)$ and $f^*: X(n)\to Y(n)$ for each positive integer n. The following is an easy consequence of the definitions:

(3.2) LEMMA. *Let* $\mathcal{U}\in Cov(X)$ *and* $\mathcal{V}\in Cov(Y)$. *If* $f^{-1}\mathcal{V}>\mathcal{U}$ *then* $f^{*-1}\langle\mathcal{V}\rangle>\langle\mathcal{U}\rangle$, *where* $f^*: 2^X\to 2^Y$. *Similarly this holds for* $f^*: C(X)\to C(Y)$ *and* $f^*: X(n)\to Y(n)$, *respectively.*

Let $\boldsymbol{p}=\{p_a : a\in A\} : X\to(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ be an approximative resolution of X.

(3.3) LEMMA. $\boldsymbol{p}^*=\{p_a^* : a\in A\} : 2^X\to 2^{(\mathcal{X},\mathcal{U})}=\{(2^{X_a}, \langle\mathcal{U}_a\rangle), p_{a',a}^*, A\}$, $\boldsymbol{p}^* : C(X)\to C(\mathcal{X}, \mathcal{U})=\{(C(X_a), \langle\mathcal{U}_a\rangle\mid C(X_a)), p_{a',a}^*, A\}$ *and* $\boldsymbol{p}^* : X(n)\to(\mathcal{X}, \mathcal{U})(n)=\{(X_a(n), \langle\mathcal{U}_a\rangle\mid X_a(n)), p_{a',a}^*, A\}$ *are approximative resolution of* 2^X, $C(X)$ *and* $X(n)$, *respectively.*

PROOF. We show the first case. In the same way we can show the others.

(1) $2^{(\mathcal{X},\mathcal{U})}$ is an approximative inverse system.

We need to show (AI1)-(AI3). Clearly (AI1) holds and (AI2) follows from

(3.2) and (AI2) for $(\mathcal{X}, \mathcal{U})$. Take any $a \in A$ and any $\mathcal{W} \in Cov(2^{X_a})$. By (3.1) there exists $\mathcal{U} \in Cov(X_a)$ such that $\mathcal{W} > \langle \mathcal{U} \rangle$. By (AI3) for $(\mathcal{X}, \mathcal{U})$ there exists $a' > a$ such that $p_{a'a}^{-1}\mathcal{U} > \mathcal{U}_{a'}$. Thus by (3.2) $p_{a'a}^{*-1}\mathcal{W} > p_{a'a}^{*-1}(\langle \mathcal{U} \rangle) > \langle \mathcal{U}_{a'} \rangle$. Then we have (AI3) and hence (1).

(2) $$\boldsymbol{p}^* : 2^X \to 2^{\mathcal{X}} \text{ is a resolution.}$$

By (I. 3.3) $\boldsymbol{p}: X \to \mathcal{X}$ is a resolution. Since all spaces are compact, $\boldsymbol{p}: X \to \mathcal{X}$ is an inverse limit by (I. 7.1). By Lemma 2 of Kodama-Spiez-Watanabe [31] $\boldsymbol{p}^* : 2^X \to 2^{\mathcal{X}}$ is an inverse limit. Hence by (I. 3.13) we have (2).

By (1), (2) and (I. 3.3) $\boldsymbol{p}^* : 2^X \to 2^{(\mathcal{X}, \mathcal{U})}$ forms an approximative resolution. ■

(3.4) LEMMA. *Let $f, g: X \to Y$ be maps and $\mathcal{V} \in Cov(Y)$. If $(f, g) < \mathcal{V}$, then $(f^*, g^*) < \langle \mathcal{V} \rangle$ for $f^*, g^* : 2^X \to 2^Y$. Similarly this holds for $f^*, g^* : C(X) \to C(Y)$ and $f^*, g^* : X(n) \to Y(n)$, respectively.*

PROOF. Take any $K \in 2^X$. Since $(f, g) < \mathcal{V}$, for each $x \in K$ there exists $V_x \in \mathcal{V}$ such that $f(x), g(x) \in V_x$. Thus $f(K) \cup g(K) \subset \cup \{V_x : x \in X\}$. Since K is compact, there exist finitely many points $x_1, x_2, \cdots, x_k \in X$ such that $f(K) \cup g(K) \subset V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_k}$. Since $f(x_i), g(x_i) \in V_{x_i}$ for $1 \leqq i \leqq k$, $f(K), g(K) \in \langle V_{x_1}, V_{x_2}, \cdots, V_{x_k} \rangle$ and hence $(f^*, g^*) < \langle \mathcal{V} \rangle$. Trivially this implies the other cases. ■

Let $\boldsymbol{q} = \{q_b : b \in B\} : Y \to (\mathcal{Y}, \mathcal{V}) = \{(Y_b, \mathcal{V}_b), q_{b'b}, B\}$ be an approximative resolution. Let $\boldsymbol{f} = \{f, f_b : b \in B\} : (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ be an approximative resolution of a map $f: X \to Y$ with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Let $\boldsymbol{f}^* = \{f, f_b^* : b \in B\} : 2^{(\mathcal{X}, \mathcal{U})} \to 2^{(\mathcal{Y}, \mathcal{V})}$, $\boldsymbol{f}^* = \{f, f_b^* | C(X_{f(b)}) : b \in B\} : C(\mathcal{X}, \mathcal{U}) \to C(\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{f}^* = \{f, f_b^* | X_{f(b)}(n) : b \in B\} : (\mathcal{X}, \mathcal{U})(n) \to (\mathcal{Y}, \mathcal{V})(n)$. Using (3.4) we can easily show the following:

(3.5) LEMMA. *$\boldsymbol{f}^* : 2^{(\mathcal{X}, \mathcal{U})} \to 2^{(\mathcal{Y}, \mathcal{V})}$ is an approximative resolution of $f^* : 2^X \to 2^Y$ with respect to $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$. This holds for $\boldsymbol{f}^* : C(\mathcal{X}, \mathcal{U}) \to C(\mathcal{Y}, \mathcal{V})$ and $\boldsymbol{f}^* : (\mathcal{X}, \mathcal{U})(n) \to (\mathcal{Y}, \mathcal{V})(n)$.* ■

(3.6) LEMMA (Wojdyslawski [47]). *Let X be a compact metric connected space. Then the following statements are equivalent:*

(i) *X is locally connected.*

(ii) *2^X is an* AR.

(iii) *$C(X)$ is an* AR. ■

(3.7) LEMMA (Ganea [18]). *Let X be a finite dimensional compact metric space. If X is an* ANR, *then so is $X(n)$ for each positive integer n.* ■

(3.8) THEOREM. (i) *If $f: X \to Y$ is an* AE-*map, then $f^* : 2^X \to 2^Y$, $f^* : C(X)$*

$\to C(Y)$ *and* $f^*: X(n) \to Y(n)$ *are* AE-*maps for each positive integer* n.

(ii) *If* f *is an* NE-*map, then* $f^*: 2^X \to 2^Y$, $f^*: C(X) \to C(Y)$ *and* $f^*: X(n) \to Y(n)$ *are* NE-*maps for each positive integer* n.

PROOF. We show (ii). In the same way we can show (i). By (I. 3.15) there exist approximative finite polyhedral resolutions $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ and $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V})$. By (I. 2.1) and (I. 3.3) $st(\boldsymbol{q}) = \{q_b : b \in B\} : Y \to st(\mathcal{Y}, \mathcal{V})$ is an approximative resolution. Then by (3.3) and (3.6) $\boldsymbol{p}^*: 2^X \to 2^{(\mathcal{X}, \mathcal{U})}$ and $st(\boldsymbol{q})^*: 2^Y \to 2^{st(\mathcal{Y}, \mathcal{V})}$ are approximative **ANR**-resolutions. Let $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ be an approximative resolution of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$, and then so is $\boldsymbol{f}$ with respect to $\boldsymbol{p}$ and $st(\boldsymbol{q})$. By (3.5) $\boldsymbol{f}^*: 2^{(\mathcal{X}, \mathcal{U})} \to 2^{st(\mathcal{Y}, \mathcal{V})}$ is an approximative resolution of $f^*: 2^X \to 2^Y$ with respect to $\boldsymbol{p}^*$ and $st(\boldsymbol{q})^*$.

We show that $\boldsymbol{f}^*: 2^{(\mathcal{X}, \mathcal{U})} \to 2^{st(\mathcal{X}, \mathcal{U})}$ satisfies (NE). Take any $b \in B$. Since $\boldsymbol{f}: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})$ satisfies (NE), there exists $a > f(b)$ satisfying (NE) for $\boldsymbol{f}$ and b. Then for each $b' > b$ there exists a map $r: X_a \to Y_{b'}$ such that $(f_b p_{a, f(b)}, q_{b', b} r) < st\mathcal{V}_b$. Since $(f_b^* p_{a, f(b)}^*, q_{b', b}^* r^*) < \langle st\mathcal{V}_b \rangle$ by (3.4), we have the required condition. Hence f^* is an NE-map. ■

(3.9) PROPOSITION. *If* X *satisfies the condition* M, *then so do* 2^X, $C(X)$ *and* $X(n)$.

In a way similar to the one used in Theorem 3 of Kodama-Spiez-Watanabe [**31**] we can easily show (3.9). ■

(3.10) COROLLARY. *Let* n *be a positive integer.*

(i) *If* X *is an* AP, *then so are* 2^X, $C(X)$ *and* $X(n)$.

(ii) *If* X *is an* AANR_C, *then so are* 2^X, $C(X)$ *and* $X(n)$. *If, in addition,* X *is connected, then* 2^X, $C(X)$ *and* $X(n)$ *are* AARs.

(iii) *If* X *is an* AANR_N, *then so are* 2^X, $C(X)$ *and* $X(n)$.

(iv) *If* X *is* AM, *then so are* 2^X, $C(X)$ *and* $X(n)$.

(i) follows from (1.21) and (3.8). (ii) follows from (II. 5.10), (i) and Corollary 1 of Kodama-Spiez-Watanabe [**31**]. (iii) follows from (i) and (II. 5.11). (iv) follows from (1.12) and (3.8). ■

A continuum which is hereditarily unicoherent and arcwise connected is a dendroid. A dendroid which satisfies the smoothness condition is a smooth dendroid (see Charatonik-Eberhart [8]). A locally connected dendroid is a dendrite. A finite tree is a finite 1-dimensional simplicial complex which is connected and does not contain any circle.

(3.11) PROPOSITION. (i) *Any dendroid is an* AAR.

(ii) *Any fan is an* AAR.
(iii) *Any smooth dendroid is an* AAR.

PROOF. We show (ii). Take an AR M which contains X. Take any $\mathcal{V}\in Cov(X)$. By Theorem 1 of Fugate [16] there exist a finite tree T in X and a retraction $r: X\to T$ such that $(jr, 1_X)<\mathcal{V}$ and $r|T=1_T$. Here $j: T\to X$ is the inclusion map. Since T is an AR, there exists a map $r': M\to T$ such that $r'|X=r$. Thus $(jr'|X, 1_X)<\mathcal{V}$ and hence X is an AAR by (II. 5.3).

Using Theorem 2 of Fugate [17], in the same way as in (ii) we can easily show (iii). (i) follows from (iii) and Charatonik-Eberhart [8]. ∎

(3.12) COROLLARY. *If X is a dendrite, fan or smooth dendroid, then 2^X, $C(X)$ are* AARs. ∎

(3.13) PROBLEM. If X is SAM, are 2^X, $C(X)$ and $X(n)$ SAM?

§ 4. *G*-products of spaces.

In this section we discuss approximative properties of G-products of spaces.

In this section all spaces are compact. Let n be a positive integer. S_n denotes the symmetric group on n letters, i.e., it consists of all permutations of $\{1, 2, \cdots, n\}$. Let G be a subgroup of S_n. Let X be a space and X^n the n-th cartesian product with the product topology. Then G can be considered as a subgroup of homeomorphisms of X^n defined as follows: $g(x_1, x_2, \cdots, x_n)=(x_{g(1)}, x_{g(2)}, \cdots, x_{g(n)})$ for $g\in G$ and $x=(x_1, x_2, \cdots, x_n)\in X^n$. X^n/G denotes the orbit space under this action with identification topology. We say that X^n/G is the n-th G-product of X. $\eta=\eta_X: X^n\to X^n/G$ denotes the quotient map. η is open and closed, because $\eta^{-1}\eta(K)=\cup\{g(K): g\in G\}$ for $K\subset X^n$, all g are homeomorphisms and G is a finite set.

Let $f, f': X\to Y$ be maps. Then f induces a map $f^n: X^n\to Y^n$ defined by $f^n(x_1, x_2, \cdots, x_n)=(f(x_1), f(x_2), \cdots, f(x_n))$. Since f^n commutes with the actions of G on X^n and Y^n, it induces a unique map $\underline{f}: X^n/G\to Y^n/G$ satisfying $\eta_Y f^n=\underline{f}\eta_X$. It is easy to show that $\underline{g}\circ\underline{f}=\underline{gf}$ and $\underline{1}_X=1_{X^n/G}$ for any map $g: Y\to Z$.

(4.1) LEMMA. *If f, f', then $\underline{f}\simeq\underline{f}'$.*

PROOF. Take a homotopy $h: X\times I\to Y$ such that $h_0=f$ and $h_1=f'$. We define a map $h^n: X^n\times I\to Y^n$ by $h^n(x_1, x_2, \cdots, x_n, t)=(h(x_1, t), h(x_2, t), \cdots, h(x_n, t))$. It induces a function $H: X^n/G\times I\to Y^n/G$ satisfying $H(\eta_X\times 1_I)=\eta_Y h^n$. Since η_X is a quotient map, by Theorem 3.3.17 of Engelking [14, p. 200] $\eta_X\times 1_I$ is also a quotient map. Thus H is continuous and $H_0=\underline{f}$, $H_1=\underline{f}'$. Hence $\underline{f}\simeq\underline{f}'$. ■

(4.2) THEOREM. *If X is a compact metric* ANR *or* AR, *then* X^n/G *is an* ANR *or an* AR, *respectively.*

PROOF. Maxwell [**36**] proved that

(1) If K is a finite simplicial complex, then so is K^n/G.

First we assume that X is a compact metric ANR and show that X^n/G is an ANR. Since G is a finite set, η is a proper (perfect) map and then by Theorem 4.4.15 of Engelking [14, p. 355] X^n/G is compact metric.

Take any $\mathcal{V}\in\mathcal{C}ov(X^n/G)$. There exists $\mathcal{U}\in\mathcal{C}ov(X)$ such that $\mathcal{U}^n=\mathcal{U}\times\mathcal{U}\times\cdots\times\mathcal{U}<\eta^{-1}\mathcal{V}$. Since X is an ANR, by Corollary 6.2 of Hu [**23**, p. 139] there exist a finite simplicial complex K and maps $f:X\to K$, $g:K\to X$ such that gf is $\mathcal{U}$-homotopic to 1_X. Take a $\mathcal{U}$-homotopy $h:X\times I\to X$ such that $h_0=gf$ and $h_1=1_X$. By (4.1) h induces a homotopy $H: X^n/G\times I\to X^n/G$ such that $H_0=\underline{h}_0=\underline{gf}$ and $H_1=\underline{h}_1=1_{X^n/G}$.

(2) H is a $\mathcal{V}$-homotopy.

Take any $x=(x_1, x_2, \cdots, x_n)\in X^n$. Since h is a $\mathcal{U}$-homotopy, there exist $U_i\in\mathcal{U}$ such that $h(x_i\times I)\subset U_i$ for all i. By the choice of $\mathcal{U}$ there exists $V\in\mathcal{V}$ such that $U_1\times\cdots\times U_n\subset\eta^{-1}V$. Then $H(\eta(x)\times I)=H\circ\eta\times 1_I(x\times I)=\eta h^n(x\times I)\subset\eta(U_1\times\cdots\times U_n)\subset V$. Hence we have (2).

By (1), (2) and Theorem 3.6 of Hu [**23**, p. 139], X^n/G is a compact metric ANR.

Next we assume that X is an AR and show that X^n/G is an AR. By the assumption X is a contractible ANR, that is, X is homotopy equivalent to a one point space $*$. By (4.1) X^n/G is homotopy equivalent to $*^n/G=*$. Thus X^n/G is a contractible ANR and hence an AR. ■

(4.3) REMARK. (4.2) was formulated by Jaworowski [**25**]. However Fedorchuk [**15**] pointed out a gap in his proof, and gave another proof. This proof depends on deep results in the theory of Q-manifolds and Fedorchuk used Schepin's theory. Our proof, which depends on (1) in (4.2), is elementary and simple.

Let $\mathcal{X}=\{X_a, p_{a',a}, A\}$ be an inverse system of compact spaces and let $\boldsymbol{p}=\{p_a: a\in A\}: X\to\mathcal{X}$ be an inverse limit. Then $\boldsymbol{p}^n=\{p_a^n: a\in A\}: X^n\to\mathcal{X}^n=\{X_a^n, p_{a',a}^n, A\}$ is an inverse limit. Moreover $\underline{\boldsymbol{p}}=\{\underline{p}_a: a\in A\}: X^n/G\to\mathcal{X}^n/G=\{X_a^n/G, \underline{p}_{a',a}, A\}$ forms a system map.

(4.4) LEMMA. $\underline{\boldsymbol{p}}: X^n/G\to\mathcal{X}^n/G$ *is an inverse limit*

PROOF. Let $\boldsymbol{q}=\{q_a : a\in A\} : Y\to\mathcal{X}^n/G$ be a system map. We need to show that there exists a unique map $q: Y\to X^n/G$ such that $\boldsymbol{q}=\underline{\boldsymbol{p}}q$, that is,

(1) $$q_a=\underline{p}_a q \quad \text{for} \quad a\in A.$$

Since $\boldsymbol{p}^n : X^n\to\mathcal{X}^n$ is an inverse limit, by Lemma 2 of Kodama-Spiez-Watanabe [31]

(2) $\boldsymbol{p}^{n*}=\{(p_a^n)^* : a\in A\} : 2^{X^n}\to 2^{\mathcal{X}^n}==\{2^{X_a^n}, (p_{a',a}^n)^*, A\}$ is an inverse limit.

Since $\eta_a=\eta_{X_a} : X_a^n\to X_a^n/G$ is open and closed, by Theorem 2 of Kuratowski [32, I. p. 165] $\eta_a^{-1} : 2^{X_a^n/G}\to 2^{X_a^n}$ is continuous. Let $j_a : X_a^n/G\to 2^{X_a^n/G}$ and $j: X^n/G\to 2^{X^n/G}$ be natural inclusion maps. From the definitions it is easy to show that

(3) $$(p_{a',a}^n)^*\eta_{a'}^{-1}j_{a'}=\eta_a^{-1}j_a\underline{p}_{a',a} \quad \text{for} \quad a'>a \quad \text{and}$$

(4) $$(\underline{p}_a^n)^*\eta_X^{-1}j=\eta_a^{-1}j_a\underline{p}_a \quad \text{for} \quad a\in A.$$

By (3) $\{\eta_a^{-1}j_aq_a : a\in A\} : Y\to 2^{\mathcal{X}^n}$ forms a system map. By (2) there exists a unique map $s: Y\to 2^{X^n}$ such that

(5) $$(p_a^n)^*s=\eta_a^{-1}j_aq_a \quad \text{for} \quad a\in A.$$

We show that

(6) $$\eta_X^* s(Y)\subset j(X^n/G).$$

To prove (6) take any $y\in Y$. Then $s(y)$ is a subset of X^n. We need to show that $\eta_X(s(y))$ is a singleton set, i.e., for $x, x'\in s(y)$, there exists $g\in G$ such that $x'=g(x)$. Since $\eta_a p_a^n(s(y))=\{q_a(y)\}$ for each $a\in A$ by (5), $\eta_a p_a^n(x)=\eta_a p_a^n(x')$. Thus there exists $g_a\in G$ such that

(7) $$g_a(p_a^n(x))=p_a^n(x').$$

For each $a\in A$ we put $G_a=\{g\in G: g(p_a^n(x))=p_a^n(x')\}$. We consider G_a as a space with discrete topology. Since G is a finite set, all G_a are compact spaces. Since $G_{a'}\subset G_a$ for $a'>a$, $\mathcal{G}=\{G_a, i_{a',a}, A\}$ forms an inverse system of compact spaces, where $i_{a',a} : G_{a'}\to G_a$ are inclusion maps for $a'>a$. Since $G_a\neq\emptyset$ for all $a\in A$ by (7), $\lim\mathcal{G}=\cap\{G_a : a\in A\}\neq\emptyset$. For each $g\in\lim\mathcal{G}$, $g\in G_a$ and then $p_a^n(x')=g(p_a^n(x))=p_a^n(g(x))$ for all $a\in A$. By (2) $x'=g(x)$. Hence we have (6).

Since j is an embedding, by (6) we obtain a continuous map $q=j^{-1}\eta_X^* s: Y\to X^n/G$. We show that q satisfies (1). Take any $a\in A$ and any $y\in Y$. By (5) $j_a\underline{p}_aq(y)=\underline{p}_a^*jq(y)=\underline{p}_a^*jj^{-1}\eta_X^*s(y)=\underline{p}_a^*\eta_a^*s(y)=\eta_a^*p_a^{n*}s(y)=\eta_a^*\eta_a^{-1}j_aq_a(y)=\eta_a(\eta_a^{-1}\{q_a(y)\})=\{q_a(y)\}=j_aq_a(y)$. Since j_a is 1-1, we have (1).

We show the uniqueness of q. We assume that $q' : Y\to X^n/G$ is a map such that $\underline{p}_aq'=q_a$ for all $a\in A$, and show that $q=q'$. For any $a\in A$ and $y\in Y$ by

(1) and (4) $p_a^{n*}\eta_X^{-1}jq'(y)=\eta_a^{-1}j_a\underline{p}_a q'(y)=\eta_a^{-1}j_a q_a(y)=\eta_a^{-1}j_a\underline{p}_a q(y)=p_a^{n*}\eta_X^{-1}jq(y)$. By (2) $\eta_X^{-1}jq'(y)=\eta_X^{-1}jq(y)$ and then $\{q'(y)\}=\eta_X\eta_X^{-1}\{q'(y)\}=\eta_X\eta_X^{-1}jq'(y)=\eta_X\eta_X^{-1}jq(y)$ $=\{q(y)\}$. Thus $q'(y)=q(y)$ and hence $q'=q$. ■

For $\mathcal{U}\in\mathcal{C}ov(X)$ we put $\mathcal{U}^n=\mathcal{U}\times\cdots\times\mathcal{U}$ (n-times)$\in\mathcal{C}ov(X^n)$. Since η_X is an open map, $\eta_X(\mathcal{U}^n)=\{\eta_X(U_2\times\cdots\times U_n): U_i\in\mathcal{U}$ for $1\leqq i\leqq n\}$ forms a covering of X^n/G.

(4.5) LEMMA. *Let $f, g: X\to Y$ be maps and $\mathcal{U}\in\mathcal{C}ov(X)$, $\mathcal{V}\in\mathcal{C}ov(Y)$.*

(i) *If $(f, g)<\mathcal{V}$, then $(\underline{f}, \underline{g})<\eta_Y(\mathcal{V}^n)$.*

(ii) *If $(f, g)<st\mathcal{V}$, then $(\underline{f}, \underline{g})<st(\eta_Y(\mathcal{V}^n))$,*

(iii) *If $f^{-1}\mathcal{V}>\mathcal{U}$, then $\underline{f}^{-1}\eta_Y(\mathcal{V}^n)>\eta_X(\mathcal{U}^n)$.*

PROOF. We show (ii). In a similar way we can easily show (i) and (iii). Take any $x=(x_1, x_2, \cdots, x_n)\in X^n$. By the assumption for each i there exists $V_i\in\mathcal{V}$ such that $f(x_i), g(x_i)\in st(V_i, \mathcal{V})$. Then there exist $V_i', V_i''\in\mathcal{V}$ such that $f(x_i)\in V_i'$, $g(x_i)\in V_i''$, $V_i'\cap V_i\neq\emptyset$ and $V_i''\cap V_i\neq\emptyset$. Thus $f^n(x)\in V'=V_1'\times\cdots\times V_n'$, $g^n(x)\in V''=V_1''\times\cdots\times V_n''$, $V\cap V'\neq\emptyset$ and $V\cap V''\neq\emptyset$, where $V=V_1\times\cdots\times V_n$. This implies that $\underline{f}\eta_X(x)=\eta_Y f^n(x)\in\eta_Z(V')$, $\underline{g}\eta_X(x)\in\eta_Y(V'')$, $\eta_Y(V')\cap\eta_Y(V)\neq\emptyset$ and $\eta_Y(V'')\cap\eta_Y(V)\neq\emptyset$. This means that $\underline{f}\eta_X(x), \underline{g}\eta_X(x)\in st(\eta_Y(V), \eta_Y(\mathcal{V}^n))$, i.e., $(\underline{f}\eta_X, \underline{g}\eta_X)<st(\eta_Y(\mathcal{V}^n))$. Since η_Y is onto, we have the required conclusion. ■

(4.6) THEOREM. *If $\boldsymbol{p}=\{p_a: a\in A\}: X\to(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ is an approximative resolution, then $\boldsymbol{p}=\{\underline{p}_a: a\in A\}: X^n/G\to(\mathcal{X},\mathcal{U})^n/G=\{(X_a^n/G,\eta_a(\mathcal{U}_a^n)), \underline{p}_{a',a}, A\}$ is an approximative resolution.*

PROOF. We show (AI1)-(AI3) for $(\mathcal{X}, \mathcal{U})^n/G$. (AI1) is obvious. (AI2) and (AI3) follow from (iii) of (4.5) and (AI2), (AI3) for $(\mathcal{X}, \mathcal{U})$. Thus $(\mathcal{X}, \mathcal{U})^n/G$ forms an approximative inverse system.

By (I. 3.3) $\boldsymbol{p}: X\to\mathcal{X}$ is a resolution and then by (I. 7.1) it is an inverse limit, By (4.4) $\boldsymbol{p}: X^n/G\to\mathcal{X}^n/G$ is an inverse limit and then by (I. 3.13) it is a resolution. Hence by (I. 3.3) $\boldsymbol{p}: X^n/G\to(\mathcal{X}, \mathcal{U})^n/G$ is an approximative resolution. ■

(4.7) COROLLARY. *If $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$ is an approximative* **ANR(CM)**-*resolution and an approximative* **POL**$_f$-*resolution, then so is $\boldsymbol{p}: X^n/G\to(\mathcal{X}, \mathcal{U})^n/G$, respectively.*

(4.7) follows from (4.2) and (4.6). ■

(4.8) THEOREM. (i) *If $f: X\to Y$ is an* AE-*map, then so is $\underline{f}: X^n/G\to Y^n/G$.*

(ii) *If f is an* NE-*map, then so is* $\underline{f}$.

PROOF. We show (ii). In a similar way we can show (i). There exist approximative finite polyhedral resolutions $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$, $\boldsymbol{q}=\{q_b: b\in B\}: Y\to(\mathcal{Y}, \mathcal{V})=\{(Y_b, \mathcal{V}_b), q_{b',b}, B\}$ and an approximative resolution $\boldsymbol{f}=\{f, f_b: q\in B\}: (\mathcal{X}, \mathcal{U})\to(\mathcal{Y}, \mathcal{V})$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$ by (I. 3.15) and (I. 4.3). By (4.7) $\underline{\boldsymbol{p}}: X^n/G\to(\mathcal{X}, \mathcal{U})^n/G$ and $\underline{\boldsymbol{q}}: Y^n/G\to(\mathcal{Y}, \mathcal{V})^n/G$ are approximative finite polyhedral resolutions. It is easy by (4.5) to show that $\underline{\boldsymbol{f}}=\{f, \underline{f}_b: b\in B\}: (\mathcal{X}, \mathcal{U})^n/G\to(\mathcal{Y}, \mathcal{V})^n/G$ forms an approximative resolution of $\underline{f}: X^n/G\to Y^n/G$ with respect to $\underline{\boldsymbol{p}}$ and $\underline{\boldsymbol{q}}$.

We show that $\underline{f}$ satisfies (NE). Take any $b\in B$. By the assumption $\boldsymbol{f}$ satisfies (NE) and then there exists $a>f(b)$ satisfying (NE) for $\boldsymbol{f}$ and b. Take any $b'>b$. Then there exists a map $r: X_a\to Y_{b'}$ such that $(f_b p_{a,f(b)}, q_{b',b}r)<st\mathcal{V}_b$. By (4.5) $(\underline{f}_b\underline{p}_{a,f(b)}, \underline{q}_{b',b}\underline{r})<st(\eta_b(\mathcal{V}_b^n))$ and then $\underline{\boldsymbol{f}}$ satisfies (NE). Hence $\underline{f}$ is an NE-map. ■

(4.9) COROLLARY. *If X is* UAM, AM *or* IAM, *then so is X^n/G, respectively.*

(4.9) follows from (II. 5.10), (1.12), (1.21) and (4.8). ■

We use the same notations $\boldsymbol{p}$, $\boldsymbol{q}$, $\underline{\boldsymbol{p}}$ and $\underline{\boldsymbol{q}}$ as in the proof of (4.8). By (I. 5.7) $H(\boldsymbol{p})=\{H(p_a): a\in A\}: X\to H(\mathcal{X})=\{X_a, H(p_{a',a}), A\}$, $H(\boldsymbol{q}): Y\to H(\mathcal{Y})$, $H(\underline{\boldsymbol{p}})=\{H(\underline{p}_a): a\in A\}: X^n/G\to H(\mathcal{X}^n/G)=\{X_a^n/G, H(\underline{p}_{a',a}), A\}$ and $H(\underline{\boldsymbol{q}}): Y^n/G\to H(\mathcal{Y}^n/G)$ are **HPOL**-expansions.

Let $f: X\to Y$ be a shaping. Let $H(\boldsymbol{f})=\{f, H(f_b): b\in B\}: H(\mathcal{X})\to H(\mathcal{Y})$ be a system map in pro-**HPOL** which represents f. For each $b'>b$ there exists $a>f(b), f(b')$ such that $f_b p_{a,f(b)}\simeq q_{b',b}f_{b'}p_{a,f(b')}$. By (4.1) $\underline{f}_b\underline{p}_{a,f(b)}\simeq \underline{q}_{b',b}\underline{f}_{b'}\underline{p}_{a,f(b')}$ and then $H(\underline{\boldsymbol{f}})=\{f, H(\underline{f}_b): b\in B\}: H(\mathcal{X}^n/G)\to H(\mathcal{Y}^n/G)$ forms a system map in pro-**HPOL**. We take another representation $H(\boldsymbol{f}')=\{f', H(f_b'): b\in B\}: H(\mathcal{X})\to H(\mathcal{Y})$ of f. Then for each $b\in B$ there exists $a>f(b), f'(b)$ such that $f_b p_{a,f(b)}\simeq f_b' p_{a,f'(b)}$. Then by (4.1) $\underline{f}_b\underline{p}_{a,f(b)}\simeq\underline{f}_b'\underline{p}_{a,f(b)}$. Thus $H(\underline{\boldsymbol{f}})$ and $H(\underline{\boldsymbol{f}}')$ are equivalent i.e., they represent the same shaping $\underline{f}: X^n/G\to Y^n/G$. It is easy from the above definition to show that $\underline{gf}=\underline{g}\circ\underline{f}$ for shapings $f: X\to Y$, $g: Y\to Z$ and 1_X induces the identity shaping of X^n/G. Hence we may define a functor GP^n: **Sh**(**COM**)$\to$**Sh**(**COM**) as follows: $GP^n(X)=X^n/G$ for a space X and $GP^n(f)=\underline{f}$ for a shaping f. We summarize as follows:

(4.10) THEOREM. *The n-th G-product induces a functor* GP^n: **Sh**(**COM**)$\to$ **Sh**(**COM**).

(4.11) COROLLARY. (i) *If X and Y have the same shape type, then X^n/G and Y^n/G have the same shape type.*

(ii) *If X is shape dominated by Y, then X^n/G is shape dominated by Y^n/G.*

(iii) *If X has trivial shape, then so does X^n/G.*

(iv) *If X is an* ANSR, *then so is X^n/G.*

(v) *If X is an* ASR, *then so X^n/G.*

(i)-(ii) follow from (4.10). (iv)-(v) follow from (II. 6.1), (4.2) and (ii). ■

(4.12) THEOREM. (i) *If X is internally movable, uniformly movable, strongly movable or movable, then so is X^n/G, respectively.*

(ii) *If X satisfies the condition M, then so does X^n/G.*

PROOF. We assume that X is movable and show that X^n/G is movable. We use the same notations as in the proof of (4.10). By the assumption for each $a \in A$ there exists $a_0 > a$ satisfying (MV) for a. Take any $a' > a$ and then there exists a map $r : X_{a_0} \to X_{a'}$ such that $p_{a',a} r \simeq p_{a_0,a}$. By (4.1) $\underline{p}_{a',a}\underline{r} \simeq \underline{p}_{a_0,a}$. Then $\mathcal{X}^n/G$ satisfies (MV) and hence X^n/G is movable.

In a way similar to the one above we can show the other assertions. ■

(4.13) COROLLARY. *If X is an* AANR_C, *an* AANR_N *or an* AAR *for* **COM**, *then so is X^n/G, respectively.*

This follows from (II. 5.10), (II. 5.11), (4.9) and (4.11). ■

§ 5. Maxwell homomorphisms.

In this section we prove a stability theorem on pro-vector spaces and using it we extend Maxwell homomorphisms to compact spaces.

In this section all spaces are compact spaces. Let F be a field. $\text{Vec}(F)$ denotes the category consisting of all vector spaces over F and all linear maps. $\dim G$ denotes the dimension of a vector space G over F. $\text{Vec}_f(F)$ denotes the full subcategory of $\text{Vec}(F)$ consisting of all finite dimensional vector spaces over F. The following is an elementary fact:

(5.1) LEMMA. *Let G and H be objects of $\text{Vec}_f(F)$ and $f : G \to H$ a linear map. If f is onto and $\dim G = \dim H$, then f is an isomorphism in $\text{Vec}_f(F)$.* ■

Let $\mathcal{G} = \{G_a, p_{a',a}, A\}$ be an object of pro-$\text{Vec}_f(F)$. That is, $\mathcal{G}$ is an inverse system on $\text{Vec}_f(F)$. Let $\boldsymbol{p} = \{p_a : a \in A\} : G = \lim \mathcal{G} \to \mathcal{G}$ be an inverse limit of $\mathcal{G}$. In general G is contained in $\text{Vec}(F)$. Using the method in the proof of Theorem 5.7 of Eilenberg-Steenrod [**13**, p. 226] and Kelly [**26**] it is not difficult to show that

(5.2) LEMMA. *If all bonding maps* $p_{a',a}:G_{a'}\to G_a$ *are onto, then all* $p_a:G\to G_a$ *are onto.* ■

We say that $\mathcal{G}$ is stable in pro-Vec(F) provided that $\mathcal{G}$ is isomorphic to a a vector space G in pro-Vec(F). In [**43**] the author discussed a stability theorem in pro-groups.

(5.3) THEOREM. *Let* $\mathcal{G}$ *be an object of pro*-Vec$_f(F)$. *Then* $\mathcal{G}$ *is stable in pro*-Vec(F) *iff the dimension of* $\lim\mathcal{G}$ *is finite.*

PROOF. Let $G=\lim\mathcal{G}$ and $\dim G=n$. First we assume that n is finite. Take any $a\in A$ and put $H^a_{a'}=p_{a',a}(G_{a'})$ for each $a'\in A$ with $a'>a$. Since $\dim G_a$ is finite, $\dim H^a_{a'}=n_{a'}$ is also finite for $a'>a$. Since $H^a_{a'}\supset H^a_{a''}$ for $a''>a'>a$, $n_{a''}\leqq n_{a'}$. Since all $n_{a'}$ are integers, there exists $k(a)>a$ such that $n_{k(a)}=n_{a'}$ for each $a'>k(a)$. Hence

(1) $$H^a_{k(a)}=H^a_{a'} \text{ for each } a'\in A \text{ with } a'>k(a).$$

(1) means that $\mathcal{G}$ satisfies the Mittag-Leffler condition (see MS [**34**]).

Let $H_a=H^a_{k(a)}$ for $a\in A$. Then by (1) it is easy to show that

(2) $$p_{a'',a'}(H_{a''})=H_{a'} \quad \text{for } a''>a'.$$

By (2) $p_{a',a}:G_{a'}\to G_a$ induces a linear map $p'_{a',a}:H_{a'}\to H_a$ for $a'>a$. Then $\mathcal{H}=\{H_a, p'_{a',a}, A\}$ forms an inverse system on Vec$_f(F)$. Let $j_a:H_a\to G_a$ be the inclusion map for $a\in A$. Then $\boldsymbol{j}=\{1_A, j_a: a\in A\}:\mathcal{H}\to\mathcal{G}$ forms a system map. Since $j_a p_{k(a),a}=p_{k(a),a}$ and $p_{k(a),a}j_{k(a)}=p'_{k(a),a}$, by Morita's diagonal theorem (see MS [**34**, p. 112])

(3) $$\boldsymbol{j}:\mathcal{H}\to\mathcal{G} \text{ is an isomorphism in pro-Vec}(F).$$

Let $\boldsymbol{p}'=\{p'_a: a\in A\}: H=\lim\mathcal{H}\to\mathcal{H}$ be an inverse limit. Then $\boldsymbol{j}$ induces a unique homomorphism $j:H\to G$ satisfying

(4) $$\boldsymbol{jp}'=\boldsymbol{p}j.$$

Since $p_{a',a}p_{a'}=p_a$ for $a'>a$, $p_a(G)\subset H_a$ for $a\in A$. Then $p_a:G\to G_a$ induce maps $p''_a:G\to H_a$ such that $j_a p''_a=p_a$ and $p'_{a',a}p''_{a'}=p''_a$ for $a'>a$. Thus $\boldsymbol{p}''=\{p''_a: a\in A\}: G\to\mathcal{H}$ forms a system map satisfying

(5) $$\boldsymbol{p}=\boldsymbol{jp}''.$$

Then there exists a unique homomorphism $h:G\to H$ such that

(6) $$\boldsymbol{p}'h=\boldsymbol{p}''.$$

Since $\boldsymbol{j}(\boldsymbol{p}''j)=(\boldsymbol{jp}'')j=\boldsymbol{p}j=\boldsymbol{jp}'$ by (4) and (5), by (3)

(7) $$p''j=p',$$

Since $p'(hj)=(p'h)j=p''j=p'1_H$ by (6) and (7), then by the uniqueness of inverse limits $hj=1_H$. Since $p(jh)=(pj)h=j(p'h)=jp''=p1_G$ by (4)-(6), then by the uniqueness $jh=1_G$ and hence

(8) $j: H\to G$ is an isomorphism and then $\dim H=\dim G=n$.

Since all bonding maps $p'_{a',a}$ are onto by (2), by (5.2) all $p'_a: H\to H_a$ are onto and then $m_a=\dim H_a\leqq\dim H=n$ for all $a\in A$. By (2) $m_{a'}\geqq m_a$ for $a'>a$. Then there exists $a_0\in A$ such that $m_{a_0}=m_a$ for each $a>a_0$. Thus by (2) $p'_{a'',a'}: H_{a''}\to H_{a'}$ is onto and $\dim H_{a''}=\dim H_{a'}$ for $a''>a'>a_0$, and then by (5.1) $p'_{a'',a'}: H_{a''}\to H_{a'}$ is an isomorphism for $a''>a'>a_0$. It follows that $p'_a: H\to H_a$ is an isomorphism for $a>a_0$. By this and Morita's diagonal theorem

(9) $p': H\to\mathcal{H}$ is an isomorphism in pro-Vec(F).

By (3), (4), (8) and (9) $p: G\to\mathcal{G}$ forms an isomorphism in pro-Vec(F). Hence $\mathcal{G}$ is stable in pro-Vec(F).

Next we assume that $\mathcal{G}$ is stable in pro-Vec(F). By Lemma 2.13 of Dydak [12] an inverse limit $p: G=\lim\mathcal{G}\to\mathcal{G}$ is an isomorphism in pro-Vec(F). Then there exists a system map $q=\{q_0\}: \mathcal{G}\to G$ satisfying $pq\simeq 1_{\mathcal{G}}$ and $qp\simeq 1_G$. Put $q_0: G_{a_0}\to G$. Since $qp\simeq 1_G$, $q_0p_{a_0}=1_G$ and then $p_{a_0}: G\to G_{a_0}$ is 1-1. Thus $\dim G=\dim p_{a_0}(G)\leqq\dim G_{a_0}<\infty$. Hence $\dim G$ is finite. ■

To show (1) in the proof of (5.3) we do not use the condition, $\dim G=n<\infty$. Thus we have the following:

(5.4) COROLLARY. *Any inverse system on* Vec$_f(F)$ *satisfies the Mittag-Leffler condition.* ■

(5.5) COROLLARY. *Let* $\mathcal{G}$ *be an inverse system on* Vec$_f(F)$. *If dimension of* $\lim\mathcal{G}$ *is finite, then there exists* $a_0\in A$ *with the following properties*:

(i) $p_a: \lim\mathcal{G}\to G_a$ *is* 1-1 *for each* $a>a_0$, *and*

(ii) *for each* $a>a_0$ *there exists* $k(a)>a$ *such that* $p_a(\lim\mathcal{G})=p_{a',a}(G_{a'})$ *for each* $a'>k(a)$. ■

$H_j(X;F)$ denotes the j-th Čech homology of a space X with coefficient F. A map $f: X\to Y$ induces a homomorphism $f_{*j}: H_j(X;F)\to H_j(Y;F)$. We say that X is of finite type with respect to F provided that there exists an interger n such that $H_j(X;F)=0$ for $j>n$ and $H_j(X;F)$ is a finite dimensional vector space over F for $j\leqq n$.

(5.6) LEMMA. *If a space* X *is of finite type with respect to* F, *then there* *exists* $\mathcal{W}\in \mathcal{C}ov(X)$ *with the following property*:

(*) *For any space* Y *and any maps* $f, g: Y\to X$ *if* $(f, g)<\mathcal{W}$, *then* $f_{*j}=g_{*j}: H_j(Y;F)\to H_j(X;F)$ *for each* j.

PROOF. We recall the definition of Čech homology. Let $\mathcal{U}, \mathcal{U}'\in \mathcal{C}ov(X)$ with $\mathcal{U}'<\mathcal{U}$. Let $p_{\mathcal{U}}: X\to N(\mathcal{U})$ be a canonical map into the nerve of $\mathcal{U}$ and $p_{\mathcal{U}',\mathcal{U}}: N(\mathcal{U}')\to N(\mathcal{U})$ a projection. $p_{\mathcal{U}',\mathcal{U}}$ is a simplicial map satisfying $p_{\mathcal{U}',\mathcal{U}}(U')\supset U'$ for each vertex $U'\in N(\mathcal{U}')$. It is well known that if $p_{\mathcal{U}',\mathcal{U}}, p'_{\mathcal{U}',\mathcal{U}}: N(\mathcal{U}')\to N(\mathcal{U})$ are projections, then $p_{\mathcal{U}',\mathcal{U}}\simeq p'_{\mathcal{U}',\mathcal{U}}$. $\boldsymbol{p}_{*j}=\{p_{\mathcal{U}*j}: \mathcal{U}\in \mathcal{C}ov(X)\}: H_j(X;F)\to\text{pro-}H_j(X;F)=\{H_j(N(\mathcal{U});F), p_{\mathcal{U}',\mathcal{U}*j}, \mathcal{C}ov(X)\}$ forms an inverse limit.

Since X is of finite type, there exists an integer n such that $H_j(X;F)=0$ for $j>n$ and $H_j(X;F)$ is a finite dimensional vector space for $j\leqq n$. For any j, $0\leqq j\leqq n$ we show the following:

(1) There exists $\mathcal{W}_j\in \mathcal{C}ov(X)$ such that for any space Y and any maps $f, g: Y\to X$ if $(f, g)<\mathcal{W}_j$, then $f_{*j}=g_{*j}: H_j(Y;F)\to H_j(X;F)$.

Since X is compact, we may assume that all coverings $\mathcal{U}$ are finite coverings and then $H_j(N(\mathcal{U});F)$ are finite dimensional vector spaces. Since $H_j(X;F)$ is a finite dimensional vector space, by (5.5) there exists $\mathcal{U}_0\in \mathcal{C}ov(X)$ such that

(2) $$p_{\mathcal{U}*j}: H_j(X;F)\to H_j(N(\mathcal{U});F) \text{ is 1-1 for } \mathcal{U}<\mathcal{U}_0.$$

Take any $\mathcal{U}=\{U_1, U_2, \cdots, U_s\}\in \mathcal{C}ov(X)$ with $\mathcal{U}<\mathcal{U}_0$. Since X is normal, $\mathcal{U}$ is shrinkable. Then there exists $\mathcal{U}'=\{U_1', U_2', \cdots, U_s'\}\in \mathcal{C}ov(X)$ such that $\bar{U}_i'\subset U_i$ for $1\leqq i\leqq s$. Since $\bar{U}_i'\cap(X-U_i)=\emptyset$, there exists $\mathcal{K}_i\in \mathcal{C}ov(X)$ such that $st(\bar{U}_i', \mathcal{K}_i)\cap st(X-U_i, \mathcal{K}_i)=\emptyset$ for each $i=1, 2, \cdots, s$. Put $\mathcal{W}_j=\mathcal{K}_1\wedge\mathcal{K}_2\wedge\cdots\wedge\mathcal{K}_s\in \mathcal{C}ov(X)$ and then

(3) $$st(\bar{U}_i', \mathcal{W}_j)\cap st(X-U_i, \mathcal{W}_j)=\emptyset \quad \text{for } i=1, 2, \cdots, s.$$

We show that $\mathcal{W}_j$ is the required covering. Take any space Y and any maps $f, g: Y\to X$ such that $(f, g)<\mathcal{W}_j$.

(4) $$g^{-1}(U_i')\subset f^{-1}(U_i) \quad \text{for } i=1, 2, \cdots, s.$$

For any $y\in g^{-1}(U_i')$ there exists $W_1\in \mathcal{W}_j$ such that $f(y), g(y)\in W_1$. Since $g(y)\in U_i'\cap W_1$, $f(y)\in st(\bar{U}_i', \mathcal{W}_j)$. By (3) $f(y)\notin st(X-U_i, \mathcal{W}_j)$ and then $f(y)\in U_i$. Thus we have (4).

Let $q_{\mathcal{V}}: Y\to N(\mathcal{V})$ and $q_{\mathcal{V}',\mathcal{V}}: N(\mathcal{V}')\to N(\mathcal{V})$ be a canonical map and a projection for $\mathcal{V}'<\mathcal{V}$. Then $\boldsymbol{q}_{*j}=\{q_{\mathcal{V}*j}: \mathcal{V}\in \mathcal{C}ov(Y)\}: H_j(Y;F)\to\text{pro-}H_j(Y;F)=\{H_j(N(\mathcal{V});F), p_{\mathcal{V}',\mathcal{V}*j}, \mathcal{C}ov(Y)\}$ forms an inverse limit. By (4) we define a simplicial map $v: N(g^{-1}\mathcal{U}')\to N(f^{-1}\mathcal{U})$ by $v(g^{-1}U_i')=f^{-1}U_i$ for each i. Then v is a projection and thus $v\simeq q_{g^{-1}\mathcal{U}',f^{-1}\mathcal{U}}$. Hence

(5) $$v_{*j}=q_{g^{-1}\mathcal{U}', f^{-1}\mathcal{U}*j}: H_j(N(g^{-1}\mathcal{U}'); F) \to H_j(N(f^{-1}\mathcal{U}); F).$$

f and g induce simplicial maps $f_\# : N(f^{-1}\mathcal{U}) \to N(\mathcal{U})$ and $g_\# : N(g^{-1}\mathcal{U}') \to N(\mathcal{U}')$ by $f_\#(f^{-1}U_i)=U_i$ and $g_\#(g^{-1}U'_i)=U'_i$ for each i. Since $U'_i \subset \bar{U}'_i \subset U_i$ for each i, we define a simplicial map $u : N(\mathcal{U}') \to N(\mathcal{U})$ by $u(U'_i)=U_i$ for each i. Since u and $p_{\mathcal{U}', \mathcal{U}}$ are projections, $u \simeq p_{\mathcal{U}', \mathcal{U}}$ and hence

(6) $$u_{*j}=p_{\mathcal{U}', \mathcal{U}*j}: H_j(N(\mathcal{U}')) \to H_j(N(\mathcal{U})).$$

By the definitions of $f_\#$, $g_\#$, u and v clearly $ug_\#=f_\# u$ and hence

(7) $$u_{*j} g_{\#*j}=f_{\#*j} v_{*j}.$$

By (5)-(7)

(8) $$p_{\mathcal{U}', \mathcal{U}*j} g_{\#j}=f_{\#*j} q_{g^{-1}\mathcal{U}', f^{-1}\mathcal{U}*j}.$$

Take any $z \in H_j(Y; F)$. By (8) $p_{\mathcal{U}*j} f_{*j}(z)=f_{\#*j} q_{f^{-1}\mathcal{U}*j}(z)=f_{\#*j} q_{g^{-1}\mathcal{U}', f^{-1}\mathcal{U}*j} q_{g^{-1}\mathcal{U}'*j}(z)=p_{'\mathcal{U}, \mathcal{U}*j} g_{\#*j} q_{g^{-1}\mathcal{U}'*j}(z)=p_{\mathcal{U}', \mathcal{U}*j} p_{\mathcal{U}'*j} g_{*j}(z)=p_{\mathcal{U}*j} g_{*j}(z)$, that is, $p_{\mathcal{U}*j} f_{*j}(z)=p_{\mathcal{U}*j} g_{*j}(z)$. By (2) $f_{*j}(z)=g_{*j}(z)$ and then $f_{*j}=g_{*j}$. Hence we have (1),

Finally we put $\mathcal{W}=\mathcal{W}_0 \wedge \mathcal{W}_1 \wedge \cdots \wedge \mathcal{W}_n \in \mathcal{C}ov(X)$. By the choice of n and (1) it is easy to show that $\mathcal{W}$ satisfies $(*)$. ■

For compact metric spaces (5.6) was proved by Dugundji [**11**].

Q denotes the field consisting of all rational numbers and put $H_j(X)=H_j(X; Q)$. We say that X is of finite type provided that X is of finite type with respect to Q.

Maxwell [**36**] defined homomorphisms $\mu_j^K : H_j(K^n/G) \to H_j(K)$ for finite polyhedra K and integers j satisfy the following conditions:

(M1) $f_{*j}\mu_j^K=\mu_j^L f_{*j}$ for any map $f : K \to L$ between finite polyhedra;

$$\begin{array}{ccc} H_j(K^n/G) & \xrightarrow{\ f_{*j}\ } & H_j(L^n/G) \\ {\scriptstyle \mu_j^K}\downarrow & & \downarrow{\scriptstyle \mu_j^L} \\ H_j(K) & \xrightarrow{\ f_*\ } & H_j(L) \end{array}$$

(M2) $\sum_{i=1}^n \pi_{i*j}^K=\pi_j^K \eta_{K*j}$ for any finite polyhedron K;

$$\begin{array}{ccc} H_j(K^n) & \xrightarrow{\ \eta_{K*j}\ } & H_j(K^n/G) \\ & \searrow^{\sum_{i=1}^n \pi_{i*j}^K} & \downarrow{\scriptstyle \mu_j^K} \\ & & H_j(K) \end{array}$$

Here $\pi_i=\pi_i^K : K^n \to K$ is the i-th projection.

We shall define homomorphisms $\mu_j^X : H_j(X^n/G) \to H_j(X)$ satisyng (M1) and (M2) for all (compact) spaces.

Let X and Y be compact spaces. Take any finite polyhedral resolution $\boldsymbol{p}=\{p_a : a\in A\} : X\to\mathcal{X}=\{X_a, p_{a',a}, A\}$ and $\boldsymbol{q}=\{q_b : b\in B\} : Y\to\mathcal{Y}=\{Y_b, q_{b',b}, B\}$. Then $\boldsymbol{p}^n=\{p^n : a\in A\} : X^n\to\mathcal{X}^n=\{X_a^n, p_{a',a}^n, A\}$ and $\underline{\boldsymbol{p}}=\{\underline{p}_a : a\in A\} : X^n/G\to\mathcal{X}^n/G=\{X_a^n/G, \underline{p}_{a',a}, A\}$ are finite polyhedral resolutions (see § 4). By (M1) for finite polyhedra the Maxwell homomorphisms $\mu_j^a=\mu_j^{X_a} : H_j(X_a^n/G)\to H_j(X_a)$ satisfy $p_{a',a*j}\mu_j^{a'}=\mu_j^a\underline{p}_{a',a*j}$ for $a'>a$. Then $\mu_j^{\boldsymbol{p}}=\{1_A, \mu_j^a : a\in A\} : H_j(\mathcal{X}^n/G)=\{H_j(X_a^n/G), \underline{p}_{a',a*j}, A\}\to H_j(\mathcal{X})=\{H_j(X_a), p_{a',a*j}, A\}$ forms a system map. By taking inverse limits we have a homomorphism $\mu_j^{\boldsymbol{p}}=\lim\mu_j^{\boldsymbol{p}} : H_j(X^n/G)=\lim H_j(\mathcal{X}^n/G)\to\lim H_j(\mathcal{X})=H_j(X)$.

(5.7) LEMMA. *$f_{*j}\mu_j^{\boldsymbol{p}}=\mu_j^{\boldsymbol{q}}\underline{f}_{*j}$ for any shaping $f : X\to Y$. Here $\underline{f}=GP^n(f)$.*

PROOF. Take any representation $\boldsymbol{f}=\{f, H(f_b) : b\in B\} : H(\mathcal{X})=\{X_a, H(p_{a',a}), A\}\to H(\mathcal{Y})=\{Y_b, H(q_{b',b}), B\}$ of a shaping f. Then $\underline{\boldsymbol{f}}=\{f, H(\underline{f}_b) : b\in B\} : H(\mathcal{X}^n/G)\to H(\mathcal{Y}^n/G)$ represents a shaping $\underline{f}=GP^n(f)$ by (4.10). Thus $\boldsymbol{f}_{*j}=\{f, f_{b*j} : b\in B\} : H_j(\mathcal{X})\to H_j(\mathcal{Y})$ and $\underline{\boldsymbol{f}}_{*j}=\{f, \underline{f}_{b*j} : b\in B\}$ $H_j(\mathcal{X}^n/G)\to H_j(\mathcal{Y}^n/G)$ form system maps on Vec(Q) by (4.1). Then $f_{*j}=\lim\boldsymbol{f}_{*j} : H_j(X)\to H_j(Y)$ and $\underline{f}_{*j}=\lim\underline{\boldsymbol{f}}_{*j} : H_j(X^n/G)\to H_j(Y^n/G)$. For any $b\in B$ $q_{b*j}f_{*j}\mu_j^{\boldsymbol{p}}=f_{b*j}p_{f(b)*j}\mu_j^{\boldsymbol{p}}=f_{b*j}\mu_j^{f(b)}\underline{p}_{f(b)*j}=\mu_j^b\underline{f}_{b*j}\underline{p}_{f(b)*j}=\mu_j^b q_{b*j}\underline{f}_{*j}=q_{b*j}\mu_j^{\boldsymbol{q}}\underline{f}_{*j}$. By the uniqueness of inverse limits we have the required one. ■

Let X be a finite polyhedron. Let $\boldsymbol{p}(X) : X\to\{X\}$ be the rudimental resolution of X. From the definition we have that

(5.8) LEMMA. *$\mu_j^K=\mu_j^{\boldsymbol{p}(K)}$ for any finite polyhedron K.* ■

When we take f as the identity shaping, from (5.7) it follows

(5.9) LEMMA. *If $\boldsymbol{p} : X\to\mathcal{X}$ and $\boldsymbol{p}' : X\to\mathcal{X}'$ are finite polyhedral resolutions, then $\mu_j^{\boldsymbol{p}}=\mu_j^{\boldsymbol{p}'}$.* ■

By (5.9) $\mu_j^{\boldsymbol{p}}$ does not depend on the choice of finite polyhedral resolutions of X and then we denote it by μ_j^X. By (5.8) our homomorphism coincides with the original Maxwell homomorphisms for finite polyhedra. By (5.7) (M1) holds for any shaping. By (M2) for finite polyhera and each $a\in A$ $p_{a*j}\mu_j^{\boldsymbol{p}}\eta_{X*j}=\mu_j^a\underline{p}_{a*j}\eta_{X*j}=\mu_j^a\eta_{a*j}p_{a*j}^n=(\sum_{i=1}^n\pi_{i*j}^{X_a})p_{a*j}^n=\sum_{i=1}^n(\pi_i^{X_a}p_a^n)_{*j}=\sum_{i=1}^n p_{a*j}\pi_{i*j}^X=p_{a*j}(\sum_{i=1}^n\pi_{i*j}^X)$. By the uniqueness of inverse limits $\mu_j^X\eta_{X*j}=\mu_j^{\boldsymbol{p}}\eta_{X*j}=\sum_{i=1}^n\pi_{i*j}^X$. This means (M2) for compact spaces. We summarize as follows:

(5.10) THEOREM. *Maxwell homomorphisms can be extended to compact spaces satisfying* (M1) *for shapings and* (M2). ■

Let X be an ANSR. Then there exist a finite polyhedron P and shapings $f: X\to P$, $g: P\to X$ such that $gf=SH(1_X)$ by (II. 6.1). By (5.10) $\underline{fg}=SH(1_{X^n/G})$. Put $W\mu_j^X=g_{*j}\mu_j^P\underline{f}_{*j}: H_j(X^n/G)\to H_j(X)$. By (5.7) $W\mu_j^X=g_{*j}\mu_j^P\underline{f}_{*j}=\mu_j^X g_{*j}\underline{f}_{*j}=\mu_j^X SH(1_{X^n/G})_{*j}=\mu_j^X$. We summarize as follows:

(5.11) LEMMA. $W\mu_j^X=\mu_j^X$ *for any* ANSR X. ■

Masih [**35**] introduced the Maxwell homomorphisms $M\mu_j^X: H_j(X^n/G)\to H_j(X)$ for any compact metric ANR X as follows: Since X is an ANR, there exist a finite polyhedron P and maps $f: X\to P$, $g: P\to X$ such that $gf\simeq 1_X$. He defines $M\mu_j^X=g_{*j}\mu_j^P\underline{f}_{*j}$. From the definitions $M\mu_j^X=W\mu_j^X$. Hence by (5.5) we have that

(5.12) COROLLARY. $M\mu_j^X=\mu_j^X$ *for any compact metric* ANR X.

Vora [**42**] introduced the Maxwell homomorphisms $V\mu_j^X: H_j(X^n/G)\to H_j(X)$ for any compact metric AANR_N X. We define a homomorphism $V\mu_j^X$ by her method.

Let X be a compact AANR_N. By (4.13) X^n/G is an AANR_N. By (II. 6.1) and (II. 6.6) X and X^n/G are shape dominated by finite polyhedra, and then they are of finite type. By (5.6) there exist $\mathcal{U}\in\mathcal{C}ov(X)$ and $\mathcal{V}\in\mathcal{C}ov(X^n/G)$ with the property in (5.6) for X and X^n/G, respectively. Since X is compact, there exists $\mathcal{U}_0\in\mathcal{C}ov(X)$ such that $\mathcal{U}_0<\mathcal{U}$ and $\mathcal{U}_0^n<\mathcal{U}^n\wedge\eta_X^{-1}\mathcal{V}$. Since X is an AP by (II. 5.10)-(II. 5.11), there exist a finite polyhedron P and maps $f: X\to P$, $g: P\to X$ such that $(gf, 1_X)<\mathcal{U}_0$. Then $(\underline{gf}, \underline{1})<\mathcal{V}$. By the choices of $\mathcal{U}$ and $\mathcal{V}$, $g_{*j}f_{*j}=1_{X*j}$ and $g_{*j}\underline{f}_{*j}=1_{X^n/G*j}$. Vora defines homomorphisms $V\mu_j^X=g_{*j}\mu_j^P\underline{f}_{*j}$: $H_j(X^n/G)\to H_j(X)$. By (5.7) $V\mu_j^X=g_{*j}\mu_j^P\underline{f}_{*j}=\mu_j^X g_{*j}\underline{f}_{*j}=\mu_j^X 1_{X^n/G*j}=\mu_j^X$. We summarize as follows:

(5.13) LEMMA. $V\mu_j^X=\mu_j^X$ *for any compact* AANR_N X. ■

(5.12) and (5.13) mean that our extension of the Maxwell homomorphisms are natural.

§6. Fixed point theorems.

In this section we discuss the Maxwell fixed point theorem for NE-maps. It implies the Lefschetz-Hopf fixed point theorem for NE-maps and fixed point theorems for hyperspaces and for cone spaces.

In this section all spaces are compact. Let X be a space of finite type and $f: X\to X^n/G$ a map. We say that a point $x\in X$ is a fixed point of f provided that for each $(x_1, x_2, \cdots, x_n) \in X^n$ with $f(x)=\eta_X(x_1, x_2, \cdots, x_n)$, there exists i, $1 \leqq i \leqq n$, such that $x=x_i$. We define a Lefschetz number $L(f)=\sum_{j=0}^{\infty}(-1)^j tr(\mu_j^X f_{*j})$ of f. Here $tr(\mu_j^X f_{*j})$ denotes the trace of the homomorphism $\mu_j^X f_{*j}: H_j(X) \to H_j(X)$. Here $H_j(X)$ is the j-th Čech homology with rational coefficient Q.

(6.1) LEMMA (Maxwell [**36**]). *Let P be a finite polyhedron and $f: P \to P^n/G$ a map. If $L(f) \neq 0$, then f has a fixed point.* ■

(6.2) THEOREM. *Let X be a compact space of finite type and let $f: X \to X^n/G$ be a map. If f is an* NE-*map and $L(f) \neq 0$, then f has a fixed point.*

Using only (6.1) we shall show (6.2). To do so we need some lemmas. For each i, $1 \leqq i \leqq n$ we put $F_i(f)=\{x \in X: x \in \pi_i \eta^{-1} f(x)\}$ and $F(f)=\cup\{F_i(f): i=1, 2, \cdots, n\}$.

(6.3) LEMMA. *$F(f)$ is the set of all fixed points of f.*

PROOF. Take any fixed poont x of f. For each $(x_1, x_2, \cdots, x_n) \in X^n$ with $f(x)=\eta(x_1, x_2, \cdots, x_n)$, there exists i such that $x=x_i$. Thus $x=x_i \in \pi_i \eta^{-1} f(x)$, that is, $x \in F_i(f) \subset F(f)$.

Take any $x \in F(f)$ and then $x \in F_i(f)$ for some i. Thus $x \in \pi_i \eta^{-1} f(x)$ and then there exists $(y_1, y_2, \cdots, y_n) \in \eta^{-1} f(x)$ such that $x=y_i$. Take any $(x_1, x_2, \cdots x_n) \in X^n$ such that $\eta(x_1, x_2, \cdots, x_n)=f(x)$. Since $\eta(x_1, x_2, \cdots, x_n)=f(x)=\eta(y_1, y_2, \cdots, y_n)$, there exists $g \in G$ such that $g(x_1, x_2, \cdots, x_n)=(y_1, y_2, \cdots, y_n)$. Thus $x=y_i=x_{g(i)}$ and hence x is a fixed point of f. ■

Let $\mathcal{V} \in \mathcal{C}ov(X)$. We put $F_i(f, \mathcal{V})=\{x \in X: st(x, \mathcal{V}) \cap \pi_i \eta^{-1} f(x) \neq \emptyset\}$ for i and $F(f, \mathcal{V})=\cup\{F_i(f, \mathcal{V}): i=1, 2, \cdots, n\}$. We say that a point of $F(f, \mathcal{V})$ is a $\mathcal{V}$-fixed point of f. Trivially $F_i(f, \mathcal{V}') \subset F_i(f, \mathcal{V})$ and $F(f, \mathcal{V}') \subset F(f, \mathcal{V})$ for $\mathcal{V}' < \mathcal{V}$. Let $j^i_{\mathcal{V}', \mathcal{V}}: \overline{F_i(f, \mathcal{V}')} \to \overline{F_i(f, \mathcal{V})}$ and $j_{\mathcal{V}'\mathcal{V}}: \overline{F(f, \mathcal{V}')} \to \overline{F(f, \mathcal{V})}$ be inclusion maps. Thus we may define inverse systems $\mathcal{F}_i(f)=\{\overline{F_i(f, \mathcal{V})}, j^i_{\mathcal{V}', \mathcal{V}}, (\mathcal{C}ov(X), \gg)\}$ and $\mathcal{F}(f)=\{\overline{F(f, \mathcal{V})}, j_{\mathcal{V}', \mathcal{V}}, (\mathcal{C}ov(X), \gg)\}$. Here $\mathcal{V}' \gg \mathcal{V}$ neans that $\mathcal{V}' < \mathcal{V}$, and then $(\mathcal{C}ov(X), \gg)$ forms a directed set.

(6.4) LEMMA. *$F_i(f)=\lim \mathcal{F}_i(f)$ and $F(f)=\lim \mathcal{F}(f)$ for each i.*

PROOF. Take any i. Since all bonding maps in $\mathcal{F}_i(f)$ and $\mathcal{F}(f)$ are inclusions, $\lim \mathcal{F}_i(f)=\cap\{\overline{F_i(f, \mathcal{V})}: \mathcal{V} \in \mathcal{C}ov(X)\}$ and $\lim \mathcal{F}(f)=\cap\{\overline{F(f, \mathcal{V})}: \mathcal{V} \in \mathcal{C}ov(X)\}$. Then we need to show that $F_i(f)=\cap\{\overline{F_i(f, \mathcal{V})}: \mathcal{V} \in \mathcal{C}ov(X)\}$ and $F(f)=\cap\{\overline{F(f, \mathcal{V})}: \mathcal{V} \in \mathcal{C}ov(X)\}$. From the definitions $F_i(f) \subset F_i(f, \mathcal{V})$ and $F(f) \subset F(f, \mathcal{V})$ for $\mathcal{V} \in \mathcal{C}ov(X)$. Thus it is sufficient to show that

$$(1) \qquad \cap\{F_i(f, \mathcal{U}) : \mathcal{U}\in \mathcal{C}ov(X)\}\subset F_i(f) \quad \text{and}$$

$$(2) \qquad \cap\{F(f, \mathcal{U}) : \mathcal{U}\in \mathcal{C}ov(X)\}\subset F(f).$$

CLAIM. For each $\mathcal{U}\in\mathcal{C}ov(X)$ there exists $\mathcal{W}<\mathcal{U}$ such that $\overline{F_i(f, \mathcal{W})}\subset F_i(f, \mathcal{U})$.

We put $k_i=\pi_i^*\eta_X^{-1}jf: X\to X^n/G\xrightarrow{j}2^{X^n/G}\to 2^{X^n}\to 2^X$. Here j is an inclusion map. Since $\eta_X^{-1}: 2^{X^n/G}\to 2^{X^n}$ is continuous by the proof of (4.4), k_i is continuous. Take any $\mathcal{U}\in\mathcal{C}ov(X)$. Since X is compact, there exists a finite covering $\mathcal{V}\in \mathcal{C}ov(X)$ such that $st\mathcal{V}<\mathcal{U}$. For each $x\in X$ we put $m(x)=\{V\in\mathcal{V} : V\cap\pi_i\eta^{-1}f(x)\neq\emptyset\}$ and $m(x)=\{V_1^x, V_2^x, \cdots, V_{n(x)}^x\}$. Since $\langle V_1^x, V_2^x, \cdots, V_{n(x)}^x\rangle$ is an open neighborhood of $\pi_i\eta^{-1}f(x)$ in 2^X, there exists an open neighborhood W_x' of x in X such that

$$(3) \qquad k_i(W_x')\subset\langle V_1^x, V_2^x, \cdots, V_{n(x)}^x\rangle \quad \text{for each} \quad x\in X.$$

We put $\mathcal{W}'=\{W_x' : x\in X\}$ and $\mathcal{W}=\mathcal{W}'\wedge\mathcal{V}\in\mathcal{C}ov(X)$.

We show that $\mathcal{W}$ has the required property. Take any $x\in\overline{F_i(f, \mathcal{W})}$. There exists $V_1\in\mathcal{V}$ with $x\in V_1$. Since $V_1\cap W_x'$ is a neighborhood of x in X, $V_1\cap W_x'\cap F_i(f, \mathcal{W})\neq\emptyset$. Take any $x_1\in V_1\cap W_x'\cap F_i(f, \mathcal{W})$ and then $st(x_1, \mathcal{W})\cap \pi_i\eta^{-1}f(x_1)\neq\emptyset$. There exists $W=V_2\cap W_{x'}'$ such that $V_2\in\mathcal{V}$, $W_{x'}'\in\mathcal{W}'$, $x_1\in W$ and $W\cap\pi_i\eta^{-1}f(x_1)\neq\emptyset$. Then there exists $x_2\in W\cap\pi_i\eta^{-1}f(x_1)$ and thus

$$(4) \qquad x_1, x_2\in V_2 \quad \text{and} \quad x_2\in\pi_i\eta^{-1}f(x_1).$$

Since x, $x_1\in V_1\cap W_x'$, then $k_i(x)$, $k_i(x_1)\in\langle V_1^x, V_2^x, \cdots, V_{n(x)}^x\rangle$ i.e.,

$$(5) \qquad \pi_i\eta^{-1}f(x_1)\subset V_1^x\cup V_2^x\cup\cdots\cup V_{n(x)}^x \quad \text{and}$$

$$(6) \qquad \pi_i\eta^{-1}f(x)\cap V_t^x\neq\emptyset \quad \text{for each} \quad t,\ 1\leqq t\leqq n(x).$$

By (4)-(5) $x_2\in V_{t_0}^x$ for some t_0, $1\leqq t_0\leqq n(x)$ and then by (6)

$$(7) \qquad x_2\in V_{t_0}^x \quad \text{and} \quad \pi_i\eta^{-1}f(x)\cap V_{t_0}^x\neq\emptyset.$$

Since x, $x_1\in V_1\cap W_x'$, by (4) and (7)

$$(8) \qquad x\in st(V_2, \mathcal{V}) \quad \text{and} \quad st(V_2, \mathcal{V})\cap\pi_i\eta^{-1}f(x)\neq\emptyset.$$

Since $st\mathcal{V}<\mathcal{U}$, by (8) $st(V_2, \mathcal{V})\subset st(x, \mathcal{U})$ and $st(x, \mathcal{U})\cap\pi_i\eta^{-1}f(x)\neq\emptyset$. Then $x\in F_i(f, \mathcal{U})$ and hence we have the Claim.

To prove (1) take any $x\notin F_i(f)$. Since $x\notin\pi_i\eta^{-1}f(x)$, $\{x\}$ and $\pi_i\eta^{-1}f(x)$ are disjoint closed subsets of X. Then there exists $\mathcal{U}\in\mathcal{C}ov(X)$ such that $st(x, \mathcal{U})\cap st(\pi_i\eta^{-1}f(x), \mathcal{U})=\emptyset$. Thus $x\notin F_i(f, \mathcal{U})$. By the Claim there exists $\mathcal{W}\in\mathcal{C}ov(X)$ such that $\overline{F_i(f, \mathcal{W})}\subset F_i(f, \mathcal{U})$. Then $x\notin\overline{F_i(f, \mathcal{W})}$ and hence $x\notin\cap\{\overline{F_i(f, \mathcal{U})} : \mathcal{U}\in\mathcal{C}ov(X)\}$. This means (1) and hence the first assertion.

To prove (2) take any $x\notin F(f)$ and then $x\notin F_i(f)$ for each i, $1\leqq i\leqq n$. Since

$F_i(f)=\cap\{\overline{F_i(f, \mathcal{V})}: \mathcal{V}\in \mathcal{C}ov(X)\}$ by the first assertion, then there exists $\mathcal{V}_i\in \mathcal{C}ov(X)$ such that $x\notin\overline{F_i(f, \mathcal{V}_i)}$ for each i. We put $\mathcal{V}=\mathcal{V}_1\wedge\mathcal{V}_2\wedge\cdots\wedge\mathcal{V}_n\in\mathcal{C}ov(X)$ and then $x\notin\overline{F_i(f, \mathcal{V})}$ for all i, $1\leqq i\leqq n$. Since $\overline{F(f, \mathcal{V})}=\cup\{\overline{F_i(f, \mathcal{V})}: i=1, 2, \cdots, n\}$, $x\notin\overline{F(f, \mathcal{V})}$ and hence $x\notin\cap\{\overline{F(f, \mathcal{V})}: \mathcal{V}\in\mathcal{C}ov(X)\}$. This means (2) and hence the second assertion. ■

Since $\mathcal{F}(f)$ is an inverse system of compact spaces, the following follows from (6.3) and (6.4).

(6.5) LEMMA. *If f has a $\mathcal{V}$-fixed point for each $\mathcal{V}\in\mathcal{C}ov(X)$, then f has a fixed point.*

(6.6) LEMMA. *Let $x=(x_1, x_2, \cdots, x_n)$, $y=(y_1, y_2, \cdots, y_n)\in X^n$. Let $\mathcal{V}\in \mathcal{C}ov(X)$ and $U=U_1\times U_2\times\cdots\times U_n\in\mathcal{V}^n$. If $\eta(x), \eta(y)\in st(\eta(U), \eta(\mathcal{V}^n))$, then for each i, $1\leqq i\leqq n$, there exist j, $1\leqq j\leqq n$, and $U^*\in\mathcal{V}$ such that $x_i, y_j\in st(U^*, \mathcal{V})$.*

PROOF. Since $\eta(x), \eta(y)\in st(\eta(U), \eta(\mathcal{V}^n))$, there exist $U'=U_1'\times\cdots\times U_n'$, $U''=U_1''\times\cdots\times U_n''\in\mathcal{V}^n$ such that

(1) $$\eta(x)\in\eta(U') \quad\text{and}\quad \eta(y)\in\eta(U''),$$

(2) $$\eta(U')\cap\eta(U)\neq\emptyset \quad\text{and}\quad \eta(U'')\cap\eta(U)\neq\emptyset.$$

By (2) there exist $s'=(s_1', s_2', \cdots, s_n')\in U'$, $s=(s_1, s_2, \cdots, s_n)\in U$, $t''=(t_1'', t_2'', \cdots, t_n'')\in U''$ and $t=(t_1, t_2, \cdots, t_n)\in U$ such that

(3) $$\eta(s')=\eta(s)\in\eta(U')\cap\eta(U) \quad\text{and}$$

(4) $$\eta(t'')=\eta(t)\in\eta(U'')\cap\eta(U).$$

By (1) there exist $x'=(x_1', x_2', \cdots, x_n')\in U'$ and $y''=(y_1'', y_2'', \cdots, y_n'')\in U''$ such that

(5) $$\eta(x)=\eta(x') \quad\text{and}\quad \eta(y)=\eta(y'').$$

By (3)-(5) there exist $g_1, g_2, g_3, g_4\in G$ such that $x=g_1x'$, $s'=g_2s$, $t=g_3t''$ and $y''=g_4y$. Take any i, $1\leqq i\leqq n$, and then

(6) $$x_i=x'_{g_1(i)},\quad s'_{g_1(i)}=s_{g_2g_1(i)},\quad t_{g_2g_1(i)}=t''_{g_3g_2g_1(i)} \quad\text{and}\quad y''_{g_3g_2g_1(i)}=y_{g_4g_3g_2g_1(i)}.$$

Since $x', s'\in U'$, $s, t\in U$ and $t'', y''\in U''$, we have that

(7) $$x'_{g_1(i)}, s'_{g_1(i)}\in U'_{g_1(i)},\quad s_{g_2g_1(i)}, t_{g_2g_1(i)}\in U_{g_2g_1(i)} \quad\text{and}$$
$$t''_{g_3g_2g_1(i)}, y''_{g_3g_2g_1(i)}\in U''_{g_3g_2g_1(i)}.$$

By (6) and (7) $x_i, y_{g_4g_3g_2g_1(i)}\in st(U_{g_2g_1(i)}, \mathcal{V})$. This means that $j=g_4g_3g_2g_1(i)$ and $U^*=U_{g_2g_1(i)}\in\mathcal{V}$ are the required index and covering. ■

(6.7) LEMMA. *Under the same conditions as in (6.2) f has a $\mathcal{V}$-fixed point for* *each* $\mathcal{U}\in Cov(X)$.

PROOF. By (I. 3.15) there exists an approximative finite polyhedral resolution $\boldsymbol{p}=\{p_a: a\in A\}: X\to(\mathcal{X}, \mathcal{U}')=\{(X_a, \mathcal{U}'_a), p_{a',a}, A\}$. By (1) in the proof of (4.2) and (4.6) $\underline{\boldsymbol{p}}=\{\underline{p}_a: a\in A\}: X^n/G\to(\mathcal{X}, \mathcal{U}')^n/G=\{(X^n_a/G, \eta_a(\mathcal{U}'^n_a)), \underline{p}_{a',a}, A\}$ forms an approximative finite polyhedral resolution. Since all X_a and X^n_a/G are finite polyhedra, there exist $\mathcal{U}''_a\in Cov(X_a)$ and $\mathcal{V}''_a\in Cov(X^n_a/G)$ such that $st^2\mathcal{U}''_a$ and $st^2\mathcal{V}''_a$ satisfy (**) in (I. 5.6). By (I. 4.4) there exist $\mathcal{U}_a\in Cov(X_a)$ for all $a\in A$ such that $\mathcal{U}_a<\mathcal{U}''_a$, $\mathcal{U}^n_a<\eta_X^{-1}\mathcal{V}''_a$ for $a\in A$ and $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ forms an approximative inverse system. By the choice of $\mathcal{U}_a$ and (I. 3.3) $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$ is an approximative $\mathbf{POL}_f$-resolution. By (4.6) $\underline{\boldsymbol{p}}: X^n/G\to(\mathcal{X}, \mathcal{U})^n/G$ is also an approximative $\mathbf{POL}_f$-resolution. By the choice of $\mathcal{U}_a$ for each $a\in A$ and for any space Y

(1) if $g, h: Y\to X_a$ are $st^2\mathcal{U}_a$-near, then $g\simeq h$ and

(2) if $g, h: Y\to X^n_a/G$ are $st^2\eta_a(\mathcal{U}^n_a)$-near, then $g\simeq h$.

By the continuity of Čech homology $H_m(X)=\lim\{H_m(X_a), p_{a',a*m}, A\}$. Since X is of finite type and all X_a are finite polyhedra, by (5.5) there exists $a_0\in A$ satisfying

(3) $p_{a*m}: H_m(X)\to H_m(X_a)$ are 1-1 for all m and all $a>a_0$ and

(4) for each $a'>a_0$ there exists $k(a')>a'$ such that $\mathrm{Im}(p_{a'*m})=\mathrm{Im}(p_{a,a'*m})$ for all m and all $a>k(a')$.

Here $\mathrm{Im}(p_{a'*m})$ denotes the image of $p_{a'*m}: H_m(X)\to H_m(X_{a'})$.

Let $\boldsymbol{f}=\{f, f_a: a\in A\}: (\mathcal{X}, \mathcal{U})\to(\mathcal{X}, \mathcal{U})^n/G$ be an approximative resolution of f with respect to $\boldsymbol{p}$ and $\underline{\boldsymbol{p}}$. Since f is an NE-map, $\boldsymbol{f}$ satisfies (NE).

Take any $\mathcal{U}\in Cov(X)$. Then by (AR1) there exists $a_1>a_0$ such that $p_{a_1}^{-1}st^2\mathcal{U}_{a_1}<\mathcal{U}$. By (NE) for $\boldsymbol{f}$ there exists $a_2>f(a_1)$ satisfying (NE) for $\boldsymbol{f}$ and a_1. Take any $a_3>a_1, a_2$. We put $A(a_3)=\{a\in A: a>a_3\}$, $F^i_a=\{x\in X_a: st(p_{a,a_1}(x), st\mathcal{U}_{a1})\cap\pi_i\eta_{a_1}^{-1}f_{a_1}p_{a,f(a_1)}(x)\neq\emptyset\}$ for each $a\in A(a_3)$ and $i=1, 2, \cdots, n$, and $F_a=\cup\{F^i_a: i=1, 2, \cdots, n\}$. It is easy to show that $p_{a',a}(F^i_{a'})\subset F^i_a$ and then $p_{a',a}(F_{a'})\subset F_a$ for $a'>a$ and all i. Thus $p_{a',a}(\bar{F}_{a'})\subset\bar{F}_a$ for $a'\geqq a$. This means that $\mathcal{F}=\{\bar{F}_a, p_{a',a}, A(a_3)\}$ forms an inverse system consisting of compact spaces.

CLAIM 1. $F_a\neq\emptyset$ for all $a\in A(a_3)$.

Take any $a\in A(a_3)$. By the choice of a_2 there exists a map $r: X_{a_2}\to X^n_{k(a)}/G$ such that

(5) $$(f_{a_1}p_{a_2,f(a_1)}, \underline{p}_{k(a),a_1}r)<st\eta_{a_1}(\mathcal{U}^n_{a_1}).$$

By (5) $(f_{a_1}p_{f(a_1)}, \underline{p}_{k(a),a_1}rp_{a_2})<st\eta_{a_1}(\mathcal{U}^n_{a_1})$. Since $\boldsymbol{f}$ is an approximative resolu-

tion of f,

(6) $$(\underline{p}_{a_1}f,\ f_{a_1}p_{f(a_1)})<\eta_{a_1}(\mathcal{V}^n_{a_1}).$$

Since $(\underline{p}_{a_1}f,\ \underline{p}_{k(a),a_1}rp_{a_2})<st^2\eta_{a_1}(\mathcal{V}^n_{a_1})$ by (6), by (2)

(7) $$\underline{p}_{a_1}f\simeq\underline{p}_{k(a),a_1}rp_{a_2}.$$

Take any integer m. By (7) $\underline{p}_{a_1*m}f_{*m}=\underline{p}_{k(a),a_1*m}r_{*m}p_{a_2*m}$ and by (M1)

(8) $$p_{a,a_1*m}p_{a*m}\mu_m f_{*m}=p_{a,a_1*m}p_{k(a),a*m}\mu^{k(a)}_m r_{*m}p_{a_2*m}.$$

By (3) and (4)

(9) $p_{a,a_1*m}|\mathrm{Im}(p_{a*m}):\mathrm{Im}(p_{a*m})\to H_m(X_{a_1})$ is 1-1 and $\mathrm{Im}(p_{a*m})=\mathrm{Im}(p_{k(a),a*m})$.

Then by (8) and (9)

(10) $$p_{a*m}\mu_m f_{*m}=p_{k(a),a*m}\mu^{k(a)}_m r_{*m}p_{a_2*m}.$$

Let $g=\underline{p}_{k(a),a}rp_{a,a_2}: X_a\to X^n_a/G$ and then by (10) and (M1)

(11) $$p_{a*m}\mu_m f_{*m}=\mu^a_m g_{*m}p_{a*m}.$$

Let $h_m=(p_{a*m})^{-1}p_{k(a),a*m}\mu^{k(a)}_m r_{*m}p_{a,a_2*m}: H_m(X_a)\to H_m(X)$. By (9) h_m is well defined and by (M1)

(12) $$p_{a*m}h_m=\mu^a_m g_{*m}.$$

By (11) and (12) $p_{a*m}h_m p_{a*m}=p_{a*m}\mu_m f_{*m}$ and then by (3)

(13) $$h_m p_{a*m}=\mu_m f_{*m}.$$

By (12) and (13) $tr(\mu_m f_{*m})=tr(h_m p_{a*m})=tr(p_{a*m}h_m)=tr(\mu^a_m g_{*m})$. Then $L(f)=\sum^{\infty}_{m=0}(-1)^m tr(\mu_m f_{*m})=\sum^{\infty}_{m=0}(-1)^m tr(\mu^a_m g_{*m})=L(g)$, that is,

(14) $$L(f)=L(g).$$

By the assumption $L(f)\neq 0$, then by (14) $L(g)\neq 0$. By (6.1) there exists a fixed point x_0 of g. Take any $x=(x_1, x_2, \cdots, x_n)\in X^n_a$ such that $\eta_a(x)=g(x_0)$. Since x_0 is a fixed point of g, there exists i_0, $1\leqq i_0\leqq n$ such that $x_0=x_{i_0}$. By the definition of g and (5) $(f_{a_1}p_{a,f(a_1)},\ \underline{p}_{a,a_1}g)<st\eta_{a_1}(\mathcal{V}^n_{a_1})$ and then there exists $U=U_1\times\cdots\times U_n\in\mathcal{V}^n_{a_1}$ such that

(15) $$f_{a_1}p_{a,f(a_1)}(x_0),\ \underline{p}_{a,a_1}g(x_0)\in st(\eta_{a_1}(U),\ \eta_{a_1}(\mathcal{V}^n_{a_1})).$$

Then $f_{a_1}p_{a,f(a_1)}(x_0)=\eta_{a_1}(y)$ for some $y=(y_1, y_2, \cdots, y_n)\in X^n_{a_1}$. Since $\underline{p}_{a,a_1}g(x_0)=\underline{p}_{a,a_1}\eta_a(x)=\eta_{a_1}p^n_{a,a_1}(x_1, x_2, \cdots, x_n)=\eta_{a_1}(p_{a,a_1}(x_1), p_{a,a_1}(x_2), \cdots, p_{a,a_1}(x_n))$, (15) means that $\eta_{a_1}(y_1, y_2, \cdots, y_n)$, $\eta_{a_1}(p_{a,a_1}(x_1), \cdots, p_{a,a_1}(x_n))\in st(\eta_{a_1}(U),\ \eta_{a_1}(\mathcal{V}^n_{a_1}))$. Thus by (6.6) there exists j, $1\leqq j\leqq n$ and $U^*\in\mathcal{V}_{a_1}$ such that

(16) $$p_{a,a_1}(x_{i_0}),\ y_j\in st(U^*,\ \mathcal{V}_{a_1}).$$

Since $x_0=x_{i_0}$ and $y_j\in\pi_j\eta_{a_1}^{-1}f_{a_1}p_{a,f(a_1)}(x_0)$, (16) means that $st(p_{a,a_1}(x_0), st\mathcal{U}_{a_1})\cap\pi_j\eta_{a_1}^{-1}f_{a_1}p_{a,f(a_1)}(x_0)\neq\emptyset$, that is, $x_0\in F_a^j\subset F_a$. Hence F_a is not empty. We have Claim 1.

CLAIM 2. All points of $\lim\mathcal{F}$ are $\mathcal{U}$-fixed points of f.

Take any $z\in F=\lim\mathcal{F}$, any $a\in A(a_3)$ and put $z_a=p_a(z)\in\bar{F}_a$. There exists $U_1^*\in\mathcal{U}_{a_1}$ such that $p_{a,a_1}(z_a)=p_{a_1}(z)\in U_1^*$. By (6) there exists $U'=U_1'\times\cdots\times U_n'\in\mathcal{U}_{a_1}^n$ such that

$$\underline{p}_{a,a_1}f(z),\ f_{a_1}p_{a,f(a_1)}(z_a)\in\eta_{a_1}(U'). \tag{17}$$

Put $W=p_{a,a_1}^{-1}(U_1^*)\cap(f_{a_1}p_{a,f(a_1)})^{-1}(\eta_{a_1}(U'))$ and then W is an open neighborhood of z_a in X_a. Since $z_a\in\bar{F}_a$, then $W\cap F_a\neq\emptyset$. Take any $w\in W\cap F_a$. Since $w\in W$, then

$$p_{a,a_1}(w)\in U_1^* \quad\text{and}\quad f_{a_1}p_{a,f(a_1)}(w)\in\eta_{a_1}(U'). \tag{18}$$

Since $w\in F_a$, there exists i_1, $1\leqq i_1\leqq n$, with $w\in F_a^{i_1}$ and then there exists $U_2^*\in\mathcal{U}_{a_1}$ such that

$$p_{a,a_1}(w)\in st(U_2^*, \mathcal{U}_{a_1}) \quad\text{and} \tag{19}$$

$$st(U_2^*, \mathcal{U}_{a_1})\cap\pi_{i_1}\eta_{a_1}^{-1}f_{a_1}p_{a,f(a_1)}(w)\neq\emptyset. \tag{20}$$

By (18) there exist $z_f=(z_1^f, z_2^f, \cdots, z_n^f)\in X^n$ and $w'=(w_1', w_2', \cdots, w_n')\in U'$ such that $f(z)=\eta(z_f)$ and $f_{a_1}p_{a,f(a_1)}(w)=\eta_{a_1}(w')$. By (20) there exists $w''=(w_1'', w_2'', \cdots, w_n'')\in X_{a_1}^n$ such that $f_{a_1}p_{a,f(a_1)}(w)=\eta_{a_1}(w'')$ and

$$w_{i_1}''\in st(U_2^*, \mathcal{U}_{a_1}). \tag{21}$$

Since $\eta_{a_1}(w'')=\eta_{a_1}(w')$, there exists $g_1\in G$ such that $g_1(w')=w''$, and then

$$w_{i_1}''=w_{g_1(i_1)}'. \tag{22}$$

By (17) there exists $z'=(z_1', z_2', \cdots, z_n')\in U'$ such that $\underline{p}_{a_1}f(z)=\eta_{a_1}(z')$. Since $\underline{p}_{a_1}f(z)=\underline{p}_{a_1}\eta(z_f)=\eta_{a_1}p_{a_1}^n(z_f)=\eta_{a_1}(p_{a_1}(z_1^f), \cdots, p_{a_1}(z_n^f))$, there exists $g_2\in G$ such that $z'=g_2(p_{a_1}(z_1^f), \cdots, p_{a_1}(z_n^f))$ and then

$$z_{g_1(i_1)}'=p_{a_1}(z_{g_2g_1(i_1)}^f). \tag{23}$$

Since $w', z'\in U'$, then $w_{g_1(i_1)}', z_{g_1(i_1)}'\in U_{g_1(i_1)}'$. By (21)-(23)

$$p_{a_1}(z_{g_2g_1(i_1)}^f)\in U_{g_1(i_1)}' \quad\text{and}\quad U_{g_1(i_1)}'\cap st(U_2^*, \mathcal{U}_{a_1})\neq\emptyset. \tag{24}$$

Since $p_{a_1}(z)\in U_1^*$, by (18) and (19)

$$p_{a_1}(z)\in U_1^* \quad\text{and}\quad U_1^*\cap st(U_2^*, \mathcal{U}_{a_1})\neq\emptyset. \tag{25}$$

By (24) and (25) $p_{a_1}(z)$, $p_{a_1}(z_{g_2g_1(i_1)}^f)\in st(st(U_2^*, \mathcal{U}_{a_1}), st\mathcal{U}_{a_1})$. Since $p_{a_1}^{-1}st^2\mathcal{U}_{a_1}<\mathcal{U}$, there exists $U^{**}\in\mathcal{U}$ such that z, $z_{g_2g_1(i_1)}^f\in p_{a_1}^{-1}st(st(U_2^*, \mathcal{U}_{a_1}), st\mathcal{U}_{a_1})\subset U^{**}$. Since

$z^f_{g_2g_1(i_1)}\in\pi_{g_2g_1(i_1)}\eta^{-1}f(z)$, we have that $st(z, \mathcal{U})\cap\pi_{g_2g_1(i_1)}\eta^{-1}f(z)\neq\emptyset$ and then $z\in F_{g_2g_1(i_1)}(f, \mathcal{U})\subset F(f, \mathcal{U})$. Hence z is a $\mathcal{U}$-fixed point of f. We have Claim 2.

Since all F_a are non-empty compact spaces by Claim 1, $F=\lim\mathcal{F}$ is not empty and hence by Claim 2 f has a $\mathcal{U}$-fixed point. Since $\mathcal{U}\in Cov(X)$ is arbitrary we have the required assertion. ■

Theorem (6.2) follows from (6.5) and (6.7). ■

(6.8) COROLLARY. *Let X be a compact space of finite type, and let $f: X\to X^n/G$ be a map. If X is approximatively movable and $L(f)\neq 0$, then f has a fixed point.*

(6.9) COROLLARY. *Let X be a compact* AANR_C *of finite type, and let $f: X\to X^n/G$ be a map. If $L(f)\neq 0$, then f has a fixed point.*

(6.10) COROLLARY. *Let X be a compact* AANR_N *and let $f: X\to X^n/G$ be a map. If $L(f)\neq 0$, then f has a fixed point.*

(6.11) COROLLARY. *Let X be a compact* ANR *and let $f: X\to X^n/G$ be a map. If $L(f)\neq 0$, then f has a fixed point.*

(6.8) follows from (1.9), (1.12) and (6.2). (6.9) follows from (I. 9.3) and (6.8). (6.10) follows from (II. 5.11) and (6.9). (6.11) follows from (II. 4.6) and (6.8). ■

We assume that $n=1$. Since S_n consists of only the identity element, G is the trivial group. Thus $X^n/G=X$ and $\eta: X^n/G\to X$ is the identity map. Then by (M2) $\mu_j: H_j(X^n/G)=H_j(X)\to H_j(X)$ must be the identity homomorphism. Hence $L(f)=\sum_{j=0}^{\infty}(-1)^j tr(\mu_j f_{*j})=\sum_{j=0}^{\infty}(-1)^j tr(f_{*j})$, that is, $L(f)$ is the usual Lefschetz number $\Lambda(f)$ of f. Thus we have the following from (6.2).

(6.12) THEOREM. *Let X be a compact space of finite type and and let $f: X\to X$ be a map. If f is an* NE-*map and $\Lambda(f)\neq 0$, then f has a fixed point.* ■

When $n=1$, (6.1) means the Lefschetz-Hopf fixed point theorem for finite polyhedra. Hence the proof of (6.2) asserts that (6.12) follows from the usual Lefschetz-Hopf fixed point theorem for finite polyhedra.

(6.13) COROLLARY. *Let X be a compact space of finite type and let $f: X\to X$ be a map. If X is approximatively movable and $\Lambda(f)\neq 0$, then f has a fixed point.*

(6.14) COROLLARY. *Let X be a compact* AANR_C *of finite type and let $f: X$ X be a map. If $\Lambda(f)\neq 0$, then f has a fixed point.*

(6.15) COROLLARY. *Let X be a compact* AANR_N *and let $f: X\to X$ be a map.*

If $\Lambda(f)\neq 0$, then f has a fixed point.

(6.16) COROLLARY. *Let X be a compact* ANR *and let $f: X\to X$ be a map. If $\Lambda(f)\neq 0$, then f has a fixed point.*

In the same way as for (6.8)-(6.11) we have (6.13)-(6.16) from (6.12). ■

(6.17) COROLLARY. *Let X be a compact space.*

(i) *If X is approximatively movable, then* Cone(X) *has the fixed point property.*

(ii) *If* $\dim X=n<\infty$ *and X is LC^{n-1}, then* Cone(X) *has the fixed point property.*

(6.18) COROLLARY. *If a compact connected space X is approximatively movable, then 2^X and $C(X)$ have the fixed point property.*

(6.19) COROLLARY. *If X is a dendrite, fan or smooth dendroid, then 2^X and $C(X)$ have the fixed point property.*

(6.17) follows from (I. 3.1), (2.5) and (6.15). (6.18) follows from (3.10) and (6.15). (6.19) follows from (3.11) and (6.15). ■

(6.20) REMARK. Kinoshita [**27**] constructed a contractible continnum X without the fixed point property. Also he showed that Cone(X) does not have the fixed point property. Knill [**28**] studied the fixed point property for cones. (6.17) is better than Theorem 2.7 of Knill [**28**]. Rogers [**41**] constructed a continuum X such that $C(X)$ does not have the fixed point property. (6.18) gives a partial answer to problems in Rogers [**41**] and Nadler [**39**]. (6.19) was proved by Fugate [**16, 17**].

(6.21) REMARK. Masih [**35**] proved (6.11) and Vora [**42**] proved (9.10) for compact metric spaces. Note that their Maxwell homomorphisms are equivalent to ours by (5.6) and (5.7). Knill [**28**] proved the Lefschetz-Hopf fixed point theorem for Q-simplicial spaces. Using his results Clapp [**9**] proved (6.14) for compact metric spaces. Granas [**22**] proved (6.15) for compact metric spaces. (6.12) was proved by Borsuk [**4**] for compact metric spaces and by Gauthier [**19**] for compact spaces. Their proofs depend on (6.16). Our proof depends only on the Lefschetz-Hopf fixed point theorem for finite polyhedra and then (6.16) is our corollary.

References

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