TSUKUBA J. MATH.
Vol. 11 No. 2 (1987). 303—339

APPROXIMATIVE SHAPE II
—GENERALIZED ANRs—

By

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§ 0. Introduction.

This paper is a continuation of [38], in which we introduced approximative shape. In this paper we introduce many approximative shape properties for spaces. These are approximative shape invariants and unify generalized absolute neighborhood retracts.

In 1931 Borsuk introduced the notions of an absolute neighborhood retract and an absolute retract, in notations ANR and AR, for metric spaces, respectively. There are many generalizations of ANRs and ARs. In 1953 Noguchi [**26**], introduced the notions of an ε-ANR and an ε-AR for compact metric spaces. Gmurczyk [**11, 12**] studied some shape properties of ε-ANRs and ε-ARs. She introduced the terms of an approximative absolute neighborhood retract in the sense of Noguchi and an approximative absolute retract, in notations $AANR_N$ and AAR, respectively, to replace Noguchi's less convenient names ε-ANR and ε-AR. Clapp [**8**] introduced an approximative absolute neighborhood retract in the sense of Clapp, in notation $AANR_C$, for compact metric spaces. Bogatyĭ [**2**] studied many properties of $AANR_N$, $AANR_C$ and AAR. Kalini [**14**] introduced these notions for compact spaces, and Powers [**28**] for metric spaces. Mardešić [**22**] introduced the notion of approximative polyhedra. Recently Gauthier [**9, 10**] introduced $AANE_N$, $AANE_C$ and AAE which are generalizations of an absolute neighborhood extensor and an absolute extensor for metric spaces.

In 1986 Borsuk introduced shape theory, which was then developed by many mathematicians. Shape theory gives us a method to investigate bad spaces and bad maps by means of the good homotopy category of polyhedra. We have many important notions in shape theory; for examples, movability (see [**5**], [**20**]), uniform movability (see [**25**]), strong movability (see [**6**], [**24**]), absolute neighborhood shape retracts (see [**4**], [**23**]) and so on (see [**19**]). These notions play fundamental roles in shape theory.

In [**38**] we introduced approximative shape. It gives us a method to investigate bad spaces and bad maps by means of the good category of polyhedra. In

Received February 4, 1986.

this paper we introduce approximative movability in § **1**, uniformly approximative movability in § **2**, approximative condition M in § **3** and strongly approximative movability in § **4**. In § **5** we show that these approximative shape properties characterize generalized ANRs. In § **6** we discuss the relationship between these approximative shape properties and shape properties. We show that approximative movability and uniformly approximative movability are equivalent for compact metric spaces, but different for compact spaces.

We assume that the reader is familiar with the theory of ANRs and with shape theory. Borsuk [**3**] and Hu [**13**] are standard textbooks for the theory of ANRs. Borsuk [**4**] and Mardešić and Segal [**19**] are standard textbooks for shape theory. For undefined notations and terminology see Hu [**13**] and Mardešić and Segal [**19**], which is quoted by MS [**19**]. We use the same notations and terminology as in [**38**]. We quote results from [**38**] as follows: for example (I.3.3) denotes theorem (3.3) in [**38**].

The author thanks Professor Y. Kodama who encouraged him to develop this theory, and also Dr. K. Sakai and Dr. A. Koyama. They carefully read the first manuscript [**37**] and gave valuable advices.

§ 1. Approximative movability.

In this section we introduce the notion of approximative movability and investigate its properties.

Let $(\mathcal{X},\mathcal{U})=\{(X_a,\mathcal{U}_a),p_{a',a},A\}$ be an approximative inverse system in **TOP**. We say that $(\mathcal{X},\mathcal{U})$ is approximatively movable, in notation AM, provided that it satisfies the following condition:

(AM) For each $a\in A$ there exists $a_0>a$ such that for each $a'>a$ there exists a map $r_{a'}: X_{a_0}\to X_{a'}$ satisfying $(p_{a',a}r_{a'},\ p_{a_0,a})<\mathcal{U}_a$.

(1.1) PROPOSITION. *Let* $(\mathcal{X},\mathcal{U})$ *and* $(\mathcal{Y},\mathcal{V})$ *be approximative inverse systems. Suppose that* $(\mathcal{Y},\mathcal{V})$ *is dominated by* $(\mathcal{X},\mathcal{U})$ *in* Appro-**TOP**. *If* $(\mathcal{X},\mathcal{U})$ *is* AM, *then so is* $(\mathcal{Y},\mathcal{V})$.

PROOF. Put $(\mathcal{Y},\mathcal{V})=\{(Y_b,\mathcal{V}_b),q_{b',b},B\}$. Let $\boldsymbol{f}=\{f,f_b: b\in B\}: (\mathcal{X},\mathcal{U})\to(\mathcal{Y},\mathcal{V})$ and $\boldsymbol{g}=\{g,g_a: a\in A\}: (\mathcal{Y},\mathcal{V})\to(\mathcal{X},\mathcal{U})$ be approximative system maps such that $[\boldsymbol{f}][\boldsymbol{g}]=[1_{(\mathcal{Y},\mathcal{V})}]$. Since $[\boldsymbol{f}][\boldsymbol{g}]=[\boldsymbol{q}(s)(\boldsymbol{fg})]$ for a 1-refinement function s of $(\mathcal{Y},\mathcal{V})$, by (I.2.7) there exists an increasing function $t: B\to B$ such that $t>1_B$ and

(1) $\boldsymbol{q}(t)(\boldsymbol{q}(s)(\boldsymbol{fg}))=:\boldsymbol{q}(t)1_{(\mathcal{Y},\mathcal{V})}$.

Let $u: B\to B$ be a 2-refinement function of $(\mathcal{Y},\mathcal{V})$. Take any $b\in B$. By (1) there exists $b_1 > tu(b), gfstu(b)$ such that

(2) $(q_{stu(b),u(b)} f_{stu(b)} g_{fstu(b)} q_{b_1, gfstu(b)},\ q_{b_1,u(b)}) < \mathcal{V}_{u(b)}$.

By the assumptions, there exists $a_0 > fstu(b)$ such that

(3) a_0 satisfies (AM) for $(\mathcal{X}, \mathcal{U})$ and $fstu(b)$.

By (AM2) there exists $b_2 > b_1, g(a_0)$ such that

(4) $(p_{a_0, fstu(b)} g_{a_0} q_{b_2, g(a_0)},\ g_{fstu(b)} q_{b_2, gfstu(b)}) < \mathcal{U}_{fstu(b)}$.

Claim. b_2 satisfies (AM) for $(\mathcal{Y}, \mathcal{V})$ and b.

Take any $b' > b$. By (AM2) there exists $a' > fstu(b), fstu(b')$ such that

(5) $(f_{stu(b)} p_{a', fstu(b)},\ q_{stu(b'), stu(b)} f_{stu(b')} p_{a', fstu(b')}) < \mathcal{V}_{stu(b)}$.

By (3) there exists a map $r_{a'}: X_{a_0} \to X_{a'}$ such that

(6) $(p_{a', fstu(b)} r_{a'},\ p_{a_0, fstu(b)}) < \mathcal{U}_{fstu(b)}$.

Put $r_{b'} = q_{stu(b), b'} f_{stu(b')} p_{a', fstu(b')} r_{a'} g_{a_0} q_{b_2, g(a_0)}: \ Y_{b_2} \to Y_{b'}$. We need to show that

(7) $(q_{b', b} r_{b'},\ q_{b_2, b}) < \mathcal{V}_b$.

By (5) and (AI2)

(8) $(q_{stu(b), u(b)} f_{stu(b)} p_{a', fstu(b)} r_{a'} g_{a_0} q_{b_2, g(a_0)},$
$q_{stu(b'), u(b)} f_{stu(b')} p_{a', fstu(b')} r_{a'} g_{a_0} q_{b_2, g(a_0)}) < \mathcal{V}_{u(b)}$.

By (6), (AM1) and (AI2)

(9) $(q_{stu(b), u(b)} f_{stu(b)} p_{a', fstu(b)} r_{a'} g_{a_0} q_{b_2, g(a_0)},$
$q_{stu(b), u(b)} f_{stu(b)} p_{a_0, fstu(b)} g_{a_0} q_{b_2, g(a_0)}) < \mathcal{V}_{u(b)}$.

By (4) and (AM1)

(10) $(q_{stu(b), u(b)} f_{stu(b)} p_{a_0, fstu(b)} g_{a_0} q_{b_2, g(a_0)},$
$q_{stu(b), u(b)} f_{stu(b)} g_{fstu(b)} q_{b_2, gfstu(b)}) < \mathcal{V}_{u(b)}$.

By (2)

(11) $(q_{stu(b), u(b)} f_{stu(b)} g_{fstu(b)} q_{b_2, gfstu(b)},\ q_{b_2, u(b)}) < \mathcal{V}_{u(b)}$.

By (8)-(11)

(12) $(q_{stu(b'), u(b)} f_{stu(b')} p_{a', fstu(b')} r_{a'} g_{a_0} q_{b_2, g(a_0)},\ q_{b_2, u(b)}) < st^2 \mathcal{V}_{u(b)}$.

Since u is a 2-refinement function, (7) follows from (12). Thus we have the Claim. Hence $(\mathcal{Y}, \mathcal{V})$ is AM. ■

(1.2) COROLLARY. *The notion of approximative movability for approximative inverse systems is an invariant property in* Appro-**TOP**. ■

Let $\boldsymbol{p} = \{p_a : a \in A\} : X \to (\mathcal{X}, \mathcal{U})$ be an approximative resolution of a space X. We say that $\boldsymbol{p}$ is approximatively movable, in notation AM, provided that $(\mathcal{X}, \mathcal{U})$

is approximatively movable.

(1.3) LEMMA. *Let* $\boldsymbol{p}: X\to(\mathcal{X},\mathcal{U})$ *and* $\boldsymbol{p}': X\to(\mathcal{X},\mathcal{U})'$ *be approximative* **AP**-*resolutions. If* $\boldsymbol{p}$ *is* AM, *then so is* $\boldsymbol{p}'$.

(1.3) follows from (I.5.1) and (1.2). ■

Let $\mathcal{X}=\{X_a, p_{a',a}, A\}$ be an inverse system in **TOP**. We say that $\mathcal{X}$ is approximatively movable, in notation AM, provided that it satisfies the following condition:

(AM)* For each $a\in A$ and for each $\mathcal{U}\in\mathcal{C}ov(X_a)$ there exists $a_0>a$ such that for each $a'>a$ there exists a map $r_{a'}: X_{a_0}\to X_{a'}$ satisfying $(p_{a',a}r_{a'}, p_{a_0,a})<\mathcal{U}$.

(1.4) PROPOSITION. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be inverse systems. Suppose that* $\mathcal{Y}$ *is dominated by* $\mathcal{X}$ *in* pro-**TOP**. *If* $\mathcal{X}$ *is* AM, *then so is* $\mathcal{Y}$.

PROOF. Put $\mathcal{Y}=\{Y_b, q_{b',b}, B\}$. Let $\boldsymbol{f}=\{f, f_b: b\in B\}: \mathcal{X}\to\mathcal{Y}$ and $\boldsymbol{g}=\{g, g_a: a\in A\}: \mathcal{Y}\to\mathcal{X}$ be morphisms of inverse systems such that $\boldsymbol{fg}$ and $1_{\mathcal{Y}}$ are equivalent (see MS [**19**, pp. 1–9]), that is,

(1) $\boldsymbol{fg}\sim 1_{\mathcal{Y}}$.

Take any $b\in B$ and any $\mathcal{V}\in\mathcal{C}ov(Y_b)$. By the assumption there exists $a_0>f(b)$ such that

(2) a_0 satisfies (AM)* for $\mathcal{X}$, $f(b)$ and $f_b^{-1}\mathcal{V}$.

By (1) and the definition of morphisms of inverse systems there exists $b_0>b$, $gf(b), g(a_0)$ such that

(3) $f_b g_{f(b)} q_{b_0,gf(b)}=q_{b_0,b}$ and

(4) $g_{f(b)} q_{b_0,gf(b)}=p_{a_0,f(b)} g_{a_0} q_{b_0,g(a_0)}$.

We show that b_0 is the required index. Take any $b'>b$. Then there exists $a'>f(b), f(b')$ such that

(5) $q_{b',b} f_{b'} p_{a',f(b')}=f_b p_{a',f(b)}$.

By (2) there exists a map $r_{a'}: X_{a_0}\to X_{a'}$ such that

(6) $(p_{a',f(b)} r_{a'}, p_{a_0,f(b)})<f_b^{-1}\mathcal{V}$.

Put $r_{b'}=f_{b'} p_{a',f(b')} r_{a'} g_{a_0} q_{b_0,g(a_0)}: Y_{b_0}\to Y_{b'}$. By (3)–(6) $(q_{b',b} r_{b'}, q_{b_0,b})<\mathcal{V}$ and then $\mathcal{Y}$ is AM. ■

(1.5) COROLLARY. *The notion of approximative movability for inverse* *systems is an invariant property in* pro-**TOP**.

(1.6) LEMMA. *Let* $(\mathcal{X}, \mathcal{U})$ *be an approximative inverse system. Then* $(\mathcal{X}, \mathcal{U})$ *satisfies* (AM) *iff* $\mathcal{X}$ *satisfies* (AM)*.

PROOF. We assume that $(\mathcal{X}, \mathcal{U})$ satisfies (AM) and show that $\mathcal{X}$ satisfies (AM)*. Take any $a \in A$ and any $\mathcal{U} \in \mathcal{C}ov(X_a)$. By (AI3) there exists $a_1 > a$ such that $p_{a_1,a}^{-1}\mathcal{U} > \mathcal{U}_{a_1}$. By the assumption there exists $a_0 > a_1$ which satisfies(AM) for $(\mathcal{X}, \mathcal{U})$ and a_1 Take any $a' > a$ and then take $a'' > a', a_1$. By the choice of a_0 there exists a map $r_{a''}: X_{a_0} \to X_{a''}$ such that $(p_{a'',a_1}r_{a''}, p_{a_0,a_1}) < \mathcal{U}_{a_1}$. Thus $(p_{a'',a}r_{a''}, p_{a_0,a}) < \mathcal{U}$. This means that a_0 and the map $p_{a'',a'}r_{a''}: X_{a_0} \to X_{a'}$ satisfy (AM)* for $\mathcal{X}$ and a. The converse assertion is trivial. ■

Let $\boldsymbol{p} = \{p_a : a \in A\} : X \to \mathcal{X}$ be a resolution. We say that $\boldsymbol{p}$ is approximatively movable, in notation AM, provided that $\mathcal{X}$ is approximatively movable.

(1.7) PROPOSITION. *Let* $\boldsymbol{p}: X \to \mathcal{X}$ *and* $\boldsymbol{q}: X \to \mathcal{Y}$ *be* **AP**-*resolutions of a space* X. *If* $\boldsymbol{p}$ *is* AM, *then so is* $\boldsymbol{q}$.

PROOF. Put $\boldsymbol{q} = \{q_b : b \in B\}$ and $\mathcal{Y} = \{Y_b, q_{b',b}, B\}$. We need to show (AM)* for $\mathcal{Y}$. Take any $b \in B$ and any $\mathcal{V} \in \mathcal{C}ov(Y_b)$. By (R2) there exists $\mathcal{V}_i \in \mathcal{C}ov(Y_b)$, $i = 1, 2, 3, 4$, such that

(1) $st\mathcal{V}_1 < \mathcal{V}, \mathcal{V}_2 < \mathcal{V}_1, \mathcal{V}_3 < \mathcal{V}_1, st\mathcal{V}_4 < \mathcal{V}_2 \wedge \mathcal{V}_3$,

(2) $\mathcal{V}_2$ satisfies (R2) for $\boldsymbol{q}$ and $\mathcal{V}_1$, and

(3) $\mathcal{V}_3$ satisfies (R2) for $\boldsymbol{p}$ and $\mathcal{V}_1$.

By (R1) for $\boldsymbol{p}$ there exist $a \in A$ and a map $h: X_a \to Y_b$ such that

(4) $(hp_a, q_b) < \mathcal{V}_4$.

Since $\mathcal{X}$ is AM, there exists $a_1 > a$ such that

(5) a_1 satisfies (AM)* for a and $h^{-1}\mathcal{V}_4$.

By (R1) for $\boldsymbol{q}$ their exists $b_1 > b$ and a map $k: Y_{b_1} \to X_{a_1}$ such that $(kq_{b_1}, p_{a_1}) < (hp_{a_1,a})^{-1}\mathcal{V}_4$. Thus $(hp_{a_1,a}kq_{b_1}, hp_a) < \mathcal{V}_4$ and then by (1) and (4)

(6) $(hp_{a_1,a}kq_{b_1}, q_{b_1,b}q_{b_1}) < \mathcal{V}_2$.

By (2) and (6) there exists $b_2 > b_1$ such that

(7) $(hp_{a_1,a}kq_{b_2,b_1}, q_{b_2,b}) < \mathcal{V}_1$.

Claim. b_2 satisfies (AM)* for $\mathcal{Y}$, b and $\mathcal{V}$.

Take any $b' > b$. By (R1) there exist $a' > a$ and a map $m: X_{a'} \to Y_{b'}$ such that $(mp_{a'}, q_{b'}) < q_{b',b}^{-1}\mathcal{V}_4$. Thus $(q_{b',b}mp_{a'}, q_b) < \mathcal{V}_4$ and then by (1) and (4)

(8) $(q_{b',b}mp_{a'}, hp_{a',a}p_{a'}) < \mathcal{V}_3$.

By (3) and (8) there exists $a'' > a'$ such that

(9) $(q_{b',b}mp_{a'',a'}, hp_{a'',a}) < \mathcal{V}_1$.

By (5) there exists a map $r_{a''} : X_{a_1} \to X_{a''}$ such that $(p_{a'',a}r_{a''}, p_{a_1,a}) < h^{-1}\mathcal{V}_4$. Thus $(hp_{a'',a}r_{a''}, hp_{a_1,a}) < \mathcal{V}_4$ and then

(10) $(hp_{a'',a}r_{a''}kq_{b_2,b_1}, hp_{a_1,a}kq_{b_2,b_1})' < \mathcal{V}_4$.

By (9)

(11) $(q_{b',b}mp_{a'',a'}r_{a''}kq_{b_2,b_1}, hp_{a'',a}r_{a''}kq_{b_2,b_1}) < \mathcal{V}_1$.

By (1), (7), (10) and (11) $(q_{b',b}mp_{a'',a'}r_{a''}kq_{b_2,b_1}, q_{b_2,b}) < \mathcal{V}$. This means that the map $mp_{a'',a'}r_{a''}kq_{b_2,b_1} : Y_{b_2} \to Y_{b'}$ gives our Claim. Hence $\mathcal{Y}$ is AM. ■

(1.8) THEOREM. *Let X be a space. Then the following conditions are equivalent:*

(i) *Any/some approximative* **AP**-*resolution of X is* AM.

(ii) *Any/some* **AP**-*resolution of X is* AM.

PROOF. By (1.3) any and some in (i) are equivalent. By (1.7) any and some in (ii) are equivalent. We show that (i) implies (ii). By (i) there exists an approximative **AP**-resolution $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ such that $(\mathcal{X}, \mathcal{U})$ satisfies (AM). Then $\mathcal{X}$ satisfies (AM)* by (1.6). Since $\boldsymbol{p}: X \to \mathcal{X}$ is an **AP**-resolution by (I.3.3), we have (ii). We show that (ii) implies (i). By (I.3.15) there exists an approximative **POL**-resolution $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$. Since $\boldsymbol{p}: X \to \mathcal{X}$ is a **POL**-resolution by (I.3.3), $\boldsymbol{p}$ is AM by (ii). Since $\mathcal{X}$ is AM, by (1.6) so is $(\mathcal{X}, \mathcal{U})$. Hence $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ is AM. Then we have (i). ■

We say that a space X is approximatively movable, in notation AM, provided that it satisfies one of the conditions in (1.8).

(1.9) THEOREM. *Let X and Y be spaces. Suppose that Y is dominated by X in* **ASh**. *If X is* AM, *then so is Y.*

PROOF. By the assumption there exist approximative shapings m, n such that $mn = \mathrm{AS}(1_Y)$. Let $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$, $\boldsymbol{p}': X \to (\mathcal{X}, \mathcal{U})' \in E(X)$ and $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V})$, $\boldsymbol{q}': Y \to (\mathcal{Y}, \mathcal{V})' \in E(Y)$. Let $\boldsymbol{f}: (\mathcal{X}, \mathcal{U})' \to (\mathcal{Y}, \mathcal{V})'$ and $\boldsymbol{g}: (\mathcal{Y}, \mathcal{V}) \to (\mathcal{X}, \mathcal{U})$ be approximative system maps such that $\langle[\boldsymbol{f}]\rangle = m$ and $\langle[\boldsymbol{g}]\rangle = n$. Since $mn = \mathrm{AS}(1_Y)$, $[1_Y]_{\boldsymbol{q}',\boldsymbol{q}}[\boldsymbol{f}][1_X]_{\boldsymbol{p},\boldsymbol{p}'}[\boldsymbol{g}] = [1_{(\mathcal{Y},\mathcal{V})}]$. This means that $(\mathcal{Y}, \mathcal{V})$ is dominated by $(\mathcal{X}, \mathcal{U})$ in Appro-**AP**. Since $(\mathcal{X}, \mathcal{U})$ is AM by (1.8), by (1.1) so is $(\mathcal{Y}, \mathcal{V})$. Hence Y is AM. ■

(1.10) COROLLARY. *The notion of approximative movability for spaces is an invariant property in* **ASh**. ■

(1.11) COROLLARY. (i) *Suppose that a space Y is dominated by a space X in* **TOP**. *If X is* AM, *then so is Y.*

(ii) *The notion of approximative movability for spaces is a topologically invariant property.*

(1.12) COROLLARY. (i) *A space X is* AM *iff so is $T(X)$.*

(ii) *A Tychonoff space X is* AM *iff so is $C(X)$.*

(iii) *A space X is* AM *iff so is $CT(X)$.*

(1.11) follows from (I.5.9) and (1.9). (1.12) follows from (I.6.8), (I.6.10) and (1.8). ■

Let $\mathcal{K}$ be a collection of spaces. We say that $(\mathcal{X}, \mathcal{U})$ is approximatively $\mathcal{K}$-movable, in notation $\mathcal{K}$-AM, provided that it satisfies the following condition:

($\mathcal{K}$-AM) For each $a \in A$ there exists $a_0 > a$ such that for each $a' > a$ and for any map $f: K \to X_{a_0}$, where $K \in \mathcal{K}$, there exists a map $f': K \to X_{a'}$ satisfying $(p_{a',a} f', p_{a_0,a} f) < \mathcal{U}_a$.

We say that $\mathcal{X}$ is approximatively $\mathcal{K}$-movable, in notation $\mathcal{K}$-AM, provided that it satisfies the following condition:

($\mathcal{K}$-AM)* For any $a \in A$ and for any $\mathcal{U} \in \mathcal{C}ov(X_a)$ there exists $a_0 > a$ such that for each $a' > a$ and for any map $f: K \to X_{a_0}$, where $K \in \mathcal{K}$, there exists a map $f': K \to X_{a'}$ satisfying $(p_{a',a} f', p_{a_0,a} f) < \mathcal{U}$.

We say that an approximative resolution $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ and a resolution $\boldsymbol{p}: X \to \mathcal{X}$ are approximatively $\mathcal{K}$-movable, in notation $\mathcal{K}$-AM, provided that they satisfy ($\mathcal{K}$-AM) and ($\mathcal{K}$-AM)*, respectively. By slight modifications of our proofs we can show (1.1)–(1.8) for approximative $\mathcal{K}$-movability. We say that a space X is approximatively $\mathcal{K}$-movable, in notation $\mathcal{K}$-AM, provided that it satisfies one of the conditions in (1.8) for approximative $\mathcal{K}$-movability. In the same way we can show the analogues of (1.9)–(1.12) for approximative $\mathcal{K}$-movability. Thus we summarize as follows:

(1.13) THEOREM. *Let $\mathcal{K}$ be a collection of spaces. All assertions* (1.1)–(1.12) *hold for approximative $\mathcal{K}$-movability.* ■

Let $\boldsymbol{D}$ be a subcategory of **TOP**. We say that a space X is approximatively $\boldsymbol{D}$-movable, in notation $\boldsymbol{D}$-AM, provided that it is approximatively Ob$\boldsymbol{D}$-movable.

Let $\mathbf{POL}^n$ be the full subcategory of **POL** consisting of all polyhedra P such that $\dim P \leq n$. We say that a space X is approximatively n-movable, in notation n-AM, provided that it is approximatively $\mathbf{POL}^n$-movable.

(1.14) COROLLARY. *All assertions* (1.1)-(1.12) *hold for approximative n-movability.* ■

Finally we show relations between approximative movability and approximative $\mathcal{K}$-movability.

(1.15) THEOREM. *Let X be a space.* (i) *If X is* AM, *then it is $\mathcal{K}$*-AM *for any collection $\mathcal{K}$ of spaces.*

(ii) *Let $\boldsymbol{K}$ be one of* **AP**, **POL** *and* **ANR**. *Then X is* AM *iff it is $\boldsymbol{K}$*-AM.

(iii) *If X is* AM, *then it is n*-AM *for each integer n.*

(iv) *Let* $\dim X \leq n$. *Then X is* AM *iff it is n*-AM.

PROOF. Since (AM) implies ($\mathcal{K}$-**AM**) for any $\mathcal{K}$, we have (i) and (iii). Let $\boldsymbol{C}$ be a full subcategory of **TOP** and $(\mathcal{X}, \mathcal{U})$ an approximative inverse system in $\boldsymbol{C}$. We easily show that if $(\mathcal{X}, \mathcal{U})$ satisfies (Ob$\boldsymbol{C}$-AM), then it satisfies (AM). This fact and (I.3.15) imply (ii) and (iv). ■

§ 2. Uniformly and internally approxlmative movabilities.

In this section we introduce the notions of uniformly approximative movability and internally approximative movability. We discuss their properties.

Let $(\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ be an approximative inverse system in **TOP**. We say that $(\mathcal{X}, \mathcal{U})$ is uniformly approximatively movable, in notation UAM, provided that it satisfies the following condition:

(UAM) For each $a \in A$ there exist $a_0 > a$ and a collection $\{r_{a'} : a' > a\}$ of maps $r_{a'} : X_{a_0} \to X_{a'}$ such that $(p_{a',a} r_{a'}, p_{a_0,a}) < \mathcal{U}_a$ and $(r_{a'}, p_{a'',a'}, r_{a''}) < \mathcal{U}_{a'}$ for $a'' > a' > a$.

In a similar ways as in (1.1) and (1.2) we can show (1.1) and (1.2) for UAM. We say that an approximative resolution $\boldsymbol{p} : X \to (\mathcal{X}, \mathcal{U})$ is uniformly approximatively movable, in notation UAM, provided that $(\mathcal{X}, \mathcal{U})$ is uniformly approximatively movable. In the same way as in (1.3) we can show (1.3) for UAM. Thus in the same way as in (1.8) we can show the following:

(2.1) THEOREM. *Let X be a space. Then the following conditions are equivalent:*

(i) *X admits an approximative* **AP**-*resolution which is* UAM.

(ii) *Any approximative* **AP**-*resolution of* X *is* UAM. ■

We say that a space X is uniformly approximatively movable, in notation UAM, provided that it satisfies one of the conditions in (2.1). In the same way as in (1.9)-(1.12) we can show these statements for UAM. Uniformly approximative movability is an invariant in **ASh**. We summarize as follows:

(2.2) THEOREM. (1.1)-(1.3) *and* (1.9)-(1.12) *hold for* UAM. ■

(2.3) LEMMA. (i) *If an approximative inverse system is* UAM, *then it is* AM.

(ii) *If a space is* UAM, *then it is* AM.

(2.3) follows from the definitions. In § **6** we shall show that, in general, the converses of (i), (ii) in (2.3) do not hold. However we show their converses for a special case.

(2.4) PROPOSITION. *Let* $(\mathcal{X}, \mathcal{U})$ *be an approximative inverse sequence. Then* $(\mathcal{X}, \mathcal{U})$ *is* AM *iff it is* UAM.

PROOF. Put $(\mathcal{X}, \mathcal{U})=\{(X_i, \mathcal{U}_i), p_{i,j}, N\}$, where N is the set of all positive integers. We assume that $(\mathcal{X}, \mathcal{U})$ is AM and show that it is UAM. By the assumption there exists a subset $A=\{a_i : i\in N\}\subset N$ such that $a_1=1<a_2<a_3<\cdots$,

(1) $p^{-1}_{a_{i+1}, a_i}\mathcal{U}_{a_i}>st\mathcal{U}_{a_{i+1}}$ for $i\in N$ and

(2) a_{i+1} satisfies (AM) for $(\mathcal{X}, \mathcal{U})$ and a_i for each $i\in N$.

By (I.2.12) $(\mathcal{X}, \mathcal{U})_A=\{(X_{a_i}, \mathcal{U}_{a_i}), p_{a_i,a_j}, A\}$ is an approximative inverse sequence and then by (I.2.1) so is $st(\mathcal{X}, \mathcal{U})_A$.

Claim. $st(\mathcal{X}, \mathcal{U})_A$ is UAM.

By (2) there exist maps $r_i : X_{a_i}\to X_{a_{i+1}}$ for $i\geq 2$ such that

(3) $(p_{a_{i+i},a_{i-1}}r_i, p_{a_i,a_{i-1}})<\mathcal{U}_{a_{i-1}}$ for each $i\geq 2$.

Take any $a_k\in A$ and put $f_i=p_{a_{i+1},a_i}r_i r_{i-1}\cdots r_{k+1} : X_{a_{k+1}}\to X_{a_i}$ for $i\geq k+1$ and $f_k=p_{a_{k+1},a_k} : X_{a_{k+1}}\to X_{a_k}$. We show that $\{f_i : i\geq k\}$ satisfies

(4) $(p_{a_i,a_j}f_i, f_j)<st\mathcal{U}_{a_j}$ for $i\geq j\geq k$ and

(5) $(p_{a_i,a_k}f_i, p_{a_{k+1},a_k})<st\mathcal{U}_{a_k}$ for $i\geq k$.

Since $f_k=p_{a_{k+1},a_k}$, (5) follows from (4). Inductively we show (4). To do so we consider the following:

P(1): $(p_{a_{i+1},a_i}f_{i+1}, f_i)<\mathcal{U}_{a_i}$ for $i\geq k$.

Q(n) : The condition (4) holds for $i \geq j \geq k$ with $i-j=n$.

First we show P(1). By (3) $(p_{a_{k+2},a_k}r_{k+1},\ p_{a_{k+1},a_k})<\mathcal{U}_{a_k}$, that is, $(p_{a_{k+1},a_k}f_{k+1},\ f_k)<\mathcal{U}_{a_k}$. Hence P(1) holds for $i=k$. Let $i\geq k+1$. By (3) $(p_{a_{i+2},a_i}r_{i+1},\ p_{a_{i+1},a_i})<\mathcal{U}_{a_i}$ and then $(p_{a_{i+2},a_i}r_{i+1}r_i\cdots r_{k+1},\ p_{a_{i+1},a_i}r_i\cdots r_{k+1})<\mathcal{U}_{a_i}$. Thus $(p_{a_{i+1},a_i}f_{i+1},\ f_i)<\mathcal{U}_{a_i}$. Then P(1) holds for $i\geq k+1$. Hence we have P(1).

Trivially Q(1) follows from P(1). We assume that Q(1), $\cdots$, Q(n−1) hold and show Q(n). Take any $i\geq k$. By the inductive assumption and P(1)

(6) $(p_{a_{i+n},a_{i+1}}f_{i+n},\ f_{i+1})<st\mathcal{U}_{a_{i+1}}$ and

(7) $(p_{a_{i+1},a_i}f_{i+1},\ f_i)<\mathcal{U}_{a_i}$.

By (1) and (6)

(8) $(p_{a_{i+n},a_i}f_{i+n},\ p_{a_{i+1},a_i}f_{i+1})<\mathcal{U}_{a_i}$.

By (7) and (8) $(p_{a_{i+n},a_i}f_{i+n},\ f_i)<st\mathcal{U}_{a_i}$. This means Q(n). Hence Q(n) holds for all n, that is, we have (4). By (4) and (5) we have the Claim.

By (I.2.12) and (I.2.14) $(\mathcal{X},\mathcal{U})$ and $st(\mathcal{X},\mathcal{U})_A$ are isomorphic in Appro-**TOP**. Hence by (1.10) for UAM $(\mathcal{X},\mathcal{U})$ is UAM. The converse follows from (2.3). ■

(2.5) THEOREM. *Let X be a compact metric space. Then X is* UAM *iff it is* AM.

(2.5) follows from (I.3.15) and (2.4). ■

In shape theory Spiez [**31**] showed that movability and uniform movability are equivalent for metric compacta. (2.5) corresponds to his result.

Let $\boldsymbol{p}=\{p_a : a\in A\} : X\to(\mathcal{X},\mathcal{U})$ be an approximative resolution of a space X. We say that $\boldsymbol{p}$ is internally approximatively movable, in notation IAM, provided that it satisfies the following condition:

(IAM) For each $a\in A$ there exist $a'>a$ and a map $r : X_{a'}\to X$ such that $(p_a r,\ p_{a',a})<\mathcal{U}_a$.

(2.6) PROPOSITION. *Let* $\boldsymbol{p} : X\to(\mathcal{X},\mathcal{U})$ *and* $\boldsymbol{q} : Y\to(\mathcal{Y},\mathcal{V})$ *be approximative* **AP**-*resolutions of spaces X and Y, respectively. Suppose that Y is dominated by X in* **TOP**. *If* $\boldsymbol{p}$ *is* IAM, *then so is* $\boldsymbol{q}$.

In a way similar to the one used in (1.1) we can show (2.6). Then we have (1.3) for IAM.

Let $\boldsymbol{p}=\{p_a : a\in A\} : X\to\mathcal{X}=\{X_a, p_{a',a}, A\}$ be a resolution of a space X. We say that $\boldsymbol{p}$ is internally approximatively movable, in notation IAM, provided that it satisfies the following condition:

(IAM)* For each $a\in A$ and for each $\mathcal{U}\in\mathcal{C}ov(X_a)$ there exist $a_0>a$ and a map $r: X_{a_0}\to X$ such that $(p_a r, p_{a_0,a})<\mathcal{U}$.

(2.7) LEMMA. *Let $\boldsymbol{p}: X\to(\mathcal{X},\mathcal{U})$ be an approximative resolution. Then $\boldsymbol{p}: X\to(\mathcal{X},\mathcal{U})$ satisfies* (IAM) *iff $\boldsymbol{p}: X\to\mathcal{X}$ satisfies* (IAM)*.

(2.7) follows from the definitions. In a way similar to the one used in (1.7) we can show (1.7) for IAM. In the same way as in (1.8) we can show the following:

(2.8) THEOREM. *Let X be a space. Then the following statements are equivalent:*

(i) *Any/some approximative* **AP**-*resolution of X is* IAM.

(ii) *Any/some* **AP**-*resolution of X is* IAM. ■

We say that a space X is internally approximatively movable, in notation IAM, provided that it satisfies one of the conditions in (2.8). By (2.6) we have (1.11) for IAM, i.e., internally approximative movability is a topological invariant.

(2.9) PROPOSITION. (i) *If a space X is* IAM, *then so is $T(X)$.*

(ii) *If a Tychonoff space X is* IAM, *then so is $C(X)$.*

(iii) *If a space X is* IAM, *then so is $CT(X)$.*

This follows from (I.6.8), (I.6.10) and (2.8). We summarize as follows:

(2.10) PROPOSITION. (1.3), (1.7) *and* (1.11) *hold for* IAM. ■

(2.11) LEMMA. (i) *If an approximative resolution $\boldsymbol{p}: X\to(\mathcal{X},\mathcal{U})$ is* IAM, *then it is* UAM.

(ii) *If a space X is* IAM, *then X is* UAM. ■

(2.12) THEOREM. *A space X is* UAM *iff $CT(X)$ is* IAM.

PROOF. First, we assume that X is UAM and show that $CT(X)$ is IAM. Since X is UAM, by (1.12) for UAM (see (2.2)) $CT(X)$ is also UAM. Let $\boldsymbol{p}=\{p_a: a\in A\}: CT(X)\to(\mathcal{X},\mathcal{U})=\{(X_a,\mathcal{U}_a), p_{a',a}, A\}$ be an approximative **POL**-resolution. Then $\boldsymbol{p}$ satisfies (UAM) and we show that $\boldsymbol{p}$ is IAM. Take any $a_1\in A$. By (AI3) there exists $a_2>a_1$ such that $p_{a_2,a_1}^{-1}\mathcal{U}_{a_1}>st^2\mathcal{U}_{a_2}$. Since $(\mathcal{X},\mathcal{U})$ is UAM, there exist $a_3>a_2$ and a collection $\{r_{a'}: a'\in A'\}$ of maps $r_{a'}: X_{a_3}\to X_{a'}$ such that

(1) $(p_{a',a_2}r_{a'},\ p_{a_3,a_2})<\mathcal{U}_{a_2}$ for $a'>a_2$ and

(2) $(p_{a'',a'}r_{a''},\ r_{a'})<\mathcal{U}_{a'}$ for $a''>a'>a_2$.

Here $A'=\{a'\in A: a'>a_2\}$. Since A' is cofinal in A, by (I.3.10) $\boldsymbol{p}_{A'}=\{p_{a'}: a'\in A'\}: CT(X)\to(\mathcal{X},\mathcal{U})_{A'}=\{(X_{a'},\mathcal{U}_{a'})p_{a'',a'},A'\}$ is an approximative **POL**-resolution. By (2) and (I.7.2) there exists a map $r: X_{a_3}\to CT(X)$ such that

(3) $(p_{a'}r,\ r_{a'})<st\mathcal{U}_{a'}$ for $a'\in A'$.

Since p_{a_2,a_2} is the identity, by (1) $(r_{a_2},p_{a_3,a_2})<\mathcal{U}_{a_2}$. Since $(p_{a_2}r,\ r_{a_2})<st\mathcal{U}_{a_2}$ by (3), $(p_{a_2}r,p_{a_3,a_2})<st^2\mathcal{U}_{a_2}$ and then by the choice of a_2 $(p_{a_1}r,\ p_{a_3,a_1})<\mathcal{U}_{a_1}$. Thus $\boldsymbol{p}: CT(X)\to(\mathcal{X},\mathcal{U})$ is IAM and hence $CT(X)$ is IAM. The converse follows from (2.11). ■

(2.13) COROLLARY. *Let X be a topologically complete Tychonoff space. Then X is* UAM *iff X is* IAM. ■

We consider the following condition for a resolution $\boldsymbol{p}: X\to\mathcal{X}$:

(C) For each $\mathcal{U}\in\mathcal{C}ov(X)$ there exist $a\in A$ and a map $r: X_a\to X$ such that $(rp_a,\ 1_X)<\mathcal{U}$.

(2.14) LEMMA. *Let $\boldsymbol{p}: X\to\mathcal{X}$ be an* **AP**-*resolution. Then $\boldsymbol{p}$ satisfies* (C) *iff it is* IAM.

PROOF. First we assume that $\boldsymbol{p}$ satisfies (C) and show that $\boldsymbol{p}$ satisfies (IAM)*. Take any $a\in A$ and any $\mathcal{U}\in\mathcal{C}ov(X_a)$. There exists $\mathcal{V}\in\mathcal{C}ov(X_a)$, such that $\mathcal{V}$ satisfies (R2) for $\boldsymbol{p}, X_a$ and $\mathcal{U}$. Since $p_a^{-1}\mathcal{V}\in\mathcal{C}ov(X)$, by the assumption there exist $a_1\in A$ and a map $r: X_{a_1}\to X$ such that $(rp_{a_1},1_X)<p_a^{-1}\mathcal{V}$. Thus $(p_arp_{a_2,a_1}p_{a_2},\ p_{a_2,a}p_{a_2})<\mathcal{V}$ for some $a_2>a,a_1$. By the choice of $\mathcal{V}$ there exists $a_3>a_2$ such that $(p_arp_{a_3,a_1},\ p_{a_3,a})<\mathcal{U}$. This means that a_3 and the map rp_{a_3,a_1}: $X_{a_3}\to X$ satisfies (IAM)* for a. Hence $\boldsymbol{p}$ is IAM.

Next we assume that $\boldsymbol{p}$ satisfies (IAM)* and show (C). Take any $\mathcal{U}\in\mathcal{C}ov(X)$. By (B1) there exist $a\in A$ and $\mathcal{V}\in\mathcal{C}ov(X_a)$ such that $p_a^{-1}\mathcal{V}<\mathcal{U}$. By the assumption there exist $a_1>a$ and a map $r: X_{a_1}\to X$ such that $(p_ar,\ p_{a_1,a})<\mathcal{V}$. Thus $(p_arp_{a_1},p_a1_X)<\mathcal{V}$ and then $(rp_{a_1},1_X)<p_a^{-1}\mathcal{V}<\mathcal{U}$. This means that a_1 and the map $r: X_{a_1}\to X$ satisfies (C) for $\mathcal{U}$. Hence $\boldsymbol{p}$ satisfies (C). ■

(2.15) COROLLARY. *Condition* (C) *does not depend on the choice of the* **AP**-*resolutions.*

(2.15) follows from (2.8) and (2.14). We say that a space X satisfies (C) provided that any/some **AP**-resolution of X satisfies condition (C). By (2.14) we have the following:

(2.16) PROPOSITION. *A space X satisfies* (C) *iff X is* IAM. ■

(2.17) THEOREM. *A space X is* IAM *iff X is an* AP.

PROOF. Take any **POL**-resolution $\boldsymbol{p}: X \to \mathcal{X}$. We assume that X is an IAM. Then $\boldsymbol{p}$ is IAM by (2.8). By (2.14) $\boldsymbol{p}$ satisfies (C). Thus for each $\mathcal{U} \in \mathcal{C}ov(X)$ there exist $a \in A$ and a map $r: X_a \to X$ such that $(rp_a, 1_X) < \mathcal{U}$. Since X_a is a polyhedron, this means that X is an AP.

Next we assume that X is an AP. By (I.3.3) $\boldsymbol{p}: X \to \mathcal{X}$ satisfies (R1) and (R2). Take any $\mathcal{U} \in \mathcal{C}ov(X)$ and then $st\mathcal{V} < \mathcal{U}$ for some $\mathcal{V} \in \mathcal{C}ov(X)$. By the assumption, there exist a polyhedron P and maps $f: X \to P$, $g: P \to X$ such that $(gf, 1_X) < \mathcal{V}$. By (R1) there exist $a \in A$ and a map $h: X_a \to P$ such that $(hp_a, f) < g^{-1}\mathcal{V}$. Thus $(ghp_a, gf) < \mathcal{V}$, and then $(ghp_a, 1_X) < st\mathcal{V} < \mathcal{U}$. This means (C). By (2.14) $\boldsymbol{p}$ is IAM. Hence X is IAM. ■

(2.18) COROLLARY. *Let X be a topologically complete Tychonoff space. Then the following statements are equivalent:*

(i) *X is an* AP.
(ii) *X is* UAM.
(iii) *X is* IAM.
(iv) *X satisfies* (C). ■

§ 3. Approximative conditions M and N.

In this section we introduce the notions of approximative condition M and approximative condition N, and investigate their properties.

Let $\boldsymbol{C}$ be a full subcategory of **AP**. Put $\mathrm{RE}(\boldsymbol{C}) = \{X \in \mathrm{Ob}\mathbf{TOP} : X$ admits a $\boldsymbol{C}$-resolution which is rigid for $\boldsymbol{C}\}$. Let $\mathbf{TOP}_{\boldsymbol{C}}$ be the full subcategory of **TOP** consisting of $\mathrm{RE}(\boldsymbol{C})$. Let $(\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ be an approximative inverse system in $\boldsymbol{C}$ and $\boldsymbol{p} = \{p_a : a \in A\}: X \to (\mathcal{X}, \mathcal{U})$ an approximative resolution of a space X.

We say that $(\mathcal{X}, \mathcal{U})$ satisfies the approximative condition M, in notation ap-M, in $\boldsymbol{C}$ provided that it satisfies the following condition:

(ap-M) There exists $a_0 \in A$ such that for each $a \in A$ there exists $a_1 > a, a_0$ and a map $r: X_{a_0} \to X_a$ in $\boldsymbol{C}$ satisfying $(rp_{a_1,a_0}, p_{a_1,a}) < \mathcal{U}_a$.

We say that $\boldsymbol{p}$ satisfies the approximative condition M, in notation ap-M, in $\boldsymbol{C}$

provided that $(\mathcal{X}, \mathcal{U})$ satisfies the approximative condition M in $\boldsymbol{C}$.

(3.1) LEMMA. *$\boldsymbol{p}$ satisfies* (ap-M) *in $\boldsymbol{C}$ iff it satisfies the following condition:*
(ap-M)$_1$ *There exists $a_0 \in A$ such that for each $a \in A$ there exists a map $r: X_{a_0} \to X_a$ in $\boldsymbol{C}$ satisfying $(rp_{a_0}, p_a) < \mathcal{U}_a$.*

(3.2) LEMMA. *Let $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ and $\boldsymbol{q}: Y \to (\mathcal{Y}, \mathcal{V})$ be approximative $\boldsymbol{C}$-resolutions of spaces X and Y, respectively. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be rigid for $\boldsymbol{C}$. Suppose that Y is dominated by X in* **TOP**. *If $\boldsymbol{p}$ satisfies* (ap-M) *in $\boldsymbol{C}$, then so does $\boldsymbol{q}$.*

(3.3) COROLLARY. *Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be approximative $\boldsymbol{C}$-resolutions of X rigid for $\boldsymbol{C}$. If $\boldsymbol{p}$ satisfies* (ap-M)$_1$ *in $\boldsymbol{C}$, then so does $\boldsymbol{p}'$.*

(3.1) follows from the definitions and (R2). We can show (3.2) in a way similar to the proof of (1.1). (3.3) follows from (3.2). ■

Let $\boldsymbol{p} = \{p_a : a \in A\} : X \to \mathcal{X} = \{X_a, p_{a',a}, A\}$ be a $\boldsymbol{C}$-resolution of a space X. We consider the following conditions:

(ap-M)* There exists $a_0 \in A$ such that for each $a \in A$ and each $\mathcal{U} \in \mathcal{C}_{ov}(X_a)$ there exist $a_1 > a, a_0$ and a map $r: X_{a_0} \to X_a$ in $\boldsymbol{C}$ satisfying $(rp_{a_1,a_0}, p_{a_1,a}) < \mathcal{U}$.

(ap-M)$^*_1$ There exists $a_0 \in A$ such that for each $a \in A$ and each $\mathcal{U} \in \mathcal{C}_{ov}(X_a)$ there exists a map $r: X_{a_0} \to X_a$ in $\boldsymbol{C}$ satisfying $(rp_{a_0}, p_a) < \mathcal{U}$.

(3.4) LEMMA. (i) *Let $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ be an approximative $\boldsymbol{C}$-resolution. Then $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ satisfies* (ap-M) *iff $\boldsymbol{p}: X \to \mathcal{X}$ satisfies* (ap-M)*.
(ii) (ap-M)* *and* (ap-M)$^*_1$ *are equivalent.*

(3.5) LEMMA. *Let a space Y be dominated by a space X in* **TOP**. *Let $\boldsymbol{p}: X \to \mathcal{X}$ and $\boldsymbol{q}: Y \to \mathcal{Y}$ be $\boldsymbol{C}$-resolutions rigid for $\boldsymbol{C}$. If $\boldsymbol{p}$ satisfies* (ap-M)*, *then so does $\boldsymbol{q}$.*

(3.4) follows from the definitions and (R2). We can show (3.5) in the same way as in (1.1). Thus in the same way as in (1.8) we have the following:

(3.6) THEOREM. *Let $X \in \mathrm{Ob}\mathbf{TOP}_{\boldsymbol{C}}$. Then the following statements are equivalent:*
(i) *Any/some approximative $\boldsymbol{C}$-resolution of X, which is rigid for $\boldsymbol{C}$, satisfies* (ap-M).
(ii) *Any/some approximative $\boldsymbol{C}$-resolution of X, which is rigid for $\boldsymbol{C}$,*

satisfies (ap-M)$_1$.

(iii) *Any/some* $\boldsymbol{C}$*-resolution of* X*, which is rigid for* $\boldsymbol{C}$*, satisfies* (ap-M)*.

(iv) *Any/some* $\boldsymbol{C}$*-resolution of* X*, which is rigid for* $\boldsymbol{C}$*, satisfies* (ap-M)$_1^*$. ■

We say that a space $X \in \mathrm{Ob}\mathbf{TOP}_C$ satisfies the approximative condition M, in notation ap-M, in $\boldsymbol{C}$ provided that it satisfies one of the conditions in (3.6).

(3.7) THEOREM. *When* $\boldsymbol{C}$ *is a full subcategory of* $\mathbf{AP}(\mathbf{CTOP}_{3.5})$*, then* (1.9)–(1.12) *for* ap-M *in* $\boldsymbol{C}$ *hold on* $\mathbf{ASh}(\mathbf{TOP}_C)$ *and* $\mathbf{TOP}_C$.

PROOF. (1.11) for ap-M follows from (3.2). (1.12) for ap-M follows from (I.6.8) and (I.6.10). (1.9) for ap-M follows from (I.6.9), (I.6.11), (I.7.8) and (1.11)–(1.12) for ap-M. (1.10) for ap-M follows from (1.9) for ap-M. ■

We say that a paracompact M-space X satisfies the approximative condition M, in notation ap-M, provided that X satisfies ap-M in ANR($\mathbf{PM}$). Since ANR($\mathbf{PM}$) is a full subcategory of $\mathbf{TOP}_{\mathrm{ANR}(\mathbf{PM})}$ by (I.3.17), the above definition is well defined.

(3.8) THEOREM. *A paracompact* M*-space* X *satisfies the approximative condition* M *iff it satisfies the condition* M (*see* [**36**]).

PROOF. By (I.3.17) there exists an ANR($\mathbf{PM}$)-resolution $\boldsymbol{p}=\{p_a : a\in A\} : X\to \mathscr{A}\mathscr{U}(X, M)=\{U_a, p_{a',a}, A\}$ such that all maps are inclusions and all U_a are ANR($\mathbf{PM}$)-open neighborhoods of X in an AR($\mathbf{PM}$) M. By (I.5.7) $H(\boldsymbol{p}) : X\to H(\mathscr{A}\mathscr{U}(X, M))$ is a $\mathbf{HTOP}$-expansion. Since all U_a have the homotopy type of polyhedra by (iii) of (I.3.17), we may assume that $H(\boldsymbol{p}) : X\to H(\mathscr{A}\mathscr{U}(X, M))$ is a $\mathbf{HPOL}$-expansion.

First we assume that X satisfies ap-M. By the assumption $\mathscr{A}\mathscr{U}(X, M)$ satisfies (ap-M). For each $a\in A$ there exists $\mathcal{V}\in\mathcal{C}_{ov}(U_a)$ satisfying (*) for $\{U_a\}$ in (I.5.5). Thus $H(\mathscr{A}\mathscr{U}(X, M))$ satisfies the following condition:

(MC)$_1$ There exists $a_0\in A$ such that for each $a\in A$ there exist $a_1 > a, a_0$ and a map $r : U_{a_0}\to U_a$ satisfying $rp_{a_1,a_0}\simeq p_{a_1,a}$.

Claim. (MC)$_1$ and (MC) given below are equivalent.

(MC) For each $a\in A$ there exists $a_0 > a$ such that for each $a' > a$ theae exist $a'' > a_0, a'$ and a map $r : U_{a_0}\to U_{a'}$ satisfying $rp_{a'',a_0}\simeq p_{a'',a'}$.

We easily show our Claim. Thus $H(\mathscr{A}\mathscr{U}(X, M))$ satisfies (MC) and hence X satisfies the condition M.

Next we assume that X satisfies the condition M. Then $H(\mathscr{A}\mathscr{U}(X, M))$

satisfies (MC) and hence satisfies $(\mathrm{MC})_1$ by the Claim. There exists $a_0\in A$ such that for each $a\in A$ there exist $a_1>a$, a_0 and a map $k: U_{a_0}\to U_a$ satisfying $kp_{a_1,a_0} \simeq p_{a_1,a}$. There exists a homotopy $h: U_{a_1}\times I\to U_a$ such that $h(x,0)=kp_{a_1,a_0}(x)=k(x)$ and $h(x,1)=p_{a_1,a}(x)=x$ for $x\in U_{a_1}$. Take $a_2>a_1$ such that $\bar{U}_{a_2}\subset U_{a_1}$ and define $H': \bar{U}_{a_2}\times I\cup U_{a_0}\times\{0\}\to U_a$ by $H'(x,t)=h(x,t)$ for $(x,t)\in\bar{U}_{a_2}\times I$ and $H'(x,0)=k(x)$ for $x\in U_{a_0}$. Then H' is well defined and then by the homotopy extension property there exists a homotopy $H: U_{a_0}\times I\to U_a$ which is an extension of H'. Define $r: U_{a_0}\to U_a$ by $r(x)=H(x,1)$ for $x\in U_{a_0}$. Thus r satisfies that $rp_{a_2,a_0}=p_{a_2,a}$. Hence $\boldsymbol{p}$ satisfies the following condition:

(ap-M)$_2$ There exists $a_0\in A$ such that for each $a\in A$ there exist $a_2>a$, a_0 and a map $r: U_{a_0}\to U_a$ satisfying $rp_{a_2,a_0}=p_{a_2,a}$.

Since (ap-M)$_2$ implies (ap-M)$_1$ for $\boldsymbol{p}$, X satisfies ap-M. ■

Let $\boldsymbol{p}: X\to(\mathcal{X},\mathcal{U})$ be an approximative $\boldsymbol{C}$-resolution of a space X. Then we say that $\boldsymbol{p}$ satisfies the condition N provided that it satisfies the following condition:

(N) There exists $a_0\in A$ such that for each $a\in A$ there exists a map $f: X_{a_0}\to X$ satisfying $(p_afp_{a_0}, p_a)<\mathcal{U}_a$.

Let $\boldsymbol{p}: X\to\mathcal{X}$ be a $\boldsymbol{C}$-resolution. We consider the following conditions:

(N)* There exists $a_0\in A$ such that for each $a\in A$ and for each $\mathcal{U}\in\mathcal{C}ov(X_a)$ there exists a map $f: X_{a_0}\to X$ satisfying $(p_afp_{a_0}, p_a)<\mathcal{U}$.

$(\mathrm{N})_1^*$ There exists $a_0\in A$ such that for each $\mathcal{U}\in\mathcal{C}ov(X)$ there exists a map $f: X_{a_0}\to X$ satisfying $(fp_{a_0}, 1_X)<\mathcal{U}$.

$(\mathrm{N})_2$ There exist $K\in\mathrm{Ob}\boldsymbol{C}$ and a map $f: X\to K$ such that for each $\mathcal{U}\in\mathcal{C}ov(X)$ there exists a map $g: K\to X$ satisfying $(gf, 1_X)<\mathcal{U}$.

(3.9) LEMMA. *Let $\boldsymbol{p}$ be a $\boldsymbol{C}$-resolution and rigid for $\boldsymbol{C}$. Then $\boldsymbol{p}$ satisfies* $(\mathrm{N})_1^*$ *iff it satisfies* $(\mathrm{N})_2$.

PROOF. Trivially $(\mathrm{N})_1^*$ implies $(\mathrm{N})_2$. We now assume $(\mathrm{N})_2$. Then there exist $K\in\mathrm{Ob}\boldsymbol{C}$ and a map $f: X\to K$ satisfying $(\mathrm{N})_2$. Since $\boldsymbol{p}$ is rigid for $\boldsymbol{C}$, there exists $a_0\in A$ and a map $h: X_{a_0}\to K$ such that $f=hp_{a_0}$. Take any $\mathcal{U}\in\mathcal{C}ov(X)$. By the assumption there exists a map $g: K\to X$ such that $(gf, 1_X)<\mathcal{U}$. Thus $(ghp_{a_0}, 1_X)<\mathcal{U}$. This means $(\mathrm{N})_1^*$. ■

Using rigidness as in the proof of (3.9) in a way similar to the one used in (3.1)-(3.3) and (3.5) we can easily show the following:

(3.10) LEMMA. (i) (3.1) *holds for* (N) *and* (N)*.

(ii) (3.2) *and* (3.3) *holds for* (N).

(iii) (N) *and* $(N)_1^*$ *are equivalent.*

(iv) (3.5) *holds for* (N)*. ■

By (3.9) and (3.10) we have the following:

(3.11) THEOREM. *Let* $X\in \mathrm{Ob}\mathbf{TOP}_C$. *Then the following statements are equivalent:*

(i) *Any/some approximative* **C**-*resolution of X, which is rigid for* **C**, *satisfies* (N).

(ii) *Any/some approximative* **C**-*resolution of X, which is rigid for* **C**, *satisfies* (N)*.

(iii) *Any/some* **C**-*resolution of X, which is rigid for* **C**, *satisfies* $(N)_1^*$.

(iv) *X satisfies* $(N)_2$. ■

We say that a space $X\in \mathrm{Ob}\mathbf{TOP}_C$ satisfies the condition N in **C** provided that X satisfies one of the conditions in (3.11). In the same way as in (3.7) using (3.10) we have the following:

(3.12) THEOREM. *When* **C** *is a full subcategory of* $\mathbf{AP}(\mathbf{CTOP}_{3.5})$, (1.10), (1.11) *and* (2.9) *hold for the condition* N *in* **C**. ■

(3.13) LEMMA. *An approximative resolution* $\boldsymbol{p}: X\to(\mathcal{X},\mathcal{U})$ *satisfies* $(\text{ap-M})_1$ *and* (IAM) *iff it satisfies* (N).

PROOF. We assume that $\boldsymbol{p}$ satisfies (N). Then there exists $a_0\in A$ satisfying (N). Take any $a\in A$. Then there exists a map $f: X_{a_0}\to X$ such that $(p_a f p_{a_0}, p_a)<\mathcal{U}_a$. Put $r=p_a f: X_{a_0}\to X_a$ and then $(rp_{a_0}, p_a)<\mathcal{U}_a$. This means $(\text{ap-M})_1$.

Since $\boldsymbol{p}\, X\to\mathcal{X}$ is a resolution by (I.3.3), there exists $\mathcal{V}\in \mathcal{C}ov(X_a)$ satisfying (R2) for $\mathcal{U}_a$. By (AI3) there exists $a'>a$ such that $p_{a',a}^{-1}\mathcal{V}>\mathcal{U}_{a'}$. By the choice of a_0 there exists a map $g: X_{a_0}\to X$ such that $(p_{a'}gp_{a_0}, p_{a'})<\mathcal{U}_{a'}$. Then by the choice of a' $(p_a g p_{a_1,a_0}, p_{a_1}, p_{a_1,a}p_{a_1})<\mathcal{V}$ for $a_1>a, a_0$. By the choice of $\mathcal{V}$ there exists $a_2>a_1$ such that $(p_a g p_{a_2,a_0}, p_{a_2,a})<\mathcal{U}_a$. This means (IAM).

Next we assume that $\boldsymbol{p}$ satisfies $(\text{ap-M})_1$ and (IAM). Then there exists $a_0\in A$ satisfying $(\text{ap-M})_1$. Take any $a\in A$. By (AI3) there exists $a_1>a$ such that $p_{a_1,a}^{-1}\mathcal{U}_a>st\,\mathcal{U}_{a_1}$. Since $\boldsymbol{p}$ satisfies (IAM), there exist $a_2>a_1$ and a map $f: X_{a_2}\to X$ such that

(1) $(p_{a_1}f, p_{a_2,a_1})<\mathcal{U}_{a_1}$.

By the choice of a_0 there exists a map $r: X_{a_0}\to X_{a_2}$ such that $(rp_{a_0}, p_{a_2})<\mathcal{U}_{a_2}$ and then

(2) $(p_{a_2,a_1}rp_{a_0},\ p_{a_1})<\mathcal{U}_{a_1}$.

By (1) $(p_{a_1}frp_{a_0},\ p_{a_2,a_1}rp_{a_0})<\mathcal{U}_{a_1}$ and then by (2) $(p_{a_1}frp_{a_0}, p_{a_1})<st\mathcal{U}_{a_1}$. By the choice of a_1 $(p_a frp_{a_0}, p_a)<\mathcal{U}_a$. This means (N). ■

(3.14) THEOREM. *Let* $X\in \mathrm{Ob}\mathbf{TOP}_{\boldsymbol{C}}$. *Then* X *satisfies the condition* N *in* $\boldsymbol{C}$ *iff* X *is* IAM *and satisfies* ap-M *in* $\boldsymbol{C}$. ■

We say that a paracompact M-space X satisfies the condition N provided that it satisfies the condition N in ANR(**PM**). By (2.18) and (3.14) we have

(3.15) COROLLARY. *Let* X *be a paracompact* M-*space. Then the following statements are equivalent:*

(i) X *satisfies the condition* N.
(ii) X *is* IAM *and satisfies* ap-M.
(iii) X *is* UAM *and satisfies* ap-M.
(iv) X *is an* AP *and satisfies* ap-M.
(v) X *satifies* (C) *and* ap-M. ■

§ 4. Strongly approximative movability and approximative contractibility.

In this section we introduce strongly approximative movability and approximative contractibility. We investigate their properties.

Let $\boldsymbol{C}$ be a full subcategory of **AP**. Let $(\mathcal{X}, \mathcal{U})=\{(X_a, \mathcal{U}_a), p_{a',a}, A\}$ be an approximative $\boldsymbol{C}$-resolution. We say that $(\mathcal{X}, \mathcal{U})$ is strongly approximatively movable, in notation SAM, in $\boldsymbol{C}$ provided that it satisfies the following condition:

(SAM) For each $a\in A$ there exists $a_0>a$ such that for each $a'>a$ there exist $a''>a', a_0$ and a map $r: X_{a_0}\to X_{a'}$ in $\boldsymbol{C}$ satisfying $(p_{a',a}r,\ p_{a_0,a})<\mathcal{U}_a$ and $(rp_{a'',a_0},\ p_{a'',a'})<\mathcal{U}_{a'}$.

Let $\boldsymbol{p}=\{p_a : a\in A\}: X\to(\mathcal{X}, \mathcal{U})$ be an approximative $\boldsymbol{C}$-resolution of a space X. We say that $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$ is strongly approximatively movable, in notation SAM, in $\boldsymbol{C}$ provided that $(\mathcal{X}, \mathcal{U})$ is SAM in $\boldsymbol{C}$.

(4.1) LEMMA. $\boldsymbol{p}: X\to(\mathcal{X}, \mathcal{U})$ *is* SAM *in* $\boldsymbol{C}$ *iff it satisfies the following condition:*

(SAM)$_1$ *For each* $a\in A$ *there exists* $a_0>a$ *such that for each* $a'>a$ *there exists a map* $r: X_{a_0}\to X_{a'}$ *in* $\boldsymbol{C}$ *satisfying* $(p_{a',a}r,\ p_{a_0,a})<\mathcal{U}_a$ *and* $(rp_{a_0}, p_{a'})<\mathcal{U}_{a'}$.

This lemma follows from the definitions and (R2). In the same way we can show (3.2) and (3.3) for $(\mathrm{SAM})_1$. Hence (3.2) and (3.3) hold for (SAM) by (4.1). ■

We consider the following conditions for a $\boldsymbol{C}$-resolution $\boldsymbol{p}: X \to \mathscr{X}$:

$(\mathrm{SAM})^*$ For each $a \in A$ and for each $\mathcal{U} \in \mathscr{C}ov(X_a)$ there exists $a_0 > a$ such that for each $a' > a$ and for each $\mathcal{U}' \in \mathscr{C}ov(X_{a'})$ there exist $a'' > a'$, a_0 and a map $r: X_{a_0} \to X_{a'}$ in $\boldsymbol{C}$ satisfying $(p_{a',a}r, p_{a_0,a}) < \mathcal{U}$ and $(rp_{a'',a_0}, p_{a'',a'}) < \mathcal{U}'$.

$(\mathrm{SAM})_1^*$ For each $a \in A$ and for each $\mathcal{U} \in \mathscr{C}ov(X_a)$ there exists $a_0 > a$ such that for each $a' > a$ and for each $\mathcal{U}' \in \mathscr{C}ov(X_{a'})$ there exist $a'' > a'$, a_0 and a map $r: X_{a_0} \to X_{a'}$ satisfying $(p_{a',a}r, p_{a_0,a}) < \mathcal{U}$ and $(rp_{a_0}, p_{a'}) < \mathcal{U}'$.

We easily show that $(\mathrm{SAM})^*$ and $(\mathrm{SAM})_1^*$ are equivalent. In the same way we show (3.4) for $(\mathrm{SAM})_1$ and $(\mathrm{SAM})_1^*$, and (3.5) for $(\mathrm{SAM})_1^*$. Thus we may summarize as in (4.2) and then in the same way as in (3.6) we have (4.3):

(4.2) LEMMA. (i) (3.2) *and* (3.3) *hold for* $(\mathrm{SAM})_1$.
(ii) (3.4) *holds for* $(\mathrm{SAM})_1$ *and* $(\mathrm{SAM})_1^*$.
(iii) (3.5) *holds for* $(\mathrm{SAM})_1$.
(iv) $(\mathrm{SAM})^*$ *and* $(\mathrm{SAM})_1^*$ *are equivalent.* ■

(4.3) THEOREM. *Let* $X \in \mathrm{Ob}\mathbf{TOP}_{\boldsymbol{C}}$. *Then the following statements are equivalent:*

(i) *Any/some approximative* $\boldsymbol{C}$*-resolution of* X, *which is rigid for* $\boldsymbol{C}$, *satisfies* (SAM).

(ii) *Any/some approximative* $\boldsymbol{C}$*-resolution of* X, *which is rigid for* $\boldsymbol{C}$, *satisfies* $(\mathrm{SAM})_1$.

(iii) *Any/some* $\boldsymbol{C}$*-resolution of* X, *which is rigid for* $\boldsymbol{C}$, *satisfies* $(\mathrm{SAM})^*$.

(iv) *Any/some* $\boldsymbol{C}$*-resolution of* X, *which is rigid for* $\boldsymbol{C}$, *satisfies* $(\mathrm{SAM})_1^*$. ■

We say that $X \in \mathrm{Ob}\mathbf{TOP}_{\boldsymbol{C}}$ is strongly approximatively movable, in notation SAM, in $\boldsymbol{C}$, provided that it satisfies one of the conditions in (4.3). In the same way as in (3.7) we have

(4.4) THEOREM. *When* $\boldsymbol{C}$ *is a full subcategory of* $\mathbf{AP}(\mathbf{CTOP}_{3.5})$, (1.9)–(1.12) *for* SAM *in* $\boldsymbol{C}$ *hold on* $\mathbf{ASh}(\mathbf{TOP}_{\boldsymbol{C}})$ *and* $\mathbf{TOP}_{\boldsymbol{C}}$, *respectively.* ■

(4.5) PROPOSITION. (i) *If* $(\mathscr{X}, \mathcal{U})$ *is* SAM *in* $\boldsymbol{C}$, *then it is* AM *and satisfies* (ap-M).

(ii) *Let* $X \in \mathrm{Ob}\mathbf{TOP}_{\boldsymbol{C}}$. *If* X *is* SAM *in* $\boldsymbol{C}$, *then it is* AM *and satisfies* ap-M *in* $\boldsymbol{C}$.

PROOF. We show (i). From the difinitions $(\mathcal{X}, \mathcal{U})$ satisfies (AM) and the following condition:

$(\text{ap-M})_3$ For each $a \in A$ there exists $a_0 > a$ such that for each $a' > a$ there exist $a'' > a'$, a_0 and a map $r: X_{a_0} \to X_{a'}$ in $\boldsymbol{C}$ satisfying $(rp_{a'',a_0}, p_{a'',a'}) < \mathcal{U}_{a'}$.

In a way similar to the one used in the Claim in the proof of (3.8) we can easily show that (ap-M) and $(\text{ap-M})_3$ are equivalent. Hence we have (i). (ii) follows from (i). ■

We say that a paracompact M-space X is strongly approximatively movable, in notation SAM, provided that X is SAM in ANR(**PM**).

(4.6) COROLLARY. *Let X be a paracompact M-space. If X is* SAM, *then it is* AM *and satisfies* ap-M. ■

(4.7) THEOREM. *A complete metric space X is* SAM *iff X is an* ANR.

PROOF. Let (X, d) be a metric space. We assume that X is complete with respect to the metric d. It is well known that X is isometric to a closed subset of a Banach space $B(X)$ (see Borsuk [**3**], Hu [**13**] and Besaga-Pelczynski [**1**]). Here $B(X)$ consists of all real bounded functions with sup norm. Since embedding is isometric, we may assume that X is a closed subset of $B(X)$ and d is the metric on $B(X)$. $B(X)$ is complete with respect to d and $B(X) \in \text{AR}$. By (I.3.17) we have an approximative resolution $\boldsymbol{p} = \{p_a : a \in A\} : X \to \mathcal{O}\mathcal{U}(X, B(X)) = \{X_a, p_{a',a}, A\}$ such that all $p_a, p_{a',a}$ are inclusion maps and all X_a are open neighborhoods of X in $B(X)$ and $\boldsymbol{p}$ is rigid for **ANR**.

Claim. $\boldsymbol{p}$ satisfies (SAM)* iff it satisfies the following condition:

$(\text{SAM})_2^*$ For each $a \in A$ and each $\mathcal{U} \in \mathcal{C}ov(X_a)$ there exists $a_0 > a$ such that for each $a' > a$ there exist $a'' > a'$, a_0 and a map $r: X_{a_0} \to X_{a'}$ satisfying $rp_{a'',a_0} = p_{a'',a'}$ and $(p_{a',a}r, p_{a_0,a}) < \mathcal{U}$.

We assume that $\boldsymbol{p}$ satisfies (SAM)*. Take any $a \in A$ and any $\mathcal{U} \in \mathcal{C}ov(X_a)$ then there exists $a_0 > a$ satisfying the condition in (SAM)* for a and $\mathcal{U}$. Take any $a' > a$ and then there exists $\mathcal{V} \in \mathcal{C}ov(X_{a'})$ satisfying (*) for $p_{a',a}^{-1}\mathcal{U}$ in (I.5.7). By the choice of a_0 there exist $a'' > a_0$, a' and a map $r: X_{a_0} \to X_{a'}$ such that

(1) $(p_{a',a}r, p_{a_0,a}) < \mathcal{U}$ and

(2) $(rp_{a'',a_0}, p_{a'',a'}) < \mathcal{V}$.

By (2) and the choice of $\mathcal{V}$ there exists a $p_{a',a}^{-1}\mathcal{U}$-homotopy $H: X_{a''} \times I \to X_{a'}$ such that $H(x,0)=p_{a'',a'}(x)=x$ and $H(x,1)=rp_{a'',a_0}(x)=r(x)$ for $x\in X_{a''}$. Take $a_2>a_1>a''$ such that $\overline{X}_{a_2}\subset X_{a_1}\subset\overline{X}_{a_1}\subset X_{a''}$. Then there exists a map $t:B(X)\to I$ such that $t(\overline{X}_{a_2})=0$ and $t(B(X)-X_{a_1})=1$. We define a map $r':X_{a_0}\to X_{a'}$ as follows: $r'(x)=H(x,t(x))$ for $x\in X_{a''}$ and $r'(x)=r(x)$ for $x\in X_{a_0}-\overline{X}_{a_1}$. Clearly r' is well defined and $r'p_{a_2,a_0}=p_{a_2,a'}$. Since H is a $p_{a',a}^{-1}\mathcal{U}$-homotopy and (1), $(p_{a',a}r',\ p_{a_0,a})<\mathcal{U}$. Hence we have $(\mathrm{SAM})_2^*$. The converse is trivial. Hence we have our Claim.

We assume that X is SAM. Then $\boldsymbol{p}$ satisfies $(\mathrm{SAM})^*$. By the Claim we can choose a subsequence $A'=\{a_i:i\in N\}\subset A$ and maps $r_i:X_{a_i}\to X_{a_{i+1}}$ for $i\geq 2$ such that $a_1=1<a_2<a_3<\cdots$,

(3) $X_i\subset U(X,(1/2)^i)$ for $i\geq 1$,

(4) $d(p_{a_{i+1},a_{i-1}}r_i,\ p_{a_i,a_{i-1}})<(1/2)^i$ for $i\geq 2$ and

(5) $r_ip_{a_{i+2},a_{i+1}}=p_{a_{i+2},a_{i+1}}$ for $i\geq 2$.

Here $U(x,\varepsilon)=\{z\in B(X):d(X,z)<\varepsilon\}$ for $\varepsilon>0$. We define maps $f_i:X_{a_2}\to X_{a_i}\subset B(X)$ for $i\geq 1$ as follows: $f_i=r_{i-1}\cdots r_2$ for $i\geq 3$ and $f_i=p_{a_2,a_i}$ for $i=1,2$. By (4) $d(f_{i-1},f_i)<(1/2)^i$ for $i\geq 2$ and then $\{f_i:i\geq 1\}$ forms a Cauchy sequence with respect to d. Since $B(X)$ is complete with respect to d, we have a (continuous) map $f:X_{a_2}\to B(X)$. Since $f_i:X_{a_2}\to X_{a_i}$, by (3) $f(X_{a_2})\subseteq X$, that is, $f:X_{a_2}\to X$. By (5) $f_i(x)=x$ for $x\in X$ for $i\geq 1$, and hence $f(x)=x$ for $x\in X$. Thus X is a retract of an ANR X_{a_2} and hence X is an ANR.

Next we assume that X is an ANR. Then trivially the rudimentary resolution $\{1_X\}:X\to\{X\}$ satisfies $(\mathrm{SAM})_1^*$. Then X is SAM. ■

(4. 8) PROBLEM. Does (4.7) hold for paracompact M-spaces or for metric spaces? Our Claim in the proof of (4.7) holds for paracompact $\boldsymbol{M}$-spaces.

Let $(\mathcal{X},\mathcal{U})$ be an approximative inverse system in **TOP**. We say that $(\mathcal{X},\mathcal{U})$ is approximatively contractible, in notation AC, provided that it satisfies the following condition:

(AC) For each $a\in A$ there exist $a'>a$ and a map $f:X_{a'}\to X_a$ such that $(f,\ p_{a',a})<\mathcal{U}_a$ and f is homotopic to a constant map.

We say that $\boldsymbol{p}:X\to(\mathcal{X},\mathcal{U})$ is approximatively contractible, in notation AC, provided that $(\mathcal{X},\mathcal{U})$ is AC. In a similar way we can show (1.1)-(1.3) for AC.

Let $\mathcal{X}$ be an inverse system in **TOP**. We say that $\mathcal{X}$ is approximatively contractible, in notation AC, provided that it satisfies the following condition:

$(\mathrm{AC})^*$ For each $a\in A$ and for each $\mathcal{U}\in\mathcal{C}ov(X_a)$ there exist $a'>a$ and a map $f:X_{a'}\to X_a$ such that $(p_{a',a},\ f)<\mathcal{U}$ and f is homotopic to a constant map.

We say that $\boldsymbol{p}: X \to \mathscr{X}$ is approximatively contractible, in notation AC, provided that $\mathscr{X}$ is AC. In a similar way we can show (1.4)-(1.7) for AC. Thus in the same way as in (1.8) we have the following:

(4.9) THEOREM. *Let X be a space. Then the following statements are equivalent:*

(i) *Any/some approximative* **AP**-*resolution of X is* AC.

(ii) *Any/some* **AP**-*resolution of X is* AC. ■

We say that a space X is approximatively contractible, in notation AC, provided that it satisfies one of the conditions in (4.9). In the same way we can show (1.9)-(1.12) for AC. We summarize as follows:

(4.10) THEOREM. (1.9)-(1.12) *hold for* AC. ■

A space X has trivial shape iff X has the shape of the one point space.

(4.11) THEOREM. *A space is* AC *iff it has the trivial shape.*

PROOF. Let X be a space. Then there exists an approximative **POL**-resolution $\boldsymbol{p}: X \to (\mathscr{X}, \mathscr{U})$ of X satisfying (**) in (I.5.6). By (I.3.3) and (I.5.7) $H(\boldsymbol{p}): X \to H(\mathscr{X})$ is a **HPOL**-expansion. It is well known that X has trivial shape iff $H(\mathscr{X})$ satisfies the following condition:

(TS) For each $a \in A$ there exists $a' > a$ such that $p_{a',a}$ is homotopic to a constant map.

We assume that X is AC. Then $(\mathscr{X}, \mathscr{U})$ satisfies (AC). Clearly (**) of (I.5.6) and (AC) imply (TS). Hence X has trivial shape. Since (TS) implies (AC), the converse also holds. ■

(4.12) COROLLARY. *An approximative polyhedron X has trivial shape iff it satisfies the following condition:*

(APT) *For each $\mathscr{U} \in \mathscr{C}_{OV}(X)$ there exists a map $f: X \to X$ such that $(f, 1_X) < \mathscr{U}$ and f is homotopic to a constant map.*

Since (AC) and (APT) for the rudimentary resolution $\{1_X\}: X \to \{X\}$ are equivalent, (4.12) follows from (4.9) and (4.11). ■

(4.13) THEOREM. *Let M be an* AR(**PM**) *and X a closed subset of M. Then X is an* AP *with trivial shape iff it satisfies following condition:*

$(\mathrm{APT})_1$ *For each $\mathscr{U} \in \mathscr{C}_{OV}(X)$ there exists a map $h: M \to X$ such that $(h|X, 1_X) < \mathscr{U}$.*

PROOF. First we assume that X is an AP with trivial shape. Take any $\mathcal{U}\in\mathcal{C}ov(X)$ and then we have $\mathcal{V}\in\mathcal{C}ov(X)$ such that $st\mathcal{V}<\mathcal{U}$. By (4.12) there exists a map $f: X\to X$ such that

(1) $(f, 1_X)<\mathcal{V}$ and f is homotopic to a constant map.

Since X is an AP, there exist an ANR K and maps $g: X\to K$, $h: K\to X$ satisfying $(hg, 1_X)<\mathcal{V}$. Thus $(hgf, 1_X)<st\mathcal{V}<\mathcal{U}$. Since M is an AR(**PM**), M is contractible. By (1) $gf: X\to K$ is homotopic to a constant, and hence by the homotopy extension property there exists a map $H: M\to K$ such that $H|X=gf$. Then $r=hH: M\to X$ has the required properties.

By (iv) in (I.3.17) and $(\mathrm{APT})_1$ X is an AP. Then $(\mathrm{APT})_1$ implies (APT) and by (4.12) X is an AP with trivial shape. ■

§ 5. Generalized absolute neighborhood retracts.

In this section we discuss generalized absolute neighborhood retracts. See § **0** for their historical development.

Let $\boldsymbol{C}$ be a subcategory of **TOP** such that Ob$\boldsymbol{C}$ is a weakly hereditary topological class (see Hu [**13**, p. 33]). Sometimes $X\in\boldsymbol{C}$ means $X\in\mathrm{Ob}\boldsymbol{C}$. ANR($\boldsymbol{C}$), AR($\boldsymbol{C}$), ANE($\boldsymbol{C}$) and AE($\boldsymbol{C}$) denote the full subcategories of **TOP** consisting of all absolute neighborhood retracts, all absolute retracts, all absolute neighborhood extensors and all absolute extensors for Ob$\boldsymbol{C}$. Let **PM** and **M** be the subcategories of **TOP** consisting of all paracompact M-spaces (see (I.3.17)) and all metric spaces, respectively. **ANR** and **AR** denote ANR(**M**) and AR(**M**), respectively. Lisica [**18**] and Mardešić and Šostak [**21**] showed the following:

(5.1) LEMMA. *ANR(**PM**)=**PM**∩ANE(**PM**), AR(**PM**)=**PM**∩AE(**PM**) and **ANR**⊂ANR(**PM**).* ■

Let Y be a space. We say that Y is an approximative absolute extensor for **PM**, in notation AAE for **PM**, provided that it satisfies the following condition:

(AAE) For any map $f: X_0\to Y$, where X_0 is any closed subspace of any paracompact M-space X, and for any $\mathcal{U}\in\mathcal{C}ov(Y)$ there exists a map $g: X\to Y$ such that $(g|X_0, f)<\mathcal{U}$.

We say that Y is an approximative absolute neighborhood extensor in the sense of Noguchi for **PM**, in notation AANE_N for **PM**, provided that it satisfies the following condition:

(AANE_N) For any map $f: X_0\to Y$, where X_0 is any closed subspace of any paracompact M-space X, there exists a neighborhood N of X_0 in X such that for any $\mathcal{U}\in\mathcal{C}_{ov}(Y)$ there exists a map $g: N\to Y$ satisfying $(g|X_0, f)<\mathcal{U}$.

We say that Y is an approximative absolute neghborhood extensor in the sense of Clapp for **PM**, in notation $AANE_C$ for **PM**, provided that it satisfies the following condition:

($AANE_C$) For any map $f: X_0\to Y$, where X_0 is a closed subspace of any paracompact M-space X, and for any $\mathcal{U}\in\mathcal{C}_{ov}(Y)$ there exist a neighborhood N of X_0 in X and a map $g: N\to Y$ satisfying $(g|X_0, f)<\mathcal{U}$.

Let X be a paracompact M-space. We say that X is an approximative absolute retract for **PM**, in notation AAR for **PM**, provided that it satisfies the following condition:

(AAR) For any closed embedding $h: X\to M$, $M\in\mathrm{Ob}\mathbf{PM}$, and for each $\mathcal{U}\in\mathcal{C}_{ov}(h(X))$ there exists a map $r: M\to h(X)$ satisfying $(r|h(X), 1_{h(X)})<\mathcal{U}$.

We say that X is an approximative absolute neighborhood retract in the sense sense of Noguchi for **PM**, in notation $AANR_N$ for **PM**, provided that it satisfies the following condition:

($AANR_N$) For any closed embedding $h: X\to M$, $M\in\mathrm{Ob}\mathbf{PM}$, there exists a neighborhood N of $h(X)$ in M such that for each $\mathcal{U}\in\mathcal{C}_{ov}(h(X))$ there exists a map $r: N\to h(X)$ satisfying $(r|h(X), 1_{h(X)})<\mathcal{U}$.

We say that X is an approximative absolute neighborhood retract in the sense of Clapp for **PM**, in notation $AANR_C$ for **PM**, provided that it satisfies the following condition:

($AANR_C$) For any closed embedding $h: X\to M$, $M\in\mathrm{Ob}\mathbf{PM}$, and for each $\mathcal{U}\in\mathcal{C}_{ov}(h(X))$ there exist a neighborhood N of $h(X)$ in M and a map $r: N\to X$ satisfying $(r|h(X), 1_{h(X)})<\mathcal{U}$.

$AANR_N$(**PM**) and $AANR_C$(**PM**) denote the full subcategories of **TOP** consisting of all $AANR_N$s and $AANR_C$s for **PM**, respectively. Similarly we may define AAE(**PM**), $AANE_N$(**PM**), $AANE_C$(**PM**) and AAR(**PM**).

(5.2) LEMMA. AAR(**PM**)=**PM**$\cap$AAE(**PM**), $AANR_N$(**PM**)=**PM**$\cap AANE_N$(**PM**) *and* $AANR_C$(**PM**)=**PM**$\cap AANE_C$(**PM**).

PROOF. We show the last one. In a similar way we can show the others. Take any $X\in AANR_C$(**PM**). Trivially $X\in$**PM**. We need to show that $X\in AANE_C$(**PM**). Take any paracompact M-space Z, its closed subspace Z_0, any map $f: Z_0\to X$ and any $\mathcal{U}\in\mathcal{C}_{ov}(X)$. By (I.3.17) there exists a $M\in$AR(**PM**) which contains X as a closed subspace. Since $X\in AANR_C$(**PM**), there exist a neighborhood N of X in M and a map $r: N \to X$ such that $(r|X, 1_X) < \mathcal{U}$. By (5.1) and Prop. 6.1 of Hu [**13**, p. 42] $\operatorname{Int} N \in \mathrm{ANE}(\mathbf{PM})$. Then there exists a neighborhood U of Z_0 in Z and a map $g': U \to \operatorname{Int} N$ such that $g'|Z_0 = f$. Thus $g = rg': U \to X$ satisfies $(g|Z_0, f) < \mathcal{U}$. Hence $X \in \mathbf{PM} \cap \mathrm{AANE}_\mathrm{C}(\mathbf{PM})$.

Next we assume that $X \in \mathbf{PM} \cap \mathrm{AANE}_\mathrm{C}(\mathbf{PM})$. Take any $Y \in \mathbf{PM}$ and a closed embedding $h: X \to Y$. Take any $\mathcal{U} \in \mathcal{C}_{ov}(h(X))$ and put $\mathcal{V} = h^{-1}\mathcal{U} \in \mathcal{C}_{ov}(X)$. Since $X \in \mathrm{AANE}_\mathrm{C}(\mathbf{PM})$ there exist a neighborhood N of $h(X)$ in Y and a map $g: N \to X$ such that $(g|h(X), h^{-1}) < \mathcal{V}$. Thus $r = hg: N \to h(X)$ satisfies $(r|h(X), 1_{h(X)}) < \mathcal{U}$. Hence $X \in \mathrm{AANR}_\mathrm{C}(\mathbf{PM})$. ■

(5.3) LEMMA. *Let $M \in \mathrm{AR}(\mathbf{PM})$ and X a closed subset of M. Then we have the following:*

(i) $X \in \mathrm{AAR}(\mathbf{PM})$ *iff it satisfies the following condition:*

$(\mathrm{AAR})_1$ *For each $\mathcal{U} \in \mathcal{C}_{ov}(X)$ there exists a map $r: M \to X$ such that $(r|X, 1_X) < \mathcal{U}$.*

(ii) $X \in \mathrm{AANR}_\mathrm{N}(\mathbf{PM})$ *iff it satisfies the following condition:*

$(\mathrm{AANR}_\mathrm{N})_1$ *There exists a neighborhood N of X in M such that for each $\mathcal{U} \in \mathcal{C}_{ov}(X)$ there exists a map $r: N \to X$ satisfying $(r|X, 1_X) < \mathcal{U}$.*

(iii) $X \in \mathrm{AANR}_\mathrm{C}(\mathbf{PM})$ *iff it satisfies the following condition:*

$(\mathrm{AANR}_\mathrm{C})_1$ *For each $\mathcal{U} \in \mathcal{C}_{ov}(X)$ there exists a neighborhood N of X in M and a map $r: N \to X$ satisfying $(r|X, 1_X) < \mathcal{U}$.*

PROOF. We show (ii). In a similar way we can show the other assertions. We assume $(\mathrm{AANR}_\mathrm{N})_1$ and show $(\mathrm{AANR}_\mathrm{N})$. Take any closed embedding $h: X \to M'$, $M' \in \mathbf{PM}$. By $(\mathrm{AANR}_\mathrm{N})_1$ there exists a neighborhood N of X in M satisfying condition in $(\mathrm{AANR}_\mathrm{N})_1$. Since $M \in \mathrm{AR}(\mathbf{PM})$, by (5.1) and Prop. 6.1 of Hu [**13**, p. 42] $\operatorname{Int} N \in \mathrm{ANE}(\mathbf{PM})$. Then there exist a neighborhood N' of $h(X)$ in M' and a map $g: N' \to \operatorname{Int} N$ such that $g|h(X) = h^{-1}$. Take any $\mathcal{U} \in \mathcal{C}_{ov}(h(X))$. By the choice of N there exists a map $r: N \to X$ such that $(r|X, 1_X) < h^{-1}\mathcal{U}$. Put $r' = hrg: N' \to h(X)$ and then it satisfies $(r'|h(X), 1_{h(X)}) < \mathcal{U}$. Hence $X \in \mathrm{AANR}_\mathrm{N}(\mathbf{PM})$. The converse holds, because $(\mathrm{AANR}_\mathrm{N})$ implies $(\mathrm{AANR}_\mathrm{N})_1$. ■

Let X be a subspace of Y. We say that X is an approximative retract of Y provided that for each $\mathcal{U} \in \mathcal{C}_{ov}(X)$ there exists a map $r: Y \to X$ such that $(r|X, 1_X) < \mathcal{U}$.

(5.4) LEMMA. *Let X be a closed subspace of a paracompact M-space Y. We assume that X is an approximative retract of Y. If Y is an* AAR, *an*

$\mathrm{AANR_N}$ *or an* $\mathrm{AANR_C}$ *for* **PM**, *then so is* X, *respectively.*

PROOF. We only show the case of $\mathrm{AANR_N}$. Take any $M\in\mathrm{AR}(\mathbf{PM})$ which contains Y as a closed subset. By (5.3) there exists a neighborhood N of Y in M satisfying $(\mathrm{AANR_N})_1$. Take any $\mathcal{U}\in\mathcal{C}ov(X)$ and then take $\mathcal{V}\in\mathcal{C}ov(X)$ with $st\mathcal{V}<\mathcal{U}$. Since X is approximative retract of Y, there exists a map $r_1: Y\to X$ such that $(r_1|X, 1_X)<\mathcal{V}$. By the choice of N there exists a map $r_2: N\to Y$ such that $(r_2|Y, 1_Y)<r_1^{-1}\mathcal{V}$. Thus $r=r_1r_2: N\to X$ satisfies $(r|X, 1_X)<st\mathcal{V}<\mathcal{U}$. Then X satisfies $(\mathrm{AANR_N})_1$. Hence $X\in\mathrm{AANR_N}(\mathbf{PM})$ by (5.3). In a similar way we can show the other assertions. ■

(5.5) THEOREM. *Let* X *be a paracompact M-space.*

(i) $X\in\mathrm{AANR_N}(\mathbf{PM})$ *iff* X *satisfies the condition* N.

(ii) $X\in\mathrm{AANR_C}(\mathbf{PM})$ *iff* X *satisfies* (C).

(iii) $X\in\mathrm{AAR}(\mathbf{PM})$ *iff* X *satisfies* $(\mathrm{APT})_1$ *in* (4.3).

PROOF. By (I.3.17) there exists an ANR(**PM**)-resolution $\boldsymbol{p}: X\to\mathcal{A}\mathcal{U}(X, M)$. Here M is an AR(**PM**) containing X as a closed subset. $(\mathrm{AANR_N})_1$ and $(\mathrm{AANR_C})_1$ in (5.3) are equivalent to $(\mathrm{N})_1^*$ for $\boldsymbol{p}$ and (C) for $\boldsymbol{p}$, respectively. Hence by (2.15), (3.11) and (5.3) we have (i) and (ii). Trivially $(\mathrm{AAR})_1$ in (5.3) and $(\mathrm{APT})_1$ in (4.3) are same. Thus we have (iii). ■

We define absolute weak neighborhood retracts and absolute weak retracts for **PM**. These notions are introduced by Bogatyĭ [**2**] for compact metric spaces. Sakai [**29**] studied these notions for metric spaces. Let $h: Y\to M$ be a closed embedding and $Y, M\in\mathbf{PM}$. Let X be a closed subspace of Y. We say that X is a weak retract of Y under h, in notation $X\in\mathrm{WR}(\mathbf{PM})(Y)_h$, provided that it satisfies the following condition:

(WR) For any neighborhood U of $h(X)$ in M there exists a map $r: h(Y)\to U$ such that $r|h(X)=1_{h(X)}$.

We say that X is a weak neighborhood retract of Y under h, in notation $X\in\mathrm{WNR}(\mathbf{PM})(Y)_h$, provided that it satisfies the following condition:

(WNR) There exists a neighborhood V of X in Y such that for any neighborhood U of $h(X)$ in M there exists a map $r: h(V)\to U$ satisfying $r|h(X)=1_{h(X)}$.

We say that X is a weak retract of Y, in notation $X\in\mathrm{WR}(\mathbf{PM})(Y)$, provided that $X\in\mathrm{WR}(\mathbf{PM})(Y)_h$ for some closed embedding $h: Y\to M$, $M\in\mathbf{PM}$. We say that X is a weak neighborhood retract of Y, in notation $\mathrm{WNR}(\mathbf{PM})(Y)$, provided that $X \in \mathrm{WNR}(\mathbf{PM})(Y)_h$ for some closed embedding $h: Y \to M$, $M \in \mathbf{PM}$. We say that $X \in \mathbf{PM}$ is an absolute weak retract, in notation $X \in \mathrm{AWR}(\mathbf{PM})$, provided that for any closed embedding $f: X \to Y$ with $Y \in \mathbf{PM}$, $f(X) \in \mathrm{WR}(\mathbf{PM})(Y)$. We say that $X \in \mathbf{PM}$ is an absolute weak neighborhood retract, in notation $X \in \mathrm{AWNR}(\mathbf{PM})$, provided that for any closed embedding $f: X \to Y$ with $Y \in \mathbf{PM}$, $f(X) \in \mathrm{WNR}(\mathbf{PM})(Y)$.

(5.6) LEMMA. (i) $X \in \mathrm{WR}(\mathbf{PM})(Y)$, *then* $X \in \mathrm{WR}(\mathbf{PM})(Y)_t$ *for any closed embedding* $t: Y \to N$, $N \in \mathrm{ANR}(\mathbf{PM})$.

(ii) *If* $X \in \mathrm{WNR}(\mathbf{PM})(Y)$, *then* $X \in \mathrm{WNR}(\mathbf{PM})(Y)_t$ *for any closed embedding* $t: Y \to N$, $N \in \mathrm{ANR}(\mathbf{PM})$.

PROOF. We only show (ii). In the same way we can show (i). Since $X \in \mathrm{WNR}(\mathrm{PM})(Y)$, there exists a closed embedding $h: Y \to M$, $M \in \mathbf{PM}$ such that $X \in \mathrm{WNR}(\mathbf{PM})(Y)_h$. Then there exists a neighborhood V of X in Y satisfying (WNR) for h. Take any neighborhood W of $t(X)$ in N. Since N is an $\mathrm{ANR}(\mathbf{PM})$, $\mathrm{Int}\, N \in \mathrm{ANE}(\mathbf{PM})$ and then there exists a neighborhood U of $h(X)$ in M and a map $F: U \to \mathrm{Int}\, W$ such that $F|h(X) = th^{-1}$. By the choice of V there exists a map $r: h(V) \to U$ such that $r|h(X) = 1_{h(X)}$. Thus $f = Frht^{-1}: t(V) \to W$ satisfies $f|t(X) = 1_{t(X)}$. Hence $X \in \mathrm{WNR}(\mathbf{PM})(Y)_t$. ■

(5.7) LEMMA. *Let X be a closed subspace of* $M \in \mathrm{AR}(\mathbf{PM})$.

(i) $X \in \mathrm{AWR}(\mathbf{PM})$ *iff it satisfies the following condition:*

(AWR) *For any neighborhood U of X in M there exists a map $r: M \to U$ such that $r|X = 1_X$.*

(ii) $X \in \mathrm{AWNR}(\mathbf{PM})$ *iff it satisfies the following condition:*

(AWNR) *There exists a neighborhood U_0 of X in M such that for any neighborhood U* of *X in M there exists a map $r: U_0 \to U$ satisfying $r|X = 1_X$.*

PROOF. We show only (ii). In the same way we can show (i). First we assume that $X \in \mathrm{AWNR}(\mathbf{PM})$. Then $X \in \mathrm{WNR}(\mathbf{PM})(M)_{1_M}$ and hence it satisfies (AWNR).

Next we assume (AWNR). Take any closed embedding $f: X \to Y$, $Y \in \mathbf{PM}$. By the assumption there exists a neighborhood U_0 of X in M satisfying (AWNR). Since $\mathrm{Int}\, U_0 \in \mathrm{ANE}(\mathbf{PM})$, there exist a neighborhood V_0 of $f(X)$ in Y and a map $F: V_0 \to \mathrm{Int}\, U_0$ such that $F|f(X) = f^{-1}$. There exists a closed embedding $h: Y \to N$, $N \in \mathrm{AR}(\mathbf{PM})$. We show that $f(X) \in \mathrm{WNR}(\mathbf{PM})(Y)_h$. Take any neighborhood W of $hf(X)$ in N. Since $\mathrm{Int}\, W \in \mathrm{ANE}(\mathbf{PM})$ there exist a neighborhood U of X in M and a map $H: U \to \mathrm{Int}\, W$ such that $H|X = hf$. By the choice of U_0 there exists a map $r: U_0 \to U$ such that $r|X=1_X$. Thus $R=HrFh^{-1}: h(V_0) \to W$ satisfies $R|hf(X)=1_{hf(X)}$. Then $f(X) \in \mathrm{WNR}(\mathbf{PM})(Y)_h$ and hence $X \in \mathrm{AWNR}(\mathbf{PM})$. ∎

(5.8) THEOREM. *Let X be a paracompact M-space. Then* $X \in \mathrm{AWR}(\mathbf{PM})$ *iff X has trivial shape.*

PROOF. Let $\boldsymbol{p}: X \to \mathscr{A}\mathcal{U}(X, M)$ be the ANR(**PM**)-resolution in (I.3.17). Here M is an AR(**PM**) which contains X as a closed subspace. We assume that X is an AWR(**PM**). Take any ANR(**PM**)-neighborhood U of X in M. By (AWR) in (5.7) there exists a map $r: M \to U$ such that $r|X=1_X$. Since $U \in \mathrm{ANR}(\mathbf{PM})$, there exists $\mathcal{U} \in \mathcal{C}_{OV}(U)$ satisfying (*) in (I.5.5). Since $r|X=1_X$, by (ii) of (I.3.17) there exists an ANR(**PM**)-neighborhood V of X in M such that $(r|V, j) < \mathcal{U}$. Here $j: V \to U$ is the inclusion map. By the choice of $\mathcal{U}$ $r|V \simeq j$. Since $M \in \mathrm{AR}(\mathbf{PM})$, M is contractible and then $r|V$ is homotopic to a constant map. Thus X is AC and hence X has trivial shape by (4.11).

Next we assume that X has trivial shape. Then X is AC by (4.11). Take any ANR(**PM**)-neighborhood U of X in M. There exists $\mathcal{U} \in \mathcal{C}_{OV}(U)$ satisfying (*) in (I.5.5). Since X is AC, there exists an ANR(**PM**)-neighborhood V of X in M and a map $f: V \to U$ such that f is homotopic to a constant map and $(f, j) < \mathcal{U}$. By the choice of $\mathcal{U}$ $f \simeq j$ and then j is homotopic to a constant map. Since $U \in \mathrm{ANR}(\mathbf{PM})$, by the homotopy extension theorem there exists a map $r: M \to U$ such that $r|X=1_X$. Thus X satisfies (AWR) in (5.7) and hence X is an AWR(**PM**). ∎

(5.9) THEOREM. *Let X be a paracompact M-space. Then* $X \in \mathrm{AWNR}(\mathbf{PM})$ *iff X satisfies* ap-M.

PROOF. Let $\boldsymbol{p}: X \to \mathscr{A}\mathcal{U}(X, M)$ be the ANR(**PM**)-resolution in (I.3.17). First we assume that $X \in \mathrm{AWNR}(\mathbf{PM})$. Then there exists an ANR(**PM**)-neighborhood U_0 of X in M satisfying (AWNR) in (5.7). Take any ANR(**PM**)-neighborhood U of X in M and any $\mathcal{U} \in \mathcal{C}_{OV}(U)$. By the choice of U_0 there exists a map $r: U_0 \to U$ such that $r|X=1_X$. By (ii) of (I.3.17) there exists an ANR(**PM**)-neighborhood V of X in M such that $(r|V, j) < \mathcal{U}$. Here $j: V \to U$ is the inclusion map. Thus $\boldsymbol{p}$ satisfies (ap-M)$^*_1$. Hence X satisfies ap-M by (3.6).

Next we assume that X satisfies ap-M. Then $\boldsymbol{p}$ satisfies (ap-M)$^*_1$ by (3.6) and then there exists an ANR(**PM**)-neiborhood U_0 of X in M satisfying (ap-M)$^*_1$ for $\boldsymbol{p}$. Take any neighborhood V of X in M. By (ii) of (I.3.17) there exists an ANR(**PM**)-neighborhood W of X in M such that $W \subset V$. There exists $\mathcal{U} \in$

$\mathcal{C}ov(W)$ satisfying (*) in (I.5.5). By the choice of U_0 there exists a map $s: U_0 \to W$ such that $(s|X, j) < \mathcal{U}$. Here $j: X \to W$ is the inclusion map. Thus $s|X \simeq j$. Since $W \in \mathrm{ANR}(\mathbf{PM})$, $W \in \mathrm{ANE}(\mathbf{PM})$ and hence by the homotopy extension theorem there exists a map $r: U_0 \to W$ such that $r|X = 1_X$. Then X satisfies (AWNR) and hence X is an AWNR by (5.7). ■

(5.10) COROLLARY. *Let X be a paracompact M-space. Then the following statements are equivalent:*

(i) *X is an* AANR$_{\mathrm{C}}$ *for* **PM**.

(ii) *X is an* AANE$_{\mathrm{C}}$ *for* **PM**.

(iii) *X is an* AP.

(iv) *X is* IAM.

(v) *X is* UAM.

(vi) *X satisfies* (C).

(5.11) COROLLARY. *Let X be a paracompact M-space. Then the following statements are equivalent:*

(i) *X is an* AANR$_{\mathrm{N}}$ *for* **PM**.

(ii) *X is an* AANE$_{\mathrm{N}}$ *for* **PM**.

(iii) *X satisfies* ap-M *and one of the conditions* (i)-(iv) *in* (5.10).

(iv) *X satisfies the condition* N.

(v) *X satisfies* $(\mathrm{N})_2$

(5.12) COROLLARY. *Let X be a paracompact M-space. Then the following statements are equivalent:*

(i) *X is an* AAR *for* **PM**.

(ii) *X is an* AAE *for* **PM**.

(iii) *X has trivial shape and satisfies one of the conditions* (i)-(vi) *in* (5.10).

(iv) *X has trivial shape and satisfies one of the conditions* (i)-(v) *in* (5.11).

(5.13) COROLLARY. AAR(**PM**) = AANR$_{\mathrm{N}}$(**PM**) ∩ AWR(**PM**) = AANR$_{\mathrm{C}}$(**PM**) ∩ AWR(**PM**) *and* AANR$_{\mathrm{N}}$(**PM**) = AANR$_{\mathrm{C}}$(**PM**) ∩ AWNR(**PM**).

(5.10) follows from (2.8), (5.2) and (5.5). (5.11) follows from (3.11), (3.15), (5.5) and (5.10). (5.12) follows (4.13), (5.5), (5.10) and (5.12). (5.13) follows from (5.8)-(5.12). ■

§ 6. Absolute neighborhood shape retracts.

In this section we discuss shape properties of $AANR_C$(**PM**), $AANR_N$(**PM**), AAR(**PM**) and so on.

Let Y be a subspace of a space X. We say that a shaping $f: X \to Y$ is a shape retraction provided that $fS(j) = S(1_Y)$. Here $j: Y \to X$ is the inclusion map and $S(j): Y \to X$ is the shaping induced by j. We say that Y is a shape retract of X provided that there exists a shape retraction $f: X \to Y$.

Let X be a paracompact M-space. We say that X is an absolute shape retract for **PM**, in notation ASR for **PM**, provided that it satisfies the following condition:

(ASR) For any closed embedding $h: X \to M$, $M \in$ **PM**, $h(X)$ is a shape retract of M.

We say that X is an absolute neighborhood shape retract for **PM**, in notation ANSR for **PM**, provided that it satisfies the following condition:

(ANSR) For any closed embedding $h: X \to M$, $M \in$ **PM**, there exists a neighborhood U of $h(X)$ in M such that $h(X)$ is a shape retract of U.

ASR(**PM**) and ANSR(**PM**) denote the full subcategories of **TOP** consisting of all ASRs and ANSRs for **PM**, respectively.

(6.1) THEOREM. *Let X be a paracompact M-space.*

(i) $X \in$ ASR(**PM**) *iff X has trivial shape.*

(ii) $X \in$ ANSR(**PM**) *iff X is strongly movable* (*see* MS [**19**]).

PROOF. Using the same way of proof as in Theorems 11 and 12 of MS [**19**, p. 233] by (I.3.17) and (5.5) we easily show (i) and

(1) $X \in$ ANSR(**PM**) iff X is shape dominated by a polyhedron.

By (1) and Theorem 4 of Watanabe [**35**] we have (ii). ■

In Bogatyĭ [**2**] introduced the notion of internal movability for compact metric spaces. This notion is not shape invariant. For arbitrary spaces we define internal movability as follows: Let $\boldsymbol{p} = \{p_a : a \in A\}: X \to \mathcal{X} = \{X_a, p_{a',a}, A\}$ be a resolution. We consider the following condition:

(IM) For each $a \in A$ there exist $a' > a$ and a map $f: X_{a'} \to X$ such that $p_a f \simeq p_{a',a}$.

(6.2) LEMMA. *Let $\boldsymbol{p}: X \to \mathcal{X}$ and $\boldsymbol{q}: X \to \mathcal{Y}$ be* **ANR**-*resolutions of a space*

X. *If* $\boldsymbol{p}$ *satisfies* (IM), *then so does* $\boldsymbol{q}$.

In a way similar to the one used in (1.7) using (I.5.5) we can show (6.2). We say that a space X is internally movable provided that X admits an **ANR**-resolution satisfying (IM). By (6.2) this property does not depend on **ANR**-resolutions. Thus for compact metric spaces Bogatyĭ's definition coincides with our definition.

(6.3) THEOREM. *Let* X *be a space,* $\mathcal{K}$ *a collection of spaces and* n *an integer.*

(i) *If* X *is* AM, *then* X *is movable.*

(ii) *If* X *is* $\mathcal{K}$-AM, *then* X *is* $\mathcal{K}$*-movable.*

(iii) *If* X *is* n-AM, *then* X *is* n*-movable.*

(iv) *If* X *is* UAM, *then* X *is uniformly movable.*

(v) *If* X *is* IAM, *then* X *is internally movable.*

PROOF. We only show (i). In a similar way we can easily show the other assertions. Let X be an approximatively movable space. There exists an approximative **POL**-resolution $\boldsymbol{p}: X \to (\mathcal{X}, \mathcal{U})$ with (**) in (I.5.6). Since $(\mathcal{X}, \mathcal{U})$ is approximatively movable, for each $a \in A$ there exists $a_0 > a$ with the following property: For each $a' > a$ there exists a map $r_{a'}: X_{a_0} \to X_{a'}$ such that $(p_{a',a} r_{a'}, p_{a_0,a}) < \mathcal{U}_a$. By (**) in (I.5.6) $p_{a',a} r_{a'} \simeq p_{a_0,a}$. This means that $H(\mathcal{X})$ is movable. Since $H(\boldsymbol{p}): X \to H(\mathcal{X})$ is a **HPOL**-expansion of X by (I.3.3) and (I.5.7), X is movable. ■

(6.4) THEOREM. *A space* X *is strongly movable iff* X *is movable and satisfies the condition* M.

PROOF. Let $\boldsymbol{p}: X \to \mathcal{X} = \{X_a, p_{a',a}, A\}$ be a **POL**-resolution of X. By (I.5.7) $H(\boldsymbol{p}): X \to H(\mathcal{X})$ is a **HPOL**-expansion of X. We assume that X is strongly movable. Thus $H(\mathcal{X})$ satisfies the following condition:

(SM) For each $a \in A$ there exists $a_0 > a$ with the following property; for each $a' > a$ there exist $a'' > a_0$, a' and a map $r: X_{a_0} \to X_{a'}$ such that $p_{a',a} r \simeq p_{a_0,a}$ and $r p_{a'',a_0} \simeq p_{a'',a'}$.

(SM) implies (M) in (3.8) and the following:

(MV) For each $a \in A$ there exists $a_0 > a$ such that for each $a' > a$ there exists a map $r: X_{a_0} \to X_{a'}$ satisfying $p_{a',a} r \simeq p_{a_0,a}$.

Hence X satisfies the condition M and is movable.

Next we assume that X is movable and satisfies the condition M. Then $H(\mathscr{X})$ satisfies (M) and (MV). By the Claim in (3.8) $H(\mathscr{X})$ satisfies $(\mathrm{M})_1$. We show that $H(\mathscr{X})$ satisfies (SM). By $(\mathrm{M})_1$ there exists $a_0 \in A$ satisfying $(\mathrm{M})_1$. Take any $a \in A$ and then take any $a_1 > a, a_0$. There exists $a_2 > a_1$ satisfying (MV) for a_1. We show that a_2 is the required index. To do so take any $a_3 > a$. By the choice of a_0 there exist $a_4 > a_0, a_3$ and a map $s: X_{a_0} \to X_{a_3}$ satisfying

(1) $s p_{a_4, a_0} \simeq p_{a_4, a_3}$.

Take any $a_5 > a_1, a_4$ and then by the choice of a_2 there exists a map $r: X_{a_2} \to X_{a_5}$ satisfying

(2) $p_{a_5, a_1} r \simeq p_{a_2, a_1}$.

We put $k = s p_{a_5, a_0} r: X_{a_2} \to X_{a_3}$. By (1) and (2) $p_{a_3, a} k = p_{a_3, a}(s p_{a_4, a_0}) p_{a_5, a_4} r \simeq p_{a_3, a} p_{a_5, a_3} r = p_{a_1, a}(p_{a_5, a_1} r) \simeq p_{a_1, a} p_{a_2, a_1} = p_{a_2, a}$, that is,

(3) $p_{a_3, a} k \simeq p_{a_2, a}$.

Take any $a_6 > a_2, a_5$. By (1) and (2) $k p_{a_6, a_2} = s p_{a_1, a_0}(p_{a_5, a_1} r) p_{a_6, a_5} \simeq s p_{a_1, a_0} p_{a_2, a_1} p_{a_6, a_2} = (s p_{a_4, a_0}) p_{a_6, a_4} \simeq p_{a_4, a_3} p_{a_6, a_4} = p_{a_6, a_3}$, that is,

(4) $k p_{a_6, a_2} \simeq p_{a_6, a_3}$.

(3) and (4) mean that $H(\mathscr{X})$ satisfies (SM) and hence X is strongly movable. ■

(6.5) COROLLARY. *Let X be a paracompact M-space.*

(i) *If X is* SAM, *then X is strongly movable.*

(ii) *If X satisfies the condition* N, *then X is strongly movable.*

PROOF. In tne same way as in (6.3) we can show (i). We show (ii). By (3.15) X is UAM and satisfies ap-M. By (6.2) X is uniformly movable, and then movable. By (3.8) X satisfies the condition M. Hence by (6.4) X is strongly movable. ■

(6.6) COROLLARY. *Let X be a paracompact M-space.*

(i) $\mathrm{AAR}(\mathbf{PM}) \subset \mathrm{ASR}(\mathbf{PM})$.

(ii) *If X is* SAM, *then* $X \in \mathrm{ANSR}(\mathbf{PM})$.

(iii) $\mathrm{AANR}_{\mathrm{N}}(\mathbf{PM}) \subset \mathrm{ANSR}(\mathbf{PM})$.

(iv) *If* $X \in \mathrm{AANR}_{\mathrm{C}}(\mathbf{PM})$, *then X is internally movable and uniformly movable.*

(i) follows from (5.12) and (6.1). (ii) follows from (6.1) and (6.5). (iii) follows from (5.11), (6.1) and (6.5). (iv) follows from (5.10) and (6.2). ■

Now we will discuss topological groups. We assume that the reader is familiar with topological groups. Pontryagin [**27**] is a good textbook for topological groups.

Let G be a compact connected abelian topological group. $Ch(G)$ denotes the character group of G. Since G is compact connected, $Ch(G)$ is a discrete and torsion free abelian group. A continuous homomorphism $h: G\to H$ induces a homomorphism $Ch(h): Ch(H)\to Ch(G)$. Let $\mathcal{G}=\{G_a : a\in A\}$ be the set of all finitely generated subgroups of $Ch(G)$. Then we have a directed system $\mathcal{G}=\{G_a, j_{a',a}, A\}$ such that $a'>a$ iff $G_{a'}\supset G_a$, and $j_{a',a}: G_a\to G_{a'}$ is the inclusion homomorphism for $a'>a$. Inclusion homomorphisms $j_a: G_a\to Ch(G)$ induce a direct limit $\boldsymbol{j}=\{j_a : a\in A\}: \mathcal{G}\to Ch(G)$. Since $Ch(G)$ is torsion free, each G_a is a free group $Z^{n(a)}$. Here Z^n is the direct sum of n-copies of the additive group Z of all integers. Thus $Ch(G_a)$ is the $n(a)$-dimentional torus $T^{n(a)}$. By taking the dual we have an inverse system $Ch(\mathcal{G})=\{Ch(G_a), Ch(j_{a',a}), A\}$. $Ch(\boldsymbol{j})=\{Ch(j_a) : a\in A\}: G\to Ch(\mathcal{G})$ forms an inverse limit. Since all $Ch(G_a)$ are polyhedra, $Ch(\boldsymbol{j}): G\to Ch(\mathcal{G})$ is a **POL**-resolution by (I.3.13).

(6.7) LEMMA (Scheffer [**30**]). *Let G be a compact connected topological group and H a locally compact abelian topological group. Then every map $f: G\to H$ with $f(e_G)=e_H$ is homotopic to exactly one continuous homomorphism $h: G\to H$. Here e_G denotes the identity element of G.* ■

(6.8) LEMMA. *Let T^n and T^m be finite dimensional tori. Then every map $f: T^n\to T^m$ is homotopic to exactly one continuous homomorphism $h: T^n\to T_m$.*

PROOF. Let e_n and e_m be the identity elements of T^n and T^m, respectively. Since T^n is arcwise connected, there exists a path $s: I\to T^m$ such that $s(0)=e_m$ and $s(1)=f(e_n)$. We define a homotopy $H: T^n\times I\to T^m$ by $H(x,t)=f(x)*s(t)^{-1}$. Here $w*z$ means the composition of w and z in the group T^m. Then $H_0=f$ and $H_1: T^n\to T^m$ is a map with $H_1(e_n)=e_m$. Therefore by (6.7) f is homotopic to exactly one continuous homomorphism $h: T^n\to T^m$. ■

(6.9) THEOREM. *Let G be a compact connected abelian topological group.*

(i) *G is* UAM *iff G is uniformly movable.*

(ii) *G is* AM *iff G is movable.*

PROOF. We show only (i). In a similar way we can show (ii). We assume that G is UM. Then for each $a\in A$ there exist $a_0>a$ and a collection $\{r_{a'} : a'>a\}$ of maps $r_{a'}: Ch(G_{a_0})\to Ch(G_{a'})$ satisfying

(1) $Ch(j_{a'',a'})r_{a''}\simeq r_{a'}$ and $Ch(j_{a',a})r_{a'}\simeq Ch(j_{a_0,a})$ for $a''>a'>a$.

By (6.8) there exist continuous homomorphisms $h_{a'}: Ch(G_{a_0})\to Ch(G_{a'})$ such that $r_{a'}\simeq h_{a'}$ for $a'>a$. By the uniquness of (6.8) and (1)

(2) $Ch(j_{a'',a'})h_{a''}=h_{a'}$ and $Ch(j_{a',a})h_{a'}=Ch(j_{a_0,a})$ for $a''>a'>a$.

Since $Ch(\boldsymbol{j}): G\to Ch(\mathscr{J})$ is an inverse limit, by (2) there exists a continuous homomorphism $h: Ch(G_{a_0})\to G$ such that $Ch(j_a)h=Ch(j_{a_0,a})$. Thus $Ch(\boldsymbol{j}): G\to Ch(\mathscr{J})$ satisfies (IAM)* and then G is IAM. Hence by (2.13) G is UAM. The converse assertion follows from (6.3). ■

(6.10) COROLLARY. *Let G be a compact connected abelian group.*

(i) *G is* UAM *iff $H^1(G:Z)$ has strong property L (see Watanabe* [**34**]).

(ii) *G is* AM *iff $H^1(G:Z)$ has property L (see Keesling* [**16**]).

Here $H^n(X:K)$ denotes the n-dimensional Čech cohomology group of a space X with coefficients K.

(6.11) COROLLARY. *Let G be a compact connected abelian group.*

(i) *If G is locally arcwise connected, then G is* UAM.

(ii) *G is locally connected iff G is* AM.

(6.12) COROLLARY. *There exists a compact connected abelian group which is* AM, *but not* UAM.

Movability and uniform movability for compact connected abelian groups are characterized by Keesling [**16**] and Watanabe [**34**]. Since $Ch(G)=H^1(G:Z)$ by Steenrod [**32**], (6.10) and (6.11) follow from these characterizations and (6.9). Watanabe [**34**] showed that Keesling's example in [**17**] is movable, but not uniformly movable. Hence by (6.9) Keesling's example is also an example for (6.12). ■

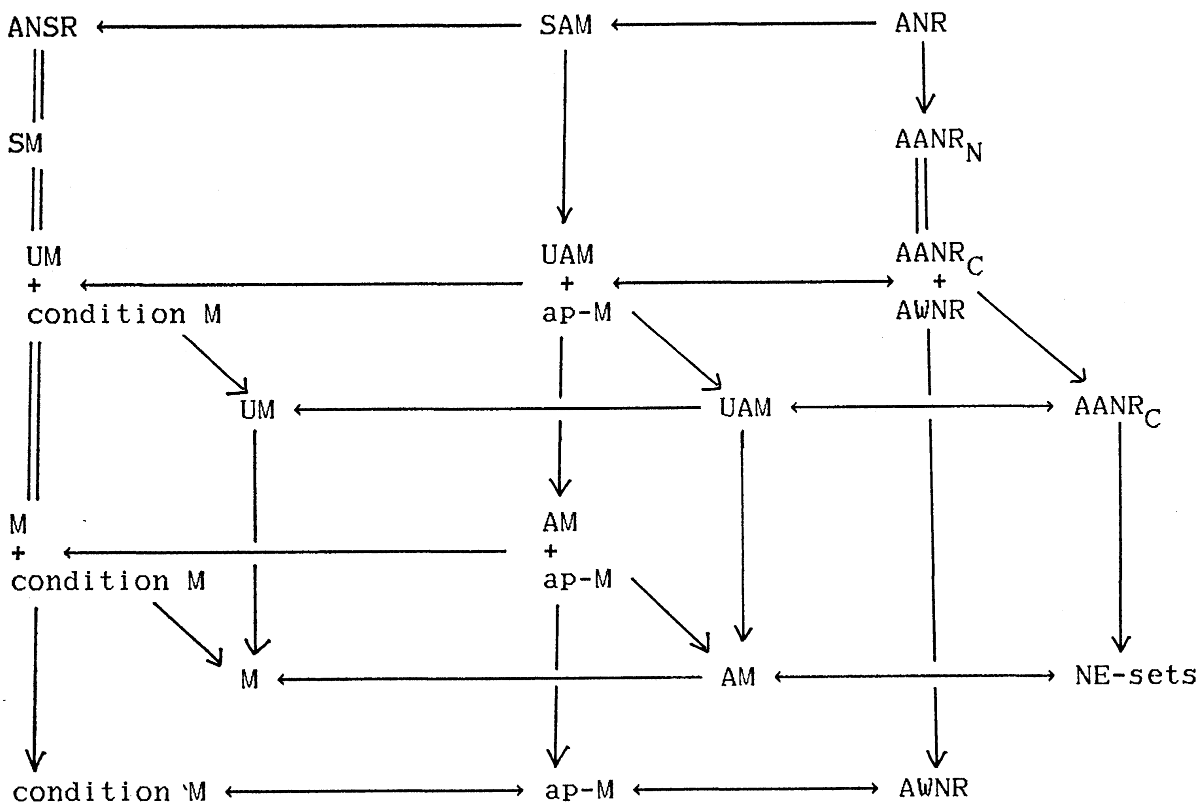
(6.13) LEMMA. *If a compact connected abelian group G is strongly movable, then G is a finite dimensional torus.*

PROOF. Since G is strongly movable, by Watanabe [**36**] G is shape dominated by a finite polyhedron P. Then $H^1(G:Z)$ is also dominated by $H^1(P:Z)$. Thus $H^1(G:Z)$ is finitely generated. By Steenrod [**32**] $H^1(G:Z)=Ch(G)$. Since G is connected, $Ch(G)$ is torsion free by Pontryagin [**27**]. Thus $Ch(G)=Z^n$ for some integer n. By Pontryagin duality G is homeomorphic to $Ch(Z^n)=T^n$. ■

(6.14) PROPOSITION. *For compact connected abelian groups the notions of* SAM, $\mathrm{AANR}_N(\mathbf{PM})$ *and* $\mathrm{ANR}(\mathbf{PM})$ *are equivalent.*

By (6.2) $\mathrm{ANR}(\mathbf{PM}) \to \mathrm{AANR}_N(\mathbf{PM}) \to \mathrm{SAM}$ and by (6.13) $\mathrm{SAM} \to \mathrm{ANR}(\mathbf{PM})$. Hence we have (6.14). ■

We established many relations between generalized ANRs and approximative shape properties. (For NE-sets see the subsequent parts of this paper.) For paracompact M-spaces we summarize these relations as follows:



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