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THE BEHAVIOR OF OSCILLATORY INTEGRALS WITH DEGENERATE STATIONARY POINTS

By

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Introduction.

Let us consider the following integral with a parameter $\sigma\in\boldsymbol{R}$:

$$I(\sigma)=\int_{\boldsymbol{R}^n} e^{-i\sigma\phi(x)}\rho(x;\sigma)dx, \tag{0.1}$$

where $\phi(x)$ is a real-valued C^∞ function and $\rho(x;\sigma)$ is a C^∞ function with an asymptotic expansion

$$\rho(x;\sigma)\sim\rho_0(x)+\rho_1(x)\sigma^{-1}+\rho_2(x)\sigma^{-2}+\cdots \quad (\text{as } |\sigma|\to\infty).$$

In the previous papers Soga [6], [7] we have studied conditions under which $I(\sigma)$ does not decrease rapidly as $\sigma\to\infty$ in the sense that

$$\sigma^m I(\sigma)\notin L^2(1, \infty).$$

for some $m\in\boldsymbol{R}$.

In this paper we shall examine the behavior of $I(\sigma)$ more precisely in the case of $n=1$ and 2; namely, we shall derive an estimate of the type

$$|I(\sigma)|\geqq\delta|\sigma|^{-\alpha} \quad \text{as} \quad |\sigma|\to\infty \tag{0.2}$$

(see Theorem 1.1 in §1). When the phase function $\phi(x)$ has only non-degenerate stationary points (in supp$[\rho]$) or is analytic, $I(\sigma)$ is expanded asymptotically as $|\sigma|\to\infty$ (cf. Hörmander [1], Varchenko [8], Kaneko [4], etc.), and then we can obtain the above estimate through that expansion. But it seems difficult to do so when all derivatives of $\phi(x)$ vanish at some points. We take this case into consideration. In our methods we do not employ the asymptotic expansion.

We can apply the above estimate to a scattering inverse problem studied by Ikawa [3]. Consider the scattering by obstacles formulated by Lax and Phillips [5]. Then the scattering matrix $\mathcal{S}(z)$ is meromorphic in the whole complex plane and analytic on the lower half plane $\{z: \operatorname{Im} z\leqq 0\}$ (cf. Chapter V of [5]). Ikawa [2], [3] examine the distribution of the poles of $\mathcal{S}(z)$ in the case where the obstacles consist of two convex bodies. In [2] it is proved that if

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the curvatures of the obstacles do not vanish anywhere $\mathcal{S}(z)$ is analytic in $\{z: \operatorname{Im} z \leqq c\}$ for some positive constant c; in [3] he shows that there exists a sequence $\{z_j\}_{j=1,2,\dots}$ of the poles of $\mathcal{S}(z)$ satisfying $\lim_{j\to\infty} \operatorname{Im} z_j=0$ and $\lim_{j\to\infty} |z_j|=\infty$ if the curvatures vanish of finite order at the points which give the minimum distance between the two bodies.

Using our result, we can extend the latter of his results above (in [3]). He reduces its proof to deriving the estimate of the type (0.2). If the degeneracy of the curvatures of the obstacles is of finite order, the phase function $\phi(x)$ has stationary points only of finite order, and so the methods of Varchenko [8], which are used for analytic phase functions, can be applied. Namely, he derives the estimate (0.2) by means of the asymptotic expansion of $I(\sigma)$ in [8]. Our main result (Theorem 1.1 in § 1) enables us to verify his conclusion also for some obstacles whose curvatures vanish of infinite order.

§ 1. Main result.

Hereafter we assume that the dimension n in (0.1) is equal to 1 or 2. Let $\operatorname{supp}[\rho(\cdot\,;\sigma)]$ and $\operatorname{supp}[\rho_j]$ ($j=0, 1, \cdots$) be contained in a compact set D, and assume that $\rho_0(x)$ is real-valued and satisfies

$$\text{(A.1)} \qquad \rho_0(y)>0 \quad \text{when} \quad \phi(y)=\min_{x\in D} \phi(x).$$

If $\partial_x\phi(x)$ vanishes at $x=y$, y is called a stationary point of $\phi(x)$, and a non-degenerate one when the Hessian matrix $\partial_x^2\phi(y)$ is non-degenerate. The stationary point y is non-degenerate if and only if every eigenvalue of $\partial_x^2\phi(y)$ is non-zero. Note that if $\phi(y)=\min\phi(x)$ and y is an inner point of D, y is a stationary point. We assume that

$$\text{(A.2)} \qquad \text{there is no stationary point in } \{x: \phi(x)>\min_{\tilde{x}\in D} \phi(\tilde{x})\}\cap D.$$

Let the dimension $n=2$, and let an inner point y of D satisfy $\phi(y)=\min_{x\in D}\phi(x)$ and $\det \partial_x^2\phi(y)=0$. Then $\partial_x^2\phi(y)$ has at least one zero-eigenvalue. If the one $\lambda_1(x)$ of the eigenvalues is equal to zero at $x=y$ and the other $\lambda_2(x)$ is not, there exist local coordinates $x=x(z)$ ($z=(z_1, z_2)$) defined near y such that

(i) $y=x(0)$,

(ii) when z_1 is fixed and only z_2 moves near 0, $\phi(x(z))$ is minimum at $z_2=0$ and $\partial_{z_2}^2\phi(x(z_1, 0))\neq 0$.

Hereafter, employing these coordinates $x(z)$, we abbreviate functions $f(x(z))$ to $f(z)$.

We assume that for each y with $\phi(y)=\min_{x\in D}\phi(x)$ one of the following conditions $(A.3)_0\sim(A.3)_2$ is satisfied:

$(A.3)_0$ either of the eigenvalues of $\partial_x^2\phi(y)$ is not equal to 0 (i.e. $\det\partial_x^2\phi(y)\neq 0$),

$(A.3)_1$ the one $\lambda_1(x)$ of the eigenvalues of $\partial_x^2\phi(x)$ is equal to 0 at $x=y$, and the other $\lambda_2(x)$ is not; furthermore, introducing the above local coordinates z, $\phi(z)$ satisfies for a constant $r_0>0$

$$(\operatorname{sgn} z_1)^j\partial_{z_1}^j\phi(z_1, 0)\geqq 0, \quad z_1\in(-r_0, r_0), \ j=2, 3, 4,$$

$$\partial_{z_1}^4\phi(\pm r_0, 0)>0,$$

$$\partial_{z_2}^2\phi(z_1, 0)\neq 0, \quad z_1\in[-r_0, r_0],$$

$(A.3)_2$ both eigenvalues of $\partial_x^2\phi(y)$ are equal to 0, and for any fixed ω ($|\omega|=1$) $\phi(x)$ satisfies for a constant $r_0>0$

$$\partial_r^j\phi(y+r\omega)\geqq 0, \quad r\in(0, r_0), \ j=2, 3, 4,$$

$$\partial_r^4\phi(y+r_0\omega)>0.$$

When $n=1$, the above case $(A.3)_1$ need not be considered, and then we assume that $(A.3)_0$ or $(A.3)_2$ is satisfied.

REMARK. If $\phi(x)$ is analytic and (A.1) is satisfied, then one of the conditions $(A.3)_0\sim(A.3)_1$ holds necessarily.

There exist C^∞ functions satisfying $(A.3)_1$ or $(A.3)_2$, but all C^∞ functions do not necessarily satisfy one of $(A.3)_0\sim(A.3)_2$.

THEOREM 1.1. *Let* (A.1) *and* (A.2) *hold, and assume that one of the conditions* $(A.3)_0\sim(A.3)_2$ *is satisfied* ($n=1, 2$). *Then we have*

$$|I(\sigma)|\geqq\delta|\sigma|^{-(n/2)+\beta} \quad \text{as} \quad |\sigma|\to\infty$$

where $\beta=0$ *in the case* $(A.3)_0$ *and* $\beta=1/4$ *in the case* $(A.3)_1$ *and* $(A.3)_2$.

§2. Proof of Theorem 1.1.

Let us note that by the assumptions $(A.3)_0\sim(A.3)_2$ the points of the different types (of $(A.3)_0\sim(A.3)_2$) are separated each other in D; that is, we can take a partition $\{\chi_j(x)\}$ of unity ($0\leqq\chi_j\leqq 1$) such that in each $\operatorname{supp}[\chi_j]$ only one of $(A.3)_0\sim(A.3)_2$ is satisfied. Furthermore we can choose the $\{\chi_j\}$ so that $\rho_0(y)\chi_j(y)>0$ for any $y\in\operatorname{supp}[\chi_j]\cap\{y:\phi(y)=\min_{x\in D}\phi(x)\}$.

If all $y\in D$ with $\phi(y)=\min_{x\in D}\phi(x)$ are of the type $(A.3)_0$, then they are isolated, and, expanding the integral $\int e^{-i\sigma\phi(x)}\rho\chi_j dx$ asymptotically (as $|\sigma|\to\infty$) by the well-known methods of stationary phases (cf. Hörmander [1], etc.), we can obtain the required estimate with $\beta=0$.

Let $(A.3)_1$ or $(A.3)_2$ be satisfied in some supp $[\chi_j]$ (e.g. for $j=1, 2, \cdots, l$). Then it suffices to show that

$$\left|\sum_{j=1}^{l}\int_{R^n} e^{-i\sigma\phi(x)}\rho_0(x)\chi_j(x)dx\right|\geqq\delta|\sigma|^{-(n/2)+\beta} \qquad (\text{as } |\sigma|\to\infty)$$

for the constant β in Theorem 1.1, because

$$\left|\int e^{-i\sigma\phi(x)}(\rho_0(x)-\rho(x\,;\sigma))dx\right|\leqq C(|\sigma|+1)^{-1}$$

and moreover

$$\left|\int e^{-i\sigma\phi(x)}\rho_0\chi_j dx\right|\leqq C(1+|\sigma|)^{-n/2}$$

if $(A.3)_0$ is satisfied in supp $[\chi_j]$. The above required estimate follows from the inequalities

$$J(\sigma)\equiv-\mathrm{Im}\int_{R^n}\{\exp i\sigma(\min_{\tilde{x}\in D}\phi(\tilde{x})-\phi(x))\}\rho_0(x)\chi_j(x)dx \tag{2.1}$$
$$\geqq\delta\sigma^{-(n/2)+\beta} \quad (\text{as } \sigma\to\infty),\ j=1, 2, \cdots, l.$$

If $(A.3)_1$ is satisfied (in the case of $n=2$), applying the methods of stationary phases in the variable z_2, we can reduce the proof to that in the case of $n=1$ under $(A.3)_2$.

From now on, let us verify (2.1) under $(A.3)_2$ only when $n=2$ since the proof in the case of $n=1$ is similar and easier. Hereafter we abbreviate $\rho_0(x)\chi_j(x)$ in (2.1) to $\rho_0(x)$, and assume for simplicity that $\phi(y)$ $(=\min_{x\in D}\phi(x))=0$.

Since $\phi(y+r\omega)$ is non-decreasing in r on $(0, r_0)$ and strictly increasing if $\phi(y+r\omega)>0$ (by $(A.3)_2$), for any small $s>0$ and any ω $(|\omega|=1)$ there exists $r(s, \omega)>0$ uniquely such that

$$\phi(y+r(s, \omega)\omega)=s.$$

We define $r(t, \omega)=0$ for $t<0$. Take a function $\chi(s)\in C_0^\infty(\boldsymbol{R}^1)$ satisfying $\chi(s)=1$ in a neighborhood of 0. Then, noting that

$$\int_{R^2}e^{-i\sigma\phi(x)}\rho_0(x)dx=\int_{-\infty}^{\infty}e^{-i\sigma s}\partial_s\left[\int_{\{x:\phi(x)\leq s\}}\rho_0(x)dx\right]ds$$

(cf. §1 of Soga [7]) and employing the polar coordinates, we have

$$\int e^{-i\sigma\phi(x)}\rho_0(x)dx=\int_{S^1}d\omega\int_0^{r(+0,\omega)}\rho_0(y+r\omega)rdr$$

$$+\int_{S^1} d\omega \int_{+0}^{+\infty} e^{-i\sigma s}\rho_0(y+r(s,\omega)\omega)r(s,\omega)\partial_s r(s,\omega)\chi(s)ds$$
$$+O(|\sigma|^{-\infty}),$$

where $|O(|\sigma|^{-\infty})|\leqq C_k(1+|\sigma|)^{-k}$ for any $k>0$. Here, we use the assumption (A.2). Therefore it follows that

$$J(\sigma)=\int_{S^1} d\omega \int_{+0}^{\infty} (\sin \sigma s)\rho_0(y+r(s,\omega)\omega)r(s,\omega)\partial_s r(s,\omega)\chi(s)ds+O(|\sigma|^{-\infty}).$$

$$g(s,\omega)=\rho_0(y+r(s,\omega)\omega)r(s,\omega)\partial_s r(s,\omega)$$

is a C^∞ function on $(0, s_0)\times S^1$ for some $s_0>0$, and has the following property:

LEMMA 2.1. *$g(s,\omega)$ is non-negative on $0<s\leqq s_0$ for some $s_0>0$, and there are constants C and $s_1(\leqq s_0)$ independent of s and ω such that*

$$\partial_s g(s,\omega)\leqq -Cs^{-5/4} \qquad on \quad (0, s_1).$$

This lemma will be proved latter.

The estimate (2.1) follows immediately from the following lemma; for, by Lemma 2.1 we can choose the $\chi(s)$ so that the function $f(s)=g(s,\omega)\chi(s)$ satisfies all assumptions in the lemma with $\varepsilon=1/4$.

LEMMA 2.2. *Let $f(s)$ be a non-negative C^1 function on $(0,\infty)$ with* supp$[f]$ $\subset(0, 1]$, *and assume that $f'(s)$ is non-positive on $(0,\infty)$ and satisfies*

$$f'(s)\leqq -Cs^{-1-\varepsilon} \qquad on \quad (0, s_1)$$

($0<\varepsilon<1$, $C>0$, $s_1>0$). *Then we have the estimate*

$$\int_0^\infty (\sin \sigma s)f(s)ds\geqq \delta\sigma^{-1+\varepsilon} \qquad for \quad \sigma\geqq\sigma_0,$$

where the positive constants δ and σ_0 depend on C, ε and s_1.

PROOF OF LEMMA 2.2. We have

$$\int_0^\infty (\sin \sigma s)f(s)ds=\sum_{k=0}^{\infty}\int_{2k\pi\sigma^{-1}}^{(2k+1)\pi\sigma^{-1}} (\sin \sigma s)(f(s)-f(s+\pi\sigma^{-1}))ds.$$

Each term in the above summation is non-negative, and furthermore if $(2k+2)\pi\sigma^{-1}\leqq s_1$ it follows that

$$\int_{2k\pi\sigma^{-1}}^{(2k+1)\pi\sigma^{-1}} (\sin \sigma s)(f(s)-f(s+\pi\sigma^{-1}))ds\geqq C\sigma^{-1}\{(2k+2)\pi\sigma^{-1}\}^{-\varepsilon-1}2\pi\sigma^{-1}$$
$$\geqq C\sigma^{-1}\int_{(2k+2)\pi\sigma^{-1}}^{(2k+4)\pi\sigma^{-1}} s^{-1-\varepsilon}ds.$$

Therefore we obtain

$$\int_0^\infty (\sin\sigma s)f(s)ds \geqq C\sigma^{-1}\int_{2\pi\sigma^{-1}}^{s_1} s^{-1-\varepsilon}ds$$
$$=(2\pi)^{-\varepsilon}C\varepsilon^{-1}\sigma^{-1+\varepsilon}-C\varepsilon^{-1}s_1^{-\varepsilon}\sigma^{-1},$$

which proves Lemma 2.2.

PROOF OF LEMMA 2.1. Fix $\omega\in S^1$ arbitrarily. It is seen that for $s>0$

$$\partial_s r(s,\omega)=\partial_r\phi(y+r(s,\omega)\omega)^{-1},$$
$$\partial_s^2 r(s,\omega)=-\partial_r\phi(y+r(s,\omega)\omega)^{-3}\partial_r^2\phi(y+r(s,\omega)\omega),$$

where $\partial_r^j\phi$ denotes the derivative $\partial_r^j\phi(y+r\omega)$ for fixed ω. From the first equality above, it is obvious that $g(s,\omega)\geqq 0$ on $(0, s_0)$ for a small s_0. Furthermore it follows from the above equalities that

$$\partial_s g(s,\omega)=-\partial_r\phi(y+r(s,\omega)\omega)^{-3}[\partial_r^2\phi(y+r(s,\omega)\omega)r(s,\omega)\rho_0(y+r(s,\omega)\omega)$$
$$-\partial_r\phi(y+r(s,\omega)\omega)\{\rho_0(y+r(s,\omega)\omega)-\partial_r\rho_0(y+r(s,\omega)\omega)r(s,\omega)\}].$$

Noting that $\phi(y+r\omega)=0$ for $r\in(0, r_+)$ (where $r_+=r(+0,\omega)$), by $(A.3)_2$ we have $\partial_r^4\phi(y+r\omega)(r-r_+)\geqq 0$ for $r\in(0, r_0)$, which yields that

$$\partial_r^2\phi(y+r\omega)(r-r_+)\geqq 2\partial_r\phi(y+r\omega), \qquad 0\leqq r\leqq r_0.$$

Therefore we obtain

$$\partial_s g\leqq -\partial_r\phi^{-2}\left[\frac{2r\rho_0}{r-r_+}-\rho_0+\partial_r\rho_0 r\right].$$

On the other hand it follows from (A.1) that

$$\mu=\inf_{0<s\leqq s_1}\left[\frac{2r(s,\omega)\rho_0(y+r(s,\omega)\omega)}{r(s,\omega)-r^+}-\rho_0(y+r(s,\omega)\omega)+\partial_r\rho_0(y+r(s,\omega)\omega)r(s,\omega)\right]>0$$

if s_1 (>0) is small enough. Hence we have

$$\partial_s g(s,\omega)\leqq -\mu\partial_r\phi(y+r(s,\omega)\omega)^{-2}, \qquad 0<s\leqq s_1.$$

Applying the following lemma to the function $\psi(r)=\phi(y+(r+r_+)\omega)$ (note that $\partial_r^j\phi(y+r_+\omega)=0$ $(0\leqq j\leqq 3)$), we obtain

$$\partial_s g(s,\omega)\leqq -\frac{\mu}{4}\phi(y+r(s,\omega)\omega)^{-5/4}=-\frac{1}{4}\mu s^{-5/4},$$

which proves the lemma.

LEMMA 2.3. *Let $\psi(r)$ be a non-negative C^4 function on $[0, a]$ $(a>0)$ such that $\psi'(r)\geqq 0$ on $[0, a]$ and $\psi(0)=\psi'(0)=\cdots=\psi'''(0)=0$. Then for a constant b satisfying $0<b\leqq a$ we have*

$$\phi'(r)\leqq 2\phi(r)^{5/8} \quad on \quad [0, b].$$

Here the constant b can be chosen independently of ϕ *if* $|\phi|_{C^4[0,a]}$ *is bounded.*

PROOF. Let $h(r)\in C^3[0, a]$ and $h(r)\geqq 0$ on $[0, a]$. Then it follows that $h(r-t)=h(r)-h'(r)t+\int_0^1(1-\theta)h''(r-\theta t)d\theta t^2\geqq 0$ for any t and r satisying $0\leqq t\leqq r\leqq a$, and so we have

$$h'(r)\leqq h(r)t^{-1}+\int_0^1 h''(r-\theta t)(1-\theta)d\theta t, \qquad 0<t\leqq r\leqq a. \tag{2.2}$$

If $h(0)=\cdots=h''(0)=0$, we have $h(r)\leqq 6^{-1}|h'''|_{C^0[0,a]}r^3$, which yields that

$$h(r)^{1/2}\leqq r \quad \text{on} \quad [0, b_1]$$

for a constant $b_1>0$. Setting $h(r)=\phi'(r)$ (or $\phi(r)$) and $t=\phi'(r)^{1/2}$ (or $\phi(r)^{1/2}$) in (2.2), we obtain

$$\phi''(r)\leqq(1+2^{-1}|\phi|_{C^4[0,a]})\phi'(r)^{1/2},$$

$$\phi'(r)\leqq(1+2^{-1}|\phi|_{C^3[0,a]})\phi(r)^{1/2}.$$

From these it follows that

$$\phi''(r-\theta t)\leqq C\phi(r-\theta t)^{1/4}\leqq C\phi(r)^{1/4}$$

(note that $\phi(r)$ is an increasing function). Insert this estimate into (2.2) with $h(r)=\phi(r)$. Then we have

$$\phi'(r)\leqq\phi(r)t^{-1}+2^{-1}\phi(r)^{1/4}t, \qquad 0<t\leqq r\leqq a. \tag{2.3}$$

The inequality $\phi(r)^{3/8}\leqq r$ holds on $[0, b_2]$ for a small b_2 (>0) since $\phi(r)\leqq C_1|\phi|_{C^4[0,a]}r^4$. Putting $t=\phi(r)^{3/8}$ in (2.3), we have

$$\phi'(r)\leqq\phi(r)^{5/8}+2^{-1}\phi(r)^{5/8},$$

which proves the lemma.

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