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LOGARITHMIC UNIFORM DISTRIBUTION OF $(\alpha n+\beta\log n)$

By

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1. Introduction

A sequence $\omega=(x_n)_{n=1}^{\infty}$ of real numbers is said to be uniformly distributed modulo 1 if the proportion of indices $n\leq N$ such that the fractional parts $\{x_n\}$ are contained in an interval $I\subseteq[0,1)$ is asymptotically equal to the length of I. Put $\chi(x;y)=1$ for $\{y\}<x$ and $\chi(x;y)=0$ otherwise; then ω is uniformly distributed if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi(x;x_n)=x \quad \text{for} \quad 0<x<1. \tag{1}$$

It is well known (cf. the monographs [1] and [3]) that (1) is equivalent to

$$\lim_{N\to\infty}D_N^*(\omega)=0,$$

where

$$D_N^*(\omega)=\sup_{0\leqslant x<1}\left|\frac{1}{N}\sum_{n=1}^{N}\chi(x;x_n)-x\right|$$

denotes the discrepancy of the sequence ω. The systematic study of uniformly distributed sequences was initiated by H. Weyl [9]. Well known examples of uniformly distributed sequences are (αn) with irrational α and $(\sqrt{n})$; $(\log n)$ is known not to be uniformly distributed, but Tsuji [8] proved that

$$\lim_{N\to\infty}\frac{1}{\sum_{n=1}^{N}\frac{1}{n}}\sum_{n=1}^{N}\frac{1}{n}\chi(x;x_n)=x \quad (0<x<1) \tag{2}$$

for $x_n=\log n$. A sequence $\omega=(x_n)$ with this property is said to be uniformly distributed with respect to the logarithmic mean. This is equivalent to

$$\lim_{N\to\infty}D_N(\omega)=0,$$

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where

$$D_N(\omega)=\sup_{0<x<1}\left|\frac{1}{\sum_{n=1}^{N}\frac{1}{n}}\sum_{n=1}^{N}\frac{1}{n}\chi(x;x_n)-x\right|$$

denotes the logarithmic discrepancy of ω (cf. [5]). In a recent article E. Hlawka [2] investigated the sequence $(\alpha n+\beta\log n)$ $(\beta\neq 0)$ with respect to the logarithmic mean. He proved upper bounds for exponential sums from which (by the inequality of Erdös-Turan) we may conclude $D_N(\alpha n+\beta\log n)\leqslant c(\beta)/\sqrt{\log N}$. In [7] the first author proved a theorem that gives upper bounds for the discrepancy with respect to general weights and remarked that these estimates in fact give $D_N(\alpha n+\beta\log n)\leqslant c(\beta)/\log N$. Unfortunately the hypotheses of the theorem are not satisfied and it remains an open problem to prove this estimate (which essentially would be best possible; cf. [5]). In the next section we prove $D_N(\alpha n+\beta\log n)\leqslant c(\alpha,\beta)/\log N$ provided α is rational or of finite approximation type. In §3 we obtain $D_N(\alpha n+\beta\log n)\leqslant c_1(\beta)(\log\log N)^2/\log N$ by means of estimates for the exponential sums

$$\sum_{n=1}^{N}\frac{1}{n}e^{2\pi ih(\alpha n+\beta\log n)}\quad(h=1,2,\cdots),$$

which are accomplished by a generalization of van der Corput's method.

Notations. As usual $[t]$ denotes the largest integer $\leqslant t$ and the fractional part is given by $\{t\}=t-[t]$; furthermore we put $\psi(t)=\{t\}-\frac{1}{2}$ and $\|t\|=\min(\{t\},1-\{t\})$. In §2 we make use of the notations $\lfloor t\rfloor$ =maximal integer $<t$ and $\lceil t\rceil$ =minimal integer $\geqslant t$. It is easy to see that $\lceil t\rceil=\lfloor t\rfloor+1$ and $\lceil t\rceil=-[-t]$.

2. Elementary Estimates

We try to establish an upper bound $c/\log N$ for $D_N(\omega)$, where $\omega=(\alpha n+\beta\log n)_{n=1}^{\infty}$. Given $\beta>0$ we will prove that we may choose an absolute constant $c=c(\beta)$ valid for a large class of α's.

THEOREM 2.1. *Assume that* $0\leqslant\{\alpha\}\leqslant c_0/\log N$ (c_0 *a given positive constant*). *Then*

$$D_N(\omega)\leqslant 2\cdot\frac{c_0+2+\beta+1/\beta+e^{1/\beta}}{\log N}$$

PROOF. Let $f(u)=\{\alpha\}u+\beta\log u-\{\alpha\}$. Then $f'(u)>0$ and $uf'(u)=\{\alpha\}u+\beta$ is increasing.

Let k be the largest integer such that $\lceil f^{-1}(k)\rceil\leqslant N$. Then we have for $0<x<1$:

$$\sum_{n=1}^{N}\frac{1}{n}\chi(x;f(n))=\sum_{j=0}^{k-1}\sum_{\lceil f^{-1}(j)\rceil\leqslant n<\lceil f^{-1}(j+1)\rceil}\frac{1}{n}\chi(x;f(n))+\sum_{\lceil f^{-1}(k)\rceil\leqslant n\leqslant N}\frac{1}{n}\chi(x;f(n)).$$

The second sum can be estimated in the following way:

$$\sum_{\lceil f^{-1}(k)\rceil \leqslant n\leqslant N}\frac{1}{n}\chi(x;f(n))\leqslant\frac{N-f^{-1}(k)+1}{f^{-1}(k)}\leqslant\frac{N}{f^{-1}(k)}\leqslant\frac{f^{-1}(k+1)}{f^{-1}(k)}$$
$$=e^{(\log f^{-1}(k+1)-\log f^{-1}(k))}=e^{1/(f^{-1}(\xi_k)f'(f^{-1}(\xi_k))}$$
$$=e^{1/(\{\alpha\}f^{-1}(\xi_k)+\beta)}\leqq e^{1/\beta},$$

where $k\leqslant\xi_k\leqslant k+1$.
Since (for positive reals $A<B$)

$$\sum_{A\leqslant n<B}\frac{1}{n}=\log\frac{B}{A}+\frac{\theta}{A}\quad(\text{with}\quad|\theta|\leqslant 1),$$

we obtain

$$S_j=\sum_{\lceil f^{-1}(j)\rceil\leqslant n<\lceil f^{-1}(j+1)\rceil}\frac{1}{n}\chi(x;f(n))=\sum_{f^{-1}(j)\leqslant n<f^{-1}(j+x)}\frac{1}{n}$$
$$=\log f^{-1}(j+x)-\log f^{-1}(j)+\frac{\theta_j}{f^{-1}(j)}\quad(|\theta_j|\leqslant 1).$$

Hence, by the mean value theorem,

$$\sum_{j=0}^{k-1}S_j=x\cdot\sum_{j=0}^{k-1}\frac{1}{f^{-1}(\xi_{x,j})f'(f^{-1}(\xi_{x,j}))}+\theta\left(1+\int_0^{f(N)}\frac{du}{f^{-1}(u)}\right)$$

for some $\xi_{x,j}$ with $j\leqslant\xi_{x,j}\leqslant j+1$. Since

$$\sum_{j=0}^{k-1}\frac{1}{f^{-1}(\xi_{x,j})f'(f^{-1}(\xi_{x,j}))}\leqslant\sum_{j=0}^{k-1}\frac{1}{f^{-1}(j)f'(f^{-1}(j))}$$
$$\leqslant\frac{1}{f'(1)}+\int_0^{f(N)}\frac{du}{f^{-1}(u)f'(f^{-1}(u))}\leqslant\frac{1}{\beta}+\int_1^N\frac{dt}{t}=\frac{1}{\beta}+\log N,$$

$$\sum_{j=0}^{k-1}\frac{1}{f^{-1}(\xi_{x,j})f'(f^{-1}(\xi_{x,j}))}\geqslant\sum_{j=0}^{k-1}\frac{1}{f^{-1}(j+1)f'(f^{-1}(j+1))}$$
$$=\frac{-1}{f'(1)}+\sum_{j=0}^{k}\frac{1}{f^{-1}(j)f'(f^{-1}(j))}\geqslant-\frac{1}{f'(1)}+\int_0^{k+1}\frac{du}{f^{-1}(u)f'(f^{-1}(u))}$$
$$\geqslant-\frac{1}{f'(1)}+\int_0^{f(N)}\frac{du}{f^{-1}(u)f'(f^{-1}(u))}\geqslant-\frac{1}{\beta}+\log N$$

and

$$0<\int_0^{f(N)}\frac{du}{f^{-1}(u)}=\int_1^N\frac{f'(t)\,\mathrm{d}t}{t}=\{\alpha\}\int_1^N\frac{dt}{t}+\beta\int_1^N\frac{dt}{t^2}\leqslant\{\alpha\}\log N+\beta\leqslant c_0+\beta,$$

we obtain

$$\left|\sum_{j=0}^{k-1} S_j - x\sum_{n=1}^{N}\frac{1}{n}\right| \leqslant 1+c_0+\beta+\frac{1}{\beta}+x\left|\sum_{n=1}^{N}\frac{1}{n}-\log N\right| \leqslant 2+c_0+\beta+\frac{1}{\beta}.$$

Hence

$$D_N(\omega)\leqslant 2D_N(f(n))\leqslant 2\cdot\frac{c_0+2+\beta+1/\beta+e^{1/\beta}}{\sum_{n=1}^{N}\frac{1}{n}},$$

thus proving the theorem.

REMARK 1. The above arguments essentially reproduce the proof of Satz 3 in [7]. (We want to note that in the estimate for $D_N^*(P, x_n)$ the integral should be replaced by $\int_1^N p'(u)f'(u)\,du$.)

REMARK 2. By Theorem 2.1. we have

$$D_N\left(\frac{n}{N}+\beta\log n;\, n=1, \cdots, N\right)\leqslant 2\cdot\frac{3+\beta+1/\beta+e^{1/\beta}}{\log N},$$

since $\log N/N\leqslant 1/e<1=c_0$.

In order to prove that $D_N(\omega)\leqslant c(\beta)/\log N$ uniformly for all α it would suffice to prove this for all rational α's. This follows immediately from

$$|D_N(\omega)-D_N(\omega')|\leqslant\varepsilon$$

where $\omega=(x_n)$, $\omega'=(x_n')$ such that $|x_n-x_n'|\leqslant\varepsilon$ (special case of [5], Satz 6). Unfortunately we did not succeed in establishing the existence of such a bound $c(\beta)$ and we must content ourselves with

THEOREM 2.2. *Let* $\alpha=p/q$ $(0<p<q;\ p, q$ *integers*) *and* $\beta>0$. *Then for* $\omega=(x_n)=((p/q)n+\beta\log n)$

$$\left|\sum_{n=q}^{N}\frac{1}{n}\chi(x;x_n)-x\sum_{n=q}^{N}\frac{1}{n}\right|\leqslant K(\beta)=\frac{2}{\beta}+2+3\frac{e^{2/\beta}}{e^{1/\beta}-1}\quad(0\leqslant x<1)$$

and

$$D_N(\omega)\leqq\frac{1+K(\beta)+\log q}{\log N}.$$

PROOF. Put $b=e^{1/\beta}>1$ and let l, k be the largest integers such that $b^l\leqslant q$, $b^k\leqslant N$, respectively. Then

$$\sum_{n=q}^{N}\frac{1}{n}\chi(x;x_n)=-\sum_{b^l\leqslant n<q}\frac{1}{n}\chi(x;x_n)+\sum_{j=l}^{k-1}\sum_{b^j\leqslant n<b^{j+1}}\frac{1}{n}\chi(x;x_n)+\sum_{b^k\leqslant n\leqslant N}\frac{1}{n}\chi(x;x_n)$$

$$=-\mathrm{I}+\mathrm{II}+\mathrm{III}.$$

We have for the first and the third term

$$\mathrm{I}\leqslant \sum_{b^l\leqslant n<b^{l+1}} \frac{1}{n}\leqslant \log b+1,\quad \mathrm{III}\leqslant \sum_{b^k\leqslant n<b^{k+1}} \frac{1}{n}\leqslant \log b+1.$$

For the remaining term we proceed as follows:

$$\mathrm{II}=\sum_{j=l}^{k-1}\sum_{r=0}^{q-1}\sum_{b^j\leqslant mq+r<b^{j+1}} \frac{1}{mq+r}\chi\left(x;\frac{rp}{q}+\log_b(mq+r)\right)$$

$$=\sum_{j=l}^{k-1}\sum_{\substack{\left\{\frac{rp}{q}\right\}<x\\ 0\leq r<q}}\left(\sum_{b^j\leqslant mq+r<b^{j-\left\{\frac{rp}{q}\right\}+x}}\frac{1}{mq+r}+\sum_{b^{j+1-\left\{\frac{rp}{q}\right\}}\leqslant mq+r<b^{j+1}}\frac{1}{mq+r}\right)$$

$$+\sum_{j=l}^{k-1}\sum_{\substack{\left\{\frac{rp}{q}\right\}\geq x\\ 0\leq r<q}}\sum_{b^{j+1-\left\{\frac{rp}{q}\right\}}\leqslant mq+r<b^{j+1+x-\left\{\frac{rp}{q}\right\}}}\frac{1}{mq+r}$$

$$=\sum_{j=l}^{k-1}\sum_{\substack{\left\{\frac{rp}{q}\right\}<x\\ 0\leq r<q}}\left(\frac{1}{q}\log b^x+\frac{\theta_{jr}+\theta'_{jr}}{b^j}\right)+\sum_{j=l}^{k-1}\sum_{\substack{\left\{\frac{rp}{q}\right\}\geq x\\ 0\leq r<q}}\left(\frac{1}{q}\log b^x+\frac{\theta''_{jr}}{b^j}\right)$$

$$=(k-l)x\log b+\frac{3\theta' q}{1-\frac{1}{b}}\frac{1}{b^l}=x(\log N-\log q)+\theta''\log b+\frac{3\theta'' b^2}{b-1},$$

where all θ's are non specified numbers with $|\theta|\leqslant 1$. Combining the estimates for I, II and III we obtain

$$\sum_{n=q}^{N}\frac{1}{n}\chi(x;x_n)=x(\log N-\log q)+\theta'\left(2\log b+1+\frac{3b^2}{b-1}\right)$$

$$=x\sum_{n=q}^{N}\frac{1}{n}+\theta\left(2\log b+2+\frac{3b^2}{b-1}\right)\quad (|\theta|\leqslant 1).$$

To prove the second part of the theorem we note that

$$\left|\sum_{n=1}^{N}\frac{1}{n}\chi(x;x_n)-x\sum_{n=1}^{N}\frac{1}{n}\right|\leqslant\left|\sum_{n=1}^{q-1}\frac{1}{n}\chi(x;x_n)-x\sum_{n=1}^{q-1}\frac{1}{n}\right|+\left|\sum_{n=q}^{N}\frac{1}{n}\chi(x;x_n)-x\sum_{n=q}^{N}\frac{1}{n}\right|$$

$$\leqslant 1+\log q+2\log b+2+\frac{3b^2}{b-1}$$

(for $N\geqslant q$, whereas the result is trivial for $N<q$).

REMARK 1. Taking $c_0=\{\alpha\}\log q$ in the last line of the proof of Theorem 2.1. we have

$$\sum_{n=1}^{q}\frac{1}{n}D_q(\omega)\leqslant 2\left(\{\alpha\}\log q+2+\beta+\frac{1}{\beta}+e^{1/\beta}\right).$$

Hence, instead of taking the trivial bound $1+\log q$, we may write

$$\left|\sum_{n=1}^{q-1}\frac{1}{n}\chi(x;x_n)-x\sum_{n=1}^{q-1}\frac{1}{n}\right|\leq\frac{1}{q}+\sum_{n=1}^{q}\frac{1}{n}D_q(\omega)\leq 1+2\left(\{\alpha\}\log q+2+\beta+\frac{1}{\beta}+e^{1/\beta}\right).$$

This gives $D_N(\omega)\leq 2(\{\alpha\}\log q+(3+\beta+1/\beta+K(\beta)+e^{1/\beta}))/\log N$. As a special case we obtain a uniform bound for all $\alpha=p/q$ with $0<p/q\leq 1/\log q$.

REMARK 2. Since

$$\left|\sum_{n=1}^{N}\frac{1}{n}\chi(x;x_n)-x\sum_{n=1}^{N}\frac{1}{n}\right|=\left|\sum_{n=1}^{N}\frac{1}{n}(\chi(x;x_n)-x)\right|\leq\left|\frac{1}{N}\sum_{n=1}^{N}(\chi(x;x_n)-x)\right|$$
$$+\sum_{n=1}^{N-1}\frac{1}{n+1}\left|\frac{1}{n}\sum_{j=1}^{n}\chi(x;x_j)-x\right|\leq\sum_{n=1}^{N-1}\frac{1}{n+1}D_n^*(\omega)+1,$$

we have

$$D_N(\omega)\leq\frac{1}{\log N}\left(\sum_{n=1}^{N-1}\frac{1}{n+1}D_n^*(\omega)+1\right), \tag{1}$$

where D_n^* denotes the usual discrepancy with respect to the arithmetic mean. In the following let α be an irrational number of finite approximation type $\eta\geq 1$, i.e. for all $\varepsilon>0$ there is a constant $c(\alpha,\varepsilon)$ such that

$$\|q\alpha\|\geq\frac{c(\alpha,\varepsilon)}{q^{\eta+\varepsilon}}$$

for all integers $q\geq 1$. For such α's rather good estimates of $D_N^*(\alpha n)$ are known. In order to utilize these we relate $D_N^*(\omega)$ to $D_N^*(\alpha n)$. For every positive integer h we have

$$\left|\sum_{n=1}^{N}e^{2\pi ih(\alpha n+\beta\log n)}\right|=\left|\sum_{n=1}^{N}e^{2\pi ih\alpha n}\cdot e^{2\pi ih\beta\log n}\right|\leq\left|\sum_{n=1}^{N}e^{2\pi ih\alpha n}\right|$$
$$+\sum_{n=1}^{N-1}2\pi h\beta(\log(n+1)-\log n)\left|\sum_{j=1}^{n}e^{2\pi ih\alpha j}\right|$$
$$\leq\frac{1}{2\|h\alpha\|}(1+2\pi\beta h\log N),$$

since $\left|\sum_{j=1}^{n}e^{2\pi ih\alpha j}\right|\leq\frac{1}{|\sin\pi h\alpha|}\leq\frac{1}{2\|h\alpha\|}$.

Now we use the inequality of Erdös and Turan (cf. [3], p. 112) together with the estimates

$$\sum_{h=1}^{m}\frac{1}{\|h\alpha\|}\leq c'(\alpha,\varepsilon)m^{\eta+\varepsilon},\quad\sum_{h=1}^{m}\frac{1}{h\|h\alpha\|}\leq c''(\alpha,\varepsilon)\cdot m^{\eta-1+\varepsilon}$$

(cf. [3], p. 123) to obtain

$$D_N^*(\omega)\leqslant c_0\left(\frac{1}{m}+\sum_{h=1}^{m}\frac{1}{h}\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ih(\alpha n+\beta\log n)}\right|\right)$$

$$\leqslant c_0\left(\frac{1}{m}+\frac{1}{N}\sum_{h=1}^{m}\frac{1}{2h\|h\alpha\|}+\frac{\log N}{N}\sum_{h=1}^{m}\frac{\pi\beta}{\|h\alpha\|}\right)$$

$$\leqslant c_1(\alpha,\varepsilon,\beta)\left(\frac{1}{m}+\frac{m^{\eta-1+\varepsilon}}{N}+\frac{\log N}{N}m^{\eta+\varepsilon}\right).$$

Choosing $m=[N^{\frac{1}{\eta+1}}]$ we obtain for every $\varepsilon>0$: $D_N^*(\omega)\leqslant c_2(\alpha,\varepsilon,\beta)N^{-\frac{1}{\eta+1}+\varepsilon}$. Inserting this in (1) yields

$$D_N(\omega)\leqslant\frac{c_3(\alpha,\beta)}{\log N} \tag{2}$$

provided that α is of finite approximation type.

The argument used in the proof of Theorem 2.1. may be refined to give the following result:

THEOREM 2.3. *Assume f to be a twice continuously differentiable (real-valued) function defined on* $[1,\infty)$ *such that*

$$f(1)=0,\ 0<c_0\leqslant f'(u)\leqslant c_1,\quad |f''(u)|\leqslant\frac{c_2}{u}\ (u\geqslant 1).$$

Then for $0\leqslant x<1$

$$\sum_{n=1}^{N}\frac{1}{n}\chi(x;f(n))=x\sum_{n=1}^{N}\frac{1}{n}+\sum_{j=0}^{[f^{-1}(N)]}\frac{\{-f^{-1}(j+x)\}-\{-f^{-1}(j)\}}{f^{-1}(j)}+O(1),$$

where the O-constant only depends on c_0, c_1, c_2.

PROOF. Let k be the largest integer such that $\lceil f^{-1}(k)\rceil\leqslant N$. Then

$$\sum_{n=1}^{N}\frac{1}{n}\chi(x;f(n))=\sum_{j=0}^{k-1}\sum_{\lceil f^{-1}(j)\rceil\leqslant n<\lceil f^{-1}(j+1)\rceil}\frac{1}{n}\chi(x;f(n))+\sum_{\lceil f^{-1}(k)\rceil\leqslant n\leqslant N}\frac{1}{n}\chi(x;f(n)).$$

The second sum can be estimated in the following way:

$$\sum_{\lceil f^{-1}(k)\rceil\leqslant n\leqslant N}\frac{1}{n}\chi(x;f(n))\leqslant\frac{N-f^{-1}(k)+1}{f^{-1}(k)}<\frac{f^{-1}(k+1)-f^{-1}(k)+1}{f^{-1}(k)}$$

$$\leqslant\frac{1}{f^{-1}(k)f'(f^{-1}(\xi_k))}+\frac{1}{f^{-1}(k)}\leqslant\frac{1}{c_0}+1.$$

Applying the formula

$$\sum_{l=1}^{m}\frac{1}{l}=\gamma+\log m+\frac{1}{2m}+O\left(\frac{1}{m^2}\right)$$

we obtain

$$\sum_{\lceil f^{-1}(j)\rceil \leqslant n< \lceil f^{-1}(j+1)\rceil} \frac{1}{n}\chi(x;f(n))=\sum_{\lceil f^{-1}(j)\rceil \leqslant n\leqslant \lfloor f^{-1}(j+x)\rfloor} \frac{1}{n}$$

$$=\frac{1}{\lceil f^{-1}(j)\rceil}+\log \lfloor f^{-1}(j+x)\rfloor -\log \lceil f^{-1}(j)\rceil$$

$$+\frac{1}{2\lfloor f^{-1}(j+x)\rfloor}-\frac{1}{2\lceil f^{-1}(j)\rceil}+O\left(\frac{1}{\lceil f^{-1}(j)\rceil^2}\right)$$

where the O–constants are absolute noes. We note, that the above formula is valid even if $\lfloor f^{-1}(j+x)\rfloor < \lceil f^{-1}(j)\rceil$. Next we observe that

$$\sum_{j=O}^{\infty}\frac{1}{\lfloor f^{-1}(j)\rfloor^2}=O(1). \tag{3}$$

Hence

$$\sum_{n=1}^{N}\frac{1}{n}\chi(x;f(n))=\sum_{j=0}^{k-1}(\log f^{-1}(j+x)-\log f^{-1}(j)$$

$$+\frac{\lceil f^{-1}(j+x)\rceil -f^{-1}(j+x)}{f^{-1}(j)}-\frac{\lceil f^{-1}(j)\rceil -f^{-1}(j)}{f^{-1}(j)}\Bigg)+O(1),$$

using Taylor's theorem, formula (3) and $\lceil t\rceil = \lfloor t\rfloor +1$. The O–constants depend on c_0, c_1, c_2. Since $\lceil t\rceil -t=-[-t]-t=\{-t\}$, the result may be written as

$$\sum_{j=0}^{k-1}(\log f^{-1}(j+x)-\log f^{-1}(j))+\sum_{j=0}^{[f^{-1}(N)]}\frac{\{-f^{-1}(j+x)\}-\{-f^{-1}(j)\}}{f^{-1}(j)}+O(1).$$

By Taylor's theorem,

$$\log f^{-1}(j+x)-\log f^{-1}(j)$$

$$=x\frac{1}{f^{-1}(j)f'(f^{-1}(j))}+\frac{x^2}{2}\left(-\frac{f^{-1}(\xi)f''(f^{-1}(\xi))+f'(f^{-1}(\xi))}{f^{-1}(\xi)^2f'(f^{-1}(\xi))^3}\right)$$

(for some ξ with $j\leqslant\xi\leqslant j+x$). The absolute value of the last expression is bounded above by

$$\frac{f^{-1}(\xi)\,|f''(f^{-1}(\xi))|+c_1}{f^{-1}(\xi)^2c_0^3}\leqslant\frac{c_2+c_1}{f^{-1}(j)^2c_0^3}=O\left(\frac{1}{f^{-1}(j)^2}\right).$$

Hence (applying (3)) we obtain by Euler's summation formula

$$\sum_{j=0}^{k-1}(\log f^{-1}(j+x)-\log f^{-1}(j))=x\int_1^{k-1}\frac{dt}{f^{-1}(t)f'(f^{-1}(t))}+O(1).$$

The proof of the theorem is now completed by observing that

$$\int_1^{k-1} \frac{dt}{f^{-1}(t)f'(f^{-1}(t))}=\log f^{-1}(k-1)-\log f^{-1}(1)=\sum_{n=1}^{N}\frac{1}{n}+O(1).$$

REMARK. The above theorem may be applied to the sequence $\omega=(\alpha n+\beta \log n)(\alpha, \beta\geqslant 0)$; we just have to consider $f(u)=(1+\{\alpha\})u+\beta \log u-(1+\{\alpha\})$. The existence of a bound $D_N(\omega)\leqslant c'(\beta)/\log N$ (uniformly in α) is equivalent to

$$\left|\sum_{j=0}^{M}\frac{\{-f^{-1}(j+x)\}-\{-f^{-1}(j)\}}{f^{-1}(j)}\right|\leqslant c(\beta) \quad \text{for all } M, \alpha \text{ and } x.$$

3. Exponential Sums

In the following we give a refinement of [2], Satz 1 in the case of the logarithmic mean.

THEOREM 3.1. *For reals α, β ($\beta\neq 0$) and positive integers h we have*

$$\left|\sum_{n=A}^{B}\frac{1}{n}\,e^{2\pi ih(\alpha n+\beta \log n)}\right|\leqslant C(\beta)\left(\frac{\sqrt{h}}{A}+\frac{1}{\sqrt{h}}\right),$$

and

$$\left|\sum_{n=1}^{N}\frac{1}{n}\,e^{2\pi ih(\alpha n+\beta \log n)}\right|\leqslant \log h+1+2C(\beta),$$

where A, B and N denote positive integers and $C(\beta)$ is a constant depending continuously on β.

PROOF. We begin by showing how to deduce the second formula form the first.

$$\left|\sum_{n=1}^{N}\frac{1}{n}\,e^{2\pi ih(\alpha n+\beta \log n)}\right|\leqslant \sum_{n=1}^{h}\frac{1}{n}+\left|\sum_{n=h+1}^{N}\frac{1}{n}\,e^{2\pi ih(\alpha n+\beta \log n)}\right|$$
$$\leqslant 1+\log h+C(\beta)\left(\frac{\sqrt{h}}{h+1}+\frac{1}{\sqrt{h}}\right)\leqslant 1+\log h+2C(\beta).$$

For the main part of the proof we require the following lemma:

LEMMA 1. *Let $A<B$ be positive reals. Then for arbitrary reals α, $\beta (\beta\neq 0)$ we have*

$$\left|\int_A^B \frac{1}{u}\,e^{2\pi ih(\alpha u+\beta \log u)}du\right|\leqslant \left(1+\frac{1}{\sqrt{\pi|\beta|}}\right)\frac{16}{\sqrt{\pi h|\mathrm{b}|}},$$
$$\left|\int_A^B \frac{1}{u^2}\,e^{2\pi ih(\alpha u+\beta \log u)}du\right|\leqslant \left(1+\frac{\sqrt{2}}{\sqrt{\pi|\beta|}}\right)\frac{32}{A\sqrt{\pi h|\beta|}}$$

for all positive integers h.

Proof of the Lemma. We may restrict ourselves to the case $\beta>0$; otherwise we may take the complex conjugate of the integral. If $\alpha u+\beta\neq 0$ for all u with $A\leq u\leq B$ then

$$\left|\int_A^B \frac{1}{u}\,e^{2\pi ih(\alpha u+\beta\log u)}du\right|=\left|\int_A^B \frac{(e^{2\pi ih(\alpha u+\beta\log u)})'}{2\pi ih(\alpha u+\beta)}\,du\right|.$$

By the second mean value theorem we obtain

$$\left|\int_A^B \frac{(e^{2\pi ih(\alpha u+\beta\log u)})'}{\alpha u+\beta}\,du\right|\leqslant\left|\int_A^B \frac{(\cos 2\pi h(\alpha u+\beta\log u))'}{\alpha u+\beta}\,du\right|$$
$$+\left|\int_A^B \frac{(\sin 2\pi h(\alpha u+\beta\log u))'du}{\alpha u+\beta}\right|\leqslant 2^3\max_{A\leqslant u\leqslant B}\frac{1}{|\alpha u+\beta|}.$$

Assume that $u_0=-\beta/\alpha\leqslant A$. Then for $0<\varepsilon<1$ we have

$$\left|\int_A^B \frac{1}{u}\,e^{2\pi ih(\alpha u+\beta\log u)}du\right|\leqslant\int_A^{(1+\varepsilon)A}\frac{du}{u}+\left|\int_{(1+\varepsilon)A}^{(1+\varepsilon)B}\frac{1}{u}\,e^{2\pi ih(\alpha u+\beta\log u)}du\right|+\int_B^{(1+\varepsilon)B}\frac{du}{u}$$
$$\leqslant 2\log(1+\varepsilon)+\frac{4}{\pi h\varepsilon\beta},$$

since

$$\max_{(1+\varepsilon)A\leqslant u\leqslant(1+\varepsilon)B}\frac{1}{|\alpha u+\beta|}\leqslant\frac{1}{|\alpha(1+\varepsilon)u_0+\beta|}=\frac{1}{\varepsilon\beta}.$$

Let us now assume that $B\leqslant u_0$. Then for $0<\varepsilon<1$ we have

$$\left|\int_A^B \frac{1}{u}\,e^{2\pi ih(\alpha u+\beta\log u)}du\right|\leqslant\int_{(1-\varepsilon)A}^{A}\frac{du}{u}+\left|\int_{(1-\varepsilon)A}^{(1-\varepsilon)B}\frac{1}{u}\,e^{2\pi ih(\alpha u+\beta\log u)}du\right|+\int_{(1+\varepsilon)B}^{B}\frac{du}{u}$$
$$\leqslant 2\log\frac{1}{1-\varepsilon}+\frac{4}{\pi h\varepsilon\beta},$$

since

$$\max_{(1-\varepsilon)A\leqslant u\leqslant(1-\varepsilon)B}\frac{1}{|\alpha u+\beta|}\leqslant\frac{1}{|\alpha(1+\varepsilon)u_0+\beta|}=\frac{1}{\varepsilon\beta}.$$

Hence for arbitrary A, B, ε with $0<A<B$, $0<\varepsilon<1$ we obtain

$$\left|\int_A^B \frac{1}{u}\,e^{2\pi ih(\alpha u+\beta\log u)}du\right|\leqslant 2\log\frac{1+\varepsilon}{1-\varepsilon}+\frac{8}{\pi h\varepsilon\beta}.$$

Choosing $\varepsilon=1/\sqrt{\pi h\beta}$ for $h\geqslant 4/\pi\beta$ we obtain the upper bound $8(\varepsilon+1/\pi h\varepsilon\beta)=16/\sqrt{\pi h\beta}$, since

$$\log\frac{1+\varepsilon}{1-\varepsilon}=\log\left(1+\frac{2}{1-\varepsilon}\right)\leqslant\frac{2\varepsilon}{1-\varepsilon}\leqslant 4\varepsilon\quad\text{for}\quad 0<\varepsilon\leqslant\frac{1}{2}.$$

For $1\leqslant h<4/\pi\beta$ we choose $\varepsilon=1/2$ and obtain the bound $8(1/2+2/\pi h\beta)$

$<8(2/\sqrt{\pi h\beta}+2/\pi\sqrt{h\beta})$. Thus the first part of the Lemma is proved.

For the second integral let us start again with the case $\alpha u+\beta\neq 0$ for all u with $A\leqslant u\leqslant B$. Then, as above, we obtain

$$\left|\int_A^B \frac{1}{u^2}\, e^{2\pi ih(\alpha u+\beta \log u)}du\right|\leqslant\frac{1}{2\pi h}\cdot 2^4 \max_{A\leqslant u\leqslant B}\frac{1}{|u(\alpha u+\beta)|},$$

since $1/u(\alpha u+\beta)$ consists of at most two monotone pieces.

If $u_0=-\beta/\alpha\leqslant A$, then for $0<\varepsilon<1$ we have

$$\left|\int_A^B \frac{1}{u^2}\, e^{2\pi ih(\alpha u+\beta \log u)}du\right|\leqslant \int_A^{(1+\varepsilon)A}\frac{du}{u^2}+\left|\int_{(1+\varepsilon)A}^{(1+\varepsilon)B}\frac{1}{u^2}\, e^{2\pi ih(\alpha u+\beta \log u)}du\right|+\int_B^{(1+\varepsilon)B}\frac{du}{u^2}$$

$$\leqslant\frac{\varepsilon}{1+\varepsilon}\cdot\frac{1}{A}+\frac{8}{(1+\varepsilon)A\pi h\varepsilon\beta}+\frac{\varepsilon}{1+\varepsilon}\cdot\frac{1}{B}\leqslant\frac{2}{(1+\varepsilon)A}\left(\varepsilon+\frac{4}{\pi h\varepsilon\beta}\right),$$

since

$$\max_{(1+\varepsilon)A\leqslant u\leqslant(1+\varepsilon)B}\frac{1}{|u(\alpha u+\beta)|}\leqslant\frac{1}{(1+\varepsilon)A}\cdot\frac{1}{|\alpha(1+\varepsilon)u_0+\beta|}=\frac{1}{(1+\varepsilon)A\varepsilon\beta}.$$

Similarly, for $B\leqslant u_0$ we obtain the bound $2/(1-\varepsilon)A\cdot(\varepsilon+4/\pi h\varepsilon\beta)$. Hence for arbitrary A, B, ε with $0<A<B$, $0<\varepsilon<1$ we may take the upper bound $4/(1-\varepsilon^2)A\cdot(\varepsilon+4/\pi h\varepsilon\beta)$. For $h\geqslant 8/\pi\beta$ we choose $\varepsilon=2/\sqrt{\pi h\beta}\leqslant 1/\sqrt{2}$ and obtain the upper bound $8/A\cdot 4/\sqrt{\pi h\beta}$; choosing $\varepsilon=1/\sqrt{2}$ for $h<8/\pi\beta$ gives the bound $8/A\cdot(1/\sqrt{2}+4\sqrt{2}/\pi h\beta)$. Since (for $1\leqslant h<8/\pi\beta$) $1+\sqrt{2/\pi\beta}>\sqrt{\pi h\beta}/4\sqrt{2}+\sqrt{2/\pi h\beta}=\sqrt{\pi h\beta}\,(1/\sqrt{2}+4\sqrt{2}/\pi h\beta)/4$, this yields the second inequality of the Lemma.

In the proof of the previous lemma we have shown the following estimates

LEMMA 2. *If* $\alpha u+\beta\neq 0$ *for* $0<A\leqslant u\leqslant B$ *then*

$$\left|\int_A^B \frac{1}{u}\, e^{2\pi ih(\alpha u+\beta \log u)}du\right|\leqslant\frac{4}{\pi h}\max_{A\leqslant u\leqslant B}\frac{1}{|\alpha u+\beta|}$$

$$\left|\int_A^B \frac{1}{u^2}\, e^{2\pi ih(\alpha u+\beta \log u)}du\right|\leqslant\frac{8}{A\pi h}\max_{A\leqslant u\leqslant B}\frac{1}{|\alpha u+\beta|}.$$

We continue now with the proof of the theorem. By Euler's summation formula

$$\sum_{n=A}^{B}\frac{1}{n}\, e^{2\pi ih(\alpha n+\beta \log n)}=\frac{\theta}{A}+\int_A^B\frac{1}{u}\, e^{2\pi ih(\alpha u+\beta \log u)}du$$

$$+\int_A^B\psi(u)\left(-\frac{1}{u^2}+\frac{1}{u}\,2\pi ih\left(\alpha+\frac{\beta}{u}\right)\right)\, e^{2\pi ih(\alpha u+\beta \log u)}du$$

for some complex number θ with $|\theta|\leqslant 1$. We may assume that $1\leqslant\alpha<2$ and $\beta>0$, since for $\beta<0$ we may take the complex conjugate of the exponential sum and the sum remains unchanged if we replace α $\beta\xi$ $\alpha+k$ for integral k. By Lemma 2

$$\left|\int_A^B \frac{1}{u}\, e^{2\pi i h(\alpha u+\beta \log u)} du\right| \leqslant \frac{4}{\pi h(\alpha+\beta)}.$$

The trivial estimate $|\psi(u)| \leqslant 1/2$ yields

$$\left|\int_A^B \frac{\psi(u)}{u^2}\, e^{2\pi i h(\alpha u+\beta \log u)} du\right| \leqslant \frac{1}{2A}.$$

Using the Fourier expansion of $\psi(u)$ we obtain

$$\int_A^B \psi(u)\frac{2\pi i h}{u}\left(\alpha+\frac{\beta}{u}\right) e^{2\pi i h(\alpha u+\beta \log u)} du$$

$$=\int_A^B \sum_{m\neq 0} \frac{2\pi i h}{m}\left(\frac{\alpha}{u}+\frac{\beta}{u^2}\right) e^{2\pi i m u}\, e^{2\pi i h(\alpha u+\beta \log u)} du$$

$$=\sum_{m\neq 0} \frac{2\pi i h}{m} I_m \quad \left(I_m=\int_A^B \left(\frac{\alpha}{u}+\frac{\beta}{u^2}\right) e^{2\pi i h\left(\left(\alpha+\frac{m}{h}\right)u+\beta \log u\right)} du\right).$$

Put

$$M_1=\left\{m \in \mathbf{Z}\colon m<-\left(1+h\left(\alpha+\frac{\beta}{A}\right)\right) \text{ or } m>-h\alpha+1,\ m\neq 0\right\},$$

$$M_2=\left\{m \in \mathbf{Z}\colon -\left(1+h\left(\alpha+\frac{\beta}{A}\right)\right) \leqslant m \leqslant -h\alpha+1\right\}.$$

For

$$m<-\left(1+h\left(\alpha+\frac{\beta}{A}\right)\right) \text{ we have } \alpha+\frac{m}{h}<-\frac{1}{h}-\frac{\beta}{A}<0$$

and $(\alpha+m/h)A+\beta<-A/h<0$. Hence for these m

$$\max_{A\leqslant u\leqslant B} \frac{1}{\left|\left(\alpha+\frac{m}{h}\right)u+\beta\right|}=\frac{1}{\left|\left(\alpha+\frac{m}{h}\right)A+\beta\right|}.$$

For $m>-h\alpha+1$ $(m\neq 0)$ we have $\alpha+m/h>0$ and

$$\max_{A\leqslant u\leqslant B} \frac{1}{\left|\left(\alpha+\frac{m}{h}\right)u+\beta\right|}=\frac{1}{\left(\alpha+\frac{m}{h}\right)A+\beta}.$$

Thus Lemma 2 gives

$$|I_m| \leqslant \left(\frac{4\alpha}{\pi h}+\frac{8\beta}{A\pi h}\right)\frac{1}{\left|\left(\alpha+\frac{m}{h}\right)A+\beta\right|} \leqslant \frac{4(\alpha+2\beta)}{A\pi\left|m+h\left(\alpha+\frac{\beta}{A}\right)\right|} \quad \text{for } m \in M_1.$$

For $m \in M_2$ we obtain

$$|I_m|\leqslant\frac{16}{\sqrt{\pi h\beta}}\left(1+\frac{\sqrt{2}}{\sqrt{\pi\beta}}\right)(\alpha+2\beta)$$

by Lemma 1. As card $(M_2)\leqslant 3+h\,\beta/A$, we derive

$$\left|\sum_{m\neq 0}\frac{2\pi ih}{m}I_m\right|\leqslant 2\pi h\left(\sum_{m\in M_1}\frac{4(\alpha+2\beta)}{\pi A\left|m+h\left(\alpha+\frac{\beta}{A}\right)\right||m|}+\frac{3+h\beta/A}{h\alpha/2}\frac{16}{\sqrt{\pi h\beta}}\right.$$
$$\left.\times\left(1+\frac{\sqrt{2}}{\sqrt{\pi\beta}}\right)(\alpha+2\beta)\right)$$

$$=\frac{8(\alpha+2\beta)}{A}\left(h\sum_{m\in M_1}\frac{1}{|m|\left|m+h\left(\alpha+\frac{\beta}{A}\right)\right|}+\frac{8(3+h\beta/A)A\pi}{\alpha\sqrt{h\beta\pi}}\left(1+\frac{2}{\pi\beta}\right)\right).$$

Put $\mu=h(\alpha+\beta/A)$, then

$$\sum_{m\in M_1}\frac{1}{|m|\,|m+\mu|}=\sum_{m>\mu+1}\frac{1}{m(m-\mu)}+\sum_{0<m<h\alpha-1}\frac{1}{m(\mu-m)}+\sum_{m>0}\frac{1}{m(m+\mu)}$$
$$\leqslant 2\sum_{m=1}^{\infty}\frac{1}{m(m+\mu)}+\sum_{0<m<h\alpha-1}\frac{1}{m(\mu-m)}.$$

We have $\mu>1$ (since $h,\ \alpha\geq 1$), and so

$$\sum_{m=1}^{\infty}\frac{1}{m(m+\mu)}\leqslant\sum_{m=1}^{\infty}\frac{1}{m(m+[\mu])}=\frac{1}{[\mu]}\sum_{m=1}^{[\mu]}\frac{1}{m}\leqslant 2\frac{1+\log\mu}{\mu}.$$

Since

$$\sum_{0<m<h\alpha-1}\frac{1}{m(\mu-m)}=\frac{1}{\mu}\sum_{0<m<h\alpha-1}\left(\frac{1}{\mu-m}+\frac{1}{m}\right)\leqslant\frac{2}{\mu}\sum_{0<m<[\mu]}\frac{1}{m}\leqslant 2\frac{1+\log\mu}{\mu},$$

we obtain

$$\sum_{m\in M_1}\frac{1}{|m|\,|m+\mu|}\leqslant 6\frac{1+\log\mu}{\mu}\leqslant 6\frac{1+\log(h(\alpha+\beta))}{h\alpha}.$$

Hence

$$\left|\sum_{m\neq 0}\frac{2\pi ih}{m}I_m\right|\leqslant\frac{8(\alpha+2\beta)}{A}\left(6h\frac{1+\log(h(\alpha+\beta)}{h\alpha}+\frac{8(3+h\beta/A)A\pi}{\alpha\sqrt{h\beta\pi}}\left(1+\frac{\sqrt{2}}{\sqrt{\pi\beta}}\right)\right),$$

and so

$$\left|\sum_{n=A}^{B}\frac{1}{n}e^{2\pi ih(\alpha n+\beta\log n)}\right|\leqslant\frac{1}{A}+\frac{4}{\pi h(\alpha+\beta)}+\frac{1}{2A}$$
$$+\frac{8(\alpha+2\beta)}{A}\left(6h\frac{1+\log(h(\alpha+\beta))}{h\alpha}+\frac{8(3+h\beta/A)A\pi}{\alpha\sqrt{h\beta\pi}}\left(1+\frac{\sqrt{2}}{\sqrt{\pi\beta}}\right)\right)\leqslant C(\beta)\cdot\left(\frac{\sqrt{h}}{A}+\frac{1}{\sqrt{h}}\right),$$

where $C(\beta)$ is a constant only depending on β. Thus the proof of the theorem is complete.

COROLLARY. *For reals α, $\beta(\beta\neq 0)$ we have*

$$D_N(\alpha n+\beta\log n)\leqslant C_1(\beta)\frac{(\log\log N)^2}{\log N}.$$

PROOF. We choose $m=[\log N]+1$ in the inequality of Erdös–Turan for the logarithmic mean (cf. [4], Th. 1):

$$D_N(x_n)\leqq 4\left(\frac{1}{m}+\sum_{h=1}^{m}\frac{1}{h}\left|\frac{1}{\sum_{n=1}^{N}\frac{1}{n}}\sum_{n=1}^{N}\frac{1}{n}e^{2\pi i h x_n}\right|\right).$$

A simple calculation yields the desired result.

Exponential sums for functions of a similar type as $\alpha n+\beta\log n$ can be related to the exponential sums considered above by the following theorem.

THEOREM 3.2. *Let $p(n)$ be positive weights and let (x_n), (a_n) be sequences of real numbers and assume that $|a_{n+1}-a_n|\leqslant c/n^{1+\delta}P(n)$ (with positive constants c and δ), where $P(n)=\sum_{k=1}^{n}p(k)$. Then we have for positive values h*

$$\left|\sum_{n=1}^{N}p(n)e^{2\pi i h(x_n+a_n)}\right|\leqslant 2P(m)+\left|\sum_{n=1}^{N}p(n)\,e^{2\pi i h x_n}\right|+2\pi c\sum_{n=m+1}^{N}\frac{1}{n^{1+\delta/2}}\left|\frac{1}{P(n)}\sum_{n=1}^{n}p(k)\,e^{2\pi i h x_k}\right|,$$

where $m=[h^{2/\delta}]+1$.

PROOF. Since $|e^{iu}-e^{iv}|\leqslant|u-v|$ we have

$$\left|\sum_{n=1}^{N}p(n)\,e^{2\pi i h(x_n+a_n)}\right|\leqslant\sum_{n=1}^{m}p(n)+\left|\sum_{n=m+1}^{N}\left(\sum_{k=1}^{n}p(k)\,e^{2\pi i h x_k}\right)(e^{2\pi i h a_n}-e^{2\pi i h a_{n+1}})\right.$$

$$\left.-\sum_{k=1}^{m}p(k)\,e^{2\pi i h x_k}\,e^{2\pi i h a_{m+1}}+\sum_{k=1}^{N}p(k)\,e^{2\pi i h x_k}\,e^{2\pi i h a_{N+1}}\right|$$

$$\leqslant 2\sum_{n=1}^{m}p(n)+\sum_{n=m+1}^{N}\left|\sum_{k=1}^{n}p(k)\,e^{2\pi i h x_k}\right|2\pi h\,|a_n-a_{n+1}|+\left|\sum_{n=1}^{N}p(n)\,e^{2\pi i h x_n}\right|$$

$$\leqslant 2\sum_{n=1}^{m}p(n)+\sum_{n=m+1}^{N}\left|\sum_{k=1}^{n}p(k)\,e^{2\pi i h x_k}\right|\frac{2\pi c\cdot n^{\delta/2}}{n^{1+\delta}P(n)}+\left|\sum_{n=1}^{N}p(n)\,e^{2\pi i h x_n}\right|,$$

thus proving the theorem.

Assuming $|a_{n+1}-a_n|\leqslant c\,p(n)/P(n)$ instead of $|a_{n+1}-a_n|\leqslant c/n^{1+\delta}P(n)$ we obtain (for $m=0$)

$$\left|\frac{1}{P(N)}\sum_{n=1}^{N}p(n)\,e^{2\pi i h(x_n+a_n)}\right|\leqslant\frac{2\pi hc}{P(N)}\sum_{n=1}^{N}p(n)\left|\frac{1}{P(n)}\sum_{k=1}^{n}p(k)\,e^{2\pi i h x_k}\right|+\left|\frac{1}{P(N)}\sum_{n=1}^{N}p(n)\,e^{2\pi i h x_n}\right|.$$

Hence $\lim_{N\to\infty} P(N)^{-1} \sum_{n=1}^{N} p(n)\, e^{2\pi i h x_n}=0$ implies $\lim_{N\to\infty} P(N)^{-1} \sum_{n=1}^{N} p(n)\, e^{2\pi i h(x_n+a_n)}=0$ provided that $\lim_{N=\infty} P(N)=\infty$. Thus Weyl's criterion (cf. [1], p. 55) gives

THEOREM 3.3. *Let $p(n)$ be positive weights with $\lim_{N\to\infty} \sum_{n=1}^{N} p(n)=\infty$. If (x_n) is uniformly distributed with respect to the weights $p(n)$ and if $|a_{n+1}-a_n|=O(p(n)/\sum_{k=1}^{n} p(k)$ then (x_n+a_n) is also uniformly distributed with respect to $p(n)$.*

COROLLARY. *For $\omega=(\alpha n+\beta \log n+a_n)$ with reals α, β $(\beta\neq 0)$ and $|a_{n+1}-a_n|\leqslant c/n^{1+\delta}$ (c, δ positive constants) we have*

$$D_N(\omega)\leqslant C(\beta, c, \delta)\frac{(\log\log N)^2}{\log N}$$

where D_N denotes the discrepancy with respect to the logarithmic mean. ($C(\beta, c, \delta)$ may be chosen to depend continuously on β.)

PROOF. From Theorem 3.1. and Theorem 3.2. we easily deduce (for $x_n=\alpha n+\beta \log n$)

$$\left|\sum_{n=1}^{N}\frac{1}{n}\, e^{2\pi i h(x_n+a_n)}\right|\leqslant c_1(\beta, c, \delta)(\log h+1).$$

As in the proof of the Corollary after Theorem 3.1., the assertion follows from the inequality of Erdös and Turan.

REMARK 1. In the special case of the arithmetic mean $p(n)=1$ the result of Theorem 3.3. can be found in [6].

REMARK 2. From Theorem 2.3. and the corollary after Theorem 3.1. it follows that (for $N\geqslant 2$)

$$\sum_{j=0}^{[f^{-1}(N)]}\frac{\{-f^{-1}(j+x)\}-\{-f^{-1}(j)\}}{f^{-1}(j)}\leqslant C(\beta)(\log\log N)^2,$$

where f^{-1} denotes the inverse function of $\alpha x+\beta\log x-\alpha$ $(1\leqslant\alpha<2, \beta>0)$ and $C(\beta)$ is a constant only depending on β. In the following we give an upper bound for $D_N(\omega^*)$, where $\omega^*=(f^{-1}(n))_{n=1}^{\infty}$. We define $\varepsilon(y)$ by

$$x=\frac{y}{\alpha}-\frac{\beta}{\alpha}\log\frac{y}{\alpha}+\varepsilon(y), \quad \alpha x+\beta\log x=y+\alpha.$$

Since $dy/dx=\alpha+\beta/x$, we obtain

$$\varepsilon'(y)=\frac{x}{\alpha x+\beta}-\frac{1}{\alpha}+\frac{\beta}{\alpha}\frac{1}{y}=\frac{\beta}{\alpha}\frac{\alpha x+\beta-y}{y(\alpha x+\beta)}=\frac{\beta}{\alpha}\frac{\beta(1-\log x)+\alpha}{y(\alpha x+\beta)};$$

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thus

$$|\varepsilon'(y)| \leqslant \frac{\beta}{\alpha} \frac{\beta \log x + \beta + \alpha}{\alpha x \cdot y} \leqslant \left(\frac{\beta}{\alpha}\right)^2 \frac{\log(y/\alpha+1)+1+\alpha/\beta}{(y+\alpha)y}(\alpha+\beta)$$

$$\leqslant \beta^2(1+\beta)\left(1+\frac{1}{\beta}\right)\frac{\log(y+1)+1}{y^2}$$

(where we have used the trivial estimates $\alpha x \leqslant y+\alpha < (\alpha+\beta)x$). By the mean value theorem we have

$$|\varepsilon(n+1)-\varepsilon(n)| \leqslant \beta(1+\beta)^2 \frac{\log(n+2)+1}{n^2} \leqslant \frac{3\beta(1+\beta)^2}{n^{3/2}}.$$

Applying the last corollary to the sequence $\omega^* = (n/\alpha - \beta/\alpha \cdot \log n + \beta/\alpha \cdot \log \alpha + \varepsilon(n))$ yields

$$D_N(\omega^*) \leqslant \max_{1 \leqslant \alpha \leqslant 2} C\left(-\frac{\beta}{\alpha}, 3\beta(1+\beta)^2, \frac{1}{2}\right)\frac{(\log\log N)^2}{\log N} = c_0(\beta)\frac{(\log\log N)^2}{\log N}.$$

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