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LINK GRAPHS OF TILED ORDERS OVER A LOCAL DEDEKIND DOMAIN

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Introduction

Let R be a local Dedekind domain with the maximal ideal πR and the quotient ring K, and let $\Lambda=(\Lambda_{ij})$ be a tiled R-order in $(K)_n$ (the full $n\times n$ matrix ring over K) between $(R)_n$ and $(\pi R)_n$ (i.e., $(R)_n\supset\Lambda\supset(\pi R)_n$). In [2, Theorem], we have obtained a procedure of determining the link graph of Λ from the quiver of the $R/\pi R$-algebra $A=\Lambda/(\pi R)_n$. As noted in [2, Remark (4)], there exist tiled R-orders Λ and Γ with the same link graph, but the quiver of $A=\Lambda/(\pi R)_n$ is different from that of $B=\Gamma/(\pi R)_n$. In this note, we shall clarify the relationship between such Λ and Γ, by proving the following

THEOREM. *Let Λ and Γ be basic tiled R-orders between $(R)_n$ and $(\pi R)_n$. Then the following statements are equivalent.*

(1) *Λ is isomorphic with Γ as rings.*

(2) *The link graphs of Λ and Γ are equal except for the numbering of the vertices.*

(3) *$\Gamma=u\Lambda u^{-1}$ for some regular element $u\in(R)_n$.*

It should be noted that the main part of the proof of the theorem follows from purely graph theoretic facts and that the result is surprisingly simple. After proving the main theorem, we shall consider the number of Λ_{ij}'s with $\Lambda_{ij}=\pi R$ and characterize hereditary basic tiled R-orders. Finally, we shall add some examples.

We shall use the same definitions and notations as in [2].

1. Some graph theoretic results

Let $\mathfrak{Q}=(\mathfrak{Q}_0, \mathfrak{Q}_1, d, r)$ be a quiver satisfying the conditions:

(A1) There is at most one arrow between any two vertices.

(A2) $\mathfrak{Q}$ has no loops and no oriented cycles.

Let $\mathcal{D}=\mathcal{D}(\mathfrak{Q})$ (resp. $\mathcal{R}=\mathcal{R}(\mathfrak{Q})$) denote the subset of $\mathfrak{Q}_0$ consisting of non-domains (resp. non-ranges) in $\mathfrak{Q}$. Since $\mathfrak{Q}$ has no loops and no oriented cycles, $\mathcal{D}\neq\phi$ and $\mathcal{R}\neq\phi$. Then, from $\mathfrak{Q}$, we can construct a new quiver $\tilde{\mathfrak{Q}}=(\tilde{\mathfrak{Q}}_0, \tilde{\mathfrak{Q}}_1, \tilde{d}, \tilde{r})$ satisfying (A1) as follows: $\tilde{\mathfrak{Q}}_0=\mathfrak{Q}_0$ and $\alpha\in\tilde{\mathfrak{Q}}_1$ if and only if $\alpha\in\mathfrak{Q}_1$ or else $\tilde{d}(\alpha)\in\mathcal{D}$ and $\tilde{r}(\alpha)\in\mathcal{R}$. We consider $\mathfrak{Q}$ as a

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subquiver of $\tilde{\mathcal{Q}}$.

It should be noted that there are quivers $\mathcal{Q}$, $\mathcal{Q}'$ satisfying (A1), (A2) and $\tilde{\mathcal{Q}}=\tilde{\mathcal{Q}}'$, but $\mathcal{Q}\neq\mathcal{Q}'$ (cf. [2, Example 2.3]).

Let $\mathcal{Q}'=(\mathcal{Q}_0', \mathcal{Q}_1', d', r')$ be another quiver satisfying (A1), (A2) and $\tilde{\mathcal{Q}}=\tilde{\mathcal{Q}}'$, and put $\mathcal{D}'=\mathcal{D}(\mathcal{Q}')$ and $\mathcal{R}'=\mathcal{R}(\mathcal{Q}')$. We shall study some relationship between $\mathcal{Q}$ and $\mathcal{Q}'$.

LEMMA 1. *Suppose that* $\mathcal{Q}\neq\mathcal{Q}'$. *Then,*

(1) $\mathcal{D}\cap\mathcal{D}'=\phi$ *and* $\mathcal{R}\cap\mathcal{R}'=\phi$.

(2) *If* $i\in\mathcal{D}$ *and* $j\in\mathcal{R}$, *then* $i\to j\in\mathcal{Q}_1'$ *and* $i\neq j$.

PROOF. (1) Assume that there is $i\in\mathcal{D}\cap\mathcal{D}'$. For each $j\in\mathcal{R}$, there is $i\to j\in\tilde{\mathcal{Q}}_1=\tilde{\mathcal{Q}}_1'$. Since $i\in\mathcal{D}'$, there are no arrows in $\mathcal{Q}_1'$ starting from i. Hence necessarily $j\in\mathcal{R}'$. So $\mathcal{R}\subset\mathcal{R}'$. Similary $\mathcal{R}'\subset\mathcal{R}$. Hence $\mathcal{R}=\mathcal{R}'$. In a similar way, we obtain $\mathcal{D}=\mathcal{D}'$. Thus we conclude that $\mathcal{Q}=\mathcal{Q}'$, a contradiction. Therefore $\mathcal{D}\cap\mathcal{D}'=\phi$. A similar argument shows $\mathcal{R}\cap\mathcal{R}'=\phi$.

(2) Let $i\in\mathcal{D}$ and $j\in\mathcal{R}$. Then, there is $i\to j$ in $\tilde{\mathcal{Q}}_1=\tilde{\mathcal{Q}}_1'$. If there is no $i\to j$ in $\mathcal{Q}_1'$, then necessarily $i\in\mathcal{D}'$ and $j\in\mathcal{R}'$ which contradict to (1). Therefore $i\to j\in\mathcal{Q}_1'$ and so $i\neq j$ by (A2).

LEMMA 2. *Let p be a path from j to i in $\tilde{\mathcal{Q}}=\tilde{\mathcal{Q}}'$ with $i\in\mathcal{D}$ and $j\in\mathcal{R}'$ (or $i\in\mathcal{D}'$ and $j\in\mathcal{R}$). Then p is in $\mathcal{Q}$ if and only if p is in $\mathcal{Q}'$.*

PROOF. Suppose that p is in $\mathcal{Q}$ and not in $\mathcal{Q}'$. Then there exists $\alpha\in\mathcal{Q}_1$ such that α appears on p, $d'(\alpha)\in\mathcal{D}'$ and $r'(\alpha)\in\mathcal{R}'$. When $i\in\mathcal{D}$ and $j\in\mathcal{R}'$, it follows from Lemma 1 (2) that there is an arrow $d'(\alpha)\to j\in\mathcal{Q}_1$. Then $\mathcal{Q}$ has an oriented cycle $j\to\cdots\to d'(\alpha)$, which contradicts to (A2). When $i\in\mathcal{D}'$ and $j\in\mathcal{R}$, there is an arrow $i\to r'(\alpha)\in\mathcal{Q}_1$ and so $r'(\alpha)\to\cdots\to i$ is in $\mathcal{Q}$, a contradiction. Therefore, in each case, if p is in $\mathcal{Q}$, then p is in $\mathcal{Q}'$. A similar argument shows the converse.

Put $\mathcal{S}=\{i\in\mathcal{Q}_0 \mid \text{there exist } b\in\mathcal{R}' \text{ and } a\in\mathcal{D} \text{ such that } i \text{ occurs in the path from } b \text{ to } a \text{ in } \mathcal{Q}\}$. Note that $i\in\mathcal{S}$ whenever $i\in\mathcal{R}'\cap\mathcal{D}$.

LEMMA 3. *Let p be a path from j to i in $\tilde{\mathcal{Q}}=\tilde{\mathcal{Q}}'$. Assume that either* (1) *$i\in\mathcal{S}$ and $j\in\mathcal{S}$ or* (2) *$i\notin\mathcal{S}$ and $j\notin\mathcal{S}$. Then p is in $\mathcal{Q}$ if and only if p is in $\mathcal{Q}'$.*

PROOF. (1) There exist $b_1, b_2\in\mathcal{R}'$ and $a_1, a_2\in\mathcal{D}$ such that i (resp. j) occurs in some path of $\mathcal{Q}$ (and $\mathcal{Q}'$, by Lemma 2) from b_1 (resp. b_2) to a_1 (resp. a_2). Thus we obtain a path $p': b_2\to\cdots\to j\to\overset{p}{\cdots}\to i\to\cdots\to a_1$ in $\tilde{\mathcal{Q}}$. Hence by Lemma 2, p is in $\mathcal{Q}\Leftrightarrow p'$ is in $\mathcal{Q}\Leftrightarrow p'$ is in $\mathcal{Q}'\Leftrightarrow p$ is in $\mathcal{Q}'$.

(2) Suppose that p is in $\mathcal{Q}'$ and not in $\mathcal{Q}$. Then there is $a\in\mathcal{D}$ which occurs in p. By (A2), we can take $b\in\mathcal{R}'$ such that there is a path $b\to\cdots\to j$ in $\mathcal{Q}'$ or $j=b$. Thus we obtain a path $b\to\cdots\to j\to\cdots\to a$ in $\mathcal{Q}'$ and so in $\mathcal{Q}$ by Lemma 2. Hence $j\in\mathcal{S}$, a contradiction. If p is in $\mathcal{Q}$ and not in $\mathcal{Q}'$, we can similarly deduce $i\in\mathcal{S}$, a contradiction. This completes the proof.

LEMMA 4. (1) *If* $i \in \mathcal{S}$ *and* $j \notin \mathcal{S}$, *then there is a path from* j *to* i *in* $\mathcal{Q}$ *and no paths from* j *to* i *in* $\mathcal{Q}'$.

(2) *If* $i \notin \mathcal{S}$ *and* $j \in \mathcal{S}$, *then there is a path from* j *to* i *in* $\mathcal{Q}'$ *and no paths from* j *to* i *in* $\mathcal{Q}$.

PROOF. (1) Since $i \in \mathcal{S}$, there is a path $b \to \cdots \to i \to \cdots \to a$ in $\mathcal{Q}$ with $b \in \mathcal{R}'$ and $a \in \mathcal{D}$. Take a path $j \to \cdots \to x$ in $\mathcal{Q}'$ with $x \in \mathcal{D}'$ (or $j = x$). Then $x \notin \mathcal{S}$ by Lemma 1 (2). Hence $j \to \cdots \to x$ is in $\mathcal{Q}$ from Lemma 3 (2). Since $x \in \mathcal{D}'$ and $b \in \mathcal{R}'$, $x \to b \in \mathcal{Q}_1$ from Lemma 1 (2). Thus we obtain a path $j \to \cdots \to x \to b \to \cdots \to i$ in $\mathcal{Q}$.

Assume that there is a path from j to i in $\mathcal{Q}'$. Take a path $y \to \cdots \to j$ in $\mathcal{Q}'$ with $y \in \mathcal{R}'$ (or $j = y$). Since $i \to \cdots \to a$ is in $\mathcal{Q}'$ by Lemma 2, we obtain a path $y \to \cdots \to j \to \cdots \to i \to \cdots \to a$ in $\mathcal{Q}'$ and so in $\mathcal{Q}$. Hence $j \in \mathcal{S}$, a contradiction.

(2) is proved by a similar way.

Though the following proposition deviates from the main subject, we state it here along the above results (cf. [2, Proposition 2.4]).

PROPOSITION 5. *Let* $\mathcal{Q}$ *and* $\mathcal{Q}'$ *be quivers satisfying* (A1), (A2) *and* $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}'$. *If* $\mathcal{Q}$ *is disconnected, then* $\mathcal{Q} = \mathcal{Q}'$.

PROOF. Let $\mathcal{U}$ and $\mathcal{V}$ be subquivers of $\mathcal{Q}$ such that $\mathcal{Q} = \mathcal{U} \dot{\cup} \mathcal{V}$ (i.e., there are no arrows between the vertices of $\mathcal{U}$ and those of $\mathcal{V}$). Since $\mathcal{U}$ and $\mathcal{V}$ also satisfy (A2), there are $i \in \mathcal{U}_0 \cap \mathcal{D}$ and $j \in \mathcal{V}_0 \cap \mathcal{R}$. Take $x \in \mathcal{R}'$ and $y \in \mathcal{D}'$ such that there are paths $p_1: x \to \cdots \to i$ (or $i = x$) and $p_2: j \to \cdots \to y$ (or $j = y$) in $\mathcal{Q}'$. It follows from Lemma 2 that p_1 and p_2 are in $\mathcal{Q}$. Hence $x \in \mathcal{U}_0$ and $y \in \mathcal{V}_0$. Since $y \in \mathcal{D}'$ and $x \in \mathcal{R}'$, it holds that $\mathcal{Q} = \mathcal{Q}'$ by Lemma 1 (2). This completes the proof.

2. Proof of the theorem

Since (1)$\Rightarrow$(2) and (3)$\Rightarrow$(1) are clear, we shall prove (2)$\Rightarrow$(3). Let $\mathcal{Q} = \mathcal{Q}(A)$ and $\mathcal{Q}' = \mathcal{Q}(B)$ be the quivers of $R/\pi R$-algebras $A = \Lambda/(\pi R)_n$ and $B = \Gamma/(\pi R)_n$ respectively. Since Λ and Γ are basic tiled R-orders between $(R)_n$ and $(\pi R)_n$, $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy the conditions (A1) and (A2). It follows from [2, Theorem] that the link graph of Λ is given by $\tilde{\mathcal{Q}}$. Using a permutation matrix in $(R)_n$ if necessary, we may assume $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}'$. Let $\mathcal{S}$ denote the set $\{i \in \mathcal{Q}_0 \mid i$ occurs in some path from b to a or $i = b = a$ for some $b \in \mathcal{R}'$ and $a \in \mathcal{D}\}$, and define

$$u = \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{pmatrix} \quad \text{where} \quad u_i = \begin{cases} 1 & \text{if } i \notin \mathcal{S} \\ \pi & \text{if } i \in \mathcal{S} \end{cases} \quad (1 \leqq i \leqq n).$$

Then, using the following lemma and Lemmas 3, 4, it is easily verified that $\Gamma = u \Lambda u^{-1}$.

LEMMA 6. *There is a path from* j *to* i *in* $\mathcal{Q}(A)$ *if and only if* $\Lambda_{ij} = R$.

PROOF. It is well-known that there is a path from j to i in $\mathfrak{Q}(A)$ iff $e_iAe_j\neq 0$. Since $\Lambda_{ij}=R$ iff $e_iAe_j\neq 0$, the lemma is proved.

One could give a direct proof of the lemma as an elementary excercise (cf. [2, Lemma 2.2]).

3. The number of Λ_{ij}'s with $\Lambda_{ij}=\pi R$

Let $\Lambda=(\Lambda_{ij})$ be a tiled R-order between $(R)_n$ and $(\pi R)_n$, and let $d(\Lambda)$ denote the number of Λ_{ij}'s with $\Lambda_{ij}=\pi R$.

COROLLARY 7. *Under the same assumption of the theorem, if Λ is isomorphic with Γ, then $d(\Lambda)=d(\Gamma)$.*

PROOF. This follows from the definition of u in the proof of the theorem.

Of course, the converse of the corollary does not hold (see Example 11), however, we should note the following proposition. A part of it follows from [3], but we shall give another proof in our context.

PROPOSITION 8. *Let $\Lambda=(\Lambda_{ij})$ be a basic tiled R-order between $(R)_n$ and $(\pi R)_n$, and put $A=\Lambda/(\pi R)_n$. Then the following statements are equivalent.*

(1) *Λ is hereditary.*
(2) *$\mathfrak{Q}(A)$ is given by $i_1\to i_2\to\cdots\to i_n$.*
(3) *If $i\neq j$, $\Lambda_{ij}=\mathrm{R}$ iff $\Lambda_{ji}=\pi R$.*
(4) *$d(\Lambda)=n(n-1)/2$.*

PROOF. (1)$\Rightarrow$(2) (cf. [3, Appendix, Theorem]) Since the link graph of Λ is connected (cf. [2, Remark (2)]), it follows from [5, IV] (or [4, Prop. 11]) and [1, §2] that the link graph of Λ is given by a cycle. Hence, by [2, Theorem], $\mathfrak{Q}(A)$ is given by $i_1\to i_2\to\cdots\to i_n$.

(2)$\Rightarrow$(1) This follows from [2, Example 3.3].

(2)$\Rightarrow$(3) This follows from Lemma 6.

(3)$\Rightarrow$(2) By Lemma 6, for any $i\neq j$, there is a path in $\mathfrak{Q}(A)$ from i to j or else j to i. Hence $\mathfrak{Q}(A)$ is connected. If there were arrows $j_1\to i\leftarrow j_2$ $(j_1\neq j_2)$ in $\mathfrak{Q}(A)_1$, then there would be a path, e.g., $j_1\to\cdots\to j_2$ in $\mathfrak{Q}(A)$, which would contradict to $j_1\to i\in\mathfrak{Q}(A)_1$. Therefore $\mathfrak{Q}(A)$ is given by $i_1\to i_2\to\cdots\to i_n$.

Since Λ is basic, if $i\neq j$ and $\Lambda_{ij}=R$, then $\Lambda_{ji}=\pi R$. Hence $d(\Lambda)\geqq n(n-1)/2$. Hence (3)$\Longleftrightarrow$(4) holds.

4. Examples

We shall list some examples to which we shall apply the above results.

EXAMPLE 9. Let

$$\Lambda=\begin{bmatrix} R & R & \pi R & \pi R & \pi R \\ \pi R & R & \pi R & \pi R & \pi R \\ R & R & R & R & \pi R \\ \pi R & R & \pi R & R & \pi R \\ R & R & \pi R & R & R \end{bmatrix} \quad\text{and}\quad \Gamma=\begin{bmatrix} R & \pi R & \pi R & \pi R & \pi R \\ \pi R & R & \pi R & \pi R & \pi R \\ R & R & R & \pi R & \pi R \\ R & R & R & R & \pi R \\ R & R & R & \pi R & R \end{bmatrix}.$$

Then the quiver $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are given by respectively

$$\begin{tikzcd} 3 & & 4 \arrow[ll] \arrow[dd] \\ & 2 \arrow[ur] \arrow[dl] & \\ 1 \arrow[uu] \arrow[rr] & & 5 \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} 1 \arrow[dr] & & 4 \\ & 3 \arrow[ur] \arrow[dl] & \\ 5 & & 2 \arrow[ul] \end{tikzcd} .$$

Hence, by [2, Theorem], the link graphs of Λ and Γ are given by respectively

$$\begin{tikzcd} 3 \arrow[dr] & & 4 \arrow[ll] \arrow[dd] \\ & 2 \arrow[ur] \arrow[dl] & \\ 1 \arrow[uu] \arrow[rr] & & 5 \arrow[ul] \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} 1 \arrow[dr] & & 4 \arrow[ll] \arrow[dd] \\ & 3 \arrow[ur] \arrow[dl] & \\ 5 \arrow[uu] \arrow[rr] & & 2 \arrow[ul] \end{tikzcd} .$$

Put $u'=\begin{bmatrix} 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \end{bmatrix}$ and $\Lambda'=u'\Lambda u'^{-1}$. Then Λ' and Γ have the same link graph

and $S=\{3, 4, 5\}$. So put

$$u''=\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \pi & & \\ & & & \pi & \\ 0 & & & & \pi \end{bmatrix} \quad\text{and}\quad u=u'u''=\begin{bmatrix} 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&\pi&0&0&0 \\ 0&0&0&\pi&0 \\ \pi&0&0&0&0 \end{bmatrix}.$$

Then $\Gamma=u\Lambda u^{-1}$.

There are tiled R-orders with the same link graph, but they are not isomorphic.

EXAMPLE 10. Let

$$\Lambda=\begin{pmatrix} R & \pi R & \pi R \\ \pi R & R & \pi R \\ \pi R & R & R \end{pmatrix} \quad \text{and} \quad \Gamma=\begin{pmatrix} R & \pi^2 R & \pi^2 R \\ \pi^2 R & R & \pi R \\ \pi^2 R & R & R \end{pmatrix}.$$

Then the link graphs of Λ and Γ are given by

$$\begin{tikzcd} & 1 \arrow[loop above] \arrow[dl] & \\ 2 \arrow[rr] & & 3. \arrow[ul] \arrow[ll, bend left] \end{tikzcd}$$

But Λ is not isomorphic with Γ.

EXAMPLE 11. Let

$$\Lambda_1=\begin{bmatrix} R & \pi R & \pi R & \pi R \\ R & R & \pi R & \pi R \\ R & \pi R & R & \pi R \\ R & \pi R & \pi R & R \end{bmatrix}, \quad \Lambda_2=\begin{bmatrix} R & \pi R & \pi R & \pi R \\ \pi R & R & \pi R & \pi R \\ R & R & R & \pi R \\ \pi R & R & \pi R & R \end{bmatrix} \quad \text{and} \quad \Lambda_3=\begin{bmatrix} R & \pi R & \pi R & \pi R \\ R & R & \pi R & \pi R \\ R & R & R & \pi R \\ \pi R & \pi R & \pi R & R \end{bmatrix}.$$

Then for each $i=1, 2, 3$, $d(\Lambda_i)=9$ and the quiver $Q(A_i)$ where $A_i=\Lambda_i/(\pi R)_4$ is given by respectively

$$\begin{tikzcd} 2 & & 3 \\ & 1 \arrow[ul] \arrow[ur] \arrow[d] & \\ & 4 & \end{tikzcd} \quad , \quad 1\to 3\leftarrow 2\to 4 \quad \text{and} \quad 1\to 2\to 3 \quad 4.$$

Moreover gl.dim $\Lambda_1=2$, gl.dim $\Lambda_2=3$ and gl.dim $\Lambda_3=\infty$(cf.[2, §3]). Hence these are mutually non-isomorphic.

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