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SOME CHARACTERIZATIONS OF A *B*-PROPERTY

By

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A topological space X has a ***B*-property** (P. Zenor[14]) if, for any monotone increasing open covering $\{U_\alpha | \alpha < \tau\}$ of X, there exists a monotone increasing open covering $\{V_\alpha | \alpha < \tau\}$ of X such that $\mathrm{cl}(V_\alpha) \subset U_\alpha$ for each $\alpha < \tau$, where $\mathrm{cl}(V_\alpha)$ denotes the closure of V_α.

B-property is weaker than the paracompactness and stronger than the countable paracompactness (M. E. Rudin [8] and [9]). So far as I know, P. Zenor was the first mathematician to introduce it as the property which characterizes the Lindelöfness of the separable regular T_1 spaces. Before now the various properties of it and its neighborhood were seen by F. Ishikawa [4], K. Chiba [2], M. E. Rudin [8], [9] and others ([6], [10], [11] and [13] etc.).

The purpose of this paper is to have some characterizations of the **B**-property and their applications. In this paper, the spaces are assumed to be regular.

THEOREM 1 *Let X be a topological space. Then the following properties are equivalent:*

(1) *X has a **B**-property.*

(2) *For any monotone increasing open covering $\{U_\alpha | \alpha < \tau\}$ of X, there exists an open covering $\{V_\alpha | \alpha < \tau\}$ of X such that*

(2-1) $V_\alpha \subset U_\alpha$ *for each* $\alpha < \tau$.

(2-2) *For each $x \in X$, there exist an open nbd (=neighborhood) 0 of x and $\alpha_0 < \tau$ such that* $0 \cap (\cup \{V_\alpha | \alpha \geq \alpha_0\}) = \phi$.

(3) *For any monotone increasing open covering $\{U_\alpha | \alpha < \tau\}$ of X, there exists an open covering $\{V_\alpha | \alpha < \tau\}$ of X such that*

(3-1) $\mathrm{cl}(V_\alpha) \subset U_\alpha$ *for each* $\alpha < \tau$.

(3-2) *For each $x \in X$, there exist an open nbd 0 of x and $\alpha_0 < \tau$ such that* $0 \cap (\cup \{V_\alpha | \alpha \geq \alpha_0\}) = \phi$.

PROOF *(1)→(3)*: Let $\{U_\alpha | \alpha < \tau\}$ be any monotone increasing open covering of X. Then we have two monotone increasing open coverings $\{T_\alpha | \alpha < \tau\}$ and $\{S_\alpha | \alpha < \tau\}$ of X such that

$$\mathrm{cl}(S_\alpha) \subset T_\alpha \subset \mathrm{cl}(T_\alpha) \subset U_\alpha \quad \text{for each} \quad \alpha < \tau.$$

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Without loss of generality, we may assume that

$$(*) \quad T_\alpha = \cup \{T_\beta \mid \beta < \alpha\}$$

for any limit ordinal $\alpha < \tau$.

Let

$$V_\alpha = \begin{cases} T_\alpha - \mathrm{cl}\,(S_{\alpha-1}) & \text{if } \alpha \text{ is non-limit} \\ \phi & \text{if } \alpha \text{ is limit} \end{cases}$$

for each ordinal $\alpha < \tau$. If we let x be any point of X, and α_0 be the first ordinal of $\{\alpha < \tau \mid x \in T_\alpha\}$, then α_0 is a non-limit ordinal by (*), and so $x \notin T_{\alpha_0 - 1} \supset \mathrm{cl}\,(S_{\alpha_0 - 1})$. Therefore we have $x \in V_{\alpha_0}$. Hence $\{V_\alpha \mid \alpha < \tau\}$ is an open covering of X.

To show that $\{V_\alpha \mid \alpha < \tau\}$ satisfies (3–2), let x be any point of X. Since $\{S_\alpha \mid \alpha < \tau\}$ is a covering of X, there exists some $\alpha_0 < \tau$ with $x \in S_{\alpha_0}$. Then, for any non-limit ordinal α with $\tau > \alpha > \alpha_0$, we have

$$S_{\alpha_0} \cap V_\alpha \subset S_{\alpha_0} - \mathrm{cl}\,(S_{\alpha-1}) \subset S_{\alpha_0} - \mathrm{cl}\,(S_{\alpha_0}) = \phi.$$

(3)→(2): Trivial.

(2)→(1): Let $\{U_\alpha \mid \alpha < \tau\}$ be any monotone increasing open covering of X. Then there is an open covering $\{V_\alpha \mid \alpha < \tau\}$ of X which satisfies (2–1) and (2–2). For each $\alpha < \tau$, we let

$$T_\alpha = \cup \{0 \mid 0\text{: open in X and } 0 \cap (\cup \{V_\beta \mid \beta \geq \alpha\}) = \phi\}.$$

It is trivial that $\{T_\alpha \mid \alpha < \tau\}$ is a monotone increasing open covering of X. For each $\alpha < \tau$, $T_\alpha \cap (\cup \{V_\beta \mid \beta \geq \alpha\}) = \phi$ and so $\mathrm{cl}\,(T_\alpha) \cap (\cup \{V_\beta \mid \beta \geq \alpha\}) = \phi$. Therefore $\mathrm{cl}\,(T_\alpha) \subset X - \cup \{V_\beta \mid \beta \geq \alpha\} \subset \cup \{V_\beta \mid \beta < \alpha\} \subset \cup \{U_\beta \mid \beta < \alpha\} \subset U_\alpha$.

As far as I know, there is no characterizations of a ***B***–property in the form that: A topological space X has a ***B***–property if and only if every open covering of X has a property P.

Then we have the following theorem:

THEOREM 2 *A topological space X has a **B**–property if and only if, for any open covering $\{U_\alpha \mid \alpha < \tau\}$ of X, there exists an open covering $\boldsymbol{V} = \{V_{\alpha\beta} \mid \beta \leq \alpha;\ \alpha < \tau\}$ of X such that*

(1) $V_{\alpha\beta} \subset U_\beta$ *for any* β, α *with* $\beta \leq \alpha$.

(2) *For each* $x \in X$, *we have an open nbd* 0 *of* x *and an ordinal* α_x *such that* $0 \cap (\cup \{V_{\alpha\beta} \mid \beta \leq \alpha;\ \alpha \geq \alpha_x\}) = \phi$.

PROOF *'if part':* Let $\boldsymbol{U} = \{U_\alpha \mid \alpha < \tau\}$ be any monotone increasing open covering of X. Then we have an open covering $\boldsymbol{V} = \{V_{\alpha\beta} \mid \beta \leq \alpha\text{: } \alpha < \tau\}$ of X with the above (1) and (2).

If we let $V_\alpha = \cup \{V_{\alpha\beta} \mid \beta \leq \alpha\}$ for each $\alpha < \tau$, then $\{V_\alpha \mid \alpha < \tau\}$ is an open covering of X such that $V_\alpha \subset U_\alpha$ for each $\alpha < \tau$ and, for each $x \in X$, there exist an open nbd 0 of x and an ordinal $\alpha_x < \tau$ such that $0 \cap (\cup \{V_\alpha \mid \alpha > \alpha_x\}) = \phi$. Therefore a proof of 'if part' is completed by theorem 1.

'only if part': Let $\boldsymbol{U}$ be any open covering of X. we may assume that $\boldsymbol{U}=\{U_\alpha \mid \alpha<\tau\}$ for some ordinal τ. If we let $U'_\alpha=\cup_{\beta<\alpha} U_\beta$ for each $\alpha<\tau$, then $\{U'_\alpha \mid \alpha<\tau\}$ is a monotone increasing open covering of X such that $U'_\alpha=\cup\{U'_\beta \mid \beta<\alpha\}$ for each limit ordinal $\alpha<\tau$. Since X has the $\boldsymbol{B}$-property, there exists a monotone increasing open covering $\boldsymbol{V}=\{V_\alpha \mid \alpha<\tau\}$ of X such that $\mathrm{cl}(V_\alpha)\subset U'_\alpha$ for each $\alpha<\tau$. Furthermore we may assume that $V_\alpha=\cup\{V_\beta \mid \beta<\alpha\}$ for each limit ordinal $\alpha<\tau$.

For each α, $\beta<\tau$ with $\beta\leq\alpha$, let

$$V_{\alpha\beta}=\begin{cases} U_\beta-\mathrm{cl}\,(V_{\alpha-1}) & \text{if } \alpha \text{ is non-limit and } \beta<\alpha \\ \phi & \text{otherwise.}\end{cases}$$

Then it is clear that $V_{\alpha\beta}\subset U_\beta$ for each α, β with $\beta\leq\alpha$.

To show that $\{V_{\alpha\beta} \mid \beta\leq\alpha\colon \alpha<\tau\}$ is a covering of X, let x be any point of X. If we let α_0 be the first ordinal of $\{\alpha \mid \alpha<\tau, x\in U'_\alpha\}$, then α_0 is non-limit and so $x\notin U'_{\alpha_0-1}\supset\mathrm{cl}\,(V_{\alpha_0-1})$. Since $U'_{\alpha_0}=\cup\{U_\beta \mid \beta<\alpha_0\}$, there is some ordinal $\beta<\alpha_0$ such that $x\in U_\beta$. Therefore $x\in U_\beta-\mathrm{cl}\,(V_{\alpha_0-1})=V_{\alpha_0\beta}$, and so $\{V_{\alpha\beta} \mid \beta\leq\alpha;\ \alpha<\tau\}$ is a covering of X.

To show that $\{V_{\alpha\beta} \mid \beta\leq\alpha\colon \alpha<\tau\}$ satisfies the condition (2) of theorem 2, let x be any point of X and $\alpha_0<\tau$ with $x\in V_{\alpha_0}$. For any α with $\alpha_0<\alpha<\tau$, we have $V_{\alpha_0}\cap(\mathrm{X}-\mathrm{cl}\,(V_\alpha))=\phi$ since $\{V_\alpha \mid \alpha<\tau\}$ is monotone increasing, and so, for any non-limit ordinal α with $\alpha>\alpha_0+1$ and any ordinal β with $\beta<\alpha$, it follows that $V_{\alpha_0}\cap V_{\alpha\beta}\subset V_{\alpha_0}\cap(\mathrm{X}-\mathrm{cl}\,(V_{\alpha-1}))=\phi$.

A topological space X is *para-Lindelöf* if every open covering of X has a locally countable open refinement. The following fact may be published elsewhere by someone.

COROLLARY 3 *If a topological space X is countably paracompact and para-Lindelöf, then X has a **B**-property.*

PROOF Let $\boldsymbol{U}=\{U_\alpha \mid \alpha<\tau\}$ be any monotone increasing open covering of X.

Case 1 $cof(\tau)$ *(=cofinality of* τ*)* $=\omega_0$. Let $\{\alpha_n \mid n<\omega_0\}$ be an increasing sequence of ordinals which converges to τ where we may assume $\alpha_0=0$. Since $\{U_{\alpha_n} \mid n<\omega_0\}$ is a countable open covering of X, there exists a locally finite open covering $\{V_n \mid n<\omega_0\}$ of X such that $V_n\subset U_{\alpha_n}$ for each $n<\omega_0$.

Let

$$V_\alpha=\begin{cases} V_n & \text{if } \alpha=\alpha_n\ (n<\omega_0) \\ \phi & \text{otherwise.}\end{cases}$$

Then $\{V_\alpha \mid \alpha<\tau\}$ is an open covering of X such that every point x of X has a nbd which intersects V_α for only finitely many $\alpha<\tau$, and so there exists some $n_0<\omega_0$ such that $0\cap V_\alpha=\phi$ for any $\alpha\geq\alpha_{n_0}$.

Case 2 $cof(\tau)>\omega_0$. We have a locally countable open covering $\boldsymbol{V}=\{V_\alpha \mid \alpha<\tau\}$ of X such that $V_\alpha \subset U_\alpha$ for each $\alpha<\tau$. For each $x\in X$, there exists an open nbd 0 of x which intersects V_α for only countably many $\alpha<\tau$, and so there exists some $\alpha_0<\tau$ such that $0\cap V_\alpha=\phi$ for any $\alpha\geq\alpha_0$ (since $\mathrm{cof}(\tau)>\omega_0$).

REMARKS (1) In (Y. Yasui [12: problem 1]), we posed the following question:

*'If a normal space X has a **B**-property, then is X paracompact?'.*

Afterword, M. E. Rudin ([8] or [9: Theorem 4]) answered negatively for this question; that is, a Navy's space S([5]), which is not paracompact, has the ***B***-property. Since C. Navy showed that the space S is countably paracompact, para-Lindelöf and normal, it can be also shown that the space S has the ***B***-property by corollary 3.

(2) In (T. Tani and Y. Yasui [10: theorem 4]), we showed that:

THEOREM 4 *Let $\{X_n | n<\omega_0\}$ be countable topological spaces. If $\Pi\{X_n | n\leq k\}$ is perfectly normal and has the **B**-property for all $k<\omega_0$, then $\Pi\{X_n | n<\omega_0\}$ has the **B**-property.*

Afterword, A. Bešlagić proved the follwing theorem:

A. Bešlagić's Theorem 5 [1: theorem 3-4] *A normal product $\Pi\{X_n | n<\omega_0\}$ is shrinking iff for all $k<\omega_0$, $\Pi\{X_n | n\leq k\}$ is shrinking.*

In this place, a topological space is shrinking if for any open covering $\{U_\alpha | \alpha\in A\}$ of X, there exists an open covering $\{V_\alpha | \alpha\in A\}$ of X such that $\mathrm{cl}(V_\alpha)\subset U_\alpha$ for each $\alpha\in A$. We shall show that, in the Bešlagić's theorem, we can replace 'be shrinking' with 'have a ***B***-property'. Though its proof is the almost same way but the last part, a characterization of ***B***-property (T. Tani and Y. Yasui [10: theorem 3]) is useful for its part and so the following theorem holds:

THEOREM 6 *Let $\{X_n | n<\omega_0\}$ be countable collection of topological spaces such that the product space $\Pi\{X_n | n<\omega_0\}$ is normal. Then $\Pi\{X_n | n<\omega_0\}$ has a **B**-property iff $\Pi\{X_n | n\leq k\}$ has a **B**-property for all $k<\omega_0$.*

PROOF (ref. A. Bešlagić [1: theorem 3-4])

Let $\{U_\alpha | \alpha<\tau\}$ be a monotone increasing open covering of $X=\Pi\{X_n | n<\omega_0\}$. If we let $U^n_\alpha=\cup\{0 | 0\text{: open in }\Pi\{X_k | k\leq n\},\ 0\times\Pi\{X_k | k>n\}\subset U_\alpha\}$ for each $n<\omega_0$ and each $\alpha<\tau$, then $\{U^n_\alpha | \alpha<\tau\}$ is a monotone increasing collection of open sets of $\Pi\{X_k | k\leq n\}$ for each $n<\omega_0$.

Furthermore if we let

$$O_n=(\cup\{U^n_\alpha | \alpha<\tau\})\times\Pi\{X_k | k>n\},$$

then we have $O_n\subset O_{n+1}$ for each $n<\omega_0$ and $X=\cup\{O_n | n<\omega_0\}$. Since X is countably paracompact ([7]), there is an increasing open covering $\{S_n | n<\omega_0\}$ of X such that $\mathrm{cl}(S_n)$

$\subset O_n$ for each $n<\omega_0$ (F. Ishikawa [4]). Let p_n be the projection from X to $\Pi\{X_k \mid k\leq n\}$ and $T_n=\Pi\{X_k \mid k\leq n\}-p_n(\Pi\{X_k \mid k<\omega_0\}-\mathrm{cl}(S_n))$ for each $n<\omega_0$, then T_n is a closed subset of $\Pi\{X_k \mid k\leq n\}$ and $T_n\subset\cup\{U^n_\alpha \mid \alpha<\tau\}$.
Since T_n has the $\boldsymbol{B}$-property, there is a monotone increasing open covering $\{V^n_\alpha \mid \alpha<\tau\}$ **of** T_n such that $\mathrm{cl}_{T_n}(V^n_\alpha)\subset U^n_\alpha$ for each $\alpha<\tau$ (where the closure of V^n_α in T_n=the closure of it in $\Pi\{X_k \mid k\leq n\}$).
We let

$$W^n_\alpha=(V^n_\alpha\cap\mathrm{Int}\,(T_n))\times\Pi\{X_k \mid k>n\}$$

for $n<\omega_0$ and $\alpha<\tau$. Then $\{W^n_\alpha \mid \alpha<\tau\}$ is a monotone increasing collection of open subsets of X such that $\mathrm{cl}\,(W^n_\alpha)\subset U_\alpha$ for each $\alpha<\tau$ and $n<\omega_0$. Since it is easy to show that $\{W^n_\alpha \mid \alpha<\tau, n<\omega_0\}$ is a covering of X, X has the $\boldsymbol{B}$-property by (T. Tani and Y. Yasui [10: theorem 3]).

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