

NOWHERE DENSELY GENERATED PROPERTIES IN TOPOLOGICAL MEASURE THEORY

By

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1. Introduction.

A topological property \mathcal{P} is said to be nowhere densely generated in a class \mathcal{C} of topological spaces if each $X \in \mathcal{C}$ has \mathcal{P} whenever every nowhere dense closed subset of X has \mathcal{P} [5]. For example, Katětov showed in [4] that the subspace of nonisolated points of a T_1 -space X is compact if each nowhere dense closed subset is compact. This means that compactness is a nowhere densely generated property in the class of T_1 -spaces without isolated points. More generally, it was showed in [1] [5] that $[\kappa, \lambda]$ -compactness is also a nowhere densely generated property in the same class. Other nowhere densely generated properties were investigated in [1], α -closed-completeness, α -compactness and pseudo- (κ, λ) -compactness. The purpose of this paper is to consider nowhere densely generated properties in topological measure theory. In this paper we examine measure-compactness, (weak) Borel measure-completeness, Borel measure-compactness, pre-Radon-ness and Radon-ness.

Terminologies and notations are due to [3]. we denote by $\mathcal{B}(X)$ ($\mathcal{B}^*(X)$) the Borel (Baire) σ -algebra in a space X . A Borel (Baire) measure μ is a σ -additive non-negative real-valued set function on $\mathcal{B}(X)$ ($\mathcal{B}^*(X)$). We assume that all measures are finite (i.e. $\mu(X) < \infty$). A measure μ which is $\mu(X) = 1$ is called a probability. A Borel measure μ is called regular (Radon) if for each $B \in \mathcal{B}(X)$ $\mu(B)$ is the supremum of measures of closed (compact) subsets contained to B . A non-empty family \mathcal{A} of sets is called directed upwards if for each $A, B \in \mathcal{A}$ there exists $C \in \mathcal{A}$ such that $A \cup B \subset C$. A Borel measure μ is called weakly τ -additive if for each directed upwards open cover \mathcal{A} of X , $\mu(X) = \sup\{\mu(U) : U \in \mathcal{A}\}$. A Borel measure μ is called τ -additive if for given open subset V and a directed upwards open cover \mathcal{A} of V , $\mu(V) = \sup\{\mu(U) : U \in \mathcal{A}\}$. Regularity and τ -additivity of Baire measures are also defined by the same way.

We assume all spaces are T_2 .

2. Nowhere densely generated properties.

DEFINITION 2.1. [3] A space X is called

- (1) measure-compact if each Baire measure in X is τ -additive.
- (2) (weakly) Borel measure-complete if each Borel measure in X is (weakly) τ -additive.
- (3) Borel measure-compact if each regular Borel measure in X is τ -additive.

These concepts are motivated by the characterization of real-compactness associated with 2-valued Baire measures. A space is realcompact if and only if each 2-valued Baire measure in the space is τ -additive.

DEFINITION 2.2. [3] A space X is called

- (1) pre-Radon if each τ -additive Borel measure in X is Radon.
- (2) Radon if each Borel measure in X is Radon.

Compact spaces are pre-Radon [3, 11.3].

A cardinal κ is called real-valued measurable if there is a discrete space X with $|X|=\kappa$ and a diffused Borel probability μ in X , where a diffused measure is a measure such that $\mu(\{x\})=0$ for any $x\in X$. By $(*)$ ($\text{cl}(*)$) we denote the condition that the cardinality of each (closed) discrete subspace is not real-valued measurable.

THEOREM 2.3. Let X be a Tychonoff space satisfying $\text{cl}(*)$, and assume that $\mathcal{B}^*(Y)=\{B\cap Y: B\in\mathcal{B}^*(X)\}$ for each closed subset Y in X . Then X is measure-compact if each nowhere dense closed subset of X is measure-compact.

PROOF. It is known that a space X is measure-compact if and only if each Baire probability in X has a nonempty support [3, 14.4], where the support of a Baire (Borel) measure μ in X is the set of all $x\in X$ such that $\mu(U)>0$ for each cozero (open) neighborhood U of x . We assume that there exists a Baire probability μ in X having the empty support. For each $x\in X$ we take a cozero neighborhood U_x of x such that $\mu(U_x)=0$. Let $\mathcal{U}=\{U_\alpha:\alpha\in A\}$ be a maximal disjoint collection of nonempty open subsets refining $\{U_x:x\in X\}$. Then $F=X-\cup\mathcal{U}$ is nowhere densely closed, so F is measure-compact. Since we can extend Baire set of F to a Baire set of X , we can consider the restricted Baire measure μ_F in F [3, 3.2], where μ_F is constructed in the following manner: $\mu_F(E)=\inf\{\mu(B): E\subset B\in\mathcal{B}^*(X)\}$ for each $E\in\mathcal{B}^*(F)$. Obviously the support of μ_F is empty, hence $\mu_F=0$. Therefore there exists $B\in\mathcal{B}^*(X)$ such that $F\subset B$ and $\mu(B)<1$. Since $\mu(X-B)>0$ and a Baire measure is always regular [3, 14.2], there exists a zero set

Z in X such that $Z \subset X - B$ and $\mu(Z) > 0$. Set $A' = \{\alpha \in A : Z \cap U_\alpha \neq \emptyset\}$ and take $x \in Z \cap U_\alpha$ for $\alpha \in A'$. Then $D = \{x_\alpha : \alpha \in A'\}$ is closed discrete in X . We define a Borel measure ν in D by the following equation, $\nu(E) = 1/\mu(Z) \cdot \mu_Z(Z \cap (\bigcup_{x_\alpha \in E} U_\alpha))$ for $E \subset D$. It is easy to show that ν is a diffused Borel probability. Hence $|D|$ is real-valued measurable. This is a contradiction.

The following two theorems are similarly proved.

THEOREM 2.4. Let X be a space satisfying (*). Then X is weakly Borel measure-complete if each nowhere dense closed subset of X is weakly Borel measure-complete.

COROLLARY 2.5. A space X is Borel measure-complete if and only if each nowhere dense closed subset of X is Borel measure-complete and (*) is satisfied.

PROOF. Note that Borel measure-completeness is equivalent to be hereditarily weakly Borel measure-complete [3, 7.4].

THEOREM 2.6. Let X be a space satisfying cl (*). Then X is Borel measure-compact if each nowhere dense closed subset of X is Borel measure-compact.

Theorem 2.3 and 2.6 generalize Corollary 2.5 in [1]. In fact, it is known that a space X is closed-complete if and only if each 2-valued regular Borel measure in X is τ -additive.

LEMMA 2.7. If X is a countable union of pre-Radon subspaces, then X is pre-Radon.

PROOF. Let μ be a τ -additive Borel measure in X and $B \in \mathcal{B}(X)$. Put $X = \bigcup_{i=1}^{\infty} X_i$, where X_i is pre-Radon. For any $\varepsilon > 0$, since the restricted Borel measure μ_{X_i} in X_i is τ -additive, we can take a compact subset K_i such that $K_i \subset B \cap X_i$ and $\mu_{X_i}(B \cap X_i - K_i) < \varepsilon/2^{i+1}$. Then $\mu(B - \bigcup_{i=1}^{\infty} K_i) \leq \sum \mu_{X_i}(B \cap X_i - K_i) \leq \varepsilon/2 < \varepsilon$. Since μ is σ -additive, $\mu(B - \bigcup_{i=1}^n K_i) < \varepsilon$ for some n . This shows that X is pre-Radon.

A space X is called locally pre-Radon if each point of X has a pre-Radon neighborhood.

LEMMA 2.8. Every locally pre-Radon space is pre-Radon.

PROOF. Let μ be a τ -additive Borel measure in a locally pre-Radon space X . It is enough to show that $\mu(U) = \sup\{\mu(K) : K \text{ is a compact subset contained to}$

U for each open set U in X [3, 6. 4]. Let U be an open set in X and for each $x \in U$, take a pre-Radon neighborhood W_x of x . We may assume that W_x is open in X and $W_x \subset U$ [3, 11. 6]. Since μ is τ -additive, for any $\varepsilon > 0$, there exists $x_1, \dots, x_n \in U$ such that $\mu(U) - \mu(W_{x_1} \cup \dots \cup W_{x_n}) < \varepsilon/2$. $W = W_{x_1} \cup \dots \cup W_{x_n}$ is pre-Radon by Lemma 2. 7 and the restricted Borel measure μ_W is τ -additive, hence μ_W is Radon. So there exists a compact set $K \subset W$ such that $\mu_W(W) - \mu_W(K) < \varepsilon/2$. These facts show that $\mu(U) - \mu(K) < \varepsilon$. Thus μ is Radon.

THEOREM 2. 9. A space X is pre-Radon if and only if the following (1) and (2) are satisfied.

- (1) Each nowhere dense closed subset of X is pre-Radon.
- (2) Each nonempty open subset of X contains a nonempty open pre-Radon set.

PROOF. A pre-Radon space obviously satisfies (1) and (2). Because each Borel subset of a pre-Radon space is pre-Radon [3, 11. 6]. We assume (1) and (2). Let \mathcal{U} be a maximal disjoint collection of nonempty open pre-Radon subsets. Since $X - \cup \mathcal{U}$ is nowhere dense closed in X , it is pre-Radon. By Lemma 2. 8, $\cup \mathcal{U}$ is pre-Radon, hence X is pre-Radon by Lemma 2. 7.

There exists a non-pre-Radon space which satisfies (1) in Theorem 2. 9, refer to [3, 5. 11].

THEOREM 2. 10. A space X is Radon if and only if the following (1), (2) and (3) are satisfied.

- (1) (*) is satisfied.
- (2) Each nowhere dense closed subset of X is Radon.
- (3) Each nonempty open subset of X contains nonempty open pre-Radon set.

PROOF. Note that Radon-ness is equivalent to be Borel measure-complete and pre-Radon. So each Radon space satisfies (1), (2) and (3). The converse follows from Corollary 2. 5 and Theorem 2. 9.

We give an application of the above theorem. Fremlin proved under $MA+2^\omega < \omega_\omega$ that a first-countable compact space of weight $< 2^\omega$ is Radon [2]. We generalize this result.

THEOREM 2. 11. [$MA+2^\omega < \omega_\omega$] A space X satisfying (*) is Radon if each nowhere dense closed subset is a first-countable compact set of weight $< 2^\omega$.

PROOF. As mentioned in the first section, the subspace of nonisolated points of X is compact. Hence X is pre-Radon by Lemma 2.7. By Fremlin's result X satisfies (2) of Theorem 2.10. Thus X is Radon by Theorem 2.10.

References

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