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ON TRACES OF SOLUTIONS OF LINEAR ELLIPTIC SYSTEMS AND THEIR APPLICATION TO THE DIRICHLET PROBLEM

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The purpose of this article is to investigate the Dirichlet problem with L^2-boundary data for elliptic systems of the form

$$(1)\qquad L_i(u_1,\cdots,u_n)=-\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n}D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u_j)$$

$$+\sum_{j=1}^{N}\sum_{\alpha=1}^{n}B_{ij}^{\alpha}(x)D_\alpha u_j+\sum_{j=1}^{N}C_{ij}(x)u_j=f_i(x)\qquad(i=1,\cdots,N),$$

$$(2)\qquad u_i(x)=\phi_i(x)\quad\text{on}\quad\partial Q(i=1,\cdots,N)$$

in a bounded domain $Q\subset R_n$ with the boundary ∂Q of the class C^2, where $\phi_i(i=1,\cdots,N)$ are given functions in $L^2(\partial Q)$ and $D_\alpha=\frac{\partial}{\partial x_\alpha}$. In recent years the Dirichlet problem with L^2-boundary data for elliptic equations has attracted attention of several authors (see [2], [3], [8] and [9], where all historical references can be found). The main difficulty in solving the Dirichlet problem with the boundary data in L^2 arises from the fact that not every function in $L^2(\partial Q)$ is the trace of some function belonging to $W^{1,2}(Q)$. Therefore the Dirichlet problem in the L^2-framework requires a proper formulation of the boundary condition (2). The central result of this work is to give proper meaning to the boundary condition (2) and then solve the Dirichlet problem in a suitable Sobolev space.

The plan of the paper is as follows. Section 1 is devoted to prelimanaries. Section 2 deals with problem of traces for solutions of (1) in $W_{\text{loc}}^{1,2}(Q)$. In particular, we obtain a sufficient condition for a solution in $W_{\text{loc}}^{1,2}(Q)$ to have an L^2-trace on boundary (see Theorem 2). The result of Section 2 provide the suitable basis for the approach to the Dirichlet problem adopted in this work. In Section 3 we discuss the existence theorem of the Dirichlet problem which is based on an energy estimate. The arguments which we give here are based partially on the references [1], [2] and [7] however they are considerably modified in order to deal with systems.

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1. In order to simplify notation we set

$$G_i(x,\ u,\ Du)=\sum_{j=1}^{N}\sum_{\alpha=1}^{n}B_{ij}^{\alpha}(x)D_\alpha u_j+\sum_{j=1}^{N}C_{ij}(x)u_j-f_i(x)$$

$(i=1,\cdots,N)$, where $u=(u_1,\cdots,u_N)$, $Du=(Du_1,\cdots,Du_N)$ and Du_i denotes the gradient of the component u_i.

Throughout we shall make the following assumptions:

(A) The system (1) is elliptic in Q, that is, there is a positive constant γ such that

$$\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}A_{ij}^{\alpha\beta}(x)\lambda_i^\alpha\lambda_j^\beta\geqq\gamma|\lambda|^2$$

for all $\lambda=(\lambda_i^\alpha)\in R_{nN}$ and $x\in Q$. The coefficients $A_{ij}^{\alpha\beta}(x)$ belong to $C^1(\bar{Q})$ and moreover

(3) For each α and β $A_{ij}^{\alpha\beta}=A_{ji}^{\alpha\beta}$ $(i,\ j=1,\cdots,N)$ in Q.

(B) The coefficients B_{ij}^{α} and C_{ij} belong to $L^\infty(Q)$ and finally f_i are in $L^2(Q)$ $(i=1,\cdots,N)$.

In the sequel we use the notion of a weak solution involving the Sobolev spaces $W_{\mathrm{loc}}^{1,2}(Q)$ and $W^{1,2}(Q)$.

A vector function $\{u_i\}$ $(i=1,\cdots,N)$ is said to be a weak solution of (1) in Q if $u_i\in W_{\mathrm{loc}}^{1,2}(Q)$ $(i=1,\cdots,N)$ and

$$\int_Q\Big[\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n}A_{ij}^{\alpha\beta}(x)D_\beta u_jD_\alpha v_i+G_i(x,\ u,\ Du)v_i\Big]dx=0 \tag{4}$$

$(i=1,\cdots,N)$ for every vector function $\{v_i\}$ $(i=1,\cdots N)$ in $W^{1,2}(Q)$ with compact support in Q.

It follows from the regularity of the boundary ∂Q that there is a number $\delta_0>0$ such that for $\delta\in(0,\ \delta_0]$ the domain

$$Q_\delta=Q\cap\{x\,;\ \min_{y\in\partial Q}|x-y|>\delta\}$$

with the boundary ∂Q_δ, possesses the following property: to each $x_0\in\partial Q$ there is a unique point $x_\delta(x_0)=x_0-\delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The above relation gives a one-to-one mapping, of class C^1 of ∂Q on ∂Q_δ. The inverse mapping to $x_0\to x_\delta(x_0)$ is given by the formula $x_0=x_\delta+\delta\nu_\delta(x_\delta)$, where $\nu_\delta(x_\delta)$ is the outward normal to ∂Q at x_δ.

Let x_δ denote an arbitrary point of ∂Q_δ. For fixed $\delta\in(0,\ \delta_0]$ let

$$A_\varepsilon=\partial Q_\delta\cap\{x\,;\ |x-x_\delta|<\varepsilon\},$$

$$B_\varepsilon=\{x\,;\ x=\bar{x}_\delta+\delta\nu_\delta(\bar{x}_\delta),\ \bar{x}_\delta\in A_\varepsilon\},$$

and

$$\frac{dS_\delta}{dS_0}(x_\delta)=\lim_{\varepsilon\to 0}\frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denotes the $n-1$ dimensional Hausdorff measure of a set A. Mikailov [7] proved that there is a positive number γ_0 such that

$$(5)\qquad \gamma_0^{-2}\leqq\frac{dS_\delta}{dS_0}\leqq\gamma_0^2$$

and

$$(6)\qquad \lim_{\delta\to 0}\frac{dS_\delta}{dS_0}(x_\delta)=1$$

uniformly with respect to $x_\delta\in\partial Q_\delta$.

Let $r(x)=\text{dist}\,(x,\partial Q)$ for $x\in\bar{Q}$. According to Lemma 1 in [5], p. 382, the distance $r(x)$ belongs to $C^2(\bar{Q}-Q_{\delta_0})$ if δ_0 is sufficently small. Denote by $\rho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties: $\rho(x)=r(x)$ for $x\in\bar{Q}-Q_{\delta_0}$, $\rho\in C^2(\bar{Q})$, $\rho(x)\geqq\frac{3}{4}\delta_0$ in Q_{δ_0}, $\gamma_1^{-1}r(x)\leqq\rho(x)\leqq\gamma_1 r(x)$ in Q for some positive constant γ_1, $\partial Q_\delta=\{x\,;\,\rho(x)=\delta\}$ for $\delta\in(0,\delta_0]$ and finally $\partial Q=\{x\,;\,\rho(x)=0\}$.

2. We commence with a theorem which plays the crucial role in our treatment of the Dirichlet problem. In this theorem we use the surface integrals

$$\int_{\partial Q}|u(x_\delta(x))|^2dS_x \quad\text{and}\quad \int_{\partial Q_\delta}|u(x)|^2dS_x$$

for a solution $u=(u_1,\cdots,u_N)$ in $W^{1,2}_{\text{loc}}(Q)$, where the values $u(x_\delta(x))$ on ∂Q and $u(x)$ on ∂Q_δ are understood in the sense of trace ([4], chapter 6). It follows from Lemma 4 in [1] that both integrals are absolutely continuous on $[\delta_1,\delta_0]$ for every $0<\delta_1<\delta_0$.

THEOREM 1. *Let* $\{u_i\}$ $i=1,\cdots,N$ *be a solution of* (1) *belonging to* $W^{1,2}_{\text{loc}}(Q)$; *then the following conditions are equivalent*

$$(\text{I})\qquad \int_{\partial Q_\delta}|u(x)|^2dS_x \quad \textit{is bounded on}\quad (0,\delta_0],$$

$$(\text{II})\qquad \int_Q|Du(x)|^2r(x)dx<\infty.$$

PROOF. To show I $\Rightarrow$ II we use as test functions in (4)

$$v_i(x)=\begin{cases} u_i(x)(\rho(x)-\delta) & \text{for} \quad x\in Q_\delta, \\ 0 & \text{for} \quad x\in Q-Q_\delta, \end{cases}$$

and on substitution in (4) we obtain

$$\int_{Q_\delta}\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x)D_\beta u_j D_\alpha u_i(\rho-\delta)dx+\int_{Q_\delta}\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x)D_\beta u_j u_i D_\alpha \rho dx$$

$$+\int_{Q_\delta} G_i(x, u, Du)u_i(\rho-\delta)dx=0, \qquad i=1,\cdots,N.$$

Let us denote the first two integrals on the left side by T_i and K_i, respectively. It follows from (A) that

$$\gamma\int_{Q_\delta}|Du(x)|^2(\rho(x)-\delta)dx\leqq\sum_{i=1}^{N}T_i.$$

Using (3) and integrating by parts we obtain

$$\sum_{i=1}^{N}K_i=\frac{1}{2}\int_{Q_\delta}\sum_{i=1}^{N}\sum_{\alpha,\beta=1}^{n}A_{ii}^{\alpha\beta}(x)D_\beta(u_i^2)D_\alpha\rho dx$$

$$+\frac{1}{2}\int_{Q_\delta}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\sum_{\alpha,\beta=1}^{n}A_{ij}^{\alpha\beta}(x)D_\beta(u_iu_j)D_\alpha\rho dx$$

$$=-\frac{1}{2}\int_{\partial Q_\delta}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}A_{ij}^{\alpha\beta}(x)u_iu_jD_\alpha\rho D_\beta\rho dS_x$$

$$-\frac{1}{2}\int_{Q_\delta}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}D_\beta(A_{ij}^{\alpha\beta}(x)D_\alpha\rho)u_iu_jdx.$$

It then follows with the help of Young's inequality that

$$\int_{Q_\delta}|Du|^2(\rho-\delta)dx\leqq C\left(\int_{Q_\delta}|u|^2dx+\int_{\partial Q_\delta}|u|^2dS_x+\int_{Q_\delta}|f|^2dx\right),$$

where $|f|^2=\sum_{i=1}^{n}f_i^2$, $C>0$ depends on n, γ and the bounds of the coefficients $A_{ij}^{\alpha\beta}$, $D_\beta A_{ij}^{\alpha\beta}$, B_{ij}^{α} and C_{ij} and the implication I$\Rightarrow$II easily follows.

To prove "II$\to$I" we first note that (II) implies that $\int_Q|u(x)|^2dx<\infty$ (Lemma 4 in [1]). From the first part of the proof we have

$$\frac{1}{2}\int_{\partial Q_\delta}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}A_{ij}^{\alpha\beta}u_iu_jD_\alpha\rho D_\beta\rho dx$$

$$=-\frac{1}{2}\int_{Q_\delta}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}D_\beta(A_{ij}^{\alpha\beta}D_\alpha\rho)u_iu_jdx+\int_{Q_\delta}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}A_{ij}^{\alpha\beta}D_\beta u_jD_\alpha u_i(\rho-\delta)dx$$

$$+\int_{Q_\delta}\sum_{i=1}^{N}G_i(x,\ u,\ Du)u_i(\rho-\delta)dx$$

and (I) follows from the ellipticity condition and assumptions (A) and (B).

As an immediate consequence we obtain

Corollary 1. *Let* $\{u_i\}$ $i=1,\cdots,N$ *be a solution of* (1). *If one of conditions* (*I*) *or* (*II*) *holds then there exist functions* $\phi_i\in L^2(\partial Q)$ $(i=1,\cdots N)$ *and a sequence* $\{\delta_\nu\}$ *tending to zero such that*

$$\lim_{\nu\to\infty}\int_{\partial Q}u_i(x_{\delta_\nu}(x))g(x)dS_x=\int_{\partial Q}\phi_i(x)g(x)dS_x$$

for each $g\in L^2(\partial Q)$.

Indeed, we note that

$$\int_{\partial Q_\delta}u_i(x)^2dS_\delta=\int_{\partial Q}u_i(x_\delta(x))^2\frac{dS_\delta}{dS_0}dS_0$$

hence by (5) and (2) $\int_{\partial Q}u_i(x_\delta(x))^2dS_x$ is bounded on $(0,\ \delta_0]$.

Consequently the result follows from the weak compactness of bounded sets in $L^2(\partial Q)$.

The main objective of this section is to prove that $\lim_{\delta\to 0}u_i(x_\delta(x))=\phi_i(x)$ $(i=1,\cdots,N)$ in $L^2(Q)$. To show this we define

$$A_i(x,\ u(x))=\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n}A_{ij}^{\alpha\beta}(x)u_j(x)D_\alpha\rho(x)D_\beta\rho(x)$$

$(i=1,\cdots N)$. We need the following lemma.

Lemma 1. *Let* $\{u_i\}$ $(i=1,\cdots,N)$ *be a solution in* $W^{1,2}_{loc}(Q)$ *of* (1) *satisfying one of the conditions* (*I*) *or* (*II*) *and let* $\phi=\{\phi_i\}$ $(i=1,\cdots,N)$ *be functions in* $L^2(\partial Q)$ *determined by Corollary* 1. *Then*

$$\lim_{\delta\to 0}\int_{\partial Q}A_i(x_\delta(x),\ u(x_\delta(x))g(x)dS_x=\int_{\partial Q}A_i(x,\ \phi(x))g(x)dS_x \tag{7}$$

$(i=1,\cdots N)$ for each $g\in L^2(\partial Q)$.

Proof. It follows from (5) and (I) that the integrals

$$\int_{\partial Q}A_i(x_\delta,\ u(x_\delta))^2dS_x\qquad(i=1,\cdots,N)$$

are bounded on $(0,\ \delta_0]$. Hence there exist functions $\Psi_i\in L^2(\partial Q)$ $(i=1,\cdots,N)$ and a sequence $\{\delta_\nu\}$ tending to zero such that

$$\lim_{\nu\to\infty}\int_{\partial Q} A_i(x_{\delta_\nu},\ u(x_{\delta_\nu}))g(x)dS_x=\int_{Q}\Psi_i(x)g(x)dS_x$$

$(i=1,\cdots,N)$ for each $g\in L^2(\partial Q)$. To prove (7) we shall prove that $\int_{\partial Q} A_i(x_\delta,\ u(x_\delta))g(x)dS_x$ $(i=1,\cdots,N)$ are continuous on $[0,\ \delta_0]$ and that

$$\Psi_i(x)=A_i(x,\ \phi(x)) \qquad (i=1,\cdots,N) \tag{8}$$

almost everywhere on ∂Q. Since $\int_{\partial Q} A_i(x_\delta,\ u(x_\delta))g(x)dS_x$ are continuous on $[\delta_1,\ \delta_0]$ for each $0<\delta_1<\delta_0$, it suffices to prove the continuity of these integrals at $\delta=0$. On the other hand we observe that the elements of $C^1(\bar{Q})$ restricted to ∂Q are dense in $L^2(\partial Q)$, so we may assume that $g=\Phi$ on ∂Q with $\Phi\in C^1(\bar{Q})$. Taking

$$v_i(x)=\begin{cases}\Phi(x)(\rho-(x)-\delta) & \text{on} \quad Q_\delta,\\ 0 & \text{on} \quad Q-Q_\delta,\end{cases}$$

$(i=1,\cdots,N)$ as test functions in (4) and integrating by parts we obtain

$$\int_{\partial Q_\delta} A_i(x,\ u(x))\Phi(x)dS_x=-\int_{Q_\delta}\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n} D_\beta(A_{ij}^{\alpha\beta}D_\alpha\rho\Phi)u_jdx$$

$$+\int_{Q_\delta}\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}D_\beta u_jD_\alpha\Phi(\rho-\delta)dx+\int_{Q_\delta} G_i(x,\ u,\ Du)\Phi(\rho-\delta)dx$$

$(i=1,\cdots,N)$. The desired continuity easily follows from (6). In order to prove (8) we note that for each $g\in C(\bar{Q})$ we have

$$\left|\int_{\partial Q} A_i(x_{\delta_\nu},\ u(x_{\delta_\nu}))g(x)dS_x-\int_{\partial Q} A_i(x,\ \phi(x))g(x)dS_x\right|$$

$$\leqq\left|\int_{\partial Q} A_i(x_{\delta_\nu},\ u(x_{\delta_\nu}))g(x)dS_x-\int_{\partial Q}\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x)u_j(x_{\delta_\nu})D_\alpha\rho(x)g(x)dS_x\right|$$

$$+\left|\int_{\partial Q}\sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x)u_j(x_{\delta_\nu})D_\alpha\rho(x)D_\beta\rho(x)g(x)dS_x-\int_{\partial Q} A_i(x,\ \phi(x))g(x)dS_x\right|$$

$$=T_i+K_i \qquad (i=1,\cdots,N)$$

We may also assume that $\{\delta_\nu\}$ is a subsequence appearing in Corollary 1. Using the Schwarz inequality we have

$$|T_i|\leqq\sup_{j,0<\delta\leqq\delta_\nu}\left|\sum_{\alpha,\beta} A_{ij}^{\alpha\beta}(x_\delta)D_\alpha\rho(x_\delta)D_\beta\rho(x_\delta)-\sum_{\alpha,\beta} A_{ij}^{\alpha\beta}(x)D_\alpha\rho(x)D_\beta\rho(x)\right|$$

$$\times\left[\int_{\partial Q}|u(x_\delta)|^2dS_x\right]^{1/2}\left[\int_{\partial Q}\Psi^2dS_x\right]^{1/2}N^{1/2}.$$

Consequently by the uniform continuity of $\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x)D_\alpha\rho(x)D_\beta\rho(x)$ $(i, j=1,\cdots,N)$ on $\bar{Q}$,

$$\lim_{\delta_\nu\to 0} T_i=0 \qquad (i=1,\cdots,N).$$

On the other hand by the weak convergence of $u_i(x_{\delta_\nu})$ to ϕ_i in $L^2(\partial Q)$ we see that

$$\lim_{\delta_\nu\to 0} K_i=0 \qquad (i=1,\cdots,N)$$

and this completes the proof.

We are now in a position to prove that $\lim_{\delta\to 0} u_i(x_\delta(x))=\phi_i(x)$ in $L^2(\partial Q)$.

For $\delta\in(0, \delta_0]$ we define the mapping x^δ: $\bar{Q}\to\bar{Q}_{\delta/2}$ by

$$x^\delta(x)=\begin{cases} x & \text{for } x\in Q_\delta, \\ y_\delta+\dfrac{1}{2}(x-y_\delta) & \text{for } x\in\bar{Q}-Q_\delta, \end{cases}$$

where y_δ denotes the nearest point on ∂Q_δ to x. Thus $x^\delta(x)=x_{\delta/2}(x)$ for each $x\in\partial Q$. Moreover x^δ is uniformly Lipschitz continuous. Note that if $u\in W_{\mathrm{loc}}^{1,2}(Q)$, then $u(x^\delta)\in W^{1,2}(Q)$.

Theorem 2. *Let* $\{u_i\}$ $(i=1,\cdots,N)$ *be a solution in* $W_{\mathrm{loc}}^{1,2}(Q)$ *of* (1) *satisfying one of the conditions* (*I*) *or* (*II*). *Let* ϕ_i $(i=1,\cdots,N)$ *be functions in* $L^2(\partial Q)$ *determined by Corollary* 1. *Then*

$$\lim_{\delta\to 0} u_i(x_\delta(x))=\phi_i(x) \qquad (i=1,\cdots,N) \text{ in } L^2(\partial Q).$$

Proof. We begin by showing that $\lim_{\delta\to 0} A_i(x_\delta, u(x_\delta))=A_i(x, \phi(x))$ $(i=1,\cdots,N)$ in $L^2(\partial Q)$. Indeed, for $\Psi\in W^{1,2}(Q)$ we have

$$\int_{\partial Q} A_i(x, \phi(x))\Psi(x)dS_x=-\int_Q \sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n} D_\beta(A_{ij}^{\alpha\beta}D_\alpha\rho\Psi)u_j dx$$

$$+\int_Q \sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x)D_\beta u_j D_\alpha\Psi\rho dx+\int_Q G_i(x, u, Du)\Psi\rho dx$$

$$\equiv\int F_i(\Psi)dx$$

$(i=1,\cdots,N)$. As $A_i(x^\delta, u(x^\delta))\in W^{1,2}(Q)$, we have

$$\int_{\partial Q} A_i(x, \phi(x))A_i(x^\delta, u(x^\delta))dS_x=\int_{Q-Q_\delta} F_i(A_i(x^\delta, u(x^\delta)))dx$$

$$+\int_{Q_\delta} F_i(A_i(x, u(x)))dx.$$

We show that

$$\lim_{\delta\to 0}\int_{Q-Q_\delta} F_i(A_i(x^\delta,\ u(x^\delta)))dx=0 \tag{9}$$

and that

$$\lim_{\delta\to 0}\int_{Q_\delta} F_i(A_i(x,\ u(x)))dx=\lim_{\delta\to 0}\|A_i(x^\delta,\ u(x^\delta))\|_2^2, \tag{10}$$

so that

$$\|A_i(x,\ \phi(x)\|_2^2=\lim_{\delta\to 0}\int_{\partial Q} A_i(x,\ \phi(x))A_i(x^\delta,\ u(x^\delta))dS_x$$

$$=\lim_{\delta\to 0}\|A_i(x^\delta,\ u(x^\delta))\|_2^2,$$

as $x^\delta(x)=x_{\delta/2}(x)$ on ∂Q. Therefore the claim will follow from the uniform convexity of $L^2(\partial Q)$.

Setting

$$v_i(x)=\begin{cases} A_i(x,\ u(x))(\rho(x)-\delta) & \text{for} \quad x\in Q_\delta \\ 0 & \text{for} \quad x\in Q-Q_\delta \end{cases}$$

in equation (4), we have

$$\lim_{\delta\to 0}\int_{Q_\delta} F_i(A_i(x,\ u(x))dx=\lim_{\delta\to 0}\Big\{-\int_{Q_\delta}\sum_{j=1}^N\sum_{\alpha,\beta=1}^n D_\beta(A_{ij}^{\alpha\beta}D_\alpha\rho A_i(x,\ u(x))u_j dx$$

$$+\int_{Q_\delta}\sum_{j=1}^N\sum_{\alpha,\beta=1}^n A_{ij}^{\alpha\beta}D_\beta u_j D_\alpha A_i(x,\ u(x))(\rho-\delta)dx+\int_{Q_\delta} G_i(x,\ u,\ Du)A_i(x,\ u(x))(\rho-\delta)dx\Big\}$$

$$=\lim_{\delta\to 0}\Big\{-\int_{Q_\delta}\sum_{j=1}^N\sum_{\alpha,\beta=1}^n D_\beta(A_{ij}^{\alpha\beta}D_\alpha\rho A_i(x,\ u(x)))u_j dx$$

$$-\int_{Q_\delta}\sum_{j=1}^N\sum_{\alpha,\beta=1}^n A_{ij}^{\alpha\beta}D_\beta u_j A_i(x,\ u(x))D_\alpha\rho dx\Big\}$$

$$=\lim_{\delta\to 0}\Big\{-\int_{Q_\delta}\sum_{j=1}^N\sum_{\alpha,\beta=1}^n D_\beta(A_{ij}^{\alpha\beta}u_j D_\alpha\rho A_i(x,\ u(x)))dx\Big\}$$

$$=\lim_{\delta\to 0}\int_{\partial Q_\delta} A_i(x,\ u(x))^2 dS_x.$$

It remains to prove (9). Note that by (A), (B) and the Young inequality we have

$$|F_i(A_i(x_\delta,\ u(x_\delta))|\leqslant C[|Du(x)||u(x)|+|u(x_\delta)||u(x)|+$$

$$+|Du(x_\delta)||Du(x)|\rho+|Du(x)||u(x_\delta)|\rho+f(x)||u(x_\delta)|\rho],$$

for some positive constant C independent of δ. Applying Lemmas 2, 3, 4, 5 and 6 from [2] (or Lemmas 8, 9, 10, 11 and 12 in [1]) we easily deduce that (9) holds and this completes the first part of the proof.

It follows from the continuity of $A_{ij}^{\alpha\beta}$ on $\bar{Q}$ and the boundedness of $u_i(x_\delta)$ in $L^2(\partial Q)$ that

$$\lim_{\delta\to 0}\int_{\partial Q}\Big[A_i(x,\ u(x_\delta))-A_i(x_\delta,\ u(x_\delta))\Big]^2 dS_x=0 \qquad (i=1,\cdots,N)$$

and therefore

$$\lim_{\delta\to 0}\int_{\partial Q}\Big[A_i(x,\ u(x_\delta))-A_i(x,\ \phi(x))\Big]^2 dS_x=0 \qquad (i=1,\cdots,N).$$

Let $A_{ij}(x)=\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x)D_\alpha\rho(x)D_\beta\rho(x)$. Since $|D\rho(x)|=1$ on ∂Q, the matrix $\{A_{ij}(x)\}$ is positively definite on ∂Q. Denote by $\{A_{ij}^{-1}(x)\}$ the inverse matrix to $\{A_{ij}(x)\}$, where $x\in\partial Q$. Consequently for each i and j we have

$$\lim_{\delta\to 0}\int_{\partial Q}\sum_{k=1}^{N} A_{ij}^{-1}(x)A_{jk}(x)u_k(x_\delta)-\sum_{k=1}^{N} A_{ij}^{-1}(x)A_{jk}(x)\phi_k(x)\Big]^2 dS_x=0$$

Hence

$$\lim_{\delta\to 0}\int_{\partial Q}[u_i(x_\delta)-\phi_i(x)]^2 dS_x=\lim_{\delta\to 0}\int_{\partial Q}\Big[\sum_{j,k=1}^{N} A_{ij}^{-1}(x)A_{jk}(x)u_k(x_\delta)-$$

$$-\sum_{j,k=1}^{N} A_{ij}^{-1}(x)A_{jk}(x)\phi(x)\Big]^2 dS_x=0$$

and this completes the proof.

3. Let us introduce the following function space

$$\tilde{W}^{1,2}(Q)=\{u\ ;\ u\in W_{\mathrm{loc}}^{1,2}(Q),\ \int_Q|Du(x)|^2 r(x)dx+\int_Q|u(x)|^2dx<\infty\}$$

Theorem 3 justifies the following approach to the Dirichlet problem for the system (1).

Let $\phi=(\phi_1,\cdots,\phi_N)$ with $\phi_i\in L^2(\partial Q)$ $(i=1,\cdots,N)$. A weak solution $u=(u_1,\cdots,u_N)$ of (1) with $u_i\in\tilde{W}^{1,2}(Q)$ $(i=1,\cdots,N)$ is a solution of the Dirichlet problem with the boundary condition (2) if

$$\lim_{\delta\to 0}\int_{\partial Q}[u_i(x_\delta)-\phi_i(x)]^2 dS_x=0 \tag{11}$$

$(i=1,\cdots,N)$.

As it stands this Dirichlet problem need not have a solution, however we shall prove that the Dirichlet problem for a modified system

$$(1_\lambda) \quad L_i(u_1,\cdots,u_N)+\lambda u_i=f_i \text{ in } Q \qquad (i=1,\cdots,N)$$

has a unique solution in $W^{1,2}(Q)$ provided the real parameter λ is sufficiently large. The existence theorem is based on the following energy estimate.

THEOREM 3. There exist positive constants λ_0, C and d such that if $u=\{u_i\}$ is a solution in $\tilde{W}^{1,2}(Q)$ of (1_λ), (2) for $\lambda\geqslant\lambda_0$ then

$$\int_Q |Du(x)|^2 r(x)dx+\int_Q |u(x)|^2 r(x)dx+\sup_{0<\delta<d}\int_{\partial Q_\delta}|u(x)|^2 dS_x\leqslant$$
$$\leqslant C\Big[\int_Q |f(x)|^2 dx+\int_{\partial Q}|\phi(x)|^2 dS_x\Big],$$

where $f=(f_1,\cdots,f_N)$.

PROOF. Taking

$$v_i(x)=\begin{cases} u_i(x)(\rho(x)-\delta) & \text{on } Q_\delta, \\ 0 & \text{on } Q-Q_\delta, \end{cases}$$

$(i=1,\cdots,N)$ as test function we obtain

$$(12) \qquad \int_{Q_\delta}\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n} A_{ij}^{\alpha\beta}(x)D_\alpha u_i D_\beta u_j(\rho-\delta)dx+\lambda\int_{Q_\delta}|u|^2(\rho-\delta)dx=$$
$$=\frac{1}{2}\int_{\partial Q_\delta}\sum_{i=1}^{n} A_i(x,\ u)u_i dS_x+\frac{1}{2}\int_{Q_\delta}\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{n} D_\beta(A_{ij}^{\alpha\beta}D_\alpha\rho)u_i u_j dx-$$
$$-\int_{Q_\delta}\sum_{i=1}^{n} G_i(x,\ u,\ Du)u_i(\rho-\delta)dx.$$

It follows from (11), that

$$\lim_{\delta\to 0}\int_{\partial Q}\sum_{i=1}^{n} A_i(x_\delta,\ u(x_\delta))u_i(x_\delta)dS_x=\int_Q\sum_{i=1}^{n} A_i(x,\ \phi(x))\phi_i(x)dS_x.$$

Hence letting $\delta\to 0$ in (11) we obtain

$$(13) \qquad \int_Q |Du|^2 r dx+\lambda\int_Q |u|^2\rho dx\leqslant C_1\Big[\int_Q |f|^2 dx+\int_{\partial Q}|\phi|^2 dS_x+\int_Q |u|^2 dx\Big],$$

where $C_1>0$ is a constant depending on n, γ and the bounds of the coefficients. It is obvious that (12) also implies that for every $0<d<\delta_0$

$$\sup_{0<\delta<d}\int_{\partial Q_\delta}|u|^2dS_x \leqslant C_2\left[\int_Q|Du|^2rdx+(\lambda+1)\int_Q u^2\rho dx\int_Q|f|^2dx\right], \tag{14}$$

where $C_2>0$ is constant of the same nature as C_1. Combining (13) and (14) we get

$$\int_Q|Du|^2rdx+\lambda\int_Q|u|^2\rho dx+\sup_{0<\delta\leq d}\int_{\partial Q_\delta}|u|^2dS_x\leqslant$$

$$\leqslant C_3\left[\int_{\partial Q}|\phi|^2dS_x+\int_Q|f|^2dx+\int_Q|u|^2dx\right]$$

for some positive constant C_3. Finally note that

$$\int_Q|u|^2dx\leqslant d\sup_{0<\delta\leq d}\int_{\partial Q_\delta}|u|^2dS_x+\frac{1}{m_d}\int_Q|u|^2\rho dx,$$

where $m_d=\inf_{Q_d}(x)$, hence taking d sufficiently small and λ sufficiently large the result follows.

To proceed further we equip $\widetilde{W}^{1,2}(Q)$ with a norm defined by

$$||u||^2_{\widetilde{W}^{1,2}}=\int_Q|u|^2dx+\int_Q|Du|^2rdx.$$

THEOREM 5. *Let $\lambda\geqslant\lambda_0$. Then for every $\phi=\{\phi_i\}$ with $\phi_i\in L^2(\partial Q)$ $(i=1,\cdots,N)$ there exists a unique solution of the Dirichlet problem* (1_λ), (2) *in* $\widetilde{W}^{1,2}(Q)$.

PROOF. The proof is similar to that of Theorem 6 in [2]. Let $\phi^m=(\phi_1^m,\cdots,\phi_N^m)$ be a sequence of functions with components in $C^1(\partial Q)$ and such that $\lim_{m\to\infty}\int_{\partial Q}|\phi^m-\phi|^2 dS_x=0$. Let u_m be a solution of the Dirichlet problem

$$L_i(u_i,\cdots,u_N)+\lambda_i u_i=f_i \text{ in } Q$$

$$u_i=\phi_i^m \text{ on } \partial Q \qquad (i=1,\cdots,N)$$

in $W^{1,2}(Q)$ ([10], Chap. 5, p. 133). Here we may assume that λ_0 is sufficiently large that the theorems on the existence of solutions in $W^{1,2}(Q)$ are applicable. It follows from the energy estimate that $\lim_{m\to\infty}u_m=u$ in $\widetilde{W}^{1,2}$ and u is a weak solution of (1_λ). According to Theorem 2 there exist $\Psi=(\Psi_1,\cdots,\Psi_N)$ with $\Psi_i\in L^2(\partial Q)$ such that

$$\lim_{\delta\to 0}\int_{\partial Q}[u_i(x_\delta)-\Psi_i(x)]^2dS_x=0 \qquad (i=1,\cdots,N).$$

It remains to show that $\phi_i\equiv\Psi_i$ $(i=1,\cdots,N)$ almost everywhere on ∂Q, the proof of which is routine.

We close by pointing out that the linear function G_i can be replaced by a non-linear function satisfying the Carathéodory conditions and the estimate

$$|G_i(x, u, Du)| \leqslant C[|u|+|Du|+f(x)], \qquad (i=1,\cdots,N)$$

where f is a non-negative function in $L^2(Q)$ and $C>0$ is a constant. Under this assumption one can easily prove the existence result analogous to Theorem 3 in [4].

References

[1] Chabrowski, J., Thompson, B., On traces of solutions of semi-linear partial differential equation of elliptic type, Ann. Polon. Math. **42** (1982) 47–73.
[2] Chabrowski, J., Thompson, H.B., On the boundary values of the solutions of linear elliptic equations, Bull. Austral. Math. Soc. **27** (1983) 1–30.
[3] Chabrowski, J., Note on the Dirichlet problem with L^2-boundary data, Manuscripta Math. **40** (1982) 91–108.
[4] Chabrowski, J., On the Dirichlet problem for a semi-linear elliptic equation with L^2-boundary data, Math. Z. **187** (1984) 171–183.
[5] Gilbarg, D., Trudinger, N.S., Elliptic partial differential equations of second order, Grundlehren der mathematischen Wissenschaften 224, Berlin, Heidelberg, New York: Springer-Verlag, 1977.
[6] Kufner, A., John, O., Fučik, S., Function spaces, Noordhoff, Leyden, Prague: Academia, 1977.
[7] Mikaĭlov, V.P., Boundary values of the solutions of elliptic equations in domains with a smooth boundary, Mat. Sb. **101** (**143**) (1976) 163–188.
[8] Mikailov, V.P., The Dirichlet problem for a second order elliptic equation, Differential'nye Uravnenija **12** (1976) 1877–1891.
[9] Mikailov, V.P., On the boundary values of the solutions of elliptic equations, Appl. Math. Optim. **6** (1980) 193–199.
[10] Morrey, Ch.B., Jr., Multiple integrals in the calculus of variations, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen Band 130, Berlin, Heidelberg, New York: Springer-Verlag (1966).

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