TSUKUBA J. MATH.
Vol. 8 No. 1 (1984). 183–188

QF-3′ RINGS AND MORITA DUALITY

By

R. R. COLBY and K. R. FULLER

In [2] we proved that a one-sided artinian ring is QF-3 if and only if its double dual functors preserve monomorphisms. Here with the aid of [3] we prove that the double dual functor over an arbitrary ring preserves monomorphisms of left modules if and only if it is a left QF-3′ ring. In view of this theorem results in [3] and [4] provide an analogue for QF-3′ rings of the Morita-Tachikawa representation theorem for QF-3 rings ([9], Chapter 5). Also we apply it to obtain a characterization of Morita duality between Grothendieck categories that serves to generalize Onodera's theorem [7] that cogenerator rings are self injective, by showing that injectivity is redundant in the classical bimodule characterization of Morita duality for categories of modules.

We denote both the dual functors $\mathrm{Hom}_R(\_, R_R)$ and $\mathrm{Hom}_R(\_, {}_RR)$ by $(\ )^*$. Recall that there is a natural transformation $\sigma: 1_{R\text{-}Mod}\longrightarrow(\ )^{**}$, defined via the usual evaluation maps $\sigma_M: M\longrightarrow M^{**}$. An R-module M is called *R-reflexive* (*R-torsionless*) in case σ_M is an isomorphism (a monomorphism). Also recall that R is left QF-3′ if the injective envelope $E({}_RR)$ of ${}_RR$ is R-torsionless.

1. THEOREM. *For any ring R, the following are equivalent:*
   (a) *R is left* QF-3′;
   (b) *The double dual functor* $(\ )^{**}$ *preserves monomorphisms in* R-Mod;
   (c) *If* $\mathrm{i}: R\longrightarrow E$ *is the inclusion of R into its injective envelope E in* R-Mod, *then* i^{**} *is a monomorphism.*

PROOF. That (b) implies (c) is immediate, and (c) implies (a) is easy (see [3], Proposition 1.2). Assume that R is a left QF-3′ ring. Since $E=E({}_RR)$ is torsionless there is a sequence

$$R\xrightarrow{i}E\xrightarrow{j}R^X$$

for some set X where i is the inclusion and j is a monomorphism.
Let $p_x: R^X\longrightarrow R$ be the canonical projections and let $b_x=p_x\circ j\circ i(1)\in R$ for each

Received September 16, 1983.

$x \in X$. Then if $K=\sum\{b_x R : x \in X\}$ it follows that the left annihilator of K in R is zero. Now suppose $\alpha : M \longrightarrow N$ is a monomorphism in R-Mod and consider the induced sequence

$$N^* \xrightarrow{\alpha^*} M^* \xrightarrow{\beta} \text{Coker } \alpha^* \longrightarrow 0.$$

If $f \in M^*$, then since E is injective there exists $\bar{f} \in \text{Hom}_R(N, E)$ such that $\bar{f} \circ \alpha = i \circ f$. Then considering the diagram

$$\begin{tikzcd} O \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "f"'] & N \arrow[d, "\bar{f}"] & & \\ & R \arrow[r, "i"] & E \arrow[r, "j"] & R^X \arrow[r, "p_x"] & R \end{tikzcd}$$

it follows easily that

$$\alpha^*(p_x \circ j \circ \bar{f}) = p_x \circ j \circ \bar{f} \circ \alpha = p_x \circ j \circ i \circ f = f b_x.$$

Hence $M^* K \subseteq \text{Im } \alpha^*$ so $(\text{Coker } \alpha^*)K = \beta(M^*)K = \beta(M^* K) = 0$. Thus if $\phi \in (\text{Coker } \alpha^*)^*$ we have $\phi(\text{Coker } \alpha^*)K = \phi((\text{Coker } \alpha^*)K) = 0$ so, since the left annihilator of K is zero, $\phi = 0$. But then since $(\text{Coker } \alpha^*)^* = 0$ we see that $M^{**} \xrightarrow{\alpha^{**}} N^{**}$ is monic.

The following theorem follows immediately from ([3], Theorem 1.4) and Theorem 1.

2. THEOREM. *For any ring R, the following are equivalent:*
(1) *R is left* QF-3$'$ *and its own maximal left quotient ring;*
(2) *The double dual functor $(\ )^{**}$ is left exact on* R-Mod;
(3) *If $0 \longrightarrow R \xrightarrow{i} E_1 \xrightarrow{j} E_2$ is exact with E_1 and E_2 injective in* R-Mod *then* $0 \longrightarrow R \xrightarrow{i^{**}} E_1^{**} \xrightarrow{j^{**}} E_2^{**}$ *is also exact.*

Let $D : \mathfrak{a} \rightleftarrows \mathfrak{a}' : D'$ be a pair of contravariant functors between abelian categories $\mathfrak{a}$ and $\mathfrak{a}'$ that are *adjoint on the right*, i.e., there are isomorphisms

$$\eta_{A,A'} : \text{Hom}_{\mathfrak{a}}(A, D'(A')) \longrightarrow \text{Hom}_{\mathfrak{a}'}(A', D(A)),$$

natural in $A \in |\mathfrak{a}|$ and $A' \in |\mathfrak{a}'|$. Associated with $\eta_{A,A'}$ are the arrows of right adjunction $\tau : 1_{\mathfrak{a}} \longrightarrow D'D$ and $\tau' : 1_{\mathfrak{a}'} \longrightarrow DD'$ defined by $\tau_A = \eta^{-1}{}_{A, D(A)}(1_{D(A)})$ and $\tau'_{A'} = \eta_{D'(A'), A'}(1_{D'(A')})$, respectively. These satisfy, for each $A \in |\mathfrak{a}|$, $A' \in |\mathfrak{a}'|$,

$$D(\tau_A) \circ \tau'_{D(A)} = 1_{D(A)} \text{ and } D'(\tau'_{A'}) \circ \tau_{D'(A')} = 1_{D'(A')}.$$

We recall that any pair of such functors $D : \mathfrak{a} \rightleftarrows \mathfrak{a}' : D'$ which are adjoint on the right are left exact ([8], Corollary 3.2.3).

We call an object A of $\mathfrak{a}$ (A' of $\mathfrak{a}'$) *reflexive* (respectively, *torsionless*) in case τ_A ($\tau'_{A'}$) is an isomorphism (respectively, a monomorphism); and we note that (as in [1], Section 23) D and D' define a duality between the *full subcategories of* *reflexive objects* $\mathfrak{a}_0 \subseteq \mathfrak{a}$ and $\mathfrak{a}'_0 \subseteq \mathfrak{a}'$. Then as in [3] we say that the pair $D: \mathfrak{a} \rightleftarrows \mathfrak{a}': D'$ defines a *Morita duality* in case D and D' are exact and the subcategories $\mathfrak{a}_0 \subseteq \mathfrak{a}$ and $\mathfrak{a}_0' \subseteq \mathfrak{a}$, are closed under subobjects and quotient objects and contain sets of generators for $\mathfrak{a}$ and $\mathfrak{a}'$, respectively.

According to ([3], Proposition 2. 3) the functors D and D' of a Morita duality are faithful as well as exact. We shall now show that these conditions imply the closure condition for reflexive objects (as is well known if $\mathfrak{a}$ and $\mathfrak{a}'$ are module categories).

3. Lemma. *Let $D: \mathfrak{a} \leftrightarrows \mathfrak{a}': D$ be a right adjoint pair of contravariant functors between abelian categories. Then D and D' are faithful if and only if all objects in $\mathfrak{a}$ and $\mathfrak{a}'$ are torsionless.*

Proof. If D is faithful and $0 \longrightarrow K \xrightarrow{f} A \xrightarrow{\tau_A} D'D(A)$ is exact, then since $D(\tau_A) \circ \tau'_{D(A)} = 1_{D(A)}$, $D(f) = 0$ so $f = 0$ also. On the other hand, if all objects of A are torsionless and $f \in \mathrm{Hom}_{\mathfrak{a}}(A, B)$, $f \neq 0$, then $D'D(f) \circ \tau_A = \tau_B \circ f \neq 0$ so $D'D(f) \neq 0$, hence $D(f) \neq 0$, so D is faithful.

4. Proposition. *A right adjoint pair of contravariant functors $D: \mathfrak{a} \rightleftarrows \mathfrak{a}': D'$ between abelian categories defines a Morita duality if and only if $\mathfrak{a}$ and $\mathfrak{a}'$ contain generating sets of reflexive objects and D and D' are faithful and exact.*

Proof. From Lemma 3 and exactness we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'D(A) & \longrightarrow & D'D(B_0) & \longrightarrow & D'D(C) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

with exact rows and columns when B_0 is reflexive. Thus the Five Lemma apples.

We don't know whether, in the presence of reflexive generating sets, the closure properties for $\mathfrak{a}_0$ and $\mathfrak{a}_0'$ imply that D and D' are exact and faithful. Of course they do if $\mathfrak{a}$ and $\mathfrak{a}'$ are the categories of modules over a pair of rings (or even if they are functor categories [10]).

We now turn to the general setting of contravariant functors $D: \mathfrak{a} \rightleftarrows \mathfrak{a}': D'$, adjoint on the right, where $\mathfrak{a}$ and $\mathfrak{a}'$ are Grothendieck categories. With $\mathfrak{a}_0$ and $\mathfrak{a}_0'$ as above we assume that these contain generators $V \in \mathfrak{a}_0$, $V' \in \mathfrak{a}_0'$. Then letting $U=V\oplus D'V'$ and $U'=V'\oplus DV$ (so that $DU\cong U'$ and $D'U'\cong U$), $R=\mathrm{End}_{\mathfrak{a}}(U)$ ($\cong\mathrm{End}_{\mathfrak{a}'}(U')^{\mathrm{op}}$), $S=\mathrm{Hom}_{\mathfrak{a}}(U,\_)$ and $S'=\mathrm{Hom}_{\mathfrak{a}'}(U',\_)$, we have, as in ([6], Theorem 8.1) and ([3], Theorem 3.1), functors

$$\begin{array}{ccc} R\text{-Mod} & \underset{S}{\overset{T}{\rightleftarrows}} & \mathfrak{a} \\ (\ )^*\downarrow\uparrow(\ )^* & & D\downarrow\uparrow D' \\ \text{Mod-}R & \underset{S'}{\overset{T'}{\rightleftarrows}} & \mathfrak{a}' \end{array}$$

where $T(T')$ is a left adjoint of $S(S')$, T and T' are exact, and TS and $T'S'$ are equivalent to the identity functors on $\mathfrak{a}$ and $\mathfrak{a}'$, respectively. Also, as in ([3], Theorem 3.1), $S'\circ D\circ T\cong(\ )^*$ and $S\circ D'\circ T'\cong(\ )^*$ so $D\circ T\cong T'\circ(\ )^*$ and $D'\circ T'\cong T\circ(\ )^*$. Thus $\mathrm{Ker}\,T'\subseteq\mathrm{Ker}(\ )^*$ and $\mathrm{Ker}\,T\subseteq\mathrm{Ker}(\ )^*$.

5. LEMMA. *Let D, D', $\mathfrak{a}$, $\mathfrak{a}'$, U, U' and R be as above. Then the following are equivalent:*

(a) *D and D' are faithful;*

(b) *U and U' are cogenerators in $\mathfrak{a}$ and $\mathfrak{a}'$, respectively;*

(c) *$\mathrm{Ker}\ T=\mathrm{Ker}(\ )^*$ and $\mathrm{Ker}\ T'=\mathrm{Ker}(\ )^*$.*

PROOF. If $\alpha\in\mathrm{Hom}_{\mathfrak{a}}(A,B)$, we have a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{a}}(A,U) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathfrak{a}'}(U',DA) \\ \mathrm{Hom}(\alpha,U)\uparrow & & \uparrow\mathrm{Hom}(U',D(\alpha)) \\ \mathrm{Hom}_{\mathfrak{a}}(B,U) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathfrak{a}'}(U',DB) \end{array}$$

Now $D(\alpha)\neq 0$ if and only if $\mathrm{Hom}(U',D(\alpha))\neq 0$ (since U' is a generator) if and only if $\mathrm{Hom}(\alpha,U)\neq 0$. It follows that (a) and (b) are equivalent. Now since $D\circ T\cong T'\circ(\ )^*$, and $D'\circ T'\cong T\circ(\ )^*$, it is clear that if D and D' are faithful, then $\mathrm{Ker}\,T=\mathrm{Ker}\,D\circ T=\mathrm{Ker}\,T'\circ(\ )^*\subseteq\mathrm{Ker}(\ )^*$ and similarly $\mathrm{Ker}\,T'\subseteq\mathrm{Ker}(\ )^*$. Thus (a) implies (c). Suppose $\mathrm{Ker}\,T=\mathrm{Ker}(\ )^*$. If $\alpha\in\mathrm{Hom}_{\mathfrak{a}}(A,B)$ and $D(\alpha)=0$, then $(S(\alpha))^*\cong S'DTS(\alpha)\cong S'D(\alpha)=0$ so $\alpha\cong TS(\alpha)=0$, also. Thus (c) implies (a).

We denote the full subcategories of R-Mod and Mod-R whose objects are the torsion modules, i.e., those modules M with $M^*=0$, by R-Tors and Tors-R, respectively. Then, if R is QF-3$'$, an R-module M is torsion if and only if $\mathrm{Hom}(M,E(R))=0$, and R-Tors and Tors-R are then localizing subcategories of R-Mod and Mod-R, respectively (see [3], Proposition 1.1).

6. THEOREM. *Every right adjoint pair of contravariant faithful functors*

$D: \mathfrak{a} \rightleftarrows \mathfrak{a}': D'$ between Grothendieck categories with reflexive generators defines a Morita duality.

Proof. Suppose that $D: \mathfrak{a} \rightleftarrows \mathfrak{a}': D', U, U', R, T$, and T' are as in Lemma 5 and that D and D' are faithful. Then by Lemma 5, U and U' are generator-cogenerators, so by Morita's ([6], Theorems 8.3 and 5.6) R is QF-3′ and its own maximal quotient ring. (See also [8], Theorem 4.13.4 or [5], Proposition 4.3.1). Thus by Theorem 2 the R-double duals are left exact. But by Lemma 5 we also have Ker T=Ker( )* so by ([8], Theorem 4.4.9) we may identify T: R-Mod$\longrightarrow \mathfrak{a}'$ and T': Mod-$R \longrightarrow \mathfrak{a}'$ with the canonical functors T: R-Mod$\longrightarrow R$-Mod$/R$-Tors and T': Mod-$R \longrightarrow$Mod-R/Tors-R and conclude that (D, D') define a Morita duality by ([3], Theorem 2.6).

Specializing Theorem 6 to the case of module categories yields the following generalization of Onodera's theorem that cogenerator rings are injective [7] and provides a new characterization of Morita duality between categories of modules.

7. Corollary. *If R and S are rings and ${}_RU_S$ is a bimodule with $S=\mathrm{End}({}_RU)$ and $R=\mathrm{End}(U_S)$ and if ${}_RU$ and U_S are cogenerators, then ${}_RU$ and U_S are injective.*

Proof. Apply Theorem 6 to the functors $D=\mathrm{Hom}_R(\_, U)$ and $D'=\mathrm{Hom}_S(\_, U)$.

8. Remarks.

(1) One can apply the technique used to prove Theorem 1 to the sequence $U \xrightarrow{i} E({}_RU) \xrightarrow{j} U^X$ to give a direct proof that if ${}_RU_S$ a balanced bimodule and ${}_RU$ and U_S are cogenerators, then ${}_RU$ is injective.

(2) In ([4], Theorem 1), conditions (i) and (ii) and the last part of (iii) easily imply that R is QF-3′ so by Theorem 1, as we speculated in ([4], Remark (a)), we can delete the first part of condition (iii) from the statement of that theorem; in view of Theorems 1 and 2 and ([3], Theorem 3.1) it now becomes an analogue for QF-3′ rings of the Morita-Tachikawa representation theorems for QF-3 rings ([9], Theorems 5.3 and 5.8).

References

[1] Anderson, F. W. and Fuller, K. R., Rings and Categories of Modules, New York-Heiderberg-Berlin: Springer-Verlag, 1974.

[2] Colby, R. R. and Fuller, K. R., Exactness of the double dual, Proc. Amer. Math. Soc. **82** (1981), 521-526.

[3] Colby, R. R. and Fuller, K. R., Exactness of the double dual and Morita duality for Grothendieck categories, J. Algebra **82** (1983), 546-558.

[4] Colby, R. R. and Fuller, K. R., On rings whose double dual functors preserve monomorphisms, J. Pure Appl. Algebra **28** (1983), 223-234.

[5] Lambek, J., Lectures on Rings and Modules, Waltham-Toronto-London: Gin/Blaisdell, 1966.

[6] Morita, K., Localization in categories of modules I, Math. Z. **114** (1970), 121–144.

[7] Onodera, T., Uber Kogeneratoren, Arch. Math. **19** (1968), 402–410

[8] Popescu, N., Abelian Categories with Applications to Rings and Modules, New York-London: Academic Press, 1973.

[9] Tachikawa, H., Quasi-Frobenius Rings and Generalizations, Springer-Verlag Lecture Notes in Math., **351** (1973).

[10] Yamagata, K., On Morita duality for additive group-valued functors, Comm. in Algebra **7** (1979), 367–392.

The University of Hawaii
Honolulu, Hawaii 96822 U. S. A.

The University of Iowa
Iowa City, Iowa 52242 U. S. A.