TSUKUBA J. MATH.
Vol. 8 No. 1 (1984). 69—75

ON CONDUCTOR OVERRINGS OF AN INTEGRAL DOMAIN

by

Akira OKABE

INTRODUCTION. Throughout this paper, R will be an integral domain with identity, and K will be the quotient field of R. By an *overring* of R we shall mean any integral domain S between R and K. A *proper overring* of R is an overring S such that $R\neq S$. Any unexplained terminology is standard as in [3] or [6].

If I is an ideal of R, then $I:_{K}I=\{x\in K|xI\subseteq I\}$ is an overring of R and furthermore it is a subring of the *ideal transform* $T(I)=\bigcup_{n=1}^{\infty}\{x\in K|xI^{n}\subseteq R\}$. We shall call $I:_{K}I$ the *conductor overring* of R with respect to I.

In [8] Nagata has shown that if I is an ideal of R and R' is an overring of R such that $R\subseteq R'\subseteq T(I)$, then there exists a one-one correspondence between the set of all prime ideals P' of R' not containing IR' and the set of all prime ideals P of R not containing I. Furthermore, this correspondence can be realized in such a manner that if P corresponds to P', then $P=P'\cap R$ and $R_{P}=R'_{P'}$. Hence, if I is an ideal of R then $P'\rightarrow P'\cap R$ is a one-one mapping from the set of all prime ideals P' of $I:_{K}I$ not containing I onto the set of all prime ideals P of R not containing I.

Our results are divided into two sections. In Section 1 we show that if I is an ideal of R then $P\rightarrow(P\cap I):_{K}I$ gives a one-one correspondence between the set of all prime ideals P of R not containing I and the set of all prime ideals P' of $I:_{K}I$ not containing I.

In Section 2 we prove that if I is an ideal of R and P is a prime ideal of R not containing I, then $I:_{K}I/((P\cap I):_{K}I)$ is isomorphic to a subring of $(I+P)/P:_{L}(I+P)/P$ with L the quotient field of R/P. As a corollary, it will be shown that if P is a prime ideal of R properly contained in an ideal I of R, then $P:_{K}I$ is not a maximal ideal of $I:_{K}I$.

1. SOME PRELIMINARY RESULTS

We first establish some general results concerning conductor overrings.

LEMMA 1.1. *Let I be an ideal of R and let S be a proper overring of R.*
(1) *I is an ideal of $I:_{K}I$.*

Received May 24, 1983.

(2) *If we set* $I_{(S)}=R:_R S$, *then* $S\subseteq I_{(S)}:_K I_{(S)}$.

(3) *If* J *is an ideal of* R *such that* $S\subseteq J:_K J$, *then* $J\subseteq I_{(S)}$.

(4) *If* I *is also an ideal of* S, *then* $S\subseteq I:_K I$ *and* $I\subseteq I_{(S)}$.

(5) $I:_R S$ *is an ideal of* R *and is contained in* I. *Furthermore if* I *is also an ideal of* S, *then* $I=I:_R S$.

(6) $I=I:_R(I:_K I)$.

(7) *If* J *is an ideal of* R *such that* $J\subset I$, *then* $J:_K I$ *is a proper ideal of* $I:_K I$.

PROOF. (1) This is trivial.

(2) We first show that $I_{(S)}$ is an ideal of S. Let $x, y\in I_{(S)}$ and $s\in S$. Since $xS\subseteq R$ and $yS\subseteq R$, $(x-y)S\subseteq xS+yS\subseteq R$, and so $x-y\in I_{(S)}$. Next, since $x\in I_{(S)}$ and $s\in S$, $xs\in R$ and moreover $sS\subseteq S$, and therefore $(xs)S=x(sS)\subseteq xS\subseteq R$. Thus $xs\in I_{(S)}$, and therefore $I_{(S)}$ is an ideal of S as we required. Then, since $I_{(S)}$ is an ideal of S, it is clear that $S\subseteq I_{(S)}:_K I_{(S)}$.

(3) By hypothesis, $JS\subseteq J\subseteq R$ and so $J\subseteq I_{(S)}$.

(4) The first assertion is evident. Next, since $S\subseteq I:_K I$, the second assertion follows immediately from (3).

(5) Let $x, y\in I:_R S$ and $r\in R$. Then $xS\subseteq I$ and $yS\subseteq I$, and so $(x-y)S\subseteq xS+yS\subseteq I$. Thus $x-y\in I:_R S$. Next, since $(rx)S=r(xS)\subseteq rI\subseteq I$, $rx\in I:_R S$. Thus $I:_R S$ is an ideal of R. Moreover, if $x\in I:_R S$ then $x=x1\in xS\subseteq I$, and hence $I:_R S\subseteq I$. Assume furthermore that I is an ideal of S. If $x\in I$, then $xS\subseteq I$ and so $I\subseteq I:_R S$. Hence we have $I=I:_R S$ as we wanted.

(6) Since I is an ideal of $I:_K I$, our assertion follows from (5).

(7) Let $x, y\in J:_K I$ and $t\in I:_K I$. Then $xI\subseteq J$ and $yI\subseteq J$, and therefore $(x-y)I\subseteq xI+yI\subseteq J$. Thus $x-y\in J:_K I$. Next, $(xt)I=x(tI)\subseteq xI\subseteq J$ and hence $xt\in J:_K I$. Therefore $J:_K I$ is an ideal of $I:_K I$. Assume that $J:_K I=I:_K I$. Then, since $1\in I:_K I=J:_K I$, $I=1I\subseteq J$ and so $I=J$, a contradiction. Therefore $J:_K I$ is a proper ideal of $I:_K I$.

REMARK 1.2. Let I be a proper ideal of R. Then $I:_K I=K$ if and only if $I=(0)$. If $I:_K I=K$, then, by (1) of Lemma 1.1, I is an ideal of a field K and hence $I=(0)$. Conversely, if $I=(0)$, then clearly $I:_K I=(0):_K(0)=K$.

PROPOSITION 1.3. *Let* I *be a nonzero ideal of* R *and let* P *be a prime ideal of* R *not containing* I. *Then*

(1) $(P\cap I):_K I=\{x\in K \mid xI\subseteq P\cap I\}$ *is a prime ideal of* $I:_K I$.

(2) $((P\cap I):_K I)\cap R=P$.

(3) *If* P' *is a prime ideal of* $I:_K I$ *such that* $P'\cap R=P$, *then* $P'=(P\cap I):_K I$.

(4) $R_P=(I:_K I)_{((P\cap I):_K I)}$.

PROOF. (1) By Lemma 1.1 (7), $(P\cap I):_K I$ is a proper ideal of $I:_K I$. We shall next prove that $(P\cap I):_K I$ is a prime ideal of $I:_K I$. To prove this, let $x, y\in I:_K I$, $xy\in(P\cap I):_K I$, and $x\notin(P\cap I):_K I$. First, since $x\notin(P\cap I):_K I$, there exists an element $t\in I$ such that $xt\notin P\cap I$. Then $xt\notin P$, because $xt\in(I:_K I)I\subseteq I$. Now, if $r\in I$, then $rt\in I$, and hence $(xt)(yr)=(xy)(tr)\in P\cap I\subseteq P$. But then, since $xt\in R\setminus P$, $yr\in(I:_K I)I\subseteq I\subset R$, $(xt)(yr)\in P$ implies that $yr\in P$. Thus $yI\subseteq P$ and therefore $yI\subseteq P\cap I$, that is, $y\in(P\cap I):_K I$. Hence it follows that $(P\cap I):_K I$ is a prime ideal of $I:_K I$.

(2) The containment $P\subseteq((P\cap I):_K I)\cap R$ is clear. To prove the reverse containment, let $x\in((P\cap I):_K I)\cap R$. Choose $t\in I\setminus P$. Then $t\in R\setminus P$, $x\in R$, and $xt\in P$ and so $x\in P$, because P is a prime ideal of R.

(3) and (4) follow from Nagata's theorem mentioned in Introduction, but here we give direct proves.

(3) First, let $x\in P'$. Since P' is an ideal of $I:_K I$, $xI\subseteq I\cap P'=I\cap(R\cap P')=I\cap P$ and so $x\in(P\cap I):_K I$. Thus $P'\subseteq(P\cap I):_K I$. Conversely, let $x\in(P\cap I):_K I$. Then $xI\subseteq P\cap I=P'\cap I$. Choose $t\in I\setminus P$. Then $xt\in P'\cap I\subseteq P'$, $t\notin P'$, and $x\in I:_K I$ and so $x\in P'$, because P' is a prime ideal of $I:_K I$. Thus we also have $(P\cap I):_K I\subseteq P'$.

To prove (4), we need the following

LEMMA 1.4. *If I is an ideal of R and P is a prime ideal of R not containing I, then $P\cap I$ is also an ideal of $I:_K I$.*

PROOF. To prove this, we need to show that $(P\cap I)(I:_K I)=P\cap I$. The containment $P\cap I\subseteq(P\cap I)(I:_K I)$ is clear. To prove the reverse containment, let $x\in(P\cap I)(I:_K I)$. Then we can write $x=\sum a_i x_i$, where $a_i\in P\cap I$ and $x_i\in I:_K I$. Now if we choose $s\in I\setminus P$, then $x_i s\in(I:_K I)I\subseteq I\subseteq R$. Hence $xs=\sum a_i(x_i s)\in P\cap I\subseteq P$. But, since $x\in(P\cap I)(I:_K I)\subseteq I\subseteq R$ and $s\in R\setminus P$, $xs\in P$ implies that $x\in P$. Thus $x\in P\cap I$, as desired.

Now let us return to the proof of (4) in Proposition 1.3. Since $P=((P\cap I):_K I)\cap R$, $R_P\subseteq(I:_K I)_{((P\cap I):_K I)}$. Conversely, let $x\in(I:_K I)_{((P\cap I):_K I)}$. Then we can write $x=a/b$, where $a\in I:_K I$ and $b\notin(P\cap I):_K I$. Since $b\notin(P\cap I):_K I$, there exists $t\in I$ such that $bt\notin P\cap I$. Then necessarily $t\notin P$. Assume the contrary. Then, since $t\in I\cap P$, $bt\in(P\cap I)(I:_K I)=P\cap I$, a contradiction. Thus $t\notin P$ and therefore $t\notin(P\cap I):_K I$. Then $x=at/bt\in R_P$, because $at\in I\subseteq R$ and $bt\in I\setminus P\subseteq R\setminus P$. This completes our proof.

COROLLARY 1.5. *Let I be an ideal of R and let P be a prime ideal of R not containing I. Then P is a prime ideal of $I:_K I$ if and only if $P=(P\cap I):_K I$. In particular, if P is properly contained in I, then P is a prime ideal of $I:_K I$ if and only if $P=P:_K I$.*

PROOF. This follows immediately from Proposition 1.3.

REMARK 1.6. If $J\subset I$ are ideals of R, then J is not necessarily an ideal of $I:_K I$. For example, let k be a field and $R=k[X^2, X^3]$ be the subring of $k[X]$. Then $K=k(X)$ is the quotient field of R. If we set $M=X^2R+X^3R$, then M is a maximal ideal of R and $M:_K M=k[X]\neq R$. Furthermore, if we take $I=X^3R\subset M$, then I is not an ideal of $M:_K M$, because $X^3\in I, X\in M:_K M$, but $X^3X=X^4\notin I$.

2. THE MAIN THEOREM

LEMMA 2.1. *Let I be an ideal of R and let P be a prime ideal of R not containing I. Then, for an element $x\in I:_K I$, $x\notin(P\cap I):_K I$ if and only if $xt\notin P\cap I$ for all $t\in I\setminus P$.*

PROOF. The "if" half is trivial. Conversely, suppose that $x\notin(P\cap I):_K I$. Then there exists $t_0\in I$ such that $xt_0\notin P\cap I$. Since $xt_0\in(I:_K I)I\subseteq I$, $xt_0\notin P$. Moreover, it follows that $t_0\notin P$. Suppose the contrary. Then, $xt_0\in(P\cap I)(I:_K I)=P\cap I$ by Lemma 1.4. But this contradicts the choice of t_0. Thus $t_0\notin P$ as required. Then, for any element $t\in I\setminus P$, $xt\in I\subseteq R$ and $(xt)t_0=(xt_0)t\notin P$, and therefore $xt\notin P$. Thus $xt\notin P\cap I$ for all $t\in I\setminus P$, and the proof is completed.

Now we shall prove the main theorem.

THEOREM 2.2. *Let I be an ideal of R and let P be a prime ideal of R not containing I. Then $I:_K I/((P\cap I):_K I)$ is isomorphic to a subring of $(I+P)/P:_L(I+P)/P$, where L is the quotient field of R/P.*

PROOF. For each $x\in I:_K I$, we shall denote its coset $x+(P\cap I):_K I$ by $\tilde{x}$, and, for each $r\in R$, we shall denote its coset $r+P$ by $\bar{r}$. Now we shall first define a mapping Φ of $I:_K I/((P\cap I):_K I)$ into $(I+P)/P:_L(I+P)/P$ as follows: For each $\tilde{x}\in I:_K I/((P\cap I):_K I)$, we set $\Phi(\tilde{x})=\overline{xt}/\bar{t}$ where $t\in I\setminus P$. Let us first show that this mapping Φ is well-defined. If t and u are any two elements of $I\setminus P$, then $\overline{t}\,\overline{xu}=\overline{txu}=\overline{xt}\,\overline{u}$ in R/P, and so $\overline{xt}/\bar{t}=\overline{xu}/\bar{u}$ in L. Next, if y is any other representative of the coset $\tilde{x}$, then $x-y\in(P\cap I):_K I$, and so, for any $t\in I\setminus P$, $(x-y)t\in P\cap I\subseteq P$. Hence $\overline{xt}=\overline{yt}$ in R/P, and so, $\overline{xt}/\bar{t}=\overline{yt}/\bar{t}$ in L. Thus $\Phi(\tilde{x})=\overline{xt}/\bar{t}$ does depend only on the coset $\tilde{x}$, and not on the choice of a representative of $\tilde{x}$ and an element t of $I\setminus P$, and therefore the mapping Φ is well-defined. Next, let us show that Φ is a ring homomorphism from $I:_K I/((P\cap I):_K I)$ into L. To prove this, let $\tilde{x}, \tilde{y}\in I:$

${}_KI/((P\cap I):{}_KI)$. Then, for any $t\in I\setminus P$, $\overline{(x+y)t}/\bar{t}=\overline{(xt+yt)}/\bar{t}=(\overline{xt}+\overline{yt})/\bar{t}=\overline{xt}/\bar{t}+\overline{yt}/\bar{t}$, and hence it follows that $\Phi(\tilde{x}+\tilde{y})=\Phi(\widetilde{x+y})=\overline{(x+y)t}/\bar{t}=\overline{xt}/\bar{t}+\overline{yt}/\bar{t}=\Phi(\tilde{x})+\Phi(\tilde{y})$. Moreover, for any $t\in I\setminus P$, $\overline{xyt^2}/\overline{t^2}=\overline{(xt)(yt)}/\bar{t}^2=(\overline{xt}/\bar{t})(\overline{yt}/\bar{t})$, and so we have $\Phi(\tilde{x}\tilde{y})=\Phi(\widetilde{xy})=\overline{xyt^2}/\overline{t^2}=(\overline{xt}/\bar{t})(\overline{yt}/\bar{t})=\Phi(\tilde{x})\cdot\Phi(\tilde{y})$. Thus Φ is a ring homomorphism from $I:{}_KI/((P\cap I):{}_KI)$ into L. We shall now proceed to prove the injectivity of Φ. For this, assume that $\Phi(\tilde{x})=0$ for some coset $\tilde{x}$. Then $\overline{xt}=0$ in R/P for all $t\in I\setminus P$, so that $xt\in P\cap I$ for all $t\in I\setminus P$. Moreover, for any $s\in P\cap I$, we have, by Lemma 1.4, $xs\in(P\cap I)(I:{}_KI)=P\cap I$. Hence it follows that $xI\subseteq P\cap I$, that is, $x\in(P\cap I):{}_KI$, and so $\tilde{x}=0$ in $I:{}_KI/((P\cap I):{}_KI)$ as we asserted. It now remains to show that the image of $I:{}_KI/((P\cap I):{}_KI)$ under Φ is actually contained in $(I+P)/P:{}_L(I+P)/P$. To prove this, let $\tilde{x}\in I:{}_KI/((P\cap I):{}_KI)$. Then, for any $\bar{r}\in(I+P)/P$ with $r\in I$, we have $\Phi(\tilde{x})\bar{r}=((\overline{xt})/\bar{t})\bar{r}=\overline{xtr}/\bar{t}=\overline{xr}\cdot\bar{t}/\bar{t}=\overline{xr}\in(I+P)/P$, where $t\in I\setminus P$. Thus we have $\Phi(\tilde{x})\in(I+P)/P:{}_L(I+P)/P$ for all $\tilde{x}\in I:{}_KI/((P\cap I):{}_KI$, and accordingly, $Im(\Phi)$ is actually contained in $(I+P)/P:{}_L(I+P)/P$. Thus our proof is complete.

REMARK 2.3. It would be worth noting that the ring homomorphism Φ defined in Theorem 2.2 is the identity mapping on R/P. In fact, by Proposition 1.3, $R/P\subseteq I:{}_KI/((P\cap I):{}_KI)$ and moreover, for any $\bar{r}\in R/P$, $\Phi(\bar{r})=\overline{rt}/\bar{t}=\bar{r}\cdot\bar{t}/\bar{t}=\bar{r}$ where $t\in I\setminus P$. Thus Φ is the identity mapping on R/P.

COROLLARY 2.4. *Let I be an ideal of R and let P be a prime ideal of R not containing I. Then*

(1) *If $I+P=R$, then $I:{}_KI/((P\cap I):{}_KI$ is isomorphic to R/P.*

(2) *P is a maximal ideal of R if and only if $(P\cap I):{}_KI$ is a maximal ideal of $I:{}_KI$.*

PROOF. (1) If $I+P=R$, then we have $(I+P)/P:{}_L(I+P)P=R/P:{}_LR/P=R/P$, where L is the quotient field of R/P, and then by Theorem 2.2 and Remark 2.3, $I:{}_KI/((P\cap I):{}_KI)$ is isomorphic to R/P as we asserted.

(2) If P is a maximal ideal of R, then $I+P=R$, since I is not contained in P. Then, by the above result (1), $I:{}_KI/((P\cap I):{}_KI)$ is a field, and accordingly, $(P\cap I):{}_KI$ is a maximal ideal of $I:{}_KI$. Conversely, assume that $(P\cap I):{}_KI$ is a maximal ideal of $I:{}_KI$. If $I+P=R$, then, by the above result (1), P is also a maximal ideal of R. Hence, to prove that P is a maximal ideal of R, it suffices to show that $I+P=R$. We shall now recall that by Theorem 2.2 and Remark 2.3, $I:{}_KI/((P\cap I):{}_KI)$ is isomorphic to an integral domain T which is an overring of R/P and is contained in $(I+P)/P:{}_L(I+P)/P$. Now, by our assumption, T is a field, and so $T=(I+P)/P:{}_L(I+P)/P=L$. Hence, if $I+P$ is a proper ideal of R,

then, by Remark 1.2, $(I+P)/P=(0)$ in R/P and therefore $I\subseteq P$, a contradiction. Therefore we have $I+P=R$, as we wanted.

COROLLARY 2.5. *If P is a prime ideal of R, then dim $P:_K P\geqq$ rank P.*

PROOF. If $P=(0)$, then, by Remark 1.2, $(0):_K(0)=K$ and hence $\dim(0):_K(0)=\operatorname{rank}(0)=0$, whence our corollary is valid. Then assume that P is a nonzero prime ideal of rank $r<\infty$, and let $(0)\subset P_1\subset P_2\subset\cdots\subset P_{r-1}\subset P$ be a chain of distinct prime ideals of R. By Proposition 1.3, $(0)=(0):_K P\subset P_1:_K P\subset\cdots\subset P_{r-1}:_K P$ is a chain of distinct proper prime ideals of $P:_K P$. Since, by Corollary 2.4, $P_{r-1}:_K P$ is not a maximal ideal of $P:_K P$, the ideal $P_{r-1}:_K P$ is properly contained in a maximal ideal M of $P:_K P$. Then

$$(0)\subset P_1:_K P\subset P_2:_K P\subset\cdots\subset P_{r-1}:_K P\subset M$$

is a chain of length r of distinct proper prime ideals of $P:_K P$. The assertion follows immediately from this fact. Lastly, we assume that rank P is infinite. Then, as in the case of finite rank, it follows from Proposition 1.3 and Corollary 2.4 that dim $P:_K P$ is infinite, and hence our proof is complete.

REMARK 2.6. If P is a finitely generated prime ideal of R, then Corollary 2.5 is evident. For, in this case, $P:_K P$ is integral over R, and accordingly dim $P:_K P=\dim R\geqq\operatorname{rank} P$.

COROLLARY 2.7. *Let (R, M) be a quasi-local domain. Then every maximal ideal of $M:_K M$ lies over M.*

PROOF. Let P be an arbitrary maximal ideal of $M:_K M$. Then we always have $P\cap R\subseteq M$. If $Q=P\cap R\neq M$, then, by Proposition 1.3, $P=Q:_K M$. But then, by Corollary 2.4, P is not a maximal ideal of $M:_K M$, the desired contradiction. Thus we have $P\cap R=M$, as asserted.

References

[1] Arnold, J. T. and Brewer, J. W., On flat overrings, ideal transforms and generalized transforms of a commutative ring, J. Algebra, **18** (1971), 254-263.

[2] Fossum, R. M., The Divisor Class Group of a Krull Domain, Springer-Verlag, New York, 1973.

[3] Gilmer, R., Multiplicative Ideal Theory, Marcel Dekker, Inc., New York, 1972.

[4] ———, Multiplicative Ideal Theory, Queen's Papers in Pure and Applied Mathematics, No. 12, Queen's Univ. Press, Kingston, Ontario, 1968.

[5] Huckaba, J. A. and Papick, I. J., When the dual of an ideal is a ring, Manuscripta Math., **37** (1982), 67-85.

[6] Kaplansky, I., Commutative Rings, Allyn and Bacon, Inc., Boston, 1970.

[7] Larsen, M. D. and McCarthy, P. J., Multiplicative theory of ideals, Academic Press, New York and London, 1971.
[8] Nagata, M., A treatise on the 14-th problem of Hilbert, Mem. Coll. Sci. Univ. Kyoto Ser. A Math., **30** (1956-57), 57-70.

Gunma Technical College
Toba-cho, Maebashi
371 Japan