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ON THE MICROLOCAL HYPOELLIPTICITY OF PSEUDODIFFERENTIAL OPERATORS

By

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§ 1. Introduction

P. Bolley and J. Camus [1] obtained some results on the microlocal hypoellipticity of differential operators with real analytic coefficients. One of their results is as follows. Let X be an open subset of $\boldsymbol{R}^n$ and $P(x, D)$ a differential operator whose coefficients are real analytic in X. Let L' be a sequence such that

$$k+1\leq L_k'\leq L_{k+1}'\leq CL_k', \quad k=0,1,2,\cdots$$

and

$$L_k''=\max(L_{[\tau k]}'^{\tau}, k^{1/(\rho-\delta)}), \quad 0\leq\delta<\rho\leq 1, \quad \tau=\frac{1}{1-\delta}.$$

Then

$$WF_{L''}(u)\subset WF_{L'}(Pu)\cup(\bigcap_{m\in\boldsymbol{R}}\Sigma_{\rho,\delta}^m(P)), \quad u\in\mathcal{D}'(X).$$

Here $WF_L(u)$ is the wave front set of u with respect to the class C^L (Cf. L. Hörmander [5]) and $\Sigma_{\rho,\delta}^m(P)$ is the complement of the set of all points $(x_0, \xi_0)\in X\times(\boldsymbol{R}^n-0)$ satisfying the following condition: There exist constants C, R and a conic neighborhood V of (x_0, ξ_0) such that for all multi-indices p, q

$$C|P(x,\xi)|\geq|\xi|^m$$

and

$$|D_\xi^p D_x^q P(x,\xi)|\leq C^{|p|+|q|}q!|\xi|^{-\rho|p|+\delta|q|}|P(x,\xi)|$$

when $(x,\xi)\in V$, $|\xi|\geq R$. Where $D_x^q=(-\sqrt{-1}\partial/\partial x)^q$.

In [1] they obtained this result by extending the theory of T. Kotake—M. S. Narasimhan [6]. In this paper we prove a more general result in which the operator P belongs to a class of pseudodifferential operators. It contains all the differential operators whose coefficients are of class C^L, not necessarily analytic. The class

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C^L is allowed to be larger than the Gevrey classes. Also, it can be quasi-analytic. Our method is different from that of [1]. We construct approximate parametrices for the transposed operator, modifying the techniques used in Chapter V of F. Treves [8].

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§ 2. Statement of the results

Let F be a pseudodifferential operator with amplitude a:

$$Fu(x)=\int e^{i\langle x-y,\xi\rangle}a(x,y,\xi)u(y)dy\bar{d}\xi\,, \qquad \bar{d}\xi=(2\pi)^{-n}d\xi\,.$$

Let L_k $(k=0,1,2,\cdots)$ be a sequence of positive numbers. We shall write $F\in I((L_k);\rho',\delta',m')$, if for every compact set $K\subset X\times X$ there exists a constant C_K with

$$|D_\xi^p D_x^q D_y^r a(x,y,\xi)|\leq C_K{}^{|p+q+r|+1}p!M_{|q+r|}\langle\xi\rangle^{m'-\rho'|p|+\delta'|q+r|}$$

when $(x,y)\in K$, $\xi\in\boldsymbol{R}^n-0$ (Cf. L. Boutet de Monvel and P. Krée [2]). Here, $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and

$$M_k=L_k^k\,. \tag{2.1}$$

Note that any differential operator with coefficients of class C^L belongs to $I((L_k);1,0,m_0)$ where m_0 is the order of the operator.

In general, the singular support of the distribution kernel of a pseudodifferential operator is contained in the diagonal ([4]), so we consider the behavior of the amplitude in the diagonal. We shall define a set

$$\Sigma^m_{\rho,\delta,s}((L_k);F)\subset X\times(\boldsymbol{R}^n-0)$$

as follows: $(x_0,\xi_0)\notin\Sigma^m_{q,\delta,s}((L_k);F)$ if and only if there exist constants C, R and a conic neighborhood V of (x_0,ξ_0) such that for all multi-indices p, q, r

$$\begin{aligned}&C|a(x,x,\xi)|\geq|\xi|^m, \qquad \text{if } |\xi|\geq R,\ (x,\xi)\in V,\\ &|(D_\xi^p D_x^q D_y^r a)(x,x,\xi)|\leq C^{|p+q+r|}p!M_{|q+r|}\langle\xi\rangle^{-\rho|p|+\delta|q+r|}|a(x,x,\xi)|\,,\\ &\qquad \text{if } |\xi|\geq R(|p+q+r|+1)^s,\ (x,\xi)\in V.\end{aligned}$$

$\Sigma^m_{\rho,\delta,s}((L_k);F)$ is a closed cone in $X\times(\boldsymbol{R}^n-0)$ and decreases when s increases. If F is a differential operator and if $L_k=k+1$, then the set $\Sigma^m_{\rho,\delta,0}((L_k);F)$ coincides with $\Sigma^m_{\rho,\delta}(F)$ of [1].

We impose the following condition on the sequence L_k:

(i) L_k satisfies that

(2.2) $$k+1\leq L_k\leq L_{k+1}\leq CL_k\,,$$

(2.3) $$\log(M_k/k!)\ \text{is convex.}$$

The condition (i) implies that the C^L is invariant under the C^L class coordinate changes ([7]). We take other sequences:

(ii) T_k and $\bar{T}_k$ are sequences of positive numbers such that

(2.4) $$T_k,\ \bar{T}_k\ \text{also satisfy (i)},$$

(2.5) $$M_{h+k}\leq C^{h+k}H_hH_k\,,\quad H_{h+k}\leq C^{h+k}\bar{H}_h\bar{H}_k\,,$$
$$\text{where } H_k=T_k^k\ \bar{H}_k=\bar{T}_k^k$$

For any L_k satisfying (2.2), such sequences T_k, $\bar{T}_k$ always exist. For example, if $M_k=e^k\cdot k!^s$ (i.e. the C^L is the Gevrey class of order s), then (i) and (ii) are fulfilled with $T_k=\bar{T}_k=L_k$. Also we can take $L_k=\exp(sk^c)$, $0<c\leq 1$, $cs\geq 1$ for instance, but the corresponding space C^L is never contained in the Gevrey class of any order.

Assuming that

(iii) $$0\leq\delta'<\rho'\leq 1\,,\qquad 0\leq\delta<\rho\leq 1\,,$$

we set

(2.6) $$\tau=\frac{1}{1-\delta}\,,\qquad \sigma=\max\left(\frac{1}{\rho'-\delta'},\ \frac{1}{\rho-\delta}\right).$$

Then we have

THEOREM. *Let $F\in I((L_k);\rho',\delta',m')$ be properly supported and the conditions* (i)-(iii) *hold. If L'_k is a sequence satisfying* (2.2), *then*

(2.7) $$WF_{L''}(u)\subset WF_{L'}(Fu)\subset\big(\bigcap_{m\in\mathbf{R}}\Sigma^m_{\rho,\delta,s}((L_k);F)\big)\,,\quad u\in\mathcal{D}'(X)\,,$$
$$\textit{where } L''_k=\max\,(L'^{\tau}_{[\tau k]},\ \bar{T}^{\sigma}_{[\sigma k]},\ k^s).$$

We prove the Theorem in § 3, constructing approximate parametrices microlocally for the transposed operator ${}^tF$.

Now we remark that the set $\Sigma^m_{\rho,\delta,s}((L_k);F)$ is independent of the lower order parts of F. In fact, we have

PROPOSITION 1. *Let L_k be a sequence of positive numbers and $G\in I((L_k);\bar{\rho},\bar{\delta},\bar{m})$, $\rho\leq\bar{\rho}$, $\bar{\delta}\leq\delta$. If $\bar{m}<m$, then*

$$\Sigma^m_{\rho,\delta,s}((L_k);F+G)=\Sigma^m_{\rho,\delta,s}((L_k);F)$$

for any s, F.

PROOF. If $(x_0,\xi_0)\notin\Sigma^m_{\rho,\delta,s}((L_k);F)$, then we have

(2.8) $$|(D_\xi^p D_x^q D_y^r g)(x, x, \xi)| \leq C|\xi|^{\bar{m}-m} C^{|p+q+r|} p! M_{|q+r|} |\xi|^{-\rho|p|+\delta|q+r|} |a(x, x, \xi)|$$

for all (x, ξ) in a conic neighborhood of (x_0, ξ_0) with $|\xi| \geq R$, where g is the amplitude of G. We take R so large that $CR^{\bar{m}-m} < 1/2$. Then $|a(x, x, \xi) + g(x, x, \xi)| \geq |a(x, x, \xi)|/2$ $(|\xi| \geq R)$, so we obtain $(x_0, \xi_0) \notin \Sigma^m_{\rho,\delta,s}((L_k); F+G)$ from (2.8). Therefore $\Sigma^m_{\rho,\delta,s}((L_k); F+G) \subset \Sigma^m_{\rho,\delta,s}((L_k); F)$. Replacing F, G by $F+G$, $-G$ respectively, we have the conclusion.

§ 3. Proof of the Theorem

Let a and b be the amplitudes of F and ${}^tF$ respectively. From the definition of ${}^tF$ we have $b(x, y, \eta) = a(y, x, -\eta)$, thus we obtain

PROPOSITION 2 (Cf. [1], Proposition 3.2).

$$\Sigma^m_{\rho,\delta,s}((L_k); {}^tF) = \{(x, -\eta);\ (x, \eta) \in \Sigma^m_{\rho,\delta,s}((L_k); F)\}.$$

If $(x_0, \xi_0) \notin \Sigma^m_{\rho,\delta,s}((L_k); F)$, then there exists a conic neighborhood V of $(x_0, -\xi_0)$ such that

(3.1) $$C|b(x, x, \eta)| \geq |\eta|^m, \quad \text{if } |\eta| \geq R,\ (x, \eta) \in V,$$

(3.2) $$|(D_\eta^p D_x^q D_y^r b)(x, x, \eta)| \leq C^{|p+q+r|} p! M_{|q+r|} |\eta|^{-\rho|p|+\delta|q+r|} |b(x, x, \eta)|,$$
$$\text{if } |\eta| \geq R(|p+q+r|+1)^s,\ (x, \eta) \in V.$$

We set

(3.3) $$G_k = \max(T_k^\sigma, k^s).$$

LEMMA 1. *Let*

$$P_k(x, \eta) = \sum_{|r|<k} (D_\eta^r d_y^r b)(x, x, \eta)/r!, \qquad k > 1\ (d_y^r = (\partial/\partial y)^r).$$

There exist constants C, $R > 0$ independent of k such that

(3.4) $$C|P_k(x, \eta)| \geq |b(x, x, \eta)| \quad \text{when } |\eta| \geq RG_k,$$

(3.5) $$|D_\eta^p D_x^q P_k(x, \eta)| \leq C^{|p+q|} p! H_{|q|} |\eta|^{-\rho|p|+\delta|q|} |b(x, x, \eta)|$$

when

(3.6) $$|\eta| \geq R(G_k + |p+q|^s), \qquad (x, \eta) \in V.$$

PROOF.

$$|D_\eta^p D_x^q (P_k(x, \eta) - b(x, x, \eta))| \leq \sum_{0<|r|<k} |(D_\eta^{p+r}(D_x + D_y)^q D_y^r b)(x, x, \eta)|/r!$$
$$\leq C^{|p+q|} p! H_{|q|} |\eta|^{-\rho|p|+\delta|q|} |b(x, x, \eta)| B(\eta)$$

where

$$B(\eta)=\sum_{0<|r|<k}\binom{p+r}{r}\left(\frac{CT_{|r|}}{|\eta|^{\rho-\delta}}\right)^{|r|}\leq 2^{|p|}\sum_{0<|r|<k}\left(\frac{2C}{R^{\rho-\delta}}\right)^{|r|}$$

in the set

$$(3.6)_0 \qquad |\eta|\geq RG_k\,, \qquad (x,\eta)\in V\,.$$

So we have

$$(3.5)' \qquad |D_\eta^p D_x^q(P_k(x,\eta)-b(x,x,\eta))|\leq\frac{1}{2}C^{|p+q|}p!H_{|q|}|\eta|^{-\rho|p|+\delta|q|}|b(x,x,\eta)|\,,$$

provided that R is large enough. Combining (3.2) with this, we have (3.5). Let $p=q=0$ in (3.5)$'$. Then we have (3.4).

LEMMA 2. *For each $k=1,2,\cdots$, we can find C^∞ functions $Q_{jk}(x,\eta)$, $j=0,1,\cdots,k-1$ such that*

$$\sum D_\eta^r P_{k-j}(x,\eta)\cdot d_x^r Q_{jk}(x,\eta)/r!=\delta_{0h}\,, \qquad h=0,1,\cdots,k-1\,,$$

in the set $(3.6)_0$, *where $\sum$ denotes the sum for all j, r with $j+|r|=h$, and d_x^r denotes $(\partial/\partial x)^r$. Moreover, in the set* (3.6), *the inequalities*

$$(3.7) \qquad |D_\eta^p D_x^q Q_{jk}(x,\eta)|\leq C^{j+|p+q|}p!H_{|q|+j}|\eta|^{-\rho|p|+\delta|q|-(\rho-\delta)j}|b(x,x,\eta)|^{-1}$$

hold, where the constants C and R are independent of j, k.

PROOF. For each k, determine recursively the functions Q_{jk} by means of the relations

$$(3.8)_0 \qquad Q_{0k}(x,\eta)=1/P_k(x,\eta)$$

and for $j=1,2,\cdots$,

$$(3.8)_j \qquad Q_{jk}(x,\eta)=-\frac{1}{P_{k-j}(x,\eta)}\sum_{0<|r|\leq j}D_\eta^r P_{k-j+|r|}(x,\eta)d_x^r Q_{j-|r|,k}(x,\eta)/r!\,.$$

We must estimate derivatives of Q_{jk}. By $(3.8)_0$ and (3.4)

$$(3.7)_0 \qquad |D_\eta^p D_x^q Q_{0k}|\leq C_0^{|p+q|}p!M_{|q|}|\eta|^{-\rho|p|+\delta|q|}|b|^{-1} \quad \text{(in the set (3.6))}$$

is certainly true when $p=q=0$. From there on we reason $(3.7)_0$ by induction on $|p+q|$, assumed to be ≥ 1. Differentiating $Q_{0k}(x,\eta)P_k(x,\eta)=1$, we have by the Leibniz formula

$$D_\eta^p D_x^q Q_{0k}=-Q_{0k}\sum'\binom{p}{p'}\binom{q}{q'}D_\eta^{p'}D_x^{q'}P_k D_\eta^{p-p'}D_x^{q-q'}Q_{0k}$$

where $\sum'$ denotes the sum for all p', q' with $|p'+q'|>0$, $p'\leq p$, and $q'\leq q$. The inductive hypothesis and (3.5) imply

$$|D_\eta^p D_x^q Q_{0k}| \leq C_0^{|p+q|} p! H_{|q|} |\eta|^{-\rho|p|+\delta|q|} A$$

where

$$A = \Sigma' \binom{q}{q'} C^{|p'+q'|} C_0^{-|p'+q'|} H_{|q'|} H_{|q-q'|} / H_{|q|}$$

with C in (3.5). Since $\binom{q}{q'} \leq \binom{|q|}{|q'|}$, we obtain, in view of (2.3),

$$A \leq \Sigma_{|p'+q'|>0} (C/C_0)^{|p'+q'|} .$$

We have $A \leq 1$, provided that C_0 is large enough in comparison to C, whence $(3.7)_0$. Therefore, it holds that

$$(3.7)_j \qquad \begin{aligned} &|D_\eta^p D_x^q Q_{jk}| \leq C_1^{|p+q|+2j} p! H_{|q|+j} |\eta|^{-(\rho-\delta)j-\rho|p|+\delta|q|} |b|^{-1} \\ &\text{where} \quad |\eta| \geq R(G_k + (j+|p+q|)^s), \quad (x, \eta) \in V, \end{aligned}$$

for $j=0$ and for all p, q. It suffices to show that $(3.7)_j$ holds for $j=1, \cdots, k$, since $G_k + (j+|p+q|)^s \leq 2^{s+1}(G_k + |p+q|^s)$ if $j \leq k$. From there on we reason by induction on j, assumed to be ≥ 1. By $(3.8)_j$, the Leibniz formula implies

$$\begin{aligned} |D_\eta^p D_x^q Q_{jk}| \leq \Sigma'' \frac{p!}{p'!p''!p'''!} \frac{q!}{q'!q''!q'''!} \frac{1}{r!} |D_\eta^{p'} D_x^{q'} Q_{0,k-j}| \\ \times |D_\eta^{p''+r} D_x^{q''} P_{k-j+|r|}| |D_\eta^{p'''} D_x^{q'''+r} Q_{j-|r|,k}| \end{aligned}$$

where Σ'' denotes the sum for all p', p'', p''', q', q'', q''', r with $p'+p''+p'''=p$, $q'+q''+q'''=q$, $0<|r| \leq j$. In view of (3.5) and $(3.7)_0$, the inductive hypothesis implies that

$$\begin{aligned} &|D_\eta^p D_x^q Q_{jk}| \leq C_1^{|p+q|+2j} p! |\eta|^{-(\rho-\delta)j-\rho|p|+\delta|q|} |b|^{-1} B, \\ &B = \Sigma'' (C_0/C_1)^{|p'+q'|} (C/C_1)^{|p''+q''+r|} (p''+r)!/p''!r! \\ &\quad \times H_{|q'|} H_{|q''|} H_{|q'''|+j} q!/q'!q''!q'''! \\ &\leq H_{|q|+j} \Sigma'' (C_0/C_1)^{|p'+q'|} (2C/C_1)^{|p''+q''+r|} \\ &\quad \times \frac{|q|!(|q'''|+j)!}{|q'''|!(|q|+j)!} . \end{aligned}$$

Since

$$\frac{|q|!(|q'''|+j)!}{|q'''|!(|q|+j)!} = \prod_{h=|q'''|+1}^{|q|} \frac{h}{h+j} \leq 1 ,$$

we have

$$B \leq H_{|q|+j} ,$$

provided that C_1 is large enough in comparison to C_0 and to C. This completes the proof.

Now we use the following fact (F. Treves [8], Chapter V).

LEMMA 3. *There is a constant C, depending only on n, such that given any open subset W of* $\boldsymbol{R}^n$, *any number* $d>0$, *any integer* $k>0$, *there is a* C^∞ *function* g_k *in* $\boldsymbol{R}^n$, *having the following properties.*

$$0\le g_k\le 1 \textit{ everywhere}, \quad g_k=1 \textit{ in } W,$$
$$g_k(x)=0 \textit{ if } \operatorname{dist}(x, W)>d,$$
$$|D^p g_k|\le (C_k/d)^{|p|} \textit{ for all } p \textit{ such that } |p|\le k.$$

Then we have

LEMMA 4. *Let* Γ, Γ' *be open cones* $\subset \boldsymbol{R}^n-0$, *such that* $\bar{\Gamma}-0\subset\Gamma'$. *For any* $R>0$, *there exist* C^∞ *functions* p_k *in* $\boldsymbol{R}^n$, *such that*

$$0\le p_k\le 1 \text{ in } \boldsymbol{R}^n$$
$$p_k(\eta)=1 \textit{ when } |\eta|>2RG_k \textit{ and } \eta\in\Gamma,$$
$$\operatorname{supp} p_k\subset\{\eta\in\Gamma'\,;\,|\eta|\ge RG_k\},$$
$$|D^p g_k(\eta)|\le (Ck/|\eta|)^{|p|} \textit{ when } |p|\le k,$$

where the constant C is independent of k.

PROOF. There exists a constant d such that $0<d<1/2$ and

$$\{\eta\,;\operatorname{dist}(\eta, W)\le d\}\subset\Gamma', \qquad \text{where } W=\{\eta\in\Gamma\,;\,|\eta|>1/2\}.$$

Let g_k be as in Lemma 3. If $r_k(\eta)=g_k(\eta/|\eta|)$, then we have

$$|D^p r_k(\eta)|\le (Ck/|\eta|)^{|p|} \quad (|p|\le k).$$

We take another W, d:

$$W=\{\eta\in\boldsymbol{R}^n\,;\,|\eta|>3RG_k/2\}, \qquad d=RG_k/2.$$

Let g_k be as in Lemma 3 and set $s_k(\eta)=g_k(\eta)$. We have

$$|D^p s_k|\le (Ck/G_k)^{|p|}.$$

Since $s_k(\eta)=1$ when $|\eta|\ge 2RG_k$, $p_k(\eta)=s_k(\eta)r_k(\eta)$ has the required properties.

Let V be as in (3.1), (3.2). We take open conic neighborhoods $\Gamma_1, \cdots, \Gamma_4$ of $-\xi_0$ and open neighborhoods $U_1, \cdots, U_4$ of x_0 such that

$$\bar{U}_1 \text{ is compact}, \quad \bar{U}_{j+1}\subset U_j, \quad \Gamma_{j+1}-0\subset\Gamma_j, \quad \bar{U}_1\times(\bar{\Gamma}_1-0)\subset V.$$

Let g_{jk}, p_{jk} be such functions as g_k, p_k in Lemma 3, Lemma 4 respectively, satisfying

$$g_{jk}=1 \text{ in } U_{j+1}, \ \operatorname{supp} g_{jk}\subset U_j,$$
$$p_{jk}(\eta)=1 \text{ when } |\eta|\ge(2j+1)RG_k \text{ and } \eta\in\Gamma_{2j},$$
$$\operatorname{supp} p_{jk}\subset\{\eta\in\Gamma_{2j-1}\,;\,|\eta|\ge 2jRG_k\}.$$

We denote by g_k, h_k, w_k, p_k, q_k the functions g_{1k}, g_{2k}, g_{3k}, p_{1k}, p_{2k} respectively.

Let Q_{jk} be as in Lemma 2 and let us set

$$Q^k(y, \zeta)=g_k(y)q_k(\zeta)\sum_{j<k}Q_{jk}(y, \zeta)\,. \tag{3.9}$$

We denote by K_k the pseudodifferential operator whose amplitude is $Q^k(x, \xi)h_k(y)$. Since ${}^tF$ and K_k are properly supported, so is $S_k={}^tFK_k-I$. We consider the pseudodifferential equation

$$Fu=f\in\mathcal{D}'(X)\,, \qquad u\in\mathcal{D}'(X)\,. \tag{3.10}$$

To prove our Theorem, it suffices to show that

$$(x_0, \xi_0)\notin WF_{L''}(u) \qquad \text{when} \quad (x_0, \xi_0)\notin WF_{L'}(f)\cup\sum\nolimits^m_{\rho,\delta,s}((L_k)\,; F)$$

for some m. Let V be as above. We may assume that

$$\{(y, -\eta)\,;\ (y, \eta)\in\bar{V}\}\cap WF_{L'}(f)=\phi\,. \tag{3.11}$$

From (3.10) we have, for any $v\in\mathcal{D}(X)$,

$$\langle u, v\rangle=\langle u, {}^tFK_kv\rangle-\langle u, S_kv\rangle=\langle f, K_kv\rangle-\langle u, S_kv\rangle\,.$$

In particular we take $v(z)=w_k(z)e^{-i\langle z, \xi\rangle}$, $\xi\in\boldsymbol{R}^n$ considered as a parameter. We have

$$\widehat{w_ku}(\xi)=\theta_k(\xi)-\langle u(x), I_k(x, \xi)\rangle$$

where

$$I_k(x, \xi)=S_kv_k(x)\,, \qquad v_k(z)=w_k(z)e^{-i\langle z, \xi\rangle}\,, \tag{3.12}$$

$$\theta_k(\xi)=\langle f, K_kv_k\rangle\,. \tag{3.13}$$

Let Γ be an open conic neighborhood of Γ_0 such that $\bar{\Gamma}-0\subset-\Gamma_4$. We shall estimate $\widehat{w_ku}(\xi)$ when $\xi\in\Gamma$.

LEMMA 5. *If* $|p|, |q|\le k$, *then*

$$|D^p_\xi D^q_yQ^k(y, \zeta)|\le C^kp!\bar{H}_{|q|}|\zeta|^{-\rho|p|+\delta|q|}|b(y, y, \zeta)|^{-1}$$

where C *is independent of* k.

PROOF. By (3.9) and (3.7) we have

$$\begin{aligned}|D^p_\xi D^q_yQ^k(y, \zeta)|\le&\sum{}'\binom{p}{p'}\binom{q}{q'}(Ck)^{|p-p'|}|D^{p-p'}_\zeta q_k(\zeta)|\\&\times C^{|q+q'|+j}p'!|\zeta|^{-\rho|p'|+\delta|q'|}|b|^{-1}B\end{aligned}$$

where $\sum'$ denotes the sum for all j, p', q' with $j<k$, $p'\le p$, $q'\le q$, and,

$$B=H_{|q'|+j}|\zeta|^{-(\rho-\delta)j}\le C^{|q'|}\bar{H}_{|q'|}(C/R^{\rho-\delta})^j\,.$$

As $k^h \leq h! e^k \leq C^k \bar{H}_h$, we have

$$|D_\xi^p D_y^q Q^k(y, \zeta)| \leq C^{|p+q|+k} p! \bar{H}_{|q|} |\zeta|^{-\rho|p|+\delta|q|} |b|^{-1},$$

if R is large enough in comparison to C.

Since $\eta^p \hat{w}_k(\eta) = \int e^{-i\langle z,\eta\rangle} D^p w_k(z) dz$, it follows that

$$|\hat{w}_k(\eta)| \leq (Ck)^j (k+|\eta|)^{-j} \quad \text{when } j \leq k,\ \eta \in \boldsymbol{R}^n. \tag{3.14}$$

In view of Peetre's inequality, it also follows that

$$|\hat{w}_k(\eta+\zeta)| \leq C^j (k+|\eta|)^{-j} (k+|\zeta|)^j \quad \text{when } j \leq k,\ \eta, \zeta \in \boldsymbol{R}^n. \tag{3.15}$$

Now we estimate (3.13). By (3.11), there exists a bounded sequence $f_J \in \mathcal{E}'$, $J=1, 2, \cdots$ such that

$$f_J = f \text{ in } U_1, \quad |\hat{f}_J(\eta)| \leq C^J M'_J \langle\eta\rangle^{-J} \quad \text{when } \eta \in -\Gamma_1.$$

Since f_J is bounded, there are constants C, n' such that

$$|\hat{f}_J(\eta)| \leq C\langle\eta\rangle^{n'} \quad \text{for any } \eta \in \boldsymbol{R}^n,\ J=1, 2, \cdots.$$

As $\operatorname{supp} K_k v_k \subset U_1$, Parseval's formula implies

$$\begin{aligned} \theta_k(\xi) &= \int \hat{f}_J(\eta) K_k v_k(-\eta) \bar{d}\eta \\ &= \int e^{i\langle y,\eta+\zeta\rangle} \hat{f}_J(\eta) Q^k(y, \zeta) \hat{w}_k(\xi+\zeta) dy d\Xi \end{aligned}$$

where $d\Xi = \bar{d}\eta \bar{d}\zeta$. We split the integral into two parts;

$$\theta_k(\xi) = \int_{CA} + \int_A = I^1 + I^2, \quad \text{say,}$$

where $A = \{(\eta, \zeta);\ \eta \in -\Gamma_1, |\zeta|/2 \leq |\eta| \leq 2|\zeta|\}$, CA is the complement of A. In the integral I^1, there exists a constant $c > 0$ such that

$$|\eta+\zeta| \geq c(|\eta|+|\zeta|)$$

So we have by integration by parts and by Lemma 5, when $J \leq k$,

$$|I^1| \leq \int_{CA} C^J (|\eta|+|\zeta|)^{-J} |\hat{f}_J(\eta)| C^k \bar{H}_J |\zeta|^{\delta J - m} |\hat{w}_k(\xi+\zeta)| d\Xi$$

where m is as in (3.1). As $|\zeta| \geq k$ in the support of Q^k, we have by (3.15)

$$|I^1| \leq C^k \bar{H}_J \langle\xi\rangle^{-N} \int \langle\eta\rangle^{-n-1} \langle\zeta\rangle^{n''} d\Xi$$

where $n'' = -(1-\delta)J + N - m + n' + n + 1$, $N \leq J$. The last integral is convergent, provided that $n'' \leq -n-1$. Therefore, we have

$$|I^1|\leq C^k \bar{H}_J\langle\xi\rangle^{-N} \quad \text{when } k\geq J\geq \tau N+C$$

for some constant C, where τ is as in (2.6). It holds by (3.15) that

$$|I^2|\leq \int_A |\hat{f}(\eta)||Q^k(y,\zeta)|C^N\langle\xi\rangle^{-N}\langle\zeta\rangle^N dyd\Xi \leq C^k M'_J\langle\xi\rangle^{-N}\int_A \langle\eta\rangle^{-J}\langle\zeta\rangle^{N-m}d\Xi .$$

If $-J+N-m\leq -2(n+1)$, then the last integral converges. Therefore we have proved that

$$|\theta_k(\xi)|\leq C^k \max(M'_J, \bar{H}_J)\langle\xi\rangle^{-N} \quad \text{when } k\geq J\geq \tau N+C .$$

Next we estimate $\langle u(x), I_k(x,\xi)\rangle$. Since $\mathrm{supp}_x I_k(x,\xi)$ is contained in a compact set K independent of k, ξ, there exist C, m'' such that

(3.16) $$|\langle u(x), I_k(x,\xi)\rangle|\leq C\Sigma_{|p|\leq m''}\sup_{x\in K}|D_x^p I_k(x,\xi)| .$$

It follows from (3.12) that

(3.17) $$\begin{aligned} I_k(x,\xi)&=B_k(x,\xi)-w_k(x)^{-i\langle x,\xi\rangle},\\ B_k(x,\xi)&={}^tFK_k v_k(x)\\ &=\int e^{i\phi}A_k(x,y,z,\eta,\zeta)dW \end{aligned}$$

where $\phi=\phi(x,y,z,\eta,\zeta)=\langle x-y,\eta\rangle+\langle y-z,\zeta\rangle-\langle z,\xi\rangle$, $dW=dydzd\Xi$,

$$A_k(x,y,z,\eta,\zeta)=b(x,y,\eta)Q^k(y,\zeta)w_k(z) .$$

We split the integral into two parts;

$$\begin{aligned} B_k(x,\xi)&=\int e^{i\phi}p_k(\eta)A_k dW+\int e^{i\phi}(1-p_k)A_k dW\\ &=I+J^1, \quad \text{say.} \end{aligned}$$

By Taylor's formula

$$b(x,y,\eta)=\Sigma_{|r|<h}\frac{(y-x)^r}{r!}(d_y^r b)(x,x,\eta)+\Sigma_{|r|=h}(y-x)^r b_r(x,y,\eta)$$

and by the relation

$$(y-x)^r e^{i\phi}=(-D_\eta)^r e^{i\phi} ,$$

we have that $I=J^2+J^3+I'$, where

$$J^2=\Sigma'\int e^{i\phi}D_\eta^r(p_k(\eta)b_r(x,y,\eta))Q^{jk}(y,\zeta)w_k(z)dW ,$$

$$Q^{jk}(y,\zeta)=g_k(y)q_k(\zeta)Q_{jk}(y,\zeta) ,$$

$$J^3=\Sigma''\int e^{i\phi}\frac{1}{r!}\Sigma_{r'<r}\binom{r}{r'}D_\eta^{r-r'}p_k(\eta)(D_\eta^{r'}d_y^r b)(x,x,\eta))Q^{jk}(y,\zeta)w_k(z)dW ,$$

$$I'=\sum_{j<k}\int e^{i\phi}p_k(\eta)P_{k-j}(x,\eta)Q^{jk}(y,\zeta)w_k(z)dW\,,$$

Σ' (resp. Σ'') denotes the sum for all j, r with $j+|r|=k$ and $j<k$ (resp. $j+|r|<k$). By the Taylor's formula

$$P_j(x,\eta)=\sum_{|r|<h}\frac{(\eta-\zeta)^r}{r!}d_\eta^r P_j(x,\zeta)+\sum_{|r|=h}(\eta-\zeta)^r P_{rj}(x,\eta,\zeta)$$

and by integration by parts, it follows that $I'=J^4+I''$, where

$$J^4=\Sigma'\int e^{i\phi}P_{r,k-j}(x,\eta,\zeta)D_y^r Q^{jk}(y,\zeta)p_k(\eta)w_k(z)dW\,,$$

$$I''=\int e^{i\phi}Z_k(x,y,z,\zeta)p_k(\eta)dW\,,$$

$$Z_k(x,y,z,\zeta)=\Sigma'' Z_{rjk}(x,y,\zeta)w_k(z)\,,$$

$$Z_{rjk}(x,y,\zeta)=D_\zeta^r P_{k-j}(x,\zeta)d_y^r Q^{jk}(y,\zeta)/r!\,.$$

Splitting the integral I'' into two parts;

$$I''=I'''+J^5\,,$$

$$J^5=\int e^{i\phi}(p_k-1)Z_k dW\,,\qquad I'''=\int e^{i\phi}Z_k dW\,,$$

and using the Fourier inversion formula, we obtain

$$I'''=\int e^{i\phi}Z_k(x,x,z,\zeta)dz\bar{d}\zeta\,,\qquad \varphi(x,z,\zeta)=\langle x,\zeta\rangle-\langle z,\xi+\zeta\rangle\,.$$

Moreover we devide the integral I''' into two parts;

$$I'''=J^6+I^{(4)}\,,$$

$$J^6=\Sigma''\int e^{i\varphi}D_\zeta^r P_{k-j}(x,\zeta)d_x^r((g_k(x)-1)Q_{jk}(x,\zeta))q_k(\zeta)w_k(z)dz\bar{d}\zeta\,.$$

By Lemma 2 we have

$$I^{(4)}=J^7+J^8+w_k(x)e^{-i\langle x,\xi\rangle}\,,$$

where

$$J^7=\int_S e^{i\langle x,\zeta\rangle}(q_k(\zeta)-1)\hat{w}_k(\xi+\zeta)\bar{d}\zeta\,,$$

$$J^8=\int_{CS}\,,\qquad \text{where } S=\{\zeta\in\boldsymbol{R}^n\,;\,|\zeta|\leq 5RG_k\}\,.$$

By (3.17) we have

$$I_k(x,\xi)=J^1+\cdots+J^8\,. \tag{3.18}$$

We shall estimate each J^j. First, note that

(3.19) $$|\eta-\zeta|\geq c(|\eta|+|\zeta|) \qquad \text{when } \eta\in\operatorname{supp}(1-p_k),\ \zeta\in\operatorname{supp} q_k$$

for some constant $c>0$. Using the operator

(3.20) $$\frac{-i}{|\eta-\zeta|^2}\sum_{j=1}^{n}(\eta_j-\zeta_j)\frac{\partial}{\partial y_j},$$

we have (by integration by parts with respect to y-variables)

$$|D_x^p J^1|\leq C^k \bar{H}_k\langle\xi\rangle^{-N} \qquad \text{if } k\geq\tau N+C,\ |p|\leq m''$$

for some constant C, where m'' is as in (3.16). Similarly it follows that

$$|D_x^p J^j|\leq C^k H_k\langle\xi\rangle^{-N} \qquad \text{if } k\geq\sigma N+C,\ j=3,5.$$

It is easily cheked that

$$|D_x^p J^7|\leq C^k G_k^N\langle\xi\rangle^{-N} \qquad \text{if } k\geq N+C.$$

Since

$$|\xi+\zeta|\geq c(|\xi|+|\zeta|) \qquad \text{if } \zeta\in\operatorname{supp}(q_k-1),\ \xi\in\Gamma,\ \zeta\in S,$$

it also follows that

$$|D_x^p J^8|\leq C^k N!\langle\xi\rangle^{-N} \qquad \text{if } k\geq N+C.$$

In the integral J^6, it holds that

$$|x-z|\geq c \qquad \text{for some constant } c>0.$$

Therefore we can use the operator

$$\frac{i}{|x-z|^2}\sum_{j=1}^{n}(x_j-z_j)\frac{\partial}{\partial\zeta_j},$$

and we get

$$|D_x^p J^6|\leq C^k H_k\langle\xi\rangle^{-N} \qquad \text{if } k\geq\sigma N+C.$$

To estimate J^2, we use the operator (3.20) on the set

$$A=\{(\eta,\zeta)\,;\ |\eta|\geq 2|\zeta| \text{ or } |\zeta|\geq 2|\eta|\}.$$

(It holds that $|\eta-\zeta|\geq(|\eta|+|\zeta|)/4$ on A.) Since $|\eta|$ is dominated by $2|\zeta|$ on the complement of A, and as

$$b_r(x,y,\eta)=\frac{|r|}{r!}\int_0^1(d_y^r b)(x,tx+(1-t)y,\eta)t^{|r|-1}dt,$$

we can get

$$|D_x^p J^2|\leq C^k H_k\langle\xi\rangle^{-N} \qquad \text{when } k\geq\sigma N+C.$$

It remains to estimate J^4. Note that

$$|t\zeta+(1-t)\eta|>c(t|\zeta|+(1-t)|\eta|) \qquad \text{if } 0\leq t\leq 1,\ \eta\in\operatorname{supp} p_k,\ \zeta\in\operatorname{supp} q_k.$$

Since

$$P_{rj}(x,\eta,\zeta)=\frac{|r|}{r!}\int_0^1 (d_\eta^r P_j)(x,(1-t)\eta+t\zeta)t^{|r|-1}dt,$$

we have for $r>0$ and $|p|\leq m''$

$$|D_x^p P_{rj}(x,\eta,\zeta)|\leq C^k(|\eta|+|\zeta|)^{m'+\delta m''}\langle\zeta\rangle^{-\rho|r|+\rho}.$$

Using the operator

$$\frac{-i}{|\eta|^2}\sum_{j=1}^n \eta_j\frac{\partial}{\partial y_j},$$

we have

$$|D_x^p J^4|\leq C^k H_k\langle\xi\rangle^{-N}, \qquad \text{if } k\geq\sigma N+C.$$

This completes the proof of the Theorem.

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