TSUKUBA J. MATH.
Vol. 7 No. 1 (1983). 147—156

THE CONVERGENCE OF MOMENTS IN THE CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT RANDOM VARIABLES

By

Ryozo YOKOYAMA

1. Introduction.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. For any two σ-fields $\mathcal{A}$ and $\mathcal{B}$ define the mixing coefficients ϕ and α and the maximal correlation coefficient ρ by

$$\phi(\mathcal{A}, \mathcal{B})=\sup|P(B|A)-P(B)| \qquad A\in\mathcal{A},\ B\in\mathcal{B},\ P(A)>0;$$

$$\alpha(\mathcal{A}, \mathcal{B})=\sup|P(A\cap B)-P(A)P(B)| \qquad A\in\mathcal{A},\ B\in\mathcal{B};$$

$$\rho(\mathcal{A}, \mathcal{B})=\sup|\mathrm{Corr}(\xi, \eta)| \qquad \xi\in L^2(\mathcal{A}),\ \eta\in L^2(\mathcal{B}).$$

Let $\{X_j: -\infty<j<\infty\}$ be a strictly stationary sequence of random variables on $(\Omega, \mathcal{F}, P)$. For integers n let $\mathcal{P}_n$ be the σ-field generated by $\{X_j: j\leqq n\}$ and $\mathcal{F}_n$ the σ-field generated by $\{X_j: j\geqq n\}$. The sequence $\{X_j\}$ is said to be *ϕ-mixing* (or *uniformly mixing*) if

$$\phi(n)\equiv\phi(\mathcal{P}_0, \mathcal{F}_n)\to 0 \qquad \text{as} \quad n\to\infty$$

(see Ibragimov [9]), *strongly mixing* if

$$\alpha(n)\equiv\alpha(\mathcal{P}_0, \mathcal{F}_n)\to 0 \qquad \text{as} \quad n\to\infty$$

(see Rosenblatt [15]) and *completely regular* if

$$\rho(n)\equiv\rho(\mathcal{P}_0, \mathcal{F}_n)\to 0 \qquad \text{as} \quad n\to\infty$$

(see Kolmogorov-Rozanov [13]).

Among these coefficients, the following inequalities always hold:

$$4\alpha(n)\leqq\rho(n)\leqq 2\phi^{1/2}(n).$$

The left-hand inequality is an easy consequence of the definitions of the coefficients $\alpha(n)$ and $\rho(n)$, and the right-hand inequality is a consequence of the Ibragimov fundamental inequality for ϕ-mixing sequences (see [11, Theorem 17.2.3, p. 309]). Thus a ϕ-mixing sequence is completely regular (the converse

Received October 27, 1982.

is false; see [11, pp. 310-314]), and a completely regular sequence is strongly mixing (the converse is false; see [16, pp. 206-209], but for Gaussian sequences complete regularity is equivalent to strong mixing; see [13]). Formulations of various mixing conditions are given by Ibragimov-Rozanov [12] for stationary Gaussian sequences in terms of the spectral density, and by Rosenblatt [16] for stationary Markov sequences in terms of the transition operator.

Let $\{X_j\}$ be a strictly stationary sequence with $EX_j=0$ and $EX_j^2<\infty$. Set

$$S_n=\sum_{j=1}^{n} X_j, \quad \sigma_n^2=ES_n^2 .$$

In numerous papers conditions are investigated which guarantee asymptotic normality of the distribution of the normed sum $\sigma_n^{-1}S_n$ (see, for example, [2, Chap. 4], [9], [10] and [11, Chap. 18]).

We are interested in knowing when the rth absolute moment of $\sigma_n^{-1}S_n$ $(r>2)$ converges to that of the normal distribution. When X_j are independent (but not necessarily identically distributed) random variables, Bernstein [1] presented a necessary and sufficient condition (the rth Lindeberg condition) for the convergence of absolute moments in the central limit theorem. Brown [4, 5] gave an alternative proof of Bernstein's result. Hall [8] extended Bernstein's theorem in both the independence and the martingale cases. For stationary ϕ-mixing and strongly mixing sequences the author [17, 18] obtained some results on the convergence of moments. Recently, in the ϕ-mixing case, the following much broader result was proved; the proof is completely different from those in [17] and [18].

THEOREM A ([19]). *Let $\{X_j\}$ be a strictly stationary sequence with $EX_j=0$ and $E|X_j|^r<\infty$ for some $r>2$. If $\phi(n)\to 0$ and $\sigma_n^2\to\infty$ as $n\to\infty$, then*

$$\lim_{n\to\infty} E|S_n/\sigma_n|^r=\int_{-\infty}^{\infty}(2\pi)^{-1/2}|u|^r\exp(-u^2/2)du .$$

In Theorem A it is not assumed that $\phi(n)\to 0$ at a specific rate, while the series-type conditions on the mixing coefficients were imposed in all the theorems of [18] (cf. [9, Theorem 1.4]). The purpose of this paper is to generalize the above ϕ-mixing result to the complete regularity case. The basic idea, which was used in [19], is a martingale representation of the sum S_n, and the proof is based on Ibragimov's moment inequality (Lemma 2 below) and a martingale result of Hall [8].

2. Statement of a result.

First we state a result of Ibragimov [10, Theorem 2.1], which generalizes an earlier result of his own [9, Theorem 1.4].

THEOREM B (Ibragimov). *Let* $\{X_j\}$ *be a strictly stationary sequence with* $EX_j=0$ *and* $EX_j^2<\infty$. (i) *If* $\lim_{n\to\infty}\rho(n)=0$ *and* $\limsup_{n\to\infty}\sigma_n^2=\infty$, *then* $\sigma_n^2=nh(n)$, *where* $h(n)$ *is a slowly varying function in the sense of Karamata.* (ii) *If in addition* $E|X_j|^r<\infty$ *for some* $r>2$, *then*

$$\lim_{n\to\infty} P\{S_n/\sigma_n<x\}=\int_{-\infty}^{x}(2\pi)^{-1/2}\exp(-u^2/2)du .$$

REMARKS. Theorem B (ii) fails if its hypothesis $E|X_j|^r<\infty$ $(r>2)$ is omitted; a counterexample is constructed by Bradley [3]. Lifshits [14] proved some central limit theorems on Markov chains under $\rho(n)\to 0$ and other slightly weaker conditions.

In this article the conditions of Theorem B, without any additional conditions, will be shown to imply the convergence of the rth absolute moments in the central limit theorem. More precisely, we shall prove

THEOREM C. *Let* $\{X_j\}$ *be a strictly stationary sequence with* $EX_j=0$ *and* $E|X_j|^r<\infty$ *for some* $r>2$. *If* $\rho(n)\to 0$ *and* $\sigma_n^2\to\infty$ *as* $n\to\infty$, *then*

$$\lim_{n\to\infty} E|S_n/\sigma_n|^r=\int_{-\infty}^{\infty}(2\pi)^{-1/2}|u|^r\exp(-u^2/2)du . \tag{1}$$

As we have remarked in Sect. 1, the ϕ-mixing condition implies the complete regularity condition, thus Theorem C contains Theorem A as a special case. For strongly mixing sequences the relation (1) holds under the conditions $EX_j=0$, $E|X_j|^{r+\delta}<\infty$ for some $r>2$ and $\delta>0$, $EX_1^2+2\sum_{j=2}^{\infty}E(X_1X_j)>0$ and $\sum_{n=1}^{\infty}n^{r/2-1}(\alpha(n))^{\delta/(r+\delta)}<\infty$ (see [18]).

3. The proof.

In the proof, limits will be taken as $n\to\infty$. The symbol K denotes a generic constant, not necessarily the same at each appearance. β_r denotes the rth absolute moment of the standard normal distribution. $I(A)$ denotes the indicator function of the event A.

For the proof of Theorem C we need a few well-known inequalities.

LEMMA 1. *Suppose that the random variables ξ and η, respectively, are measurable with respect to $\mathcal{P}_k$ and $\mathcal{F}_{k+n}$;*

1) *If $E\xi^2<\infty$ and $E\eta^2<\infty$, then*

$$|E(\xi\eta)-E\xi\cdot E\eta|\leqq(E\xi^2)^{1/2}(E\eta^2)^{1/2}\rho(n); \tag{2}$$

2) *if $|\xi|\leqq B$ a.s. and $E|\eta|^s<\infty$ for some $s>1$, then*

$$|E(\xi\eta)-E\xi\cdot E\eta|\leqq 6B(E|\eta|^s)^{1/s}(\alpha(n))^{1-1/s}. \tag{3}$$

The inequality (2) is an immediate consequence of the definition of the coefficient $\rho(n)$. The inequality (3) is due to Davydov [7]. The following inequality, due to Ibragimov [10], is fundamental to our proof.

LEMMA 2. *Under the assumptions of Theorem C, there exists a constant C such that*

$$E|S_n|^r\leqq C\sigma_n^r \qquad \text{for all } n\geqq 1. \tag{4}$$

We shall divide the sum S_n into three parts:

$$S_n=S_n'+S_n''=\sigma_n T_n+\sigma_n T_n'+S_n'',$$

and show that $\sigma_n^{-1}S_n''$ and T_n' are asymptotically negligible, while the rth absolute moment of T_n converges to β_r, where the variable T_n will be chosen to be a martingale.

The first step is to represent the sum S_n in the form

$$S_n=\sum_{j=1}^{k} y_j+\sum_{j=1}^{k+1} z_j=S_n'+S_n'',$$

where

$$y_j=\sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} X_i, \qquad 1\leqq j\leqq k;$$

$$z_j=\sum_{i=jp+(j-1)q+1}^{j(p+q)} X_i, \qquad 1\leqq j\leqq k;$$

$$z_{k+1}=\sum_{i=k(p+q)+1}^{n} X_i,$$

$p=p(n)$ and $q=q(n)\in\{1, 2, \cdots, n\}$ and satisfy the following conditions:

(5) a) $p\to\infty$, $q\to\infty$, $n^{-1}p\to 0$, $p^{-1}q\to 0$,
b) $n^{1+\beta}q^{1-\beta}p^{-2}\to 0$ for some $\beta>0$,
c) $np^{-1}\rho^{2/r}(q)\to 0$,

and $k=k(n)=[n/(p+q)]$. Here $[a]$ denotes the greatest integer $\leqq a$. Such systems of p and q actually exist. In fact, if we set

$$\lambda(n)=\max\{\rho^{1/r}([n^{1/4}]),\ (\log n)^{-1}\},$$

$$p=\max\{[n\rho^{2/r}([n^{1/4}])(\lambda(n))^{-1}],\ [n^{3/4}(\lambda(n))^{-1}]\},$$

$$q=[n^{1/4}],$$

then all the conditions in (5) are satisfied;

a) $p\to\infty,\quad q\to\infty,\quad n^{-1}p\to 0,\quad p^{-1}q\to 0,$

b) $n^{1+\beta}q^{1-\beta}p^{-2}\leqq n^{-(1-3\beta)/4}\lambda^2(n)\to 0,$ if $\beta\leqq 1/3,$

c) $np^{-1}\rho^{2/r}(q)\leqq\lambda(n)\to 0.$

Now we break the sum S_n' into two parts. We denote by $\mathcal{L}_{nj}$ the σ-fields $\mathcal{P}_{jp+(j-1)q}$, and define the random variables

$$Y_{nj}=y_{nj}-E\{y_{nj}|\mathcal{L}_{n,j-1}\},\qquad 1\leqq j\leqq k,$$

where $y_{nj}=y_j/\sigma_n$. Then $\{Y_{nj},\ \mathcal{L}_{nj}:\ 1\leqq j\leqq k\}$ is trivially a martingale difference sequence for each $n\geqq 1$. Let

$$T_n=\sum_{j=1}^{k}Y_{nj},\quad T_n'=S_n'/\sigma_n-T_n=\sum_{j=1}^{k}E\{y_{nj}|\mathcal{L}_{n,j-1}\}.$$

Then $S_n=\sigma_nT_n+\sigma_nT_n'+S_n''$.

The theorem will be proved in three stages:

(i) $E|S_n''/\sigma_n|^r\to 0,$

(ii) $E|T_n'|^r\to 0,$

(iii) $E|T_n|^r\to\beta_r.$

In view of (i), (ii), (iii) and the inequality:

$$|(E|S_n/\sigma_n|^r)^{1/r}-(E|T_n|^r)^{1/r}|^r\leqq 2^{r-1}(E|T_n'|^r+E|S_n''/\sigma_n|^r),$$

the assertion of the theorem follows.

PROOF OF (i). Since $\sigma_n^2=nh(n)$, where $h(n)$ is a slowly varying function (Theorem B), using Lemma 2, Minkowski's inequality and stationarity, and arguing as in [11, p. 337], we obtain

$$\begin{aligned}E|S_n''/\sigma_n|^r&\leqq\sigma_n^{-r}(k(E|z_1|^r)^{1/r}+(E|z_{k+1}|^r)^{1/r})^r\\&\leqq K(k\sigma_q/\sigma_n+\sigma_{q'}/\sigma_n)^r\\&=K\Big(\Big(\frac{k^2qh(q)}{nh(n)}\Big)^{1/2}+\Big(\frac{q'h(q')}{nh(n)}\Big)^{1/2}\Big)^r\to 0,\end{aligned}$$

where $q'=n-k(p+q)$ is the number of terms in z_{k+1}, and (i) is proved.

Before proving (ii) and (iii), we note that under the requirements imposed on p, q and k,

(6) $$k\sigma_p^2/\sigma_n^2=1+o(1).$$

In fact,

$$E(S'_n/\sigma_n)^2=k\sigma_p^2/\sigma_n^2+2\sum_{j=2}^{k}(k-j+1)E(y_{n1}y_{nj})$$

by stationarity. Since y_{n1} is measurable with respect to $\mathcal{P}_p$ and y_{nj}, $2\leqq j\leqq k$, are measurable with respect to $\mathcal{F}_{p+q}$, applying the inequality (2),

$$\sum_{j=2}^{k}(k-j+1)|E(y_{n1}y_{nj})|\leqq(k\sigma_p/\sigma_n)^2\rho(q).$$

Moreover, by condition (5),

$$k\rho(q)\sim np^{-1}\rho(q)\leqq np^{-1}\rho^{2/r}(q)\to 0.$$

Hence

(7) $$E(S'_n/\sigma_n)^2=(k\sigma_p^2/\sigma_n^2)(1+o(1)).$$

On the other hand,

(8) $$\begin{aligned}E(S'_n/\sigma_n)^2&=E(S_n/\sigma_n)^2+E(S''_n/\sigma_n)^2-2E(S_nS''_n/\sigma_n^2)\\&=1+o(1)\end{aligned}$$

by (i). The equality (6) now follows from (7) and (8).

PROOF OF (ii). For simplicity we put

$$w_{nj}=E\{y_{nj}|\mathcal{L}_{n,j-1}\},\qquad 1\leqq j\leqq k,$$

and because of the stationarity, we put

$$a_n=E|y_{nj}|^r,\qquad 1\leqq j\leqq k.$$

By Hölder's inequality,

$$\begin{aligned}E|w_{nj}|^r&=E(w_{nj}w_{nj}|w_{nj}|^{r-2})\\&=E(E\{y_{nj}w_{nj}|w_{nj}|^{r-2}|\mathcal{L}_{n,j-1}\})\\&=E(y_{nj}w_{nj}|w_{nj}|^{r-2})\\&\leqq(E|y_{nj}w_{nj}|^{r/2})^{2/r}(E|w_{nj}|^r)^{1-2/r},\end{aligned}$$

so that we have

$$E|w_{nj}|^r\leqq E|y_{nj}w_{nj}|^{r/2}.$$

Since w_{nj} is measurable with respect to $\mathcal{P}_{(j-1)p+(j-2)q}$ and y_{nj} is measurable with respect to $\mathcal{F}_{(j-1)(p+q)}$ for each $1\leqq j\leqq k$, using (2) and Jensen's inequality,

$$\begin{aligned}E|y_{nj}w_{nj}|^{r/2}&\leqq(E|y_{nj}|^r)^{1/2}(E|w_{nj}|^r)^{1/2}\rho(q)+E|y_{nj}|^{r/2}E|w_{nj}|^{r/2}\\&\leqq a_n\rho(q)+a_n^{1/2}E|w_{nj}|^{r/2}.\end{aligned}$$

Using (2) and Jensen's inequality again,

$$\begin{aligned} E|w_{nj}|^{r/2} &= E(y_{nj}w_{nj}|w_{nj}|^{r/2-2}I(|w_{nj}|>0)) \\ &\leqq (Ey_{nj}^2)^{1/2}(E|w_{nj}|^{r-2})^{1/2}\rho(q) \\ &\leqq a_n^{1/2}\rho(q)\,. \end{aligned}$$

Combining the estimates above, we find that

$$E|w_{nj}|^r \leqq 2a_n\rho(q)\,. \tag{9}$$

We obtain from Minkowski's inequality, (4)-(6) and (9) that

$$\begin{aligned} E|T_n'|^r &\leqq \Big(\sum_{j=1}^k (E|w_{nj}|^r)^{1/r}\Big)^r \\ &\leqq 2k^r a_n\rho(q) \\ &\leqq K(k\sigma_p^2/\sigma_n^2)^{r/2}k^{r/2}\rho(q)\to 0\,, \end{aligned}$$

and hence (ii) is proved.

PROOF OF (iii). Define w_{nj} and a_n as before. For simplicity of notation we also define

$$u_{nj}=y_{nj}^2-Ey_{nj}^2\,, \qquad 1\leqq j\leqq k,$$

$$v_{nj}=E\{u_{nj}|\mathcal{L}_{n,j-1}\}\,, \qquad 1\leqq j\leqq k$$

and (because of the stationarity)

$$b_n=E|u_{nj}|^{r/2}, \qquad 1\leqq j\leqq k.$$

Now by stationarity,

$$\sum_{j=1}^k EY_{nj}^2=k\sigma_p^2/\sigma_n^2-\sum_{j=1}^k E(y_{nj}w_{nj})\,,$$

and using (2),

$$\sum_{j=1}^k E(y_{nj}w_{nj})\leqq(k\sigma_p^2/\sigma_n^2)\rho(q)\,.$$

Thus, taking account of (6), we see that

$$\sum_{j=1}^k EY_{nj}^2=1+o(1)\,.$$

Therefore, according to Hall's [8] theorem, the proof of (iii) will be complete if we can show that

$$\max_{j\leq k} E\{Y_{nj}^2|\mathcal{L}_{n,j-1}\}\to 0 \qquad \text{in probability,} \tag{10}$$

$$\sum_{j=1}^k E|Y_{nj}|^r\to 0 \tag{11}$$

and

(12) $$E\left|\sum_{j=1}^{k} E\{Y_{nj}^2 \mid \mathcal{L}_{n,j-1}\}-1\right|^{r/2} \to 0 .$$

However, (11) immediately implies the conditional Lindeberg condition:

$$\text{for all } \varepsilon>0, \quad \sum_{j=1}^{k} E\{Y_{nj}^2 I(|Y_{nj}|>\varepsilon) \mid \mathcal{L}_{n,j-1}\} \to 0 \quad \text{in probability.}$$

Hence (10) is a consequence of (11) combined with (12) (see Brown [6, Theorem 1 and Lemma 1]). We have from Jensen's inequality, (4) and (6) that

$$\begin{aligned}\sum_{j=1}^{k} E|Y_{nj}|^r &\leqq 2^{r-1}\sum_{j=1}^{k}(E|y_{nj}|^r+E|w_{nj}|^r)\\ &\leqq 2^r k a_n\\ &\leqq K(k\sigma_p^2/\sigma_n^2)^{r/2}k^{-r/2+1}\to 0 ,\end{aligned}$$

and thus (11) holds.

Our goal is to show that (12) holds. Now,

$$\begin{aligned}&E\left|\sum_{j=1}^{k} E\{Y_{nj}^2 \mid \mathcal{L}_{n,j-1}\}-1\right|^{r/2}\\ &=E\left|\sum_{j=1}^{k} E\{y_{nj}^2 \mid \mathcal{L}_{n,j-1}\}-\sum_{j=1}^{k} w_{nj}^2-1\right|^{r/2}\\ &\leqq 2^{r/2-1}\left\{E\left|\sum_{j=1}^{k} E\{y_{nj}^2 \mid \mathcal{L}_{n,j-1}\}-1\right|^{r/2}+E\left(\sum_{j=1}^{k} w_{nj}^2\right)^{r/2}\right\}.\end{aligned}$$

Making use of the inequality (9), and arguing just as in the proof of (ii), we get

$$\begin{aligned}E\left(\sum_{j=1}^{k} w_{nj}^2\right)^{r/2} &\leqq 2k^{r/2}a_n\rho(q)\\ &\leqq K(k\sigma_p^2/\sigma_n^2)^{r/2}\rho(q)\to 0 .\end{aligned}$$

Moreover, we have from (6) and Minkowski's inequality that

$$\begin{aligned}&E\left|\sum_{j=1}^{k} E\{y_{nj}^2 \mid \mathcal{L}_{n,j-1}\}-1\right|^{r/2}\\ &\sim E\left|\sum_{j=1}^{k} E\{y_{nj}^2 \mid \mathcal{L}_{n,j-1}\}-k\sigma_p^2/\sigma_n^2\right|^{r/2}\\ &=E\left|\sum_{j=1}^{k} v_{nj}\right|^{r/2}\\ &\leqq\left(\sum_{j=1}^{k}(E|v_{nj}|^{r/2})^{2/r}\right)^{r/2}.\end{aligned}$$

Consequently, to prove (12) it is sufficient to show that

(13) $$\sum_{j=1}^{k}(E|v_{nj}|^{r/2})^{2/r}\to 0 .$$

We shall separate the proof of (13) in three cases; $r>4$, $r=4$ and $2<r<4$.

Suppose first that $r>4$. By replacing y_{nj}, w_{nj}, a_n and r in the proof of (9) by u_{nj}, v_{nj}, b_n and $r/2$ respectively, we deduce that

$$E|v_{nj}|^{r/2}\leqq 2b_n\rho(q)\,.$$

Since

$$b_n\leqq 2^{r/2-1}\{E|y_{nj}|^r+(Ey_{nj}^2)^{r/2}\}\leqq 2^{r/2}a_n\,,$$

then, by virtue of (4) and (6), we see that

$$(14)\qquad \sum_{j=1}^{k}(E|v_{nj}|^{r/2})^{2/r}\leqq k(2b_n\rho(q))^{2/r}$$
$$\leqq k(2^{r/2+1}a_n\rho(q))^{2/r}$$
$$\leqq K(k\sigma_p^2/\sigma_n^2)\rho^{2/r}(q)\to 0\,,$$

and thus (13) is proved for the case $r>4$.

When $r=4$, using (2) and Jensen's inequality, we get

$$E|v_{nj}|^{r/2}=E(u_{nj}v_{nj})$$
$$\leqq(Eu_{nj}^2)^{1/2}(Ev_{nj}^2)^{1/2}\rho(q)$$
$$\leqq Eu_{nj}^2\rho(q)\,.$$

Hence (13) also holds for $r=4$.

Finally, we assume that $2<r<4$. By Hölder's inequality,

$$(15)\qquad E|v_{nj}|^{r/2}=E(u_{nj}v_{nj}|v_{nj}|^{r/2-2}I(|v_{nj}|>0))$$
$$\leqq E(|u_{nj}|^{2-r/2}|u_{nj}v_{nj}|^{r/2-1})$$
$$\leqq b_n^{4/r-1}(E|u_{nj}v_{nj}|^{r/4})^{2-4/r}.$$

Using (2) and Jensen's inequality, and noting that $r/4<1$,

$$(16)\qquad E|u_{nj}v_{nj}|^{r/4}\leqq(E|u_{nj}|^{r/2})^{1/2}(E|v_{nj}|^{r/2})^{1/2}\rho(q)+E|u_{nj}|^{r/4}E|v_{nj}|^{r/4}$$
$$\leqq b_n\rho(q)+b_n^{1/2}(E|v_{nj}|)^{r/4}.$$

Since $\alpha(n)\leqq\rho(n)$, applying the inequality (3) with $\xi=v_{nj}|v_{nj}|^{-1}I(|v_{nj}|>0)$, $\eta=u_{nj}$ and $s=r/2$,

$$(17)\qquad E|v_{nj}|=E(u_{nj}v_{nj}|v_{nj}|^{-1}I(|v_{nj}|>0))$$
$$\leqq 6(E|u_{nj}|^{r/2})^{2/r}(\rho(q))^{1-2/r}.$$

Inserting the inequalities (16) and (17) into (15), we have

$$E|v_{nj}|^{r/2}\leqq b_n^{4/r-1}\{b_n\rho(q)+6^{r/4}b_n(\rho(q))^{r/4-1/2}\}^{2-4/r}$$
$$\leqq Kb_n(\rho(q))^{(r-2)^2/2r}.$$

Just as in (14), we obtain that for $2<r<4$,

$$\sum_{j=1}^{k}(E|v_{nj}|^{r/2})^{2/r}\leqq K(k\sigma_p^2/\sigma_n^2)(\rho(q))^{(r-2)^2/r^2}\to 0\,,$$

and hence (13) follows as desired.

The proof of Theorem C is now complete.

References

[1] Bernstein, S., Quelques remarques sur le théorème limite Liapounoff. Dokl. Akad. Nauk SSSR **24** (1939), 3-8.

[2] Billingsley, P., Convergence of Probability Measures, Wiley, New York, 1968.

[3] Bradley, R. C., A remark on the central limit question for dependent random variables. J. Appl. Probability **17** (1980), 94-101.

[4] Brown, B. M., Moments of a stopping rule related to the central limit theorem. Ann. Math. Statist. **40** (1969), 1236-1249.

[5] Brown, B. M., Characteristic functions, moments and the central limit theorem. Ann. Math. Statist. **41** (1970), 658-664.

[6] Brown, B. M., Martingale central limit theorems. Ann. Math. Statist. **42** (1971), 59-66.

[7] Davydov, Yu. A., Convergence of distributions generated by stationary stochastic processes. Theor. Probability Appl. **13** (1968), 691-696.

[8] Hall, P., The convergence of moments in the martingale central limit theorem. Z. Wahrscheinlichkeitstheorie verw. Gebiete **44** (1978), 253-260.

[9] Ibragimov, I. A., Some limit theorems for stationary processes. Theor. Probability Appl. **7** (1962), 349-382.

[10] Ibragimov, I. A., A note on the central limit theorem for dependent random variables. Theor. Probability Appl. **20** (1975), 135-141.

[11] Ibragimov, I. A. and Linnik, Yu. V., Independent and Stationary Sequences of Random Variables, Wolters-Noordhoff, Groningen, 1971.

[12] Ibragimov, I. A. and Rozanov, Yu. A., Gaussian Random Processes, Springer-Verlag, New York, 1978.

[13] Kolmogorov, A. N. and Rozanov, Yu. A., On strong mixing conditions for stationary Gaussian processes. Theor. Probability Appl. **5** (1960), 204-208.

[14] Lifshits, B. A., On the central limit theorem for Markov chains. Theor. Probability Appl. **23** (1978), 279-296.

[15] Rosenblatt, M., A central limit theorem and a strong mixing condition. Proc. Nat. Acad. Sci. USA **42** (1956), 43-47.

[16] Rosenblatt, M., Markov Processes. Structure and Asymptotic Behavior, Springer-Verlag, Berlin, 1971.

[17] Yokoyama, R., Convergence of moments in the central limit theorem for stationary ϕ-mixing sequences. Tsukuba J. Math. **3**, No. 2 (1979), 1-6.

[18] Yokoyama, R., Moment bounds for stationary mixing sequences. Z. Wahrscheinlichkeitstheorie verw. Gebiete **52** (1980), 45-57.

[19] Yokoyama, R., The convergence of moments in the central limit theorem for stationary ϕ-mixing processes. Analysis Math. (to appear)

Department of Mathematics
Osaka Kyoiku University
Tennoji, Osaka 543