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ON CERTAIN CURVES OF GENUS THREE WITH MANY AUTOMORPHISMS

By

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Introduction.

Let k be an algebraically closed ground field. When C is a complete nonsingular curve of genus g and G is a subgroup of its automorphism group $\mathrm{Aut}(C)$, we call the pair (C, G) an AM curve of genus g (AM stands for " automorphism ").

In Part I, we consider the AM curve $(K, \mathrm{Aut}(K))$, where K is the plane curve defined by $x_1x_2^3+x_2x_3^3+x_3x_1^3$ (in $\mathrm{char}(k)\neq 7$). It is known [7] that $\sharp\mathrm{Aut}(K)$ attains the Hurwitz's bound: $84(g-1)$ with $g=3$, in case $\mathrm{char}(k)>g+1$ with $g=3$. To determine $(K, \mathrm{Aut}(K))$, we use the fact that $\mathrm{Aut}(C)$ of a nonsingular quartic plane curve C is canonically identified with a subgroup of $PGL(3, k)$. We shall show in particular that when $\mathrm{char}(k)=3$, $(K, \mathrm{Aut}(K))$ is isomorphic to the AM curve $(K_4, PSU(3, 3^2))$, where K_4 is defined by $x_1^4+x_2^4+x_3^4$ and $PSU(3, 3^2)$ is a simple subgroup of $PGL(3, k)$ of order 6048. We note that it is the maximum order among the automorphism groups of (complete nonsingular) curves of genus 3 [8].

In Part II we consider the families of AM curves (C, G) of genus 3, where G is isomorphic to the symmetric group of degree 4, $\mathfrak{S}_4$. (We note that $\mathrm{Aut}(K)$ contains such subgroups.) In § 1, we shall determine " normal forms " of such AM curves. In § 2 we shall determine the isomorphism classes in the above normal forms. In § 3, using these results, we explain the relations between the subgroups of Teichmüller modular group Mod(3) which are isomorphic to $\mathfrak{S}_4$ and their representations on the spaces of holomorphic differentials. In fact, for an AM Riemann surface (W, G) (similarly defined as in the case of AM curves), we obtain naturally a subgroup (denoted by $M(W, G)$) of the Teichmüller modular group Mod(3), which is isomorphic to G. Also we obtain a subgroup (denoted by $\rho(W, G)$) of $GL(3, \boldsymbol{C})$ which is the image of the representation of G on the space of holomorphic differentials. We shall prove:

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THEOREM. *Let (W, G) be an AM Riemann surface of genus three. Assume that G is isomorphic to $\mathfrak{S}_4$. Then we have:*

(1) $M(W, G)$ *is* Mod(3)-*conjugate to either* MG_{24} *or* MH_{24}, $\rho(W, G)$ *is* $GL(3, \boldsymbol{C})$-*conjugate to either* G_{24} *or* H_{24}.

(2) $M(W, G)\sim MG_{24}$ (*resp.* MH_{24}) *if and only if* $\rho(W, G)\sim G_{24}$ (*resp.* H_{24}).

MG_{24} and MH_{24} (resp. G_{24} and H_{24}) in the above are certain subgroups of Mod(3) (resp. $GL(3, \boldsymbol{C})$), which are explained in (3.1) of Part II.

Contents.

Part I. On the automorphism group of Klein's quartic curve.

§1. Notations and theorem.

1.1. Let k be an algebraically closed base field of characteristic $p\geqq 0$. A curve will mean a complete nonsingular curve over k. If C is a nonhyperelliptic curve of genus 3, then its canonical embedding is a quartic plane curve. Conversely, any (nonsingular) quartic plane curve is nonhyperelliptic of genus 3, and its embedding into the ambient projective plane is canonical.

Let C' and C be two quartic plane curves. We denote by $\mathrm{Lin}(C', C)$ the set of automorphisms of the ambient projective plane which induce isomorphisms of C' onto C. Then it is known that the natural mapping of $\mathrm{Lin}(C', C)$ into $\mathrm{Iso}(C', C)$ is a bijection.

Considering a system of homogeneous coordinates, we put

$$\boldsymbol{P}^2=\mathrm{Proj}(k[x_1, x_2, x_3]\}.$$

Then we may identify the group of automorphisms of $\boldsymbol{P}^2$, $\mathrm{Aut}(\boldsymbol{P}^2)$, with a projective linear group, $PGL(3, k)$. In fact, if a matrix (a_{ij}) represents an element of $PGL(3, k)$, its corresponding automorphism (of $\boldsymbol{P}^2$) is defined by:

$$(x_1, x_2, x_3) \mapsto \left(\sum_{j=1}^{3} a_{1j}x_j, \sum_{j=1}^{3} a_{2j}x_j, \sum_{j=1}^{3} a_{3j}x_j \right).$$

If C is a quartic plane curve in $\boldsymbol{P}^2 = \mathrm{Proj}(k[x_1, x_2, x_3])$ the automorphism group of C, $\mathrm{Aut}(C)$, is always considered as a subgroup of $PGL(3, k)$. For a matrix $T=(a_{ij})$ in $M(3, k)$, T^* denotes the homomorphism of the graded k-algebra $k[x_1, x_2, x_3]$, defined by: $x_i \mapsto \sum_{j=1}^{3} a_{ij}x_j$ $(i=1, 2, 3)$. And when T is an element of $GL(3, k)$ and H is a subset or an element of $GL(3, k)$, we denote $T^{-1}\cdot H\cdot T$ by $T^*(H)$.

We use the same notation for a quartic curve and a generator of its homogeneous ideal of definition. And we denote an element of $PGL(3, k)$ by its representatives when there is no fear of confusion. Then, for example, if C is a quartic curve and H is a subset of $\mathrm{Aut}(C)$, then for any element T of $PGL(3, k)$, $T^*(C)$ is well-defined as a plane curve, and $T^*(H)$ is also well-defined as a subset of $\mathrm{Aut}(T^*(C))$.

1.2. Notations. We fix a primitive 7-th root ζ of unity in k (if exists), and we denote: (cf. [1])

$$\beta_1 := \zeta^5+\zeta^2, \quad \beta_2 := \zeta^3+\zeta^4, \quad \beta_3 := \zeta^6+\zeta,$$

$$\gamma_1 := \zeta^5-\zeta^2, \quad \gamma_2 := \zeta^3-\zeta^4, \quad \gamma_3 := \zeta^6-\zeta,$$

$$\theta_1 := \zeta+\zeta^2+\zeta^4, \quad \theta_2 := \zeta^6+\zeta^5+\zeta^3 \quad \text{and}$$

$$\alpha_1 := \beta_3+\beta_1, \quad \alpha_2 := \beta_1+\beta_2, \quad \alpha_3 := \beta_2+\beta_3.$$

It is immediate to see:

(1) $\beta_1^2=\beta_2+2,\ \beta_2^2=\beta_3+2,\ \beta_3^2=\beta_1+2,\ \beta_1\beta_2=\beta_1+\beta_3,\ \beta_2\beta_3=\beta_2+\beta_1,\ \beta_3\beta_1=\beta_3+\beta_2,$

(2) β_1, β_2 and β_3 are the distinct three roots of the equation $\beta^3+\beta^2-2\beta+1=0$,

(3) θ_1 and θ_2 are the distinct two roots of the equation $(2\theta+1)^2+7=0$,

(4) $\beta_1\gamma_1=\gamma_2,\ \beta_2\gamma_2=\gamma_3,\ \beta_3\gamma_3=\gamma_1,\ \alpha_1\gamma_1=\gamma_3,\ \alpha_2\gamma_2=\gamma_1,\ \alpha_3\gamma_3=\gamma_2.$

Next we define distinguished elements and a subgroup of $GL(3, k)$ as follows: (cf. [3, p. 444])

$\lambda := D(\zeta^2, \zeta^4, \zeta)$, $\sigma_i := \gamma_i\cdot(\theta_1-\theta_2)^{-1}\cdot S(\alpha_i, \beta_i, 1)$, $(i=1, 2, 3)$

$$\tau := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \text{ where } D(a, b, c) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \text{ and } S(a, b, c) = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}.$$

And $G_K := \langle \lambda, \tau, \sigma \rangle$, where $\sigma := \sigma_1$.

1.2.1. LEMMA. *The followings hold in* $GL(3, k)$:

(1) *the order of* λ (*resp.* τ, σ) *is* 7 (*resp.* 3, 2).

(2) $\sigma_1=\tau\sigma_2$, $\sigma_2=\tau\sigma_3$, $\sigma_3=\tau\sigma_1$, $\sigma_1\tau=\sigma_2$, $\sigma_2\tau=\sigma_3$, $\sigma_3\tau=\sigma_1$,

(3) $\tau\lambda\tau^{-1}=\lambda^2$,

(4) "*defining relation*" $\sigma_i\lambda^{-2i}\sigma_i=\lambda^{2i}\sigma_i\lambda^{2i}$ $(i=1, 2, 3)$.

PROOF. These are followed from above by direct calculation.

1.3. LEMMA. *Assume that* $\mathrm{char}(k)\neq 7$. *There is an isomorphism of* $PSL(2, 7)$ *onto* G_K *sending* $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ $\left(\textit{resp.}\ \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 3\\ 1 & 0\end{bmatrix}\right)$ *to* λ (*resp.* τ, σ). *Hence the natural homomorphism of* G_K *into* $PGL(3, k)$ *is injective.*

PROOF. We have known that the followings are defining relations for $PSL(2, 7)$:

$$x^7=y^3=1,\quad y^{-1}xy=x^2,\quad t^2=1,\quad t^{-1}yt=y^{-1}\ \text{ and }\ (xt)^3=1.$$

If we take (in $PSL(2, 7)$)

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}^{-1} \text{ and } \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}0 & 3\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$$

in lieu of x, y and z, then these satisfy the above relations. From (1.2.1) λ, τ^{-1} and $\tau^{-1}\sigma\tau$ also satisfy the relations. Therefore there is a surjective homomorphism as in the statement of the Lemma. Since $PSL(2, 7)$ is a simple group (of order 168), this is an isomorphism. Then the latter part is obvious. Q. E. D.

1.4. A couple (C, G) of a curve C and its automorphism group G shall be called an AM curve. An isomorphism of AM curves of (C', G') onto (C, G) is an isomorphism of curves $T: C'\to C$ such that $G'=T^{-1}GT$. In this case we denote (C', G') by $T^*(C, G)$ or $(T^*(C), T^*(G))$, and also write $(C', G')\cong(C, G)$.

The purpose of this part is to prove the following theorem:

1.4.1. THEOREM. *When* $\mathrm{char}(k)\neq 3$ (*resp.* $\mathrm{char}(k)=3$), $(K, \mathrm{Aut}(K))$ *is isomorphic* (*as AM curves*) *to* (K, G_K) (*resp.* $(K_4, PSU(3, 3^2))$). *Moreover when* $\mathrm{char}(k)=2$, $(K, \mathrm{Aut}(K))$ *is isomorphic to* $(K_2, PSL(3, 2))$.

In the above, K denotes the plane curve defined by $x_1x_2^3+x_2x_3^3+x_3x_1^3$, in case $\mathrm{char}(k)\neq 7$. K_4 denotes the curve $x_1^4+x_2^4+x_3^4$ and K_2 denotes the curve $x_1^4+x_2^4+x_3^4+x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2+x_1x_2x_3(x_1+x_2+x_3)$. And $PSU(3, 3^2)$ denotes the injective image in $PGL(3, k)$ (in case $\mathrm{char}(k)=3$) of

$$SU(3, 3^2)=\{A\in SL(3, 3^2)\,|\,{}^tA\cdot A^{(3)}=I\},$$

where $A^{(l)}:=(a_{ij}^l)$ if $A=(a_{ij})$. It is known as a simple group of order $2^5\cdot 3^3\cdot 7=6048$. $PSL(3, 2)$ denotes the injective image in $PGL(3, k)$ (in case $\mathrm{char}(k)=2$) of a finite general linear group $GL(3, 2)$. It is known as a simple group of order $2^3\cdot 3\cdot 7=168$.

A part of proof. First we note that in case where $\mathrm{char}(k)=7$, K is a singular plane curve, so we omit this case. Now it follows that $\lambda^*(K)=K$, $\tau^*(K)=K$ and $\sigma^*(K)=K$ in $k[x_1, x_2, x_3]$ by direct calculation using (1.2). So G_K is contained in $\mathrm{Aut}(K)$ (in $PGL(3, k)$). On the other hand, when $\mathrm{char}(k)\neq 2$ or 3, it follows from [7] that $\sharp\mathrm{Aut}(K)\leqq 84(g-1)$ with $g=3$. Thus we get that $\mathrm{Aut}(K)=G_K$ in these cases.

The excluded cases are settled in § 2, (2.2.1) and § 3, (3.1.1).

§ 2. The case char$(k)=2$.

Throughout this section we assume that $\mathrm{char}(k)=2$. First we write down rather general notations for the use in Part II.

2.1. Notations. We define distinguished subgroups of $GL(3, 2)$:

$$G_8:=\langle R_+, R_-\rangle,\ G_{24}(+):=\langle S_+, R_+R_-\rangle,\ G_{24}(-):=\langle S_-, R_+R_-\rangle$$

where

$$R_+:=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\quad R_-:=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\quad S_+:=\begin{pmatrix}1&0&0\\0&0&1\\0&1&1\end{pmatrix}\quad S_-:=\begin{pmatrix}1&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

Here we have known that G_8 is a 2-Sylow subgroup of $GL(3, 2)$ and that $G_{24}(+)$ and $G_{24}(-)$ are isomorphic to the symmetric group of degree 4, $\mathfrak{S}_4$.

Also we define distinguished families of AM curves as follows:

$F_8:=$the set of AM curves $(C(a, b), G_8)$ (with parameters a and b)
$F_{24}(+):=$the set of AM curves $(C(a, a), G_{24}(+))$
$F_{24}(-):=$the set of AM curves $(C(1, b), G_{24}(-))$

where

$$C(a, b):=x_1^4+ax_2^4+bx_3^4+x_1^2x_2^2+ax_2^2x_3^2+x_3^2x_1^2+x_1x_2x_3(x_1+x_2+x_3).$$

When G is a subgroup of $GL(3, k)$ (in any characteristic) we denote by $F(G)$ the set of (nonsingular) quartic AM curves (C, G). Forgetting automorphism groups, we also use the above each family as the set of corresponding curves.

Now we prove a lemma which characterize the curve K_2.

2.1.1. LEMMA. *We have*: $F_{24}(+)=F(G_{24}(+))$ *and* $F_{24}(-)=F(G_{24}(-))$. *Hence* $F(PSL(3, 2))=\{K_2\}$.

PROOF. Comparing coefficients we see easily that $F(\langle R_+R_-\rangle)$=the set of curves $C(a, b, c_2, c_3)$, where $C(a, b, c_2, c_3):=x_1^4+ax_2^4+bx_3^4+(x_1^2x_2^2+c_2x_2^2x_3^2+c_3x_3^2x_1^2)+x_1x_2x_3(x_1+x_2+x_3)+(1+c_3)x_2^3x_3+(1+c_3+a+c_2)x_2x_3^3+(1+c_3)x_1x_3^3$ with a, b, c_2 and c_3 in k. Again comparing coefficients as for S_+ (resp. S_-), we get that $F(\langle S_+, R_+R_-\rangle)$=the set of curves of the form $C(a, a, a, 1)$ i.e. $F_{24}(+)$, and that $F(\langle S_-, R_+R_-\rangle)$=the set of curves of the form $C(1, b, 1, 1)$ i.e. $F_{24}(-)$. Since $\langle S_+, S_-, R_+R_-\rangle$ is equal to $PSL(3, 2)$, it follows from these facts that $F(PSL(3, 2))=F_{24}(+)\cap F_{24}(-)=\{C(1, 1)$ i.e. $K_2\}$. Q.E.D.

2.2. We shall prove (2.2.1) using (2.2.2).

2.2.1. PROPOSITION. $(K, \mathrm{Aut}(K))\cong(K_2, PSL(3, 2))$.

2.2.2. LEMMA. *Let C be a curve in F_8, and let T be an element of $GL(3, k)$. If $T^*(C)$ is again a curve in F_8, then T is contained in $PSL(3, 2)$ (in $PGL(3, k)$).*

PROOF of (2.2.2). Let $C=C(a, b)$ and $T=(a_{ij})$ be as above. We denote $T^{(2)}:=(a_{ij}^2)$, $\Delta:=(\Delta_{ij})$ where Δ_{ij} are the cofactors of the matrix (a_{ij}), and put ${}^t\Delta\cdot T^{(2)}=(b_{ij})$. Then we have (in $k[x_1, x_2, x_3]$):

$$\begin{aligned}T^*(x_1x_2x_3(x_1+x_2+x_3))&=T^*(x_1^2x_2x_3+x_2^2x_3x_1+x_3^2x_1x_2)\\&=(a_{11}^2x_1^2+a_{12}^2x_2^2+a_{13}^2x_3^2)(a_{21}x_1+a_{22}x_2+a_{23}x_3)(a_{31}x_1+a_{32}x_2+a_{33}x_3)\\&\quad+(a_{21}^2x_1^2+a_{22}^2x_2^2+a_{23}^2x_3^2)(a_{31}x_1+a_{32}x_2+a_{33}x_3)(a_{11}x_1+a_{12}x_2+a_{13}x_3)\\&\quad+(a_{31}^2x_1^2+a_{32}^2x_2^2+a_{33}^2x_3^2)(a_{11}x_1+a_{12}x_2+a_{13}x_3)(a_{21}x_1+a_{22}x_2+a_{23}x_3).\end{aligned}$$

Thus we have:

(the coefficient of $x_1^2x_2x_3$ in $T^*(C(a, b))$)
$=$(the coefficient of $x_1^2x_2x_3$ in $T^*(x_1x_2x_3(x_1+x_2+x_3))$)
$=a_{11}^2\Delta_{11}+a_{21}^2\Delta_{21}+a_{31}^2\Delta_{31}=b_{11}$.

Similarly we have:

(the coefficient of $x_2^2x_2x_3$ (resp. $x_3^2x_2x_3$) in $T^*(C(a, b)))=b_{12}$ (resp. b_{13}).
(the coefficient of $x_1^2x_3x_1$ (resp. $x_2^2x_3x_1$, $x_3^2x_3x_1$, $x_1^2x_1x_2$, $x_2^2x_1x_2$, $x_3^2x_1x_2$) in $T^*(C(a, b)))=b_{21}$ (resp. b_{22}, b_{23}, b_{31}, b_{32}, b_{33}).

Since $T^*(C(a, b))$ is a curve in F_8, we have that ${}^t\Delta\cdot T^{(2)}=(b_{ij})=I$ in $PGL(3, k)$. On the other hand we have ${}^t\Delta\cdot T=I$ in $PGL(3, k)$. It follows that $T=T^{(2)}$ in $PGL(3, k)$. This means that T is contained in $PSL(3, 2)$. Q.E.D.

PROOF of (2.2.1). It follows from (2.1.1) and (2.2.2) that $\mathrm{Aut}(K_2)=PSL(3, 2)$. On the other hand it is easy to see that $S(\beta_1, \alpha_1, 1)^*K=K_2$ (as curves) by (1.2). Thus we conclude that $(K, \mathrm{Aut}(K))$ is isomorphic (as AM curves) to $(K_2, PSL(3, 2))$. Q. E. D.

Also from (2.1.1), (2.2.1) and (2.2.2) we get:

2.2.3. REMARK. $G_{24}(+)$ and $G_{24}(-)$ are not $PGL(3, k)$-conjugate to each other.

§ 3. The case char$(k)=3$.

In this section we assume that char$(k)=3$.

3.1. We shall prove (3.1.1) using (3.1.2).

3.1.1. PROPOSITION. $(K, \mathrm{Aut}(K))\cong(K_4, PSU(3, 3^2))$.

3.1.2. LEMMA. *Let T be an element of $GL(3, k)$ such that $T^*(K_4)$ is in F_{24}. Then T is contained in $PSU(3, 3^2)$ (in $PGL(3, k)$), and $T^*(K_4)=K_4$.*

In the above, F_{24} denotes (in general when char$(k)\neq 2$), the set of AM curves $(C(a), G_{24})$ where $C(a)$ is a plane curve defined by: $x_1^4+x_2^4+x_3^4+a(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2)$, $a\in k$, and G_{24} is a subgroup $\langle R, S\rangle$ of $GL(3, k)$, with

$$R=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \quad\text{and}\quad S=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$$

PROOF of (3.1.2). Let $T=(a_{ij})$ and ${}^tT\cdot T^{(3)}=(b_{ij})$. First we note that:

$$\begin{aligned}T^*(K_4)&=(a_{11}x_1+a_{12}x_2+a_{13}x_3)^4+(a_{21}x_1+a_{22}x_2+a_{23}x_3)^4+(a_{31}x_1+a_{32}x_2+a_{33}x_3)^4\\&=b_{11}x_1x_1^3+b_{12}x_1x_2^3+b_{13}x_1x_3^3+b_{21}x_2x_1^3+b_{22}x_2x_2^3+b_{23}x_2x_3^3\\&\quad+b_{31}x_3x_1^3+b_{32}x_3x_2^3+b_{33}x_3x_3^3\,.\end{aligned}$$

Hence it follows by the assumption that $T^*(K_4)=K_4$ and that ${}^tT\cdot T^{(3)}=I$ (in $PGL(3, k)$). Then we have also that $T^{(3)*}(K_4)=K_4$, so that ${}^tT^{(3)}\cdot T^{(9)}=I$ i.e. ${}^tT^{(9)}\cdot T^{(3)}=I$. Hence we get that $T=T^{(9)}$ (in $PGL(3, k)$). Put $c^{-8}\cdot T=T^{(9)}$ in $GL(3, k)$ with some c in k. Then we have that cT is in $GU(3, 3^2)$ and so that $(\det(cT))^5\cdot cT$ is in $SU(3, 3^2)$. Q. E. D.

PROOF of (3.1.1.). It also follows from the above proof that $PSU(3, 3^2)$ is contained in $\mathrm{Aut}(K_4)$. So we have that $\mathrm{Aut}(K_4)=PSU(3, 3^2)$. On the other hand it is easy to see that $S(\beta_1, \alpha_1, 1)^*(K)=K_4$ by (1.2). Thus we conclude that $(K, \mathrm{Aut}(K))$ is isomorphic to $(K_4, PSU(3, 3^2))$. Q. E. D.

3.2. REMARK. In the similar line (as in (3.1)) we also have that $\mathrm{Aut}(X_{q+1})$ is isomorphic to $PU(3, q^2)$, if $\mathrm{char}(k)=p$ is positive and $q=p^n>3$ with $n\geqq 1$. In the above, X_{q+1} denotes the (nonsingular) plane curve (of genus $2^{-1}\cdot q(q-1)$) defined by: $x_1^{q+1}+x_2^{q+1}+x_3^{q+1}$. Hence the order of $\mathrm{Aut}(X_{q+1})$ is $(q^3+1)q^3(q^2-1)$. Moreover if $(3, q+1)=1$, then $PU(3, q^2)=PSU(3, q^2)$ is a simple group. Here we note that this curve is isomorphic to the curve defined by: $y^q+y=x^{q+1}$, (e.g. [8, p. 528]).

Part II. On curves of genus three which have automorphism groups isomorphic to $\mathfrak{S}_4$.

§1. Normal forms.

The purpose of this section is to prove the following theorem:

1.1. THEOREM. *Let (C, G) be an AM curve of genus three. Assume that G is isomorphic to $\mathfrak{S}_4$. Then there is an isomorphism T (of AM curves) such that:*

(i) *$T^*(C, G)$ is in F_{24}, hF_{24} or hF'_{24}, when* $\mathrm{char}(k)\neq 2$, *or*
(ii) *$T^*(C, G)$ is in $F_{24}(+)$ or $F_{24}(-)$, when* $\mathrm{char}(k)=2$.

In the above we denote:

F_{24}=the set of AM curves $(C(a), G_{24})$ (with a parameter a), (3.1 of Part I),
$hF_{24}=\{$the AM curve $(C^*, hG_{24})\}$,
$hF'_{24}=\{$the AM curve $(C^*, hH_{24})\}$,

where C^* denotes the hyperelliptic curve (in case where $\mathrm{char}(k)\neq 2$ or 3) defined by: $y^2=x^8+14x^4+1$, and $hG_{24}=\langle A_4\cdot J, T_3\rangle$, $hH_{24}=\langle A_4, T_3\rangle$. In the above we denote by J (resp. A_4, T_3) the automorphism of C^* defined by $(x, y)\mapsto(x, -y)$ (resp. (ix, y), $(-i(x-1)\cdot(x+1)^{-1}, -4y(x+1)^{-4})$), ($i$ denotes $\sqrt{-1}$).

1.2. The case: $\mathrm{char}(k)\neq 2$ and C is nonhyperelliptic. Then we may assume that (C, G) is a quartic plane AM curve. Since it is obvious that $F(G_{24})=F_{24}$ (cf. (2.1 of Part I)), it suffices to show:

1.2.1. LEMMA. *Assume that* $\mathrm{char}(k)\neq 2$. *Let H be a subgroup of $PGL(3, k)$*

which is isomorphic to $\mathfrak{S}_4$. *Then* H *is* $PGL(3, k)$*-conjugate to* G_{24}.

PROOF. We denote by $\boldsymbol{P}$-PGL (resp. $\boldsymbol{D}$-PGL) the set of elements of $PGL(3, k)$ which are represented by (a_{ij}), where $a_{31}=a_{32}=a_{13}=a_{23}=0$ (resp. $a_{ij}=0$ if $i\neq j$). Also we denote $\langle S^2, RS^2R^{-1}\rangle$ by G_4.

Let $V=\langle A_1, A_2\rangle$ be the (unique) normal subgroup of H of order 4. We may assume that $A_1=S^2$ by the Jordan's canonical form. Then A_2 is contained in $\boldsymbol{P}$-PGL, which is equal to the centralizer of S^2 in $PGL(3, k)$, $C_{PGL}(S^2)$. Since $A_2^2=I$ (in $PGL(3, k)$), there is an element T in $\boldsymbol{P}$-PGL such that $T^*(A_2)$ is in $\boldsymbol{D}$-PGL. Thus we get that $T^*(V)=\langle T^*(A_1), T^*(A_2)\rangle=G_4$. So we may assume that V is equal to G_4.

Next it is easy to show that $C_{PGL}(G_4)=\boldsymbol{D}$-$PGL$ and that the normalizer of G_4 in $PGL(3, k)$, $N_{PGL}(G_4)$ eguals to $\langle R, S'\rangle\cdot C_{PGL}(G_4)$, where $S'=S^2\cdot RSR$. Therefore H contains an element of the form RD, where $D=D(\alpha, \beta, 1)$ (cf. (1.2 of Part I)). Let v be a solution of the equation $\alpha\beta v^3=1$. Then we have that $D(\beta v^2, v, 1)^*(RD)=R$ (in $PGL(3, k)$). Thus we may assume that R belongs to H.

Since H is isomorphic to $\mathfrak{S}_4$, we have that $N_H(\langle R\rangle)=\langle R, S'D'\rangle$ for some $D'=D(\gamma, \delta, 1)$. It follows from $(S'D')^2=I$ that $\gamma\delta=1$. And it follows from $S'D'\cdot R(S'D')^{-1}=R^{-1}$ that $\gamma^2=\delta$. Then we have that $D'^*(S'D')=S'$. Since this D' is in $C_{PGL}\langle S^2, R\rangle$, we get that $D'^*(H)=\langle D'^*(S^2), D'^*(R), D'^*(S'D')\rangle=G_{24}$. This completes the proof of (1.2.1), and hence the theorem (1.1) in case where $\mathrm{char}(k)\neq 2$ and C is nonhyperelliptic.

1.3. The case: C is hyperelliptic.

First we show:

1.3.1. LEMMA. *Assume that* $\mathrm{char}(k)\neq 2$.

(1) *Let* $\underline{H}$ *be an abelian subgroup of* $PGL(2, k)$ *of type* $(2, 2)$. *Then* $\underline{H}$ *is* $PGL(2, k)$*-conjugate to* $\underline{H}_4$, *where* $\underline{H}_4$ *denotes* $\langle \underline{A}^2, \underline{B}\rangle$.

(2) $N_{PGL(2,k)}(\underline{H}_4)$ *is equal to* $\langle \underline{A}, \underline{T}_3\rangle$ *and is isomorphic to* $\mathfrak{S}_4$.

In the above we denote $\begin{bmatrix} i & 0 \\ 0 & 1\end{bmatrix}$ $\left(\text{resp. } \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, \begin{bmatrix} 1 & -1 \\ i & i\end{bmatrix}\right)$ by $\underline{A}$ (resp. $\underline{B}$, $\underline{T}_3$). Also we shall denote $\begin{bmatrix} \alpha & 0 \\ 0 & \beta\end{bmatrix}$ by $D(\alpha, \beta)$.

PROOF. (1) Let $\underline{H}=\langle \underline{A}_1, \underline{A}_2\rangle$. We may assume that $\underline{A}_1=\underline{A}^2$ by the Jordan's canonical form. Then $\underline{A}_2$ is of the form $D(\alpha, 1)\underline{B}$. Put $\underline{T}=D(\beta, 1)\underline{B}$ with $\beta^2=\alpha$. Then we have that $\underline{T}^{-1}\cdot\underline{H}\underline{T}=\langle \underline{T}^{-1}\underline{A}_1\underline{T}, \underline{T}^{-1}\underline{A}_2\underline{T}\rangle=\langle \underline{A}^2, \underline{B}\rangle=\underline{H}_4$.

(2) It is easy to show that $C_{PGL(2,k)}(\underline{H}_4)=\underline{H}_4$. Since we have that $\underline{B}'\underline{A}^2\underline{B}'^{-1}$

$=\underline{A}^2$, $\underline{B}'\underline{B}\underline{B}'^{-1}=\underline{A}^2\underline{B}$, where $\underline{B}'=\underline{A}^2\underline{T}_3\underline{A}\underline{T}_3$ and that $\underline{T}_3^{-1}\underline{A}^2\underline{T}_3=\underline{B}$, $\underline{T}_3\underline{A}^2\underline{T}_3^{-1}=\underline{A}^2\underline{B}$, it follows that $N_{PGL(2,k)}(\underline{H}_4)=\langle\underline{T}_3, \underline{B}'\rangle\cdot C_{PGL(2,k)}(\underline{H}_4)$. Therefore we have that $N_{PGL(2,k)}(\underline{H}_4)=\langle\underline{T}_3, \underline{A}\rangle$, since $\langle\underline{A}^2, \underline{B}, \underline{T}_3, \underline{B}'\rangle=\langle\underline{T}_3, \underline{A}\rangle$. Since $(\underline{A}^{-1})^4=(\underline{T}_3\underline{A})^2=(\underline{A}^{-1}\underline{T}_3\underline{A})^3=I$, and since $\sharp N_{PGL(2,k)}(\underline{H}_4)=24$, we have an isomorphism of $\mathfrak{S}_4$ onto $N_{PGL(2,k)}(\underline{H}_4)$. Q. E. D.

Next we shall show the theorem (1.1) in case where C is hyperelliptic. In this case we have a natural exact sequence $\langle J\rangle\rightarrow\mathrm{Aut}(C)\rightarrow PGL(2, k)$. Since G is isomorphic to $\mathfrak{S}_4$, we have that the image $\underline{G}$ of G in $PGL(2, k)$ is also isomorphic to $\mathfrak{S}_4$. Thus char(k) must be different from 2, because there is no elements of order 4 in $PGL(2, k)$ in case char$(k)=2$. Then C is determined by $f(x, z)$, where $f(x, z)$ is a homogeneous form of degree 8 which is a semi-invariant with respect to $\underline{G}$. Then we may assume by (1.3.1) that $\underline{G}=N_{PGL(2,k)}(\underline{H}_4)$. Since $f(x, z)$ is a semi-invariant for $\underline{A}$, we have that $f(x, z)=\alpha x^8+\beta x^4z^4+\gamma z^8$ for some α, β and γ. Moreover since $f(x, z)$ is a semi-invariant for $\underline{B}$, we have that Case 1: $\alpha+\gamma=0$, $\beta=0$, or Case 2: $\alpha=\gamma$. In Case 1, $f(x, z)$ cannot be a semi-invariant for $\underline{T}_3$. So Case 1 does not happen. In Case 2, since $f(x, z)$ is a semi-invariant for $\underline{T}_3$, we have that $14\alpha=\beta$ i. e. $f(x, z)=\alpha(x^8+14x^4z^4+z^8)$. Thus we see that C is defined by $y^2=x^8+14x^4+1$. Since $\underline{G}=\langle\underline{A}, \underline{T}_3\rangle$, and since A_4 and T_3 are automorphisms of C, we have that G is contained in $\langle A_4, T_3, J\rangle$. On the other hand T_3 is in G, because there are no element of order 6 in $\mathfrak{S}_4$. Thus we obtain that $G=\langle A_4J, T_3\rangle$ or $\langle A_4, T_3\rangle$. This completes the proof of the fact that (C, G) isomorphic to (C^*, hG_{24}) or (C^*, hH_{24}), in case where C is hyperelliptic.

1.4. The case: char$(k)=2$. Then we may assume that C is nonhyperelliptic. And it follows from the Jordan's canonical form that we may assume that R_+R_- is in G. Then C equals to some $C(a, b, c_2, c_3)$ in $F(\langle R_+R_-\rangle)$ (cf. (2.1.1 of Part I). If $T=\begin{pmatrix}1&\alpha&\beta\\0&1&\alpha\\0&0&1\end{pmatrix}$ (α, β in k), then T is in $C_{PGL}(R_+R_-)$ and $T^*(C)=C(a', b', c_2', c_3')$ in $F(\langle R_+R_-\rangle)$, where $c_2'=c_2+c_3(\alpha^2+\alpha)+\alpha^4+\alpha^3+\beta^2+\beta$, and $c_3'=c_3+\alpha^2+\alpha$. For suitable choice of α and β, we get that $T^*(C)$ is a curve in F_8. Hence we may assume that C is in F_8 with R_+R_- in G. It follows from (2.2.2 of Part I) that Aut(C) is contained in $PSL(3, 2)$. It is easy to see that $C_{PSL(3,2)}\langle(R_+R_-)^2\rangle=G_8$. So we have that G_8 is contained in G. Therefore the normal subgroup of G of order 4 is either $\langle R_+, (R_+R_-)^2\rangle$ or $\langle R_-, (R_+R_-)^2\rangle$. Since $N_{PSL(3,2)}\langle R_+, (R_+R_-)^2\rangle=G_{24}(+)$, and $N_{PSL(3,2)}\langle R_-, (R_+R_-)^2\rangle=G_{24}(-)$, we have that $G=G_{24}(+)$ or $G_{24}(-)$. On the other hand, since $F(G_{24}(+))=F_{24}(+)$ and $F(G_{24}(-))=F_{24}(-)$ (2.1.1 of Part I), we get that (C, G) is a member of $F_{24}(+)$ or $F_{24}(-)$. This completes the proof of (1.1) in case where $\mathrm{char}(k)=2$.

§ 2. Isomorphism classes.

The purpose of this section is to prove the following theorem:

2.1. THEOREM. *Assume that* $\mathrm{char}(k)\neq 2$. *Let* $C(a)$ *and* $C(a')$ *be two curves in* F_{24}, *where* $a\neq 3\theta_1$ *or* $3\theta_2$. *Then* $C(a)$ *is isomorphic to* $C(a')$ *if and only if* $a=a'$.

PROOF. To prove the "only if" part, we assume that $C(a)\cong C(a')$ and $a\neq a'$. First it is easy to see that $C_{PGL}(G_{24})=\{I\}$. Since any automorphism of $\mathfrak{S}_4$ is an inner automorphism, we also have that $N_{PGL}(G_{24})=G_{24}$. Therefore by the assumption it follows that $\mathrm{Aut}(C(a))$ contains strictly G_{24}. Then we apply a result on the classification of nonhyperelliptic AM curves of genus three [5], and it follows that $C(a)$ is isomorphic to K or K_4.

(1) The case: $C(a)\cong K_4$. When $\mathrm{char}(k)=3$, it follows from (3.1.2 of Part I) that $a=0$, where this is the excluded value. When $\mathrm{char}(k)\neq 3$, we note that $\sharp\mathrm{Aut}(K_4)=96$, and that $C_{\mathrm{Aut}(K_4)}(S^2)$ is a 2-Sylow subgroup of $\mathrm{Aut}(K_4)$ with $\langle D(i, i, -1)\rangle$ as its center. So any 2-Sylow subgroup of $\mathrm{Aut}(K_4)$ has a cyclic subgroup of order 4 as its center. Since $C_{PGL}\langle S^2, RS^2R^{-1}\rangle$ is contained in $\boldsymbol{D}$-PGL, $\mathrm{Aut}(C(a))$ contains an element of $\boldsymbol{D}$-PGL of order 4. Then we have at any rate that $a=0$. Also we have that $a'=0$. These lead to a contradiction to the assumption on a and a'.

(2) The case: $C(a)\cong K$. We may assume that $\mathrm{char}(k)\neq 3$, by (1.4.1 of Part I). If we denote by S_0 (resp. $\bar{S}_0$) $S(\zeta^6\alpha_3, \zeta^4\beta_1, 1)$ (resp. $S(\zeta\alpha_3, \zeta^3\beta_1, 1)$) (cf. (1.2 of Part I)) then by direct calculations we see that $S_0^*(K)=C(3\theta_1)$ (in F_{24}) and $\bar{S}_0^*(K)=C(3\theta_2)$ (in F_{24}). Let T be an isomorphism of K onto $C(a)$. Then $T^*(G_{24})$ is G_K-conjugate to either $S_0^{-1*}(G_{24})$ or $\bar{S}_0^{-1*}(G_{24})$, since $G_K=\mathrm{Aut}(K)$ (1.4.1 of Part I) and G_K is isomorphic to $PSL(2, 7)$. Hence replacing T if necessary, we may assume that TS_0 or $T\bar{S}_0$ is contained in $N_{PGL}(G_{24})=G_{24}$, which is contained in $\mathrm{Aut}(C(a))$. Thus we have at any rate that $a=3\theta_1$ or $3\theta_2$, which are the excluded values. This completes the proof of (2.1).

2.2. REMARK. We have an analogous result for the case $\mathrm{char}(k)=2$, by (2.2.2 of Part I):

Assume that $\mathrm{char}(k)=2$. *Let* $C(a, b)$ *and* $C(a', b')$ *be two curves in* F_8. *Then* $C(a, b)$ *is isomorphic to* $C(a', b')$ *if and only if* $a=a'$ *and* $b=b'$.

§ 3. Subgroups of Mod(3) which are isomorphic to $\mathfrak{S}_4$ and their representations.

In this section we work in the category of (compact) Riemann surfaces.

3.1. Notations and theorem.

3.1.1. Let W_0 be a fixed Riemann surface of genus 3. For each Riemann surface W of genus 3, we consider the pairs (W, α), where α are homotopy classes of orientation-preserving (or shortly o.p.) homeomorphisms of W_0 onto W. Two such pairs (W, α) and (W', α') are said to be conformally equivalent if there is a conformal mapping of W onto W' which is an element of $\alpha'\alpha^{-1}$. We denote by $\langle W, \alpha\rangle$ the equivalence class of (W, α). And the set of these classes is called the Teichmüller space $T(3)$ of genus 3. $T(3)$ becomes a metric space [9], and moreover a (simply connected) complex manifold of dimension $3g-3$ with $g=3$ [2].

Let $G(W_0)$ be the group of o.p. homeomorphisms of W_0. Each c in $G(W_0)$ defines a well-defined permutation c^* of $T(3)$ sending $\langle W, \alpha\rangle$ to $\langle W, \alpha\cdot c^{-1}\rangle$. In fact this c^* is a biholomorphic mapping. And so we have a group homomorphism of $G(W_0)$ into $\mathrm{Aut}(T(3))$, the group of biholomorphic mappings of $T(3)$. We denote its image by Mod(3). For $\langle W, \alpha\rangle$ in $T(3)$, we have a natural group homomorphism (denoted by M_α) of $\mathrm{Aut}(W)$ into Mod(3) defined by $\sigma\mapsto(\alpha^{-1}\sigma\alpha)^*$. It is known that M_α defines an isomorphism of $\mathrm{Aut}(W)$ and the isotropy subgroup of Mod(3) at $\langle W, \alpha\rangle$ (e.g. [6, p. 16, Corollary]). For an *AM* Riemann surface (W, G) (defined as in (1.4 of Part I)), taking a homotopy class α of W_0 onto W, we define a homomorphism (denoted by $M(W, \ )$) of $\mathrm{Aut}(W)$ into Mod(3) as above. Then we note that its image $M(W, G)$ is determined up to Mod(3)-conjugacy.

3.1.2. For an *AM* Riemann surface (W, G) of genus 3, taking a basis $\varphi_1, \varphi_2, \varphi_3$ of the space of holomorphic differentials, we define a representation, $\rho(W, \ )$, of $\mathrm{Aut}(W)$ on the space which is defined by: $\rho(W, \sigma)=(a_{ij})$ in $GL(3, \boldsymbol{C})$, where $\sigma^*(\varphi_i)=\sum_{j=1}^{3} a_{ij}\varphi_j$ $(\sigma\in\mathrm{Aut}(W))$. Then we note that the image $\rho(W, G)$ of G is determined up to $GL(3, \boldsymbol{C})$-conjugacy.

The purpose of this section is to prove the following theorem:

3.1.3. THEOREM. *Let (W, G) be an AM Riemann surface of genus three. Assume that G is isomorphic to $\mathfrak{S}_4$. Then we have:*

(1) *$M(W, G)$ is* Mod(3)*-conjugate to either MG_{24} or MH_{24}, $\rho(W, G)$ is $GL(3, \boldsymbol{C})$-conjugate to either G_{24} or H_{24}.*

(2) *$M(W, G) \sim MG_{24}$ (resp. MH_{24}) if and only if $\rho(W, G) \sim G_{24}$ (resp. H_{24}).*

In the above we denote by MG_{24} (resp. MH_{24}) the subgroup $M(C^*, hG_{24})$ (resp. $M(C^*, hH_{24})$) of Mod(3). And we denote by G_{24} (resp. H_{24}) the subgroup $\langle R, S\rangle$ (resp. $\langle R, -S\rangle$) of $GL(3, \boldsymbol{C})$ (cf. (3.1 of Part I)).

3.2. Our proof is based on the following several lemmas:

3.2.1. LEMMA. *Let $(C(a), G_{24})$ is an AM Riemann surface in F_{24}. Then $\rho(C(a), G_{24})$ is $GL(3, \boldsymbol{C})$-conjugate to G_{24}.*

PROOF. Let $F(x_1, x_2, x_3)$ be the homogeneous polynomial defining $C(a)$. And we denote by x and y the functions on $C(a)$, x_1/x_3 and x_2/x_3. Since $C(a)$ is a nonsingular plane curve which meets the line defined by $x_3=0$ transversally, the differentials $xF_2^{-1}dx$, $yF_2^{-1}dx$ and $F_2^{-1}dx$ form a basis of the space of holomorphic differentials, where $F_2=F_2(x, y)=\left(\frac{\partial}{\partial x_2}F\right)(x, y, 1)$. If $\rho(C(a), \ )$ is the representation with respect to this basis, then we have that $\rho(C(a), S)=S$, since $S^*(xF_2^{-1}dx)=-yF_2^{-1}dx$, $S^*(F_2^{-1}dx)=F_2^{-1}dx$ and $S^*(yF_2^{-1}dx)=x\cdot F_2^{-1}dx$. On the other hand we have that $R^*(F_2^{-1}dx)=(4x^{-3}+2a((yx^{-1})^2x^{-1}+x^{-1}))^{-1}d(yx^{-1})=(4+2a(x^2+y^2))^{-1}x(xdy-ydx)=xF_2^{-1}dx$, since $F_1(x, y)dx+F_2(x, y)dy=0$. Hence we also have that $R^*(xF_2^{-1}dx)=yx^{-1}R^*(F_2^{-1}dx)=yF_2^{-1}dx$, and that $R^*(yF_2^{-1}dx)=x^{-1}R^*(F_2^{-1}dx)=F_2^{-1}dx$. Thus we get that $\rho(C(a), R)=R$. Therefore we conclude that $\rho(C(a), G_{24})=G_{24}$. Q. E. D.

3.2.2. LEMMA. *Let C^* be the hyperelliptic surface in* (1.1). *Then $\rho(C^*, hG_{24})$ (resp. $\rho(C^*, hH_{24})$) is $GL(3, C)$-conjugate to G_{24} (resp. H_{24}).*

PROOF. Let $\rho(C^*, \ )$ be the representation of Aut(C^*) with respect to the basis: $i(x^2-1)y^{-1}\cdot dx$, $(x^2+1)y^{-1}\cdot dx$ and $2ixy^{-1}\cdot dx$. First it is obvious that $\rho(C^*, J)=-I$. Next it follows easily that:

$$(A_4J)^*(i(x^2-1)y^{-1}dx)=i^2(-x^2-1)(-y)^{-1}dx=-(x^2+1)y^{-1}dx\,,$$

$$(A_4J)^*((x^2+1)y^{-1}dx)=i(x^2-1)y^{-1}dx\,, \quad \text{and}$$

$$(A_4J)^*(2ixy^{-1}dx)=2ixy^{-1}dx\,.$$

Hence we obtain that $\rho(C^*, A_4J)=S$ and $\rho(C^*, A_4)=-S$. We also have that:

$$T_3^*(y^{-1}dx)=i(x+1)^2(2y)^{-1}dx\,, \quad T_3^*(xy^{-1}dx)=(x^2-1)(2y)^{-1}dx \quad \text{and}$$

$$T_3^*(x^2\cdot y^{-1}dx)=-i(x-1)^2(2y)^{-1}dx\,.$$

Hence we obtain that:

$$T_3^*(i(x^2-1)y^{-1}dx)=(x^2+1)y^{-1}dx\,,\quad T_3((x^2+1)y^{-1}dx)=2ixy^{-1}dx \quad \text{and}$$
$$T_3^*(2ixy^{-1}dx)=i(x^2-1)y^{-1}dx\,.$$

Therefore it follows that $\rho(C^*, T_3)=R$. Combining these results, we have that $\rho(C^*, hG_{24})=G_{24}$ and $\rho(C^*, hH_{24})=H_{24}$. Q.E.D.

3.2.3. REMARK. G_{24} and H_{24} are not $GL(3, \boldsymbol{C})$-conjugate are each other, since $\langle S\rangle$ and $\langle -S\rangle$ are not conjugate.

3.3. Now we prove the following proposition:

3.3.1. PROPOSITION. *Let $C(a)$ and $C(a')$ be two Riemann surfaces in F_{24}. Then there exists an orientation-preserving homeomorphism f of $C(a)$ onto $C(a')$ such that $f\cdot A=A\cdot f$ for each automorphism A in G_{24}.*

PROOF. We shall prove this proposition in several steps.

Step 1. We denote by $\boldsymbol{C}^{\cdot}$ a Zariski-open subset $\{a\,|\,C(a)\in F_{24}\}$ of $\boldsymbol{C}$. We fix an element a_0 of $\boldsymbol{C}^{\cdot}$. Let L be a topological embedding of $\boldsymbol{R}$ to $\boldsymbol{C}^{\cdot}$ such that $L(0)=a_0$. For $\varepsilon>0$, we denote by L_ε the restriction of L to the open interval $(-\varepsilon, \varepsilon)$. And we also denote by L_ε its image in $\boldsymbol{C}^{\cdot}$.

Then it suffices to show:

CLAIM. *There exists an $\varepsilon>0$ such that for any a in L_ε, there is an o.p. homeomorphism f_a of $C(a_0)$ to $C(a)$ with the property that $f_a\cdot A=A\cdot f_a$ for each A in G_{24}.*

If we prove this Claim, then we obtain a desired mapping after composing of finitely many such mappings as in the Claim.

In the following we shall prove this Claim.

Step 2. Let a_0 and L be as above. If $n_1(a)$ and $n_2(a)$ are the two solutions (in $\boldsymbol{C}$) of the equation: $n^2+2an+(a+2)=0$, then we denote $N_i'(a)=1+2(n_i(a)+1)^2\cdot n_i(a)^{-1}$ $(i=1, 2)$. If ε is sufficiently small, then we may assume that the mapping N_i' of L_ε to $\boldsymbol{C}$ is continuous, since $N_1'(a)$ and $N_2'(a)$ are distinct (and different from 0) for each a in $\boldsymbol{C}^{\cdot}$.

Next we choose a quasi-conformal mapping ϕ of $\boldsymbol{P}^1$ onto $\boldsymbol{P}^1$ such that $\phi(0)=0$, $\phi(\infty)=\infty$, $\phi(N_1'(a_0))=1$ and $\phi(N_2'(a_0))=i$. We denote the continuous mapping $\psi N_i'$ by N_i.

Let C be the complex subspace of $\boldsymbol{P}^2\times L_\varepsilon$ defined by the locus of the equation:

$$x_1^4+x_2^4+x_3^4+a(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2)=0\,.$$

Then we have the following Claim:

CLAIM. (1) If we define the continuous mapping π of C onto $\boldsymbol{P}^1\times L_\varepsilon$ by sending (x_1, x_2, x_3, a) to $(\psi(1+(x_1^2+x_2^2)(x_2^2+x_3^2)(x_3^2+x_1^2)(x_1x_2x_3)^{-2}, a)$, then it is the quotient mapping of C onto C/G_{24}.

(2) The o. p. continuous mapping $\pi_a: \pi^{-1}(a)\to\boldsymbol{P}^1$ (the fiber of π over a) is the natural mapping of $C(a)$ onto $C(a)/G_{24}$.

(3) The branch points of π_a are 0, ∞, $N_1(a)$ and $N_2(a)$.

PROOF. We have (1) and (2) from the fact that the holomorphic mapping of $C(a)$ to $\boldsymbol{P}^1$ defined by $(x_1, x_2, x_3)\mapsto 1+(x_1^2+x_2^2)(x_2^2+x_3^2)(x_3^2+x_1^2)(x_1x_2x_3)^{-2}$ is the quotient mapping $C(a)\to C(a)/G_{24}$.

Since G_{24} is isomorphic to $\mathfrak{S}_4$, it is easy to see that the branch points are the images of the following 4 points of $C(a)$; $(1, \omega, \omega^2)$: a fixed point of R (in $C(a)$), where ω is a solution of the equation $\omega^2+\omega+1=0$, $(*, 1, 0)$: a fixed point of S^2, $(1, 1, \sqrt{n_i(a)})$: a fixed point of S^2RSR $(i=1, 2)$. These images are in fact 0, ∞, $N_1(a)$ and $N_2(a)$. Q. E. D.

Step 3. We define a mapping g of $\boldsymbol{P}^1\times L_\varepsilon$ into $\boldsymbol{P}^1\times L_\varepsilon$ by $(P, a)\mapsto (Re(P)N_1(a)+Im(P)N_2(a), a)$ (if $P\neq\infty$), and $(\infty, a)\mapsto(\infty, a)$. If ε is sufficiently small, then it follows easily that:

(1) g is a homeomorphism such that $g(0, a)=(0, a)$, $g(\infty, a)=(\infty, a)$ and $g(N_i(a_0), a)=(N_i(a), a)$ $(i=1, 2)$.

(2) the fiber of g over a (denote it by g_a) is an o. p. homeomorphism.

Step 4. $B(a)$ denotes the set $\{(Q, a)$ in $\boldsymbol{P}^1\times L_\varepsilon | Q$ is a branch point of π_a: $C(a)\to\boldsymbol{P}^1\}$, and B denotes the union $\bigcup_{a\in L_\varepsilon} B(a)$. Since the action of G_{24} on $C\backslash\pi^{-1}B$ is fixed-point free, the restriction of π to $C\backslash\pi^{-1}B$ into $\boldsymbol{P}^1\times L_\varepsilon\backslash B$ is surjective and locally homeomorphic.

For a point P of $C(a_0)\backslash\pi_{a_0}^{-1}B(a_0)$ and a in L_ε, let $L(P, a)$ be the lifting with initial point P (considered as a point of C) of the $\boldsymbol{R}$-curve from $[0, t_a]$ to $\boldsymbol{P}^1\times L_\varepsilon$ (where $L(t_a)=a$) defined by $t\mapsto g(\pi_{a_0}(P), L(t))$. Then we have a homeomorphism (denoted by f) of $(C(a_0)\backslash\pi_{a_0}^{-1}B(a_0))\times L_\varepsilon$ onto $C\backslash\pi^{-1}B$, sending (P, a) to the end point of $L(P, a)$. This mapping has the property that $f(AP, a)=Af(P, a)$ for any automorphism A in G_{24}, since $Af(P, a)$ is the end point of the $\boldsymbol{R}$-curve $AL(P, a)$ which is equal to $L(AP, a)$.

It is obvious that f can be uniquely extended to a homeomorphism (again denoted by f) of $C(a_0)\times L_\varepsilon$ onto C, and that f has the property that $f(AP, a)=Af(P, a)$, because $C\to L_\varepsilon$ is a proper mapping.

Step 5. The fiber (denoted by f_a) of f over $a\in L_\varepsilon$ is the desired homeomorphism of $C(a_0)$ onto $C(a)$ with the property that $f_a A=Af_a$ for each A in G_{24}. The fact that f_a is orientation-preserving is followed from (2) of Claim in Step 2 and from (2) of Step 3. Q. E. D. of (3.3.1).

3.3.2. COROLLARY. *Let* $(C(a), G_{24})$ *and* $(C(a'), G_{24})$ *be two AM Riemann surfaces in* F_{24}. *Then* $M(C(a), G_{24})$ *and* $M(C(a'), G_{24})$ *are* Mod(3)-*conjugate to each other.*

PROOF. Let f be as in (3.3.1). If we take a homotopy class α of W_0 onto $C(a)$, then we have that $M_{f\alpha}(A)=((f\cdot\alpha)^{-1}A(f\cdot\alpha))^*=(\alpha^{-1}\cdot f^{-1}Af\cdot\alpha)^*=M_\alpha(f^{-1}Af)=M_\alpha(A)$. Thus we have that $M(C(a), G_{24})\sim M(C(a'), G_{24})$. Q. E. D.

3.4. Proof of the theorem: Let (W, G) be as in (3.1.3).

First we note by (3.2.1), (3.2.2) and (1.1) that $\rho(W, G)$ is $GL(3, \boldsymbol{C})$-conjugate to either G_{24} or H_{24}, and that $\rho(W, G)\sim G_{24}$ (resp. H_{24}) if and only if (W, G) is an element of F_{24} or hF_{24} (resp. of hF'_{24}), up to isomorphisms of AM Riemann surfaces.

For the rest of this section we shall prove the similar results as above concerning the subgroups of Mod(3). In general, when H is a finite subgroup of Mod(3), we denote by $T(3)^H$ the fixed point set $\{\langle W', \alpha\rangle \mid c^*(\langle W', \alpha\rangle)=\langle W', \alpha\rangle$ for all c^* in $H\}$. If $\langle W', \alpha\rangle$ is an element of $T(3)^H$, we consider the AM Riemann surface (W', G') where $G'=M_\alpha^{-1}(H)$, and we denote by $d(H)$ the number: $3\cdot$(genus of $W'/G')-3+\#$(branch points for $W'\to W'/G'$). Then it follows from [4] that $T(3)^H$ is a simply connected submanifold (of $T(3)$) of dimension $d(H)$. Since the genus of C^*/hG_{24} (resp. C^*/hH_{24}) is 0 (resp. 0) and $\#$ (branch points for $C^*\to C^*/hG_{24}$ (resp. C^*/hH_{24})) is 4 (resp. 3), we have by definition that $d(MG_{24})=1$ (resp. $d(MH_{24})=0$). Thus in particular it follows that MG_{24} is not Mod(3)-conjugate to MH_{24}. Since Mod(3) acts on $T(3)$ properly discontinuously, it follows from the classification (1.1) and (2.1) that $T(3)^{MG_{24}}$ contains an element $\langle W, \alpha\rangle$ such that $(W, M_\alpha^{-1}(MG_{24}))$ is an AM Riemann surface in F_{24} up to isomorphisms. Hence by (3.3.2) we have that $M(C(a), G_{24})$

is conjugate to MG_{24} for any AM Riemann surface $(C(a), G_{24})$ of F_{24}. Thus we obtain that $M(W, G)$ is Mod(3)-conjugate to either MG_{24} or MH_{24}, and that $M(W, G) \sim MG_{24}$ (resp. MH_{24}) if and only if (W, G) is an element of F_{24} or hF_{24} (resp. of hF'_{24}), up to isomorphisms of AM Riemann surfaces.

The above two results completes the proof of (3.1.1).

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