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CLASS GROUPS OF GROUP RINGS WHOSE COEFFICIENTS ARE ALGEBRAIC INTEGERS

By

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Let R be the ring of integers of an algebraic number field k. Let Λ be an R-order in a finite dimensional semisimple k-algebra A. We mean by the class group of Λ the class group defined by using locally free left Λ-modules and denote it by $C(\Lambda)$. We define $D(\Lambda)$ to be the kernel of the natural subjection $C(\Lambda)\rightarrow C(\Omega)$, where Ω is a maximal R-order in A containing Λ, and denote by $d(\Lambda)$ the order of $D(\Lambda)$. $C(\Omega)$ is isomorphic to a (narrow) ideal class group of the center of A, which is a product of the ideal class groups of algebraic number fields with modulus some real infinite primes. Hence, in a sense, we may concentrate on $D(\Lambda)$.

Let G be a finite group and let RG be the group ring of G with coefficients in R. Then RG can be regarded as an R-order in the semisimple k-algebra kG. We define $T(RG)$ to be the kernel of the natural surjection $C(RG)\rightarrow G(R)\oplus C(RG/(\Sigma_{G}))$, where $\Sigma_{G}=\sum_{g\in G}g\in RG$, and denote by $t(RG)$ the order of $T(RG)$. Then $T(RG)\cong\mathrm{Ker}\,(D(RG)\rightarrow D(RG/(\Sigma_{G})))$. Throughout this paper, C_{n} denotes the cyclic group of order n and p stands for a rational prime.

Much investigation has been done on $D(\boldsymbol{Z}G)$ and $T(\boldsymbol{Z}G)$ (cf. [8]), but the results seem to depend on the speciality of $\boldsymbol{Z}$.

The purpose of this paper is to study $D(RG)$ for the case where $R\neq\boldsymbol{Z}$. In §1 we give some basic results on $D(RG)$ and $T(RG)$. In §2~§4 we assume that R is the ring of integers in a quadratic field. We first give some results on $D(RC_{p^{e}})$, and next examine the structure of $D(RC_{p})$.

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§1.

For a ring S, $U(S)$ denotes its unit group. For an abelian group A and a positive integer q, $A^{(q)}$ denotes the q-part of A and $A^{(q')}$ denotes the maximal

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subgroup of A whose order is coprime to q. In the case where $G=C_n$, we denote Σ_n instead of Σ_G. Let k be an algebraic number field and let R be the ring of integers of k. Let $\Phi_n(X)$ be the cyclotomic polynomial of degree n. Write $R[X]/(\Phi_n(X))=R[\zeta_n]$ (resp. $k[X]/(\Phi_n(X))=k[\zeta_n]$) where ζ_n denotes the class of X in $R[X]/(\Phi_n(X))$ (resp. $k[X]/(\Phi_n(X))$).

PROPOSITION 1.1. *$d(RC_{p^e})=|T(RC_p)^{(p')}|^e\cdot p^{f(e)}\circ\prod_{i=1}^{e} d(R[\zeta_{p^i}])$ for some integer $f(e)\geqq 0$.*

PROOF. Let $e\geqq 1$. From the pullback diagrams

$$\begin{array}{ccc} RC_{p^{e+1}} & \longrightarrow & R[X]/(\Phi_{p^{e+1}}(X))=R[\zeta_{p^{e+1}}] \\ \downarrow & & \downarrow \\ RC_{p^e} & \longrightarrow & (R/pR)C_{p^e} \end{array}\quad,\qquad \begin{array}{ccc} RC_p & \longrightarrow & R[\zeta_p] \\ \downarrow & & \downarrow \\ R & \longrightarrow & R/pR \end{array},$$

we have an exact sequence

$$0\longrightarrow K\longrightarrow D(RC_{p^{e+1}})\longrightarrow D(RC_{p^e})\oplus D(R[\zeta_{p^{e+1}}])\longrightarrow 0$$

and a commutative diagram with exact rows

$$\begin{array}{ccccccc} U(RC_{p^e})\oplus U(R[\zeta_{p^{e+1}}]) & \longrightarrow & U((R/pR)C_{p^e}) & \longrightarrow & K & \longrightarrow & 0 \\ \downarrow\varphi' & & \downarrow\varphi & & \downarrow & & \\ U(R)\oplus U(R[\zeta_p]) & \longrightarrow & U(R/pR) & \longrightarrow & T(RC_p) & \longrightarrow & 0, \end{array}$$

where the vertical maps are induced by the norm maps. Since φ is bijective on the p'-parts and Coker φ' is a p-group, we see that $K^{(p')}\cong T(RC_p)^{(p')}$. Hence, by induction on e, we have the equality as desired.

COROLLARY 1.2. *Suppose that p is unramified in R. Then*

i) *$D(RC_p)$ $(=T(RC_p))$ is a p'-group.*

ii) *If $d(RC_p)=1$, then $D(RP)$ is a p-group for every p-group P.*

PROOF. i) Since p is unramified in R, $U(R/pR)$ is a p'-group and $R[\zeta_{p^i}]$ is a Dedekind domain for every $i\geqq 1$. The assertion follows from these facts. ii) If $d(RC_p)=1$, then $D(RC_{p^e})$ is a p-group by (1.1). Then, by the induction theorem of Artin ([1, § 1]), we see that $D(RP)$ is a p-group for every p-group P.

PROPOSITION 1.3. i) *$T(RC_n)\cong\bigoplus_{p|n}T(RC_{p^{e_p}})$ where $p^{e_p}\|n$ for each $p|n$.*

ii) *There is an exact sequence*

$$0\longrightarrow P_e\longrightarrow T(RC_{p^e})\longrightarrow T(RC_p)\longrightarrow 0,$$

where P_e is a p-group whose exponent divides p^{e-1} (resp. p^e) if p is unramified in R (resp. ramified in R).

iii) *Let G be a finite group of order n. If $p|t(RG)$, then $p|n$ or $p|t(RC_q)$ for some prime factor q of n.*

PROOF. i) Let $\tilde{\mathcal{M}}=R\oplus\mathcal{M}$ be a maximal R-order in $kC_n\cong k\oplus kC_n/(\Sigma_n)$ containing RC_n. By ([2, Theorem 1]), we have

$$D(RC_n)\cong\frac{\prod_{p|n}U(\tilde{\mathcal{M}}_p)}{U(\tilde{\mathcal{M}})\prod_{p|n}U(R_pC_n)},$$

where $\tilde{\mathcal{M}}_p=\boldsymbol{Z}_p\otimes_{\boldsymbol{Z}}\tilde{\mathcal{M}}$ and $R_p=\boldsymbol{Z}_p\otimes_{\boldsymbol{Z}}R$. Since R_p can be embedded in $\mathcal{M}_p$, the map $U(\tilde{\mathcal{M}}_p)=U(R_p)\times U(\mathcal{M}_p)\to U(\mathcal{M}_p)$; $(x, y)\mapsto yx^{-1}$, induces an isomorphism

$$D(RC_n)\cong\frac{\prod_{p|n}U(\mathcal{M}_p)}{U(\mathcal{M})\prod_{p|n}u(R_pC_n)},$$

where $u(R_pC_n)=\{x\,|\,(1, x)\in U(R_pC_n)\hookrightarrow U(R_p)\times U(R_pC_n/(\Sigma_n))\}$. On the other hand, we have

$$D(RC_n/(\Sigma_n))\cong\frac{\prod_{p|n}U(\mathcal{M}_p)}{U(\mathcal{M})\prod_{p|n}U(R_pC_n/(\Sigma_n))}.$$

Hence we get

$$T(RC_n)\cong\frac{U(\mathcal{M})\prod_{p|n}U(R_pC_n/(\Sigma_n))}{U(\mathcal{M})\prod_{p|n}u(R_pC_n)}.$$

For each $p|n$, let e_p be the integer such that $p^{e_p}\|n$. Since $R_pC_n/(\Sigma_n)\cong R_pC_{p^{e_p}}/(\Sigma_{p^{e_p}})\oplus(\bigoplus_{d|n/p^{e_p},\, d\neq 1}R_pC_{p^{e_p}}[\zeta_d])$ and $R_pC_n\cong R_pC_{p^{e_p}}\oplus(\bigoplus_{d|n/p^{e_p},\, d\neq 1}R_pC_{p^{e_p}}[\zeta_d])$, we see that

$$T(RC_n)\cong\prod_{p|n}\frac{U(\mathcal{M}(p))U(R_pC_{p^{e_p}}/(\Sigma_{p^{e_p}}))}{U(\mathcal{M}(p))n(R_pC_{p^{e_p}})},$$

where $\mathcal{M}(p)$ is a maximal R-order in $kC_p/(\Sigma_{p^{e_p}})$ containing $RC_{p^{e_p}}/(\Sigma_{p^{e_p}})$. This shows that $T(RC_n)\cong\bigoplus_{p|n}T(RC_{p^{e_p}})$.

ii) Let $\mathcal{O}_i$ be the maximal R-order in $k[\zeta_{p^i}]$ containing $R[\zeta_{p^i}]$, $1\leqq i\leqq e$. Then $\mathcal{M}=\bigoplus_{i=1}^{e}\mathcal{O}_i$ is a maximal R-order in $kC_{p^e}/(\Sigma_{p^e})$ containing $RC_{p^e}/(\Sigma_{p^e})$, and we have

$$T(RC_{p^e})\cong\frac{U(\mathcal{M})U(R_pC_{p^e}/(\Sigma_{p^e}))}{U(\mathcal{M})u(R_pC_{p^e})}\quad\text{and}\quad T(RC_p)\cong\frac{U(\mathcal{O}_1)U(R[\zeta_p])}{U(\mathcal{O}_1)u(R_pC_p)}.$$

The natural surjection $C_{p^e}\to C_{p^e}/C_{p^{e-1}}\cong C_p$ induces the surjection $T(RC_{p^e})\to T(RC_p)$; $(x, y)\mapsto x$ where $(x, y)\in U(\mathcal{O}_{1,p}\oplus(\bigoplus_{i=2}^{e}\mathcal{O}_{i,p}))$. Set $P_e=\mathrm{Ker}(T(RC_{p^e})\to T(RC_p))$. Each $\alpha\in P_e$ is represented by an element $(x, y)\in U(R_pC_{p^e}/(\Sigma_{p^e}))$ such that $x=uv$ for some $u\in U(\mathcal{O}_1)$ and $(1, v)\in U(R_pC_p)$. Let $f(\bar{\sigma})=\sum_{i=0}^{p^e-2}b_i\bar{\sigma}^i=(x, y)$ $\in U(R_pC_{p^e}/(\Sigma_{p^e}))$, where $b_i\in R_p$ and $\bar{\sigma}$ denotes the image of a generator σ of C_{p^e} in $R_pC_{p^e}/(\Sigma_{p^e})$, and let $f(\sigma)=\sum_{i=0}^{p^e-2}b_i\sigma^i\in R_pC_{p^e}$. Then $x=f(\zeta_p)\equiv\sum_{i=0}^{p^e-2}b_i\equiv uv$ $\equiv u\ (\mathrm{mod}\,(\zeta_p-1)\mathcal{O}_{1,p})$, and so we see that $f(1)=\sum_{i=0}^{p^e-2}b_i\in U(R_p)$. Hence $f(\sigma)\in U(R_pC_{p^e})$. Then $\alpha=\overline{(x, y)}=\overline{(f(1), f(1))}$, because $(x^{-1}f(1), y^{-1}f(1))\in u(R_pC_{p^e})$. Thus we know that

$$P_e\subseteq N=\left\{\rho_x=(x, x)\in T(RC_{p^e})\,\middle|\, \begin{array}{l} x\in U(R_p),\ x\equiv u\ (\mathrm{mod}\,(\zeta_p-1)\mathcal{O}_{1,p}) \\ \text{for some } u\in U(\mathcal{O}_1)\end{array}\right\}.$$

It is easily verified that

$$(*)\qquad p^eR\oplus p^{e-1}(\zeta_p-1)R[\zeta_p]\oplus p^{e-2}(\zeta_p-1)R[\zeta_{p^2}]\oplus\cdots\oplus(\zeta_p-1)R[\zeta_{p^e}]\subseteq RC_{p^e}.$$

Let $\rho_x\in N$ and $x\equiv u\ (\mathrm{mod}\,(\zeta_p-1)\mathcal{O}_{1,p})$, $u\in U(\mathcal{O}_1)$. If p is unramified in R, then $\mathcal{O}_i=R[\zeta_{p^i}]$ and $u^{-1}x\in 1+(\zeta_p-1)\mathcal{O}_{1,p}$, and hence $(u^{-1}x)^{p^{e-1}}\in 1+(\zeta_p-1)p^{e-1}\mathcal{O}_{1,p}$. By force of (*), we know that $\rho_x^{p^{e-1}}=1$ in $T(RC_{p^e})$. Thus we see that $\exp(P_e)\,|\,p^{e-1}$. Even if p is ramified in R, $(u^{-1}x)^p\in 1+(\zeta_p-1)R_p[\zeta_p]$, and so we have $\exp(P_e)\,|\,p^e$.

iii) By the induction theorem of Artin ([1, § 1]), we have that $T(RG)^{(n')}\cong\sum_C T(RC)^{(n')}$, where C ranges over all cyclic subgroups of G. The result follows from i) and ii).

REMARK 1.4. By force of (*) above, if p is unramified in R, we can see that the exponent of $D(RC_{p^e})^{(p)}$ divides p^{e-1}. Further assume that R is the ring of integers of a real algebraic number field k and $p\geqq 5$. Then $\exp(D(RC_{p^e})^{(p)})=p^{e-1}$.

In fact, let τ denote the endomorphism of RC_{p^e} induced by $\sigma\mapsto\sigma^{-1}$, where $C_{p^e}=\langle\sigma\rangle$. Then $D(RC_{p^e})$ can be regarded as a $\langle\tau\rangle$-module. For every $\langle\tau\rangle$-module M, we put $M^-=\{m\in M\,|\,m^\tau=m^{-1}\}$. Let V be the kernel of the natural surjection $D(RC_{p^{e+1}})^{(p)}\to D(RC_{p^e})^{(p)}$. Then, along the almost same line as in ([4]), we can show that $V^-\cong\bigoplus_{a=1}^{e}(\boldsymbol{Z}/p^a\boldsymbol{Z})^{v_a}$, where $v_a=(1/2)[k:\boldsymbol{Q}](p-1)^2p^{e-a-1}$ for $a<e$ and $v_e=(1/2)[k:\boldsymbol{Q}](p-1)-g$, g is the number of prime ideals in R over p.

PROPOSITION 1.5. *Suppose that p is unramified in R. Then*

$$D(RC_{p^e})^{(p')}\cong D(RC_p)^e \qquad (direct\ sum).$$

PROOF. Let $\mathcal{O}_i=R[\zeta_{p^i}]$, $1\leqq i\leqq e$. Then $\mathcal{O}_i$ is a Dedekind domain and $\bigoplus_{i=1}^{e}\mathcal{O}_i$ is a maximal R-order in $kC_{p^e}/(\Sigma_{p^e})$ containing $RC_{p^e}/(\Sigma_{p^e})$, and the product p_i of all prime ideals over p in $\mathcal{O}_i$ equals $(1-\zeta_{p^i})$, $1\leqq i\leqq e$. Hence we get

$$D(RC_{p^e})\cong \prod_{i=1}^{e} U(\mathcal{O}_{i,p})\Big/\prod_{i=1}^{e} U(\mathcal{O}_i)u(R_pC_{p^e})$$
$$=\left[\prod_{i=1}^{e}\frac{U(R/pR)}{\varphi_i(U(\mathcal{O}_i))}\right]\times\left[\frac{\prod_{i=1}^{e}(1+p_i\mathcal{O}_{i,p})}{\prod_{i=1}^{e}U^1(\mathcal{O}_i)u(R_pC_{p^e})}\right],$$

where φ_i is induced by the natural surjection $\mathcal{O}_i\to\mathcal{O}_i/p_i\cong R/pR$ and $U^1(\mathcal{O}_i)=\operatorname{Ker}\varphi_i=U(\mathcal{O}_i)\cap(1+p_i\mathcal{O}_{i,p})$. Then it is easily seen that the former factor is isomorphic to $D(RC_{p^e})^{(p')}$. On the other hand, $|D(RC_{p^e})^{(p')}|=d(RC_p)^e$ by (1.1) and (1.2), and so we have

$$U(R/pR)/\varphi_i(U(\mathcal{O}_i))\cong D(RC_p)\,, \qquad 1\leqq i\leqq e\,.$$

Thus we complete the proof.

§ 2.

Hereafter, let k denote $\boldsymbol{Q}(\sqrt{m})$, a quadratic field, where m is a square-free integer, and R be the ring of integers of k. We write $w_m=\sqrt{m}$ (resp. $\sqrt{m}+1/2$) if $m\not\equiv 1 \pmod 4$ (resp. $m\equiv 1 \pmod 4$).

Let $\mathcal{O}_i$ be the maximal R-order in $k[\zeta_{p^i}]$ and p_i be the product of all the prime ideals over p in $\mathcal{O}_i$, $1\leqq i\leqq e$. Then

$$D(RC_{p^e})\cong \prod_{i=1}^{e} U(\mathcal{O}_{i,p})\Big/\prod_{i=1}^{e} U(\mathcal{O}_i)u(R_pC_{p^e})$$
$$\cong\left[\prod_{i=1}^{e}\frac{U(\mathcal{O}_i/p_i)}{\varphi_i(U(\mathcal{O}_i))}\right]\times\left[\frac{\prod_{i=1}^{e}(1+p_i\mathcal{O}_{i,p})}{\prod_{i=1}^{e}U^1(\mathcal{O}_i)u(R_pC_{p^e})}\right],$$

where $\varphi_i: U(\mathcal{O}_i)\to U(\mathcal{O}_i/p_i)$ is the natural map and $U^1(\mathcal{O}_i)=\operatorname{Ker}\varphi_i$, $1\leqq i\leqq e$. It is easily seen that the latter factor is isomorphic to $D(RC_{p^e})^{(p)}$.

PROPOSITION 2.1. *Let p be unramified in R, i.e. $p\nmid m$ if $p\neq 2$ and $m\equiv 1$ (mod 4) if $p=2$. Then*

$$\exp(D(RC_{p^e})^{(p)})\,|\,p^{e-1} \quad and \quad D(RC_{p^e})^{(p')}\cong D(RC_p)^e\,.$$

PROOF. This is a special case of (1.4) and (1.5).

We write $p^*=(-1)^{p-1/2}p$.

PROPOSITION 2.2. *Let* $p|m$, *and* $m\neq p^*$ *if* $p\neq 2$, *and let* $m\not\equiv 1$ (*mod* 4) *and* $m\neq -1, \pm 2$ *if* $p=2$. *Then*

i) *The exponent of* $D(RC_{p^e})^{(p)}$ *divides*

$$\begin{cases} 2^{e+1} & \text{if } p=2,\ m\equiv 2 \pmod 4 \text{ and } e>1, \text{ or} \\ p^e & \text{otherwise.} \end{cases}$$

ii) *For the case* $p\neq 2$ *and* $m=np^*$,

$$D(RC_{p^e})^{(p')}\cong D(R'C_p)^e \quad \text{where } R'=\boldsymbol{Z}[w_n].$$

iii) *For the case* $p=2$ *and* $m=-n$ *where* $n\equiv 1$ (mod 4),

$$D(RC_{2^e})^{(2')}\cong D(R'C_2)^{e-1} \quad \text{where } R'=\boldsymbol{Z}[w_n].$$

iv) *For the case* $p=2$ *and* $m=2n$ *or* $-2n$ *where* $n\equiv 1$ (mod 4),

$$D(RC_{2^e})^{(2')}\cong\begin{cases} 0 & \text{if } e=1, 2 \\ D(R'C_2)^{e-2} & \text{if } e\geqq 3, \end{cases}$$

where $R'=\boldsymbol{Z}[w_n]$.

PROOF. If $p\neq 2$ and $m=np^*$, then we see that $\mathcal{O}_i=\boldsymbol{Z}[w_n, \zeta_{p^i}]$, $p_i=(1-\zeta_{p^i})$ and $p\mathcal{O}_i\subseteqq R[\zeta_{p^i}]$, $1\leqq i\leqq e$. Hence we get that $\exp(D(RC_{p^e})^{(p)})\,|\,p^e$ and $D(RC_{p^e})^{(p')}\cong D(R'C_{p^e})^{(p')}\cong D(R'C_p)^e$, where $R'=\boldsymbol{Z}[w_n]$.

If $p=2$, $m=-n$ and $n\equiv 1$ (mod 4), then we see that $\mathcal{O}_1=R$ and $\mathcal{O}_i=\boldsymbol{Z}[w_n, \zeta_{2^i}]$ for $i\geqq 2$. Then, it is easy to see that $\exp(D(RC_{2^e})^{(2)})\,|\,2^e$ and $D(RC_2)=0$. For $e\geqq 2$, we have

$$D(RC_{2^e})^{(2')}\oplus D(R'C_2)\cong D(R'C_{2^e})^{(2')},$$

where $R'=\boldsymbol{Z}[w_n]$, and so, by (2.1),

$$D(RC_{2^e})^{(2')}\cong D(R'C_2)^{e-1}.$$

If $p=2$, $m=2n$ or $-2n$ and $n\equiv 1$ (mod 4), then $\mathcal{O}_1=R$, $\mathcal{O}_2=\boldsymbol{Z}[\sqrt{m}, \sqrt{-1}, \sqrt{m}+\sqrt{-1}/2]$ and $\mathcal{O}_i=\boldsymbol{Z}[w_n, \zeta_{2^i}]$ for $i\geqq 3$. The assertion can be shown similarly for this case.

PROPOSITION 2.3. *Let* $m=p^*$ *if* $p\neq 2$ *and let* $m=-1, \pm 2$ *if* $p=2$. *Then the exponent of* $D(RC_{p^e})$ *divides* p^e. *Especially, it divides* p^{e-1} *if* $p=3, 5$ *or* $p=2$ *and* $m=-1$.

PROOF. Put $\mathcal{O}_i=\boldsymbol{Z}[\zeta_{p^i}]$, $1\leqq i\leqq e$. If $p\neq 2$, then $R\oplus\bigoplus_{i=1}^{e}(\mathcal{O}_i\oplus\mathcal{O}_i)$ is a maximal R-order in kC_{p^e} containing RC_{p^e}, and so we have

$$D(RC_{p^e}) \cong \frac{\prod_{i=1}^{e} U(\mathcal{O}_{i,p} \oplus \mathcal{O}_{i,p})}{\prod_{i=1}^{e} U(\mathcal{O}_i \oplus \mathcal{O}_i) u(R_p C_{p^e})}$$

$$\cong \frac{\prod_{i=1}^{e} \{(1+\pi_i \mathcal{O}_{i,p}) \times (1+\pi_i \mathcal{O}_{i,p})\}}{\{\prod_{i=1}^{e} U^1(\mathcal{O}_i) \times U^1(\mathcal{O}_i)\} u(R_p C_{p^e})},$$

where $\pi_i = \zeta_{p^i} - 1$ is a prime element of $\mathcal{O}_{i,p}$ and $U^1(\mathcal{O}_i) = U(\mathcal{O}_i) \cap (1+\pi_i \mathcal{O}_{i,p})$. Hence $D(RC_{p^e})$ is a p-group. It is easy to see that $u(R_p C_{p^e})$ contains $\prod_{i=1}^{e} \{(1+\pi^{t_i} \mathcal{O}_{i,p}) \times (1+\pi^{t_i} \mathcal{O}_{i,p})\}$, where $\pi = \pi_1$ and $t_i = 1+(p-1/2)+(p-1)(e-i)$. The assertion follows from the facts that $U^1(\mathcal{O}_i) \ni \zeta_{p^i} = 1+\pi_i$ for each p^i and $U(\mathcal{O}_i) \ni 3+(\zeta_{p^i}+\zeta_{p^i}^{-1}-2)$ unless $p=3$ and that $(1+\pi_i{}^k \mathcal{O}_{i,p})^{p^{i-1}p^m} \subseteqq 1+\pi^{(p-1)m+k} \mathcal{O}_{i,p}$ for every $1 \leqq k \leqq p-1$, $m \geqq 1$ and $i \geqq 1$. The assertion for $p=2$ can be similarly shown.

§ 3.

Let R be the ring of integers of $k=\boldsymbol{Q}(\sqrt{m})$. In the case $m>0$, we denote a fundamental unit of R by ε_m. ε_m can be written as $a+b\sqrt{m}$, $a, b \in \boldsymbol{Z}$ or $(a+b\sqrt{m})/2$, $a, b \in \boldsymbol{Z}$, $2 \nmid ab$.

Here we investigate $D(RC_p)$ more precisely.

There is an exact sequence

$$0 \longrightarrow D(RC_2) \longrightarrow C(RC_2) \longrightarrow C(R) \oplus C(R) \longrightarrow 0$$

and $T(RC_2) = D(RC_2)$. Further we have easily

PROPOSITION 3.1.

$D(RC_2)$	$m<0$	$m>0$
$\boldsymbol{Z}/3\boldsymbol{Z}$	$m \equiv 5 \pmod 8$ *and* $m<-3$	$m \equiv 5 \pmod 8$ *and* $\varepsilon_m \in \boldsymbol{Z}[\sqrt{m}]$
$\boldsymbol{Z}/2\boldsymbol{Z}$	$m \equiv 2$ *or* $3 \pmod 4$ *and* $m<-1$	$m \equiv 2$ *or* $3 \pmod 4$ *and* $2 \mid b$
0	$m \equiv 1 \pmod 8$, $m=-1$ *or* -3	$m \equiv 1 \pmod 8$, $m \equiv 5 \pmod 8$ *and* $\varepsilon_m \notin \boldsymbol{Z}[\sqrt{m}]$ *or* $m \equiv 2$ *or* $3 \pmod 4$ *and* $2 \nmid b$

From now on, p is assumed to be an odd prime. From the pullback diagram

$$\begin{array}{ccc} RC_p & \longrightarrow & R \\ \downarrow & & \downarrow \\ R[\bar{\sigma}]=RC_p/(\Sigma_p) & \underset{\tilde{\phi}}{\longrightarrow} & R/pR=\boldsymbol{F}_p[\sqrt{m}], \end{array} \qquad \text{where } C_p=\langle\sigma\rangle,$$

we have exact sequences

$$U(R[\bar{\sigma}]) \xrightarrow{\phi} U(\boldsymbol{F}_p[\sqrt{m}]) \xrightarrow{\xi} T(RC_p) \longrightarrow 0$$

$$0 \longrightarrow T(RC_p) \longrightarrow D(RC_p) \longrightarrow D(R[\bar{\sigma}]) \longrightarrow 0.$$

Here ϕ is the restriction of the canonical surjection $\tilde{\phi}$: $R[\bar{\sigma}]\rightarrow R[\bar{\sigma}]/(\bar{\sigma}-1)$, $\xi(\tilde{\phi}(x))=$*the class of the ideal* (x, Σ_p) and

$$\boldsymbol{F}_p[\sqrt{m}]=\begin{cases} \boldsymbol{F}_p\oplus\boldsymbol{F}_p & \text{if } \left(\frac{m}{p}\right)=1 \\ \boldsymbol{F}_{p^2} & \text{if } \left(\frac{m}{p}\right)=-1, \end{cases}$$

where $\left(\frac{m}{p}\right)$ is the quadratic residue symbol.

Let $p\nmid m$ and let r be an element of $R[\bar{\sigma}]=R[\zeta_p]$ such that

i) if $\left(\frac{m}{p}\right)=1$, then $\phi(r)=(a, 1)\in U(\boldsymbol{F}_p)\oplus U(\boldsymbol{F}_p)$, where a is a generator of $U(\boldsymbol{F}_p)$,

ii) if $\left(\frac{m}{p}\right)=-1$, then $\phi(r)$ is a generator of $U(\boldsymbol{F}_{p^2})$.

Noticing that $\phi(U(\boldsymbol{Z}[\zeta_p]))=U(\boldsymbol{F}_p)$, we have

LEMMA 3.2. *In the case $p\nmid m$, $D(RC_p)$ $(=T(RC_p))$ is a cyclic group generated by the class of (r, Σ_p), where r is given as above. Its order divides $p-1$ (resp. $p+1$) if $\left(\frac{m}{p}\right)=1$ (resp. $\left(\frac{m}{p}\right)=-1$).*

For an imaginary abelian field K, let K_0 be the maximal real subfield of K. Denote by U (resp. U_0) the group of units in the ring of integers of K (resp. K_0) and denote by W the group of roots of unity contained in K. Then the unit index Q_K of K is defined by the index $[U:WU_0]$. It is known that $Q_K=1$ or 2. (cf. [3, § 20-26])

Assume that $p\nmid m$ and $m<0$. Let $K=\boldsymbol{Q}(\zeta_p, \sqrt{m})$ and $K_1=\boldsymbol{Q}(\zeta_p+\zeta_p^{-1})$. Let $\mathrm{Gal}(K/\boldsymbol{Q})=\langle\sigma, \tau\,|\,\sigma^{p-1}=\tau^2=1,\ \sigma\tau=\tau\sigma\rangle$, $\mathrm{Gal}(K/K_0)=\langle\sigma^{p-1/2}\tau\rangle$ and $\mathrm{Gal}(K/K_1)=\langle\sigma^{p-1/2}, \tau\rangle$. The characters of K are given as follows:

i) the characters of K/K_0;

$$\begin{cases} \sigma \longmapsto \zeta_{p-1}^{i} \\ \tau \longmapsto 1, \quad 1 \leqq i \leqq p-1 \text{ and } 2 \nmid i. \end{cases}$$

$$\begin{cases} \sigma \longmapsto \zeta_{p-1/2}^{j} \\ \tau \longmapsto -1, \quad 1 \leqq j \leqq \frac{p-1}{2}. \end{cases}$$

ii) the characters of K_0;

$$\begin{cases} \sigma \longmapsto \zeta_{p-1}^{i} \\ \tau \longmapsto -1, \quad 1 \leqq i \leqq p-1 \text{ and } 2 \nmid i. \end{cases}$$

$$\begin{cases} \sigma \longmapsto \zeta_{p-1/2}^{j} \\ \tau \longmapsto 1, \quad 1 \leqq j \leqq \frac{p-1}{2}. \end{cases}$$

Then we see that K/K_0 is unramified at p. Since we can compute the absolute discriminants of K_1 and K_0, we see that the discriminant $d_{K_0/K_1}=(\pi^2 m^*)$, where $\pi=\zeta_p-\zeta_p^{-1}$ and $m^*=\begin{cases} m & \text{if } m\equiv 1 \pmod 4 \\ 4m & \text{otherwise} \end{cases}$. Thus, (p) is totally ramified in K_0/Q, and so there is a unique prime ideal $\mathfrak{P}$ over (p) in K_0. It is easy to see that $\mathfrak{P}=(\pi^2, \pi\sqrt{m})$.

PROPOSITION 3.3. *Assume that* $p \nmid m$ *and* $m<0$. *Let* $K=\boldsymbol{Q}(\zeta_p, \sqrt{m})$ *and let* $\mathcal{O}$ *be the ring of integers of* K. *Then the following conditions are equivalent.*

i) $Q_K=2$.

ii) $\mathfrak{P}$ *is a principal ideal in* K_0.

iii) *There exists a unit of* $\mathcal{O}$ *of the form* $(\pi x+\sqrt{m}y)/2$, *where* $x, y\in \boldsymbol{Z}[\zeta_p+\zeta_p^{-1}]$.

PROOF. It is easy to see that i) is equivalent to

i′) $K=K_0(\sqrt{\varepsilon})$ for some unit ε of the ring $\mathcal{O}_0$ of integers of K_0.

On the other hand, we have

$$K=K_0(\zeta_p)=K_0(\sqrt{(\zeta_p-\zeta_p^{-1})^2}) \quad \text{and}$$

$$((\zeta_p-\zeta_p^{-1})^2)=\mathfrak{P}^2 \qquad \text{as ideals in } \mathcal{O}_0.$$

Thus we get the equivalence between i) and ii). Let $K_1=\boldsymbol{Q}(\zeta_p+\zeta_p^{-1})$ and $\mathcal{O}_1$ be the ring of integers of K_1. Then $K_0=K_1(\pi\sqrt{m^*})$, and so $\mathcal{O}_1$ has an integral basis with respected to $\mathcal{O}_0$, since $d_{K_0/K_1}=(\pi^2 m^*)$ ([6]). As the discriminant of $\pi\sqrt{m}$ with respected to K_0/K_1 is equals to $4\pi^2 m$, we see that $[\mathcal{O}_0 : \mathcal{O}_1[\pi\sqrt{m}]]=1$ or 2. Thus every element of $\mathcal{O}_0$ can be written uniquely with the form

$(x+\pi\sqrt{m}y)/2$, $x, y \in \mathcal{O}_1$. Assume the condition ii). Let $\alpha=(x'+\pi\sqrt{m}y)/2$, x', $y \in \mathcal{O}_1$, be a generator of $\mathcal{P}$. Since $\mathcal{P}\mathcal{O}=\pi\mathcal{O}$, π must divide x', and so we have $\pi^2 | x'$. Thus α can be writtn as $(\pi^2 x+\pi\sqrt{m}y)/2$. Then it is clear that $(\pi x+\sqrt{m}y)/2$ is a unit of $\mathcal{O}$, hence we have iii). Conversely, if there exists such a unit ε in $\mathcal{O}$, then $\pi\varepsilon$ is an element of K_0 and generates $\mathcal{P}$. This completes the proof.

THEOREM 3.4. *Assume that* $p \nmid m$ *and* $m<0$. *Let* $K=\boldsymbol{Q}(\zeta_p, \sqrt{m})$. *Then*

i) *The case where* $m \neq -1, -3$:

$$d(RC_p)=\begin{cases} p-1 & \text{if } \left(\frac{m}{p}\right)=1 \text{ and } Q_K=1 \\ \frac{p-1}{2} & \text{if } \left(\frac{m}{p}\right)=1 \text{ and } Q_K=2 \\ p+1 & \text{if } \left(\frac{m}{p}\right)=-1 \text{ and } Q_K=1 \\ \frac{p+1}{2} & \text{if } \left(\frac{m}{p}\right)=-1 \text{ and } Q_K=2 . \end{cases}$$

ii) *The case where* $m=-1$:

$$d(RC_p)=\begin{cases} \frac{p-1}{4} & \text{if } p\equiv 1 \pmod 4 \\ \frac{p+1}{4} & \text{if } p\equiv 3 \pmod 4 . \end{cases}$$

iii) *The case where* $m=-3$:

$$d(RC_p)=\begin{cases} \frac{p-1}{6} & \text{if } p\equiv 1 \pmod 3 \\ \frac{p+1}{6} & \text{if } p\equiv -1 \pmod 3 . \end{cases}$$

PROOF. There is an exact sequence

$$U(\mathcal{O}) \xrightarrow{\phi} U(\boldsymbol{F}_p[\sqrt{m}]) \longrightarrow D(RC_p) \longrightarrow 0 .$$

Since (p) is totally ramified in $K_0/\boldsymbol{Q}$, we see that $\phi(U(\mathcal{O}_0)) \subseteq U(\boldsymbol{F}_p)$. On the other hand, we have $\phi(U(\mathcal{O}_1))=U(\boldsymbol{F}_p)$. Let $m\neq -1, -3$. If $Q_K=1$, then $U(\mathcal{O})=\langle\zeta_p\rangle U(\mathcal{O}_0)$. Thus we have

$$d(RC_p)=\begin{cases} p-1 & \text{if } \left(\frac{m}{p}\right)=1 \\ p+1 & \text{if } \left(\frac{m}{p}\right)=-1 . \end{cases}$$

If $Q_K=2$, then $U(\mathcal{O})=\langle\zeta_p,\ \varepsilon=(\pi x+my)/2\rangle U(\mathcal{O}_0)$, for some $\varepsilon\in U(\mathcal{O})$ by (3.3). $\phi(\varepsilon)\notin U(\boldsymbol{F}_p)$ and $\varepsilon^2\in\langle\zeta_p\rangle U(\mathcal{O}_0)$, hence we see that

$$d(RC_p)=\begin{cases}\dfrac{p-1}{2} & \text{if } \left(\dfrac{m}{p}\right)=1\\[2mm] \dfrac{p+1}{2} & \text{if } \left(\dfrac{m}{p}\right)=-1\,.\end{cases}$$

Let $m=-1$. Then $K=\boldsymbol{Q}(\zeta_p,\ \zeta_4)$ and $U(\mathcal{O})=\langle\zeta_p,\ \sqrt{-1},\ \varepsilon=1-\sqrt{-1}\,\zeta_p\rangle U(\mathcal{O}_0)$. $\phi(\varepsilon)$ is of order 4 in $U(\boldsymbol{F}_p[\sqrt{-1}])/U(\boldsymbol{F}_p)$. Thus we see that

$$d(RC_p)=\begin{cases}\dfrac{p-1}{4} & \text{if } p\equiv1 \pmod 4\\[2mm] \dfrac{p+1}{4} & \text{if } p\equiv3 \pmod 4\,.\end{cases}$$

Let $m=-3$. Then $K=\boldsymbol{Q}(\zeta_p,\ \zeta_3)$ and $U(\mathcal{O})=\langle\zeta_p,\ \zeta_3,\ \varepsilon=1-\zeta_3\zeta_p\rangle U(\mathcal{O}_0)$. $\phi(\varepsilon)$ is of order 6 in $U(\boldsymbol{F}_p[\sqrt{-3}])/U(\boldsymbol{F}_p)$. Since $\left(\dfrac{-3}{p}\right)=\left(\dfrac{p}{3}\right)$, we see that

$$d(RC_p)=\begin{cases}\dfrac{p-1}{6} & \text{if } p\equiv1 \pmod 3\\[2mm] \dfrac{p+1}{6} & \text{if } p\equiv-1 \pmod 3\,.\end{cases}$$

REMARK 3.5. 1) Assume that $p\equiv3 \pmod 4$ and $m\neq-1,\ -3$. Let $M=\boldsymbol{Q}(\sqrt{-p},\ \sqrt{m})$ and let $\varepsilon>0$ be a fundamental unit of $M_0=\boldsymbol{Q}(\sqrt{-mp})$. Then the following conditions are equivalent.

i) $Q_K=2$ ii) $Q_M=2$ iii) $\sqrt{-\varepsilon}\in M$.

2) Assume that $p\equiv1 \pmod 4$ and $m=-q$, where q is a prime and $q\equiv3 \pmod 4$. Then $Q_K=2$.

PROOF. 1) The equivalence between ii) and iii) is clear. By [3, Satz 29], ii) implies i). Let $\mathfrak{p}$ be the unique prime ideal over (p) in M_0. Then it is easy to see that ii) is equivalent to the condition that $\mathfrak{p}$ is principal. If $Q_K=2$, then we can take a generator α of $\mathfrak{P}$. Since $N_{K_0/M_0}(\mathfrak{P})=\mathfrak{p}$, we see that $N_{K_0/M_0}(\alpha)$ generates $\mathfrak{p}$. This establishes 1). 2) We may assume that $q\neq3$. Let b be a primitive root modulo q, and let $\varepsilon=\prod_{i=0}^{q-3/2}(1-\zeta_q^{b^{2i}}\zeta_p)$. Then $\varepsilon\in U(\mathcal{O})$ and $\phi(\varepsilon)=\tilde{\phi}(\prod_{i=0}^{q-3/2}(1-\zeta_q^{b^{2i}}))=\tilde{\phi}(\pm\sqrt{-q})\notin U(\boldsymbol{F}_p)$. Hence $\varepsilon\notin\langle\zeta_p\rangle U(\mathcal{O}_0)$, and so $Q_K=2$.

The next result is a special case of [3, Satz 22]. We give a direct proof based on the idea of T. Miyata ([7, (2.6)]).

LEMMA 3.6. *Suppose that* $m>0$ *and* $w_m \notin \boldsymbol{Z}[\zeta_p]$. *Then* $U(\boldsymbol{Z}[w_m, \zeta_p]) = \langle\zeta_p\rangle U(\boldsymbol{Z}[w_m, \zeta_p+\zeta_p^{-1}])$.

PROOF. Let τ be the complex conjugation. For every $u \in U(\boldsymbol{Z}[w_m, \zeta_p])$, $(u^\tau/u)(u^\tau/u)^\tau=1$. So we see that u^τ/u is a root of unity in $U(\boldsymbol{Z}[w_m, \zeta_p])$. Since u^τ/u is mapped to 1 by the map ψ: $U(\boldsymbol{Z}[w_m, \zeta_p]) \to U(\boldsymbol{F}_p[w_m])$, $u^\tau/u=\zeta_p{}^i$ for some i. Then there is an integer j such that $(\zeta_p{}^j u)^\tau=\zeta_p{}^j u$. Hence we see that $U(\boldsymbol{Z}[w_m, \zeta_p])=\langle\zeta_p\rangle U(\boldsymbol{Z}[w_m, \zeta_p+\zeta_p^{-1}])$.

Let $p \nmid m$, $m>0$, $N_{\boldsymbol{Q}(\sqrt{m})/\boldsymbol{Q}}(\varepsilon_m)=-1$ and $p\equiv 1 \pmod 4$. Then a system of fundamental units of $\boldsymbol{Z}[w_p, w_m]$ is given as one of the following three types ([5, Satz 11]):

(a) ε_p, ε_m and ε_{pm},

(b) ε_p, ε_m and $\sqrt{\varepsilon_{pm}}$ (in this case, $N_{\boldsymbol{Q}(\sqrt{pm})/\boldsymbol{Q}}(\varepsilon_{pm})=1$), or

(c) ε_p, ε_m and $\sqrt{\varepsilon_p\varepsilon_m\varepsilon_{pm}}$ (in this case, $N_{\boldsymbol{Q}(\sqrt{pm})/\boldsymbol{Q}}(\varepsilon_{pm})=-1$).

THEOREM 3.7. *Suppose that* $p \nmid m$ *and* $m>0$. *Then*

i) *If* $N_{\boldsymbol{Q}(\sqrt{m})/\boldsymbol{Q}}(\varepsilon_m)=1$, *then* $D(RC_p)^{(2)}\neq 0$.

ii) *If* $p\equiv 3 \pmod 4$ *and* $N_{\boldsymbol{Q}(\sqrt{m})/\boldsymbol{Q}}(\varepsilon_m)=-1$, *then* $D(RC_p)^{(2)}=0$.

iii) *If* $p\equiv 1 \pmod 4$ *and* $N_{\boldsymbol{Q}(\sqrt{m})/\boldsymbol{Q}}(\varepsilon_m)=-1$, *then* $D(RC_p)^{(2)}\neq 0$ *when the type of fundamental units of* $\boldsymbol{Z}[w_p, w_m]$ *is* (a) *or* (b), *and* $D(RC_p)^{(2)}=0$ *when the type of fundamental units of* $\boldsymbol{Z}[w_p, w_m]$ *is* (c) *and* $p\equiv 5 \pmod 8$.

PROOF. Let φ: $U(\boldsymbol{Z}[w_m, \zeta_p+\zeta_p^{-1}]) \to U(\boldsymbol{F}_p[\sqrt{m}])$ be the restriction of Ψ to $U(\boldsymbol{Z}[w_m, \zeta_p+\zeta_p^{-1}])$. Then, by force of (3.5), $D(RC_p)\cong \operatorname{Coker}\varphi$. There is a commutative diagram with surjective vertical maps

$$\begin{array}{ccc}
U(\boldsymbol{Z}[w_m, \zeta_p+\zeta_p^{-1}]) & \xrightarrow{\ \varphi\ } & U(\boldsymbol{F}_p[\sqrt{m}]) \\
N_1\big\downarrow & & \big\downarrow N_1' \\
U(\boldsymbol{Z}[w_m])\supseteq \operatorname{Im} N_1 & \xrightarrow{\ \varphi'\ } & U(\boldsymbol{F}_p[\sqrt{m}])^{p-1/2} \\
N_2\big\downarrow & & \big\downarrow N_2' \\
U(\boldsymbol{Z})\supseteq \operatorname{Im} N_2 & \xrightarrow{\ \varphi''\ } & U(\boldsymbol{F}_p)^{p-1/2},
\end{array}$$

where $N_1=N_{\boldsymbol{Q}(\sqrt{m}, \zeta_p+\zeta_p^{-1})/\boldsymbol{Q}(\sqrt{m})}$, $N_2=N_{\boldsymbol{Q}(\sqrt{m})/\boldsymbol{Q}}$, $N_1'(x)=x^{p-1/2}$, $N_2'=N_{\boldsymbol{F}_p[\sqrt{m}]/\boldsymbol{F}_p}$ and φ' and φ'' are the restrictions of φ to $\operatorname{Im} N_1$ and $\operatorname{Im} N_2$ respectively. If $N_2(\varepsilon_m)=1$, then $\operatorname{Im} N_2\circ N_1=\{1\}$, and so $2 \mid |\operatorname{Coker}\varphi|$. For the case where $N_2(\varepsilon_m)=-1$, $p\equiv 3 \pmod 4$ and $\left(\frac{m}{p}\right)=1$, $\varphi'\circ N_1(\varepsilon_m)=(\bar{1}, \overline{-1})$ or $(\overline{-1}, \bar{1})$ in $U(\boldsymbol{F}_p[\sqrt{m}])\cong U(\boldsymbol{F}_p)\times U(\boldsymbol{F}_p)$. Since $|U(\boldsymbol{F}_p[\sqrt{m}])^{(2)}|=4$, this shows that $(\operatorname{Coker}\varphi)^{(2)}=0$. For the case where $N_2(\varepsilon_m)=-1$, $p\equiv 3 \pmod 4$ and $\left(\frac{m}{p}\right)=-1$, $U(\boldsymbol{F}_p[\sqrt{m}])^{(2)}=\langle\varphi(\varepsilon_m)\rangle^{(2)}$, because $\varepsilon_m^{p+1} \equiv -1 \pmod{p}$, and therefore we see that $(\mathrm{Coker}\,\varphi)^{(2)}=0$.

To prove iii), we form the following commutative diagram with surjective vertical maps:

$$\begin{array}{ccc}
U(\boldsymbol{Z}[w_m, \zeta_p+\zeta_p^{-1}]) & \xrightarrow{\quad\varphi\quad} & U(\boldsymbol{F}_p[\sqrt{m}]) \\
N_1 \downarrow & & \downarrow \\
U(\boldsymbol{Z}[w_m, w_p]) \supseteqq \mathrm{Im}\,N_1 & \longrightarrow & U(\boldsymbol{F}_p[\sqrt{m}])^{p-1/4} \\
N_2 \downarrow & & \downarrow \\
U(\boldsymbol{Z}[w_{pm}]) \supseteqq \mathrm{Im}\,N_2 & \longrightarrow & U(\boldsymbol{F}_p)^{p-1/4} \\
N_3 \downarrow & & \downarrow \\
U(\boldsymbol{Z}) \supseteqq \mathrm{Im}\,N_3 & \xrightarrow{\quad\varphi'\quad} & U(\boldsymbol{F}_p)^{p-1/2},
\end{array}$$

where N_i, $i=1$, 2 and 3, are the norm maps and the other maps are natural. For the case of type (a), $\mathrm{Im}\,N_2 \subseteqq \langle -1, \varepsilon_{pm}^2\rangle$, and for the case of type (b), $\mathrm{Im}\,N_2 \subseteqq \langle -1, \varepsilon_{pm}\rangle$ and $N_3(\varepsilon_{pm})=1$. Hence, for either case, $\mathrm{Im}\,\varphi' \circ N_3 \circ N_2 \circ N_1 = \{1\}$, and so $2 \mid |\mathrm{Coker}\,\varphi|$. If $p\equiv 5 \pmod{8}$, $U(\boldsymbol{F}_p[\sqrt{m}])^{(2)}=(U(\boldsymbol{F}_p[\sqrt{m}])^{p-1/4})^{(2)}$. Now consider the case of type (c). If $\left(\frac{m}{p}\right)=1$, $U(\boldsymbol{F}_p[\sqrt{m}])^{p-1/4}=\boldsymbol{Z}/4\boldsymbol{Z}\oplus\boldsymbol{Z}/4\boldsymbol{Z}$. Since $\varphi' \circ N_3 \circ N_2 \circ N_1(\sqrt{\varepsilon_p\varepsilon_m\varepsilon_{pm}})=\varphi' \circ N_3 \circ N_2(\sqrt{\varepsilon_p\varepsilon_m\varepsilon_{pm}}^{p-1/4})=\varphi' \circ N_3(\pm\varepsilon_{pm}{}^{p-1/4})=\varphi'(-1)=\overline{-1}$, $\varphi(\sqrt{\varepsilon_p\varepsilon_m\varepsilon_{pm}}^{p-1/4})$ is of type $(\overline{\pm 1}, c)$ or $(c, \overline{\pm 1})$ in $U(\boldsymbol{F}_p)\times U(\boldsymbol{F}_p)$, where c is of order 4 in $U(\boldsymbol{F}_p)$. Hence $(\mathrm{Coker}\,\varphi)^{(2)}=0$, because $\mathrm{Im}\,\varphi \supseteqq \{(a, a) \mid a \in U(\boldsymbol{F}_p)\}$. If $\left(\frac{m}{p}\right)=-1$, $U(\boldsymbol{F}_p[\sqrt{m}])^{p-1/4}\cong\boldsymbol{Z}/4(p+1)\boldsymbol{Z}$. We see that the order of $(\sqrt{\varepsilon_p\varepsilon_m\varepsilon_{pm}}^{p+1/2})$ is 8, because $\varepsilon_m^{p+1}\equiv -1 \pmod{p}$. This shows that $(\mathrm{Coker}\,\varphi)^{(2)}=0$, and thus the proof is completed.

REMARK 3.8. For the case where the type of fundamental units of $\boldsymbol{Z}[w_p, w_m]$ is (c) and $p\equiv 1 \pmod{8}$, we do not know whether $D(RC_p)^{(2)}=0$ or not.

PROPOSITION 3.9. *Suppose that* $p \mid m$ *and write* $m=np^*$. *Then*

i) $D(RC_p)\cong T(RC_p)\oplus D(RC_p/(\Sigma_p))$.

ii) $T(RC_p)\cong\begin{cases} \boldsymbol{Z}/p\boldsymbol{Z} & \textit{if } m<-3 \textit{ or } m>0 \textit{ and } p \mid b \\ 0 & \textit{otherwise.}\end{cases}$

iii) $D(RC_p/(\Sigma_p))^{(p)}$ *is an elementary p-group of rank* $\leqq (p-3)/2$. *Especially, if* $n=1$, *then* $D(RC_p/(\Sigma_p))$ *is an elementary p-group of rank* $\leqq \max(0, (p-7)/2)$.

PROOF. ii) There is a commutative diagram

$$\begin{array}{ccc} U(R[\bar{\sigma}]) & \xrightarrow{\psi} & U(\boldsymbol{F}_p[\sqrt{m}])\cong \boldsymbol{Z}/p(p-1)\boldsymbol{Z} \longrightarrow T(RC_p) \longrightarrow 0 \\ N\downarrow & & \downarrow N' \\ U(R)\supseteqq \operatorname{Im} N & \xrightarrow{\psi'} & U(\boldsymbol{F}_p[\sqrt{m}])^{p-1}\cong \boldsymbol{Z}/p\boldsymbol{Z}, \end{array}$$

where $N(f(\bar{\sigma}))=\prod_{i=1}^{p-1} f(\bar{\sigma}^i)$ for every $f(\bar{\sigma})\in U(R[\bar{\sigma}])$, $N'(x)=x^{p-1}$ for every $x\in U(\boldsymbol{F}_p[\sqrt{m}])$ and ψ' is the restriction of ψ to $\operatorname{Im} N$. Then $\operatorname{Coker}\psi\cong \boldsymbol{Z}/p\boldsymbol{Z}$ or 0, and $\operatorname{Coker}\psi\cong \boldsymbol{Z}/p\boldsymbol{Z}$ if and only if $\operatorname{Coker}\psi'\cong \boldsymbol{Z}/p\boldsymbol{Z}$. If $m>0$, then $\psi'\circ N(\varepsilon_m)=\psi'(\varepsilon_m^{p-1})=\bar{1}$ if and only if $p|b$. If $m<-3$, then $U(R)=\{\pm 1\}$, and hence $\operatorname{Coker}\psi'\cong \boldsymbol{Z}/p\boldsymbol{Z}$. For $m=-3$, we can compute directly that ψ is surjective.

i) The conclusion follows from ii) and (2.2 i).

iii) Let $n\neq 1$. Then we can write as

$$D(RC_p/(\Sigma_p))^{(p)}\cong \frac{1+p\bar{S}_p}{U^1(\bar{S})(1+pS_p)},$$

where $S=\boldsymbol{Z}[w_{np^*}, \zeta_p]$, $\bar{S}=\boldsymbol{Z}[w_n, \zeta_p]$, p is the unique prime ideal over p in S and $\bar{p}=p\bar{S}$. Then the conclusion follows from (2.2 i) and the fact that $\left|\frac{1+\bar{p}\bar{S}_p}{1+pS_p}\right|=p^{p-3/2}$. Next assume that $n=1$. By force of (2.3), we may assume that $p\geqq 7$. Then

$$D(RC_p/(\Sigma_p))\cong \frac{(1+\pi\mathcal{O}_p)\times(1+\pi\mathcal{O}_p)}{\{U^1(\mathcal{O})\times U^1(\mathcal{O})\}\{U(R_pC_p/(\Sigma_p))\cap((1+\pi\mathcal{O}_p)\times(1+\pi\mathcal{O}_p))\}},$$

where $\pi=\zeta_p-1$ and $\mathcal{O}=\boldsymbol{Z}[\zeta_p]$. The map $U(R_pC_p/(\Sigma_p))\cap((1+\pi\mathcal{O}_p)\times(1+\pi\mathcal{O}_p)) \hookrightarrow (1+\pi\mathcal{O}_p)\times(1+\pi\mathcal{O}_p) \xrightarrow{\varphi} (1+\pi\mathcal{O}_p)$ is surjective where $\varphi(x, y)=x$. Since $U^1(\mathcal{O})$ contains $1+\pi$ and $1+\pi^2-\pi^3\zeta_p^{-1}$, each element of $D(RC_p/(\Sigma_p))$ has a representative of the form $(1, 1+\pi^3x)\in(1+\pi\mathcal{O}_p)\times(1+\pi\mathcal{O}_p)$, $x\in\mathcal{O}_p$. The conclusion follows from this, because $u(R_pC_p/(\Sigma_p))\supseteqq\{1\}\times(1+\pi^{p-1/2}\mathcal{O}_p)$.

REMARK 3.10. If $p=5$ and $n>1$, then $D(RC_5/(\Sigma_5))^{(5)}\cong \boldsymbol{Z}/5\boldsymbol{Z}$. In fact, since $U(\bar{S})=\langle\zeta_5\rangle U(\boldsymbol{Z}[w_n, w_5])$, it is easy to see that $U^1(\bar{S})=U^1(S)\subseteqq 1+pS_p$. On the other hand, there are examples of n for which $D(RC_5/(\Sigma_5))^{(5)}=0$, e.g. $n=-1$, -3, -7 or -11.

§4.

In this section, we shall determine completely the structure of $D(RC_3)$.

LEMMA 4.1. *Let $m>0$ and $3\nmid m$. Put $\varepsilon_m=(a+b\sqrt{m})/2$. Then*

i) *$3 \nmid a$ or $3 \nmid b$.*

ii) *If $m \equiv 1 \pmod 3$, then $3 | ab$.*

iii) *$N_{k/Q}(\varepsilon_m)=1$ if and only if $m \equiv 1 \pmod 3$ and $3 \nmid a$ or $m \equiv -1 \pmod 3$ and $3 | ab$.*

PROOF. The results follow from the facts that $N_{k/Q}(\varepsilon_m) \equiv a^2-b^2 \pmod 3$ if $m \equiv 1 \pmod 3$ and that $N_{k/Q}(\varepsilon_m) \equiv a^2+b^2 \pmod 3$ if $m \equiv -1 \pmod 3$.

We can refine (3.4) and (3.7) as follows.

THEOREM 4.2. *Suppose that $3 \nmid m$. Then $(\sqrt{m}, \Sigma_3)$ (resp. $(-1+\sqrt{m}, \Sigma_3)$) is a Representative of a generator of $D(RC_3)$ if $(m/p)=1$ (resp. $(m/p)=-1$), and*

$D(RC_3)$	$m<0$	$m>0$
0	$m \equiv 1 \pmod 3$ *and* (A), *or* $m=-1$	$N_{k/Q}(\varepsilon_m)=-1$
$\boldsymbol{Z}/2\boldsymbol{Z}$	$m \equiv 1 \pmod 3$ *and not* (A), *or* $m \equiv -1 \pmod 3$, $m \neq -1$ *and* (A)	$m \equiv 1 \pmod 3$ *and* $3 \vert b$, *or* $m \equiv -1 \pmod 3$, $3 \vert a$
$\boldsymbol{Z}/4\boldsymbol{Z}$	$m \equiv -1 \pmod 3$ *and not* (A)	$m \equiv -1 \pmod 3$ *and* $3 \vert b$

where, for $m<0$, (A) means the property that $\sqrt{-\varepsilon_{-3m}} \in U(\boldsymbol{Z}[w_m, \zeta_3])$.

PROOF. For the case $m<0$, the result follows from (3.4), since the condition (A) is equivalent to the condition $Q_K=2$, where $K=Q(\zeta_3, \sqrt{m})$. For the case $m>0$, we see that $D(RC_3)=U(\boldsymbol{F}_3[\sqrt{m}])/\varphi(U(\boldsymbol{Z}[w_m]))$ by (3.6), and so $D(RC_3)$ is a 2-qroup. Therefore the result follows from (3.7 ii) and (4.1).

For the case $m=-3n$, we have, by (3.9) and (2.2),

$$D(RC_3) \cong T(RC_3) \oplus D(RC_3/(\Sigma_3)) \text{ and}$$

$$D(RC_3/(\Sigma_3)) \cong \begin{cases} 0 & \text{if } n=1 \\ D(R'C_3) & \text{if } n \neq 1, \text{ where } R'=\boldsymbol{Z}[w_m]. \end{cases}$$

Hence we have

THEOREM 4.3. *Suppose that $3 | m$ and wrie $m=-3n$. Then*

$D(RC_3)$	$n<0$	$n>0$
0	$n\equiv 1 \pmod 3$, *(A) and* $3\nmid d$, *or* $n=-1$	$n=1$
$Z/2Z$	$n\equiv 1 \pmod 3$, *not (A) and* $3\nmid d$, *or* $n\equiv -1 \pmod 3$, $n\neq -1$, *(A) and* $3\nmid d$	
$Z/3Z$	$n\equiv 1 \pmod 3$, *(A) and* $3\mid d$	$n\neq 1$ *and* $N(\varepsilon_n)=-1$
$Z/4Z$	$n\equiv -1 \pmod 3$, *not (A) and* $3\mid d$	
$Z/2Z\oplus Z/3Z$	$n\equiv 1 \pmod 3$, *not (A) and* $3\mid d$, *or* $n\equiv -1 \pmod 3$, $n\neq -1$, *not (A) and* $3\mid d$	$n\equiv 1 \pmod 3$, $n\neq 1$ *and* $3\mid b$, *or* $n\equiv -1 \pmod 3$, *and* $3\mid a$
$Z/3Z\oplus Z/4Z$	$n\equiv -1 \pmod 3$, *not (A) and* $3\mid d$	$n\equiv -1 \pmod 3$, $3\mid b$

where $\varepsilon_n=(a+b\sqrt{n})/2$ *if* $n>1$, $\varepsilon_m=(c+d\sqrt{m})/2$ *if* $m>0$, *(A) means the property that* $\sqrt{-\varepsilon_m}\in U(Z[w_n, \zeta_3])$ *if* $m>0$, *and* $N=N_{Q(\sqrt{n})/Q}$.

Next we want to know representatives of generators of $D(RC_3)$ in the case of $3|m$. Since $T(RC_3)$ is generated by the class of $(1+\sqrt{m}, \Sigma_3)$, we have only to consider $D(RC_3/(\Sigma_3))$. Write $m=-3n$ assume that $n\neq\pm 1$. Let $R=Z[w_{-3n}]$, $S=R[\zeta_3]$ and $\bar{S}=Z[w_n, \zeta_3]$. Then we see that $\bar{S}$ is the integral closure of S. Put $p=(\sqrt{-3}, \sqrt{-3n})$ (resp. $(\sqrt{-3}, 1+(1+\sqrt{-3n})/2)$) if $n\not\equiv 1 \pmod 4$ (resp. $n\equiv 1 \pmod 4$). Then we see that p is a unique prime ideal of S which contains $p^2=(\sqrt{-3})p$ and $p\bar{S}=(\sqrt{-3})$. First we note

LEMMA 4.4. *An invertible ideal C of S such that $C\bar{S}$ is principal in $\bar{S}$ is isomorphic to some p-primary invertible ideal of S not contained in p^2.*

PROOF. Let C' be an invertible ideal of S such that $C'\cong C^{-1}$. Since p is a unique non-invertible prime ideal of S, we have $S[1/3]=\bar{S}[1/3]$. Hence, there is $c'\in C'$ such that $C'S[1/3]=(c')$ in $S[1/3]$, and so there is a p-primary invertible ideal g of S such that $(c')=C'g$ in S. Since $p^2=(\sqrt{-3})p$, g is isomorphic to a p-primary invertible not contained in p^2. Since $C\cong C'^{-1}\cong g$, this completes the proof.

Put $a=(3, \sqrt{-3n})$ (resp. $(3, 1+(1+\sqrt{-3n})/2)$) if $n\not\equiv 1 \pmod 4$ (resp. $n\equiv 1 \pmod 4$). Then $a\bar{S}=(\sqrt{-3})$ and $a^2=(3)$.

LEMMA 4.5. *The following statements are equivalent*:

i) $\mathfrak{a}$ *is non-principal in* S.

ii) *In the case where* $n<0$, $U(\boldsymbol{Z}[w_n, \zeta_3])=\langle -1, \zeta_3, \varepsilon_{-3n}\rangle$, *and in the case where* $n>0$, $\varepsilon_n=(a+b\sqrt{n})/2$, $3\nmid a$, $3|b$.

PROOF. Let $n\not\equiv 1 \pmod 4$. If $\mathfrak{a}$ is principal, then we can write as $\mathfrak{a}=(3x+y\sqrt{-3n})$ for some $x, y\in \boldsymbol{Z}[\zeta_3]$. Further we see that $\sqrt{-3}\nmid y$ and $(x, y)=(1)$. Since $3\in \mathfrak{a}$,

$$(v'+z\sqrt{-3n})(3x+y\sqrt{-3n})=3 \qquad \text{for some } v', z\in \boldsymbol{Z}[\zeta_3].$$

Hence we have that $3xz+yv'=0$, and so $v'=3v$ for some $v\in \boldsymbol{Z}[\zeta_3]$. Then the equality $xz+yz=0$ implies that $v=ux$ and $z=-uy$ for some $u\in \boldsymbol{Z}[\zeta_3]$. Thus $u(3x-y\sqrt{-3n})(3x+y\sqrt{-3n})=3$, and so $u(x\sqrt{-3}+y\sqrt{n})(x\sqrt{-3}-y\sqrt{n})=-1$ in $\bar{S}$. Hence there is a unit of type $x\sqrt{-3}+y\sqrt{n}$ $(x, y\in \boldsymbol{Z}[\zeta_3])$ in $\bar{S}$. Conversely, if there is a unit in $\bar{S}$ of the above type, we see that $\mathfrak{a}=(3x+y\sqrt{-3n})$. If $n<0$, then there is a unit of the above type when and only when the unit index of $Q(\zeta_3, w_n)$ is 2, i.e. $U(\bar{S})=\langle -1, \zeta_3, \sqrt{-\varepsilon_{-3n}}\rangle$. If $n>0$, then $U(\bar{S})=\langle -1, \zeta_3, \varepsilon_n\rangle$ by (3.6), $\varepsilon_n=a+b\sqrt{n}$, and there is a unit in $\bar{S}$ of the above when and only when $3|a$ and $3\nmid b$. We can similarly prove the assertion for the case where $n\equiv 1 \pmod 4$.

LEMMA 4.6. *Assume that* $n\equiv -1 \pmod 3$. *Put* $\mathfrak{g}=(3, \sqrt{-3n}+\sqrt{-3})$ (*resp.* $(3, \sqrt{-3}+(3+\sqrt{-3n})))$ *if* $n\not\equiv 1 \pmod 4$ (*resp.* $n\equiv 1 \pmod 4$)). *Then* $\mathfrak{g}$ *is a* $\mathfrak{p}$-*primary invertible ideal in* S *such that* $\mathfrak{g}^2=(\sqrt{-3})\mathfrak{a}$ *and* $\mathfrak{g}\nsubseteq \mathfrak{p}^2$. *Further*, $\mathfrak{g}$ *is principal in* S *if and only if* $n>0$ *and* $3\nmid ab$, *where* $\varepsilon_n=(a+b\sqrt{n})/2$.

PROOF. Let $n\not\equiv 1 \pmod 4$. The first statement is obvious, so we have only to show the second one. If $\mathfrak{g}$ is principal in S, then $\mathfrak{g}=(3(x+y\sqrt{-3n})+(z+v\sqrt{-3n})(\sqrt{-3}+\sqrt{-3n}))$ for some $x, y, z, v\in \boldsymbol{Z}[\zeta_3]$, and so $\mathfrak{g}=(s\sqrt{-3}+t\sqrt{-3n})$, where $s=-\sqrt{-3}x+z+\sqrt{-3}nv$ and $t=3y+z+\sqrt{-3}v$. Since $\mathfrak{g}\nsubseteq \mathfrak{p}^2$, we have $\sqrt{-3}\nmid z$, and hence $\sqrt{-3}\nmid st$ and $(s, t)=(1)$. Since $3\in \mathfrak{g}$, $3=u(s\sqrt{-3}+t\sqrt{-3n})(s\sqrt{-3}-t\sqrt{-3n})$ for some $u\in \boldsymbol{Z}[\zeta_3]$. Hence $u(s+t\sqrt{n})x(s-t\sqrt{n})=-1$ in $\bar{S}$ and so

(*) $\quad s+t\sqrt{n}\in U(\boldsymbol{Z}[\sqrt{n}, \zeta_3]) \qquad$ where $s, t\in \boldsymbol{Z}[\zeta_3]$ and $\sqrt{-3}\nmid st$.

If $n<0$, then $U(\boldsymbol{Z}[\sqrt{n}, \zeta_3])=\langle -1, \zeta_3, \varepsilon_{-3n}\rangle$ or $\langle -1, \zeta_3, \sqrt{-\varepsilon_{-3n}}\rangle$, where $\sqrt{-\varepsilon_{-3n}}=x\sqrt{-3}+y\sqrt{n}$ for some $x, y\neq 0$. Therefore (*) is impossible, and hence $\mathfrak{g}$ is non-principal. If $n>0$, then $U(\boldsymbol{Z}[\sqrt{n}, \zeta_3])=\langle -1, \zeta_3, \varepsilon_n\rangle$ where $\varepsilon_n=a+b\sqrt{n}$. This shows that (*) is possible if and only if $3\nmid ab$. We can similarly prove the assertion for the case where $n\equiv 1 \pmod 4$.

Combining (4.3), (4.5) and (4.6), we have

THEOREM 4.7. *Suppose that* $3|m$ *and* $m\neq\pm 3$. *Let*

$$a=\begin{cases}(3, \sqrt{m}) & \text{if } m\not\equiv 1 \pmod 4\\ \left(3, 1+\dfrac{1+\sqrt{m}}{2}\right) & \text{if } m\equiv 1 \pmod 4,\end{cases}$$

and

$$g=\begin{cases}(3, \sqrt{-3}+\sqrt{m}) & \text{if } m\not\equiv 1 \pmod 4\\ \left(3, \sqrt{-3}+\dfrac{3+\sqrt{m}}{2}\right) & \text{if } m\equiv 1 \pmod 4.\end{cases}$$

Then $D(RC_3/(\Sigma_3))$ *is generated by the class of* a (*resp.* g) *if* $m/3\equiv -1 \pmod 3$ (*resp.* $m/3\equiv 1 \pmod 3$)).

REMARK 4.8. We can also determine the structure of $D(RC_5)$ for the case that k is a real quadratic field. Let $R=\boldsymbol{Z}[w_m]$, $S=\boldsymbol{Z}[w_m, w_5]$, where $5\nmid m>0$, and $\varepsilon_m=(a+b\sqrt{m}/2)$. Then a system of fundamental units of S is given as one of the following:

(*a*) $\varepsilon_5, \varepsilon_m, \varepsilon_{5m}$,

(*b*) $\varepsilon_5, \varepsilon_m, \sqrt{\varepsilon_{5m}}$ (in this case $N(\varepsilon_{5m})=1$ and $\left(\frac{m}{5}\right)=1$),

(*c*) $\varepsilon_5, \varepsilon_m, \sqrt{\varepsilon_5\varepsilon_m\varepsilon_{5m}}$ (in this case $N(\varepsilon_m)=N(\varepsilon_{5m})=-1$, or $N(\varepsilon_m)=1$, $\left(\frac{m}{5}\right)=-1$ and $5\nmid b$), or

(*d*) $\varepsilon_5, \varepsilon_m, \sqrt{\varepsilon_m\varepsilon_{5m}}$ (in this case $N(\varepsilon_m)=N(\varepsilon_{5m})=1$, $\left(\frac{m}{5}\right)=-1$ and $5\nmid b$),

where, for a sequare-free positive integer d, $N(\varepsilon_d)=N_{\boldsymbol{Q}(\sqrt{d})/\boldsymbol{Q}}(\varepsilon_d)$.

We have a following table:

$D(RC_5)$	$\left(\frac{m}{5}\right)=1$	$\left(\frac{m}{5}\right)=-1$
0	$N(\varepsilon_m)=-1$ *and* (*c*)	$N(\varepsilon_m)=-1$, (*c*) *and* $5\nmid b$
$\boldsymbol{Z}/2\boldsymbol{Z}$	(*a*) *and* $5\nmid b$, *or* (*b*)	(*a*) *and* $5\nmid b$, *or* $N(\varepsilon_m)=1$ *and* (*c*) *or* (*d*)
$\boldsymbol{Z}/3\boldsymbol{Z}$		$N(\varepsilon_m)=-1$, (*c*) *and* $5\mid b$
$\boldsymbol{Z}/4\boldsymbol{Z}$	(*a*) *and* $5\mid b$	
$\boldsymbol{Z}/6\boldsymbol{Z}$		(*a*) *and* $5\mid b$

where (*a*) means that the type of fundamental units of S is (*a*).

Further, let $R'=\boldsymbol{Z}[w_{5m}]$ and $\varepsilon_{5m}=(c+d\sqrt{5m})/2$. Then

$$D(R'C_5)\cong D(RC_5)\oplus \boldsymbol{Z}/5\boldsymbol{Z}\oplus T(R'C_5)$$

and

$$T(R'C_5)\cong\begin{cases} 0 & \text{if } 5\nmid d \\ \boldsymbol{Z}/5\boldsymbol{Z} & \text{if } 5\mid d\,. \end{cases}$$

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