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ON THE CAUCHY PROBLEM FOR A SEMI-LINEAR HYPERBOLIC SYSTEM AND ITS TRAVELING WAVE-LIKE SOLUTIONS

By

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Introduction

We consider the Cauchy problem of the following system of semi-linear partial differential equations for $u(x,t)$ and $v(x,t)$:

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}+\lambda\dfrac{\partial u}{\partial x}=-uv+g(u)\varepsilon\,, \\[2ex] \dfrac{\partial v}{\partial t}+\mu\dfrac{\partial v}{\partial x}=uv+h(v)\varepsilon, \quad (x,t)\in R\times R_+\,, \end{cases}$$

with the initial data

$$(2)\qquad \begin{cases} u(x,0)=\phi(x)\,, \\ v(x,0)=\psi(x), \quad x\in R\,, \end{cases}$$

where $R=(-\infty,+\infty)$ and $R_+=(0,+\infty)$; $\lambda,\mu(\lambda\neq\mu)$ and ε are real constants; g and h are real-valued and real analytic functions at the origin with radii ρ_1 and ρ_2 respectively, that is to say

$$(3)\qquad \begin{cases} g(u)=\sum_{k=0}^{\infty}a_k u^k, \quad h(v)=\sum_{k=0}^{\infty}b_k v^k\,; \\[1ex] \limsup\limits_{k\to\infty}\sqrt{|a_k|}=\dfrac{1}{\rho_1}, \quad \limsup\limits_{k\to\infty}\sqrt{|b_k|}=\dfrac{1}{\rho_2}\,. \end{cases}$$

Without loss of generality we may assume that $0<\rho_1\leqq\rho_2$, and we suppose that

$$(4)\qquad \phi(x),\ \psi(x)\geqq 0,\ x\in R\,;\ \phi(x),\ \psi(x)\in\mathfrak{B}^1(R)\,,$$

where by $\mathfrak{B}^1(R)$ we mean the function space of all real-valued C^1-functions which are bounded on R together with their first derivatives. From now on by $C^1(S)$ we mean the function space of all real-valued continously differentiable functions defined on S.

The system (1)-(2) has an ecological meaning when both g and h are some

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polynomials of degree one, namely $g(u)=a_1u$ and $h(v)=b_1v$. If $a_1>0$, $b_1<0$ and $\varepsilon>0$, then u and v in (1) represent *prey* and *predator* respectively, and the system (1)-(2) describes what is called *prey-predator equations.* The constants $a_1\varepsilon$ and $b_1\varepsilon$ represent the rate of natural multiplication of prey without predator and the rate of natural extinction of predator without prey respectively (see Yamaguti and Niizeki [9]).

In this paper we will investigate on the following three matters.

The first is to obtain the solutions, which belong to $C^1(\Omega_T)$, of the Cauchy problem (1)-(2) in the following form (see THEOREM 4.4):

$$(5)\qquad \begin{cases} u(x,t)=\sum_{l=0}^{\infty} u_l(x,t)\varepsilon^l\,, \\ v(x,t)=\sum_{l=0}^{\infty} v_l(x,t)\varepsilon^l,\ (x,t)\in\Omega_T\,, \end{cases}$$

where u_l and v_l will be introduced in §1, and Ω_T is defined by

$$(6)\qquad \Omega_T=R\times[0,T],\ T>0\,.$$

The representation (5) shows that the solutions of the Cauchy problem (1)-(2) can be described as analytic functions of ε.

The second is as follows: The solutions of semi-linear hyperbolic system of partial differential equations of two independent variables can be constructed by the method of successive approximation (see Nagumo [4]). In this case, in general, we need to take the absolute values of the initial data sufficiently small according to T. In this paper, however, it will be shown that if g and h in (1) are entire functions over R then we can take initial data independent of T (see REMARK 4.5).

And the third is to show that for some initial data ϕ and ψ the Cauchy problem (1)-(2) has traveling wave-like solutions for sufficiently small ε (see THEOREM 5.3).

Now, in case that $g(u)=a_0+a_1u+a_2u^2$ and $h(v)=b_0+b_1v+b_2v^2$ and in case that $g(u)=\sum_{k=0}^{n} a_ku^k$ and $h(v)=\sum_{k=0}^{n} b_kv^k$, where n is an arbitrary positive integer, we investigated in detail in Niizeki [5] and [6] respectively.

§1. Preliminaries and notations

u_l and v_l in (5) are, in truth, the solutions of the following semi-linear system (1.1) ($l=0$) or the linear system (1.2) ($l\geqq 1$) of partial differential equations for u_l and v_l: in case $l=0$

$$
(1.1)\quad \begin{cases} \dfrac{\partial u_0}{\partial t}+\lambda\dfrac{\partial u_0}{\partial x}=-u_0v_0\,, \\[2ex] \dfrac{\partial v_0}{\partial t}+\mu\dfrac{\partial v_0}{\partial x}=u_0v_0,\ (x,t)\in R\times R_+\,, \\[2ex] u_0(x,0)=\phi(x),\ v_0(x,0)=\psi(x),\ x\in R\,, \end{cases}
$$

where $\phi(x)$ and $\psi(x)$ are the initial data in (2), and in case $l\geqq 1$

$$
(1.2)\quad \begin{cases} \dfrac{\partial u_l}{\partial t}+\lambda\dfrac{\partial u_l}{\partial x}=-\sum_{i=0}^{l}u_{l-i}v_i+\sum_{k=0}^{\infty}a_kU_{l-1}^{(k)}(x,t)\,, \\[2ex] \dfrac{\partial v_l}{\partial t}+\mu\dfrac{\partial v_l}{\partial x}=\sum_{i=0}^{l}u_{l-i}v_i+\sum_{k=0}^{\infty}b_kV_{l-1}^{(k)}(x,t),\ (x,t)\in\Omega_T\,, \\[2ex] u_l(x,0)=0,\ v_l(x,0)=0,\ x\in R\,, \end{cases}
$$

where

$$
(1.3)\quad \begin{cases} U_0^{(0)}(x,t)=V_0^{(0)}(x,t)\equiv 1\,, \\ U_l^{(0)}(x,t)=V_l^{(0)}(x,t)\equiv 0\ (l\geqq 1)\,, \\ U_l^{(k)}(x,t)=\sum_{i=0}^{l}u_i(x,t)U_{l-i}^{(k-1)}(x,t)\ (k\geqq 1,\ l\geqq 0)\,, \\ V_l^{(k)}(x,t)=\sum_{i=0}^{l}v_i(x,t)V_{l-i}^{(k-1)}(x,t)\ (k\geqq 1,\ l\geqq 0),\ (x,t)\in\Omega_T\,. \end{cases}
$$

The properties of solutions of (1.1) are investigated in detail in Yoshikawa and Yamaguti [10]. The convergency of the series appearing in (1.2)

$$
(1.4)\qquad \sum_{k=0}^{\infty}a_kU_{l-1}^{(k)}(x,t),\ \sum_{k=0}^{\infty}b_kV_{l-1}^{(k)}(x,t),\ (x,t)\in\Omega_T
$$

will be examined in (1.20) for $l=1$ and in REMARK 2.4 for $l\geqq 2$. The system (1.1) and (1.2) can be formally obtained by substituting (5) into (1) and collecting terms with the same power in ε.

Now, in view of (4) there exist positive constants M and $\tilde{M}$ such that

$$
(1.5)\qquad 0\leqq\phi(x),\ \psi(x)\leqq M;\ \left|\frac{d}{dx}\phi(x)\right|,\ \left|\frac{d}{dx}\psi(x)\right|\leqq\tilde{M},\ x\in R\,.
$$

PROPOSITION 1.1. *For any $T>0$, the solutions u_0 and v_0 of the Cauchy problem (1.1) are nonnegative and bounded over Ω_T.*

PROOF. We remark here that the system (1.1) has real-valued global solutions which belong to $C^1(R\times R_+)$ (see Hashimoto [2] or Hirota [3]). Now, from (1.1) we have

$$(1.6)\quad \begin{cases} u_0(x,t)=\phi(x-\lambda t)\exp\left(-\int_0^t v_0(x-\lambda t+\lambda s,s)ds\right), \\ v_0(x,t)=\psi(x-\mu t)\exp\left(\int_0^t u_0(x-\mu t+\mu s,s)ds\right). \end{cases}$$

Hence u_0 and v_0 are nonnegative since ϕ and ψ are nonnegative. Next, from (1.5) and (1.6) we have

$$(1.7)\quad u_0(x,t)\leqq M,\ v_0(x,t)\leqq Me^{MT},\ (x,t)\in\Omega_T,$$

which shows that u_0 and v_0 are bounded over Ω_T. Q.E.D.

In connection with the above proposition, we define r_0 by

$$(1.8)\quad r_0=Me^{MT}.$$

Furthermore, for every solution u_l and v_l ($l\geqq 0$) of the Cauchy problem (1.1) and (1.2) we put

$$(1.9)\quad \begin{cases} \tilde{u}_l(x,t)=\dfrac{\partial}{\partial x}u_l(x,t), & \tilde{v}_l(x,t)=\dfrac{\partial}{\partial x}v_l(x,t), \\ \bar{u}_l(x,t)=\dfrac{\partial}{\partial t}u_l(x,t), & \bar{v}_l(x,t)=\dfrac{\partial}{\partial t}v_l(x,t). \end{cases}$$

Now, we will give here Harr's inequality (see Petrovski [7]), which will be often used later in the following form.

Let us consider the system of linear partial differential equations

$$\frac{\partial u_1}{\partial t}+c_1\frac{\partial u_1}{\partial x}=a_{11}(x,t)u_1+a_{12}(x,t)u_2+b_1(x,t),$$

$$\frac{\partial u_2}{\partial t}+c_2\frac{\partial u_2}{\partial x}=a_{12}(x,t)u_1+a_{22}(x,t)u_2+b_2(x,t),\ (x,t)\in\Omega_T,$$

with the initial data $u_1(x,0)=\phi_1(x)$ and $u_2(x,0)=\phi_2(x)$ ($x\in R$). Here, c_1 and c_2 are real constants and $a_{ij}(x,t)$ ($1\leqq i,j\leqq 2$) and $b_i(x,t)$ ($1\leqq i\leqq 2$) are continuous and bounded over Ω_T, and $\phi_1(x)$ and $\phi_2(x)$ are continuous and bounded over R. Further we put

$$A=\max_{1\leqq i,j\leqq 2}\{\sup_{(x,t)\in\Omega_T}|a_{ij}(x,t)|\},$$

$$B=\max_{1\leqq i\leqq 2}\{\sup_{(x,t)\in\Omega_T}|b_i(x,t)|\},$$

$$C=\max_{1\leqq i\leqq 2}\{\sup_{x\in R}|\phi_i(x)|\},$$

then we have Haar's inequality:

$$|u_1(x,t)|,\ |u_2(x,t)| \leqq Ce^{2AT} + \frac{B}{2A}(e^{2AT}-1),\ (x,t)\in\Omega_T\,. \tag{1.10}$$

Under these preparations we prove the following propositions.

PROPOSITION 1.2. For any $T>0$, $\tilde{u}_0, \tilde{v}_0, \bar{u}_0$ and $\bar{v}_0$ are all bounded over Ω_T.

PROOF. Differentiating (1.1) with respect to x we obtain the following system (1.11) of partial differential equations for $\tilde{u}_0$ and $\tilde{v}_0$ with the initial data (1.12):

$$\left\{\begin{aligned} &\frac{\partial\tilde{u}_0}{\partial t}+\lambda\frac{\partial\tilde{u}_0}{\partial x}=-v_0\tilde{u}_0-u_0\tilde{v}_0\,, \\ &\frac{\partial\tilde{v}_0}{\partial t}+\mu\frac{\partial\tilde{v}_0}{\partial x}=v_0\tilde{u}_0+u_0\tilde{v}_0,\ (x,t)\in R\times R_+\,, \end{aligned}\right. \tag{1.11}$$

$$\tilde{u}_0(x,0)=\frac{d}{dx}\phi(x),\ \tilde{v}_0(x,0)=\frac{d}{dx}\psi(x),\ x\in R\,. \tag{1.12}$$

Applying Haar's inequality (1.10) to the Cauchy problem (1.11)-(1.12) and using (1.5) and (1.7) we have

$$|\tilde{u}_0(x,t)|,\ |\tilde{v}_0(x,t)| \leqq \tilde{M}\exp(2MTe^{MT}),\ (x,t)\in\Omega_T\,. \tag{1.13}$$

Hence $\tilde{u}_0$ and $\tilde{v}_0$ are bounded over Ω_T.

Next, by considering (1.9) for $l=0$ the first and the second expressions in (1.1) can be rewritten by

$$\bar{u}_0+\lambda\tilde{u}_0=-u_0v_0,\quad \bar{v}_0+\mu\tilde{v}_0=u_0v_0\,.$$

Hence, from (1.7) and (1.13) we have

$$|\bar{u}_0(x,t)| \leqq |u_0|\,|v_0|+|\lambda|\,|\tilde{u}_0| \leqq M^2e^{MT}+|\lambda|\tilde{M}\exp(2MTe^{MT}),\ (x,t)\in\Omega_T\,.$$

Similarly we have

$$|\bar{v}_0(x,t)| \leqq M^2e^{MT}+|\mu|\tilde{M}\exp(2MTe^{MT}),\ (x,t)\in\Omega_T\,.$$

Therefore when $\eta=\max\{1, |\lambda|, |\mu|\}$ we have

$$|\bar{u}_0(x,t)|,\ |\bar{v}_0(x,t)| \leqq M^2e^{MT}+\eta\tilde{M}\exp(2MTe^{MT}),\ (x,t)\in\Omega_T\,. \tag{1.14}$$

Hence the proposition now follows at once. Q.E.D.

In connection with (1.13) and (1.14) we will define $\tilde{r}_0$ by

$$\tilde{r}_0=M^2e^{MT}+\eta\tilde{M}\exp(2MTe^{MT})\,. \tag{1.15}$$

PROPOSITION 1.3. *For any positive number p, there exists a positive number δ_T*

such that if $M \leqq \delta_T$ *then the inequality*

(1.16) $$r_0 \leqq p$$

holds. Here δ_T *depends on* T, *and for example we may take* δ_T *as*

$$\delta_T = \frac{1}{T} \log \frac{1+\sqrt{1+4pT}}{2},$$

then we remark that if $p \to +\infty$ *then* $\delta_T \to +\infty$.

PROOF. The proof is easily performed, so we omit it. Q. E. D.

Now, for all integers $k \geqq 0$ we will define q_k by

(1.17) $$q_k = \max\{|a_k|,\ |b_k|\},$$

where a_k and b_k ($k \geqq 0$) are given in (3). Then we have the following proposition.

PROPOSITION 1.4. *The radius of convergence of* $\sum_{k=0}^{\infty} q_k z^k$ *is equal to* ρ_1, *where* ρ_1 *is defined in* (3).

PROOF. The proof is obvious, so we omit it. Q. E. D.

Now, by PROPOSITION 1.3 we can choose M, which is introduced in (1.5), so small that the inequality

(1.18) $$r_0 \leqq \frac{\rho_1}{2}$$

holds, where r_0 is defined in (1.8). We remark here that if M is chosen so that the inequality

(1.19) $$M \leqq \frac{1}{T} \log \frac{1+\sqrt{1+2\rho_1 T}}{2}$$

holds, then (1.18) holds. From now on we suppose that M is chosen so that (1.18) holds. Then in view of PROPOSITION 1.4 we have

(1.20) $$\sum_{k=0}^{\infty} q_k r_0{}^k < +\infty .$$

Hence, for $l=1$ both of the series in (1.4) converge uniformly over Ω_T.

Here, we define a constant L_T by

(1.21) $$L_T = \frac{1}{2r_0} \{\exp(2r_0 T) - 1\},\ T>0,$$

which will be used in the definition of r_l in (2.1).

§ 2. The estimates of $u_l(x, t)$ and $v_l(x, t)$

For every integer $k \geqq 0$ and $l \geqq 0$, we will define r_l and $K_l^{(k)}$ inductively by means of the following relations:

$$(2.1) \qquad \begin{cases} r_l = (\sum_{i=1}^{l-1} r_{l-i} r_i) L_T + \sum_{k=0}^{\infty} q_k K_{l-1}^{(k)} \quad (l \geqq 1), \\ K_0^{(0)} = 1, \ K_l^{(0)} = 0 \quad (l \geqq 1), \\ K_l^{(k+1)} = \sum_{i=0}^{l} r_i K_{l-i}^{(k)} \quad (k \geqq 0, \ l \geqq 0), \end{cases}$$

where we put $\sum_{i=1}^{l-1} r_{l-i} r_i = 0$ for $l=1$ and the validity of definition of r_1 follows immediately from (1.20) since $K_0^{(k)} = r_0{}^k$, and the convergency of $\sum_{k=1}^{\infty} q_k K_{l-1}^{(k)}$ in the definition of r_l will be shown in the proof of PROPOSITION 2.1.

PROPOSITION 2.1. *The implicit function $w(z)$, which satisfies $w(0) = r_0$, determined by the equation*

$$(2.2) \qquad F(z, w) = z \sum_{k=0}^{\infty} q_k w^k + L_T (w - r_0)^2 - (w - r_0) = 0,$$

$$(z, w) \in \{(z, w) \mid |z| < +\infty, \ |w| < \rho_1\},$$

has the expression

$$(2.3) \qquad w(z) = \sum_{l=0}^{\infty} r_l z^l, \quad |z| < \rho_0,$$

where the sequence $\{r_l\}_{l=0}^{\infty}$ is given in (1.8) *and* (2.1), *and ρ_0 is some positive constant.*

PROOF. Since by PROPOSITION 1.4 the right-hand side of $F(z, w)$ converges on the domain $\{(z, w) \mid |z| < +\infty, \ |w - r_0| < \rho_1 - r_0\}$ and since $F(0, r_0) = 0$ and $\dfrac{\partial F}{\partial w}(0, r_0) = -1$, by the existence theorem for implicit function (see Tsuji [8]), $w(z)$ has the following expansion with the radius of convergence ρ:

$$(2.4) \qquad w(z) = \sum_{l=0}^{\infty} c_l z^l, \quad |z| < \rho,$$

where $c_0 = r_0$ and ρ is some positive constant. Now, by using the Weierstrass' double series theorem we will define $E_l^{(k)}$ for every pair (k, l) of integers $k \geqq 0$ and $l \geqq 0$ by

$$(2.5) \qquad w(z)^k = (\sum_{l=0}^{\infty} c_l z^l)^k = \sum_{l=0}^{\infty} E_l^{(k)} z^l, \quad |z| < \rho.$$

Hence we have

$$(2.6) \qquad \begin{cases} E_0^{(0)} = 1, \ E_l^{(0)} = 0 \quad (l \geqq 1), \\ E_l^{(k+1)} = \sum_{i=0}^{l} c_i E_{l-i}^{(k)} \quad (k \geqq 0, \ l \geqq 0). \end{cases}$$

Substituting (2.4) into (2.2) and using (2.5), we have

(2.7) $$\sum_{l=1}^{\infty}(\sum_{k=0}^{\infty} q_k E_{l-1}^{(k)})z^l + L_T \sum_{l=2}^{\infty}(\sum_{i=1}^{l-1} c_i c_{l-i})z^l = \sum_{l=1}^{\infty} c_l z^l, \ |z|<\rho.$$

(2.8) $$c_l = \sum_{k=0}^{\infty} q_k E_{l-1}^{(k)} + L_T(\sum_{i=1}^{l-1} c_i c_{l-i}) \ (l \geqq 1),$$

where the right-hand side of (2.8) is well-defined. Therefore we have

(2.9) $$c_l = r_l \ (l \geqq 0), \ E_l^{(k)} = K_l^{(k)} \ (k \geqq 0, \ l \geqq 0)$$

comparing (2.1) with (2.6) and (2.8). Hence we have

(2.10) $$\sum_{k=0}^{\infty} q_k K_l^{(k)} < +\infty \ (l \geqq 0),$$

and we see that r_l $(l \geqq 1)$ defined in (2.1) are well-defined. Q. E. D.

We are now in a position to prove the main proposition of this section.

PROPOSITION 2.2. *For $U_l^{(k)}(x,t)$ and $V_l^{(k)}(x,t)$ defined in* (1.3) *we have the following estimates*:

(2.11) $$|U_l^{(k)}(x,t)|, \ |V_l^{(k)}(x,t)| \leqq K_l^{(k)} L_T{}^l \ (l \geqq 0, \ k \geqq 0), \ (x,t) \in \Omega_T.$$

PROOF. Let us prove this through the following four steps.

(i) From (1.3), (1.7), (1.8) and (2.1) it is obvious that the estimates (2.11) hold for $k=0$ and $l \geqq 0$ and for $k=1$ and $l=0$.

(ii) We suppose that (2.11) hold for $1 \leqq k \leqq s$ and $0 \leqq l \leqq n$. Then from (1.3) and (2.1) we have

$$|U_l^{(s+1)}| \leqq \sum_{i=0}^{l} |u_i| \, |U_{l-i}^{(s)}| \leqq (\sum_{i=0}^{l} r_i K_{l-i}^{(s)}) L_T{}^l = K_l^{(s+1)} L_T{}^l \ (0 \leqq l \leqq n),$$

where by (1.3) and (2.1) we have $U_i^{(1)} = u_i$ and $K_i^{(1)} = r_i$. Similarly we have

$$|V_l^{(s+1)}| \leqq K_l^{(s+1)} L_T{}^l \ (0 \leqq l \leqq n).$$

Therefore by an induction process on k the estimates (2.11) hold for $k \geqq 0$ and $0 \leqq l \leqq n$.

(iii) From (1.17), (2.10) and (ii), we see that the system (1.2) has meaning for $l=n+1$. Therefore applying Haar's inequality (1.10) to (1.2) for $l=n+1$ and using (ii) and (2.1) we have

$$\begin{aligned} |U_{n+1}^{(1)}| = |u_{n+1}| &\leqq L_T{}^{n+2} \sum_{i=1}^{n} r_{n+1-i} r_i + L_T{}^{n+1} (\sum_{k=0}^{\infty} q_k K_n^{(k)}) \\ &= L_T{}^{n+1} \{ L_T \sum_{i=1}^{n} r_{n+1-i} r_i + \sum_{k=0}^{\infty} q_k K_n^{(k)} \} = r_{n+1} L_T{}^{n+1} = K_{n+1}^{(1)} L_T{}^{n+1}. \end{aligned}$$

Similarly we have $V_{n+1}^{(1)} \leqq K_{n+1}^{(1)} L_T{}^{n+1}$. Hence the estimates (2.11) hold for $k=1$ and $l=n+1$.

(iv) From (i), (ii) and (iii) we easily see that (2.11) hold for all integers $k \geqq 0$ and $l \geqq 0$. Q. E. D.

REMARK 2.3. From (1.3) we see that $U_l^{(1)} = u_l$ and $V_l^{(1)} = v_l$, and from (2.1) we see that $K_l^{(1)}=r_l$. Therefore putting $k=1$ in (2.11) we have

$$|u_l(x,t)|,\ |v_l(x,t)|\leqq r_l L_T{}^l,\ (x,t)\in\Omega_T\,, \tag{2.12}$$

which will be used in proving PROPOSITION 3.6, LEMMA 5.1 and 5.2.

REMARK 2.4. In view of (2.10) and (2.11) we see that both $\sum_{k=0}^{\infty} a_k U_{l-1}^{(k)}(x,t)$ and $\sum_{k=0}^{\infty} b_k V_{l-1}^{(k)}(x,t)$ $(l\geqq 1)$ appearing in (1.2) converge uniformly over Ω_T.

§ 3. The estimates of $\tilde{u}_l$, $\tilde{v}_l$, $\bar{u}_l$, and $\bar{v}$

The purpose of this section is to estimate $\tilde{u}_l$, $\tilde{v}_l$, $\bar{u}_l$ and $\bar{v}_l$ $(l\geqq 1)$, which are defined in (1.9), in the same manner as in PROPOSITION 2.2.

For any pair (k,l) of integers $k\geqq 0$ and $l\geqq 0$ we will inductively define $\tilde{U}_l^{(k)}(x,t)$ and $\tilde{V}_l^{(x,t)}$ on Ω_T by means of the following relations:

$$\left\{\begin{array}{l} \tilde{U}_l^{(0)}=\tilde{V}_l^{(0)}=0 \quad (l\geqq 0)\,, \\ \tilde{U}_l^{(k)}=k\sum_{i=0}^{l}\tilde{u}_i U_{l-i}^{(k-1)} \quad (k\geqq 1,\ l\geqq 0)\,, \\ \tilde{V}_l^{(k)}=k\sum_{i=0}^{l}\tilde{v}_i V_{l-1}^{(k-1)} \quad (k\geqq 1,\ l\geqq 0)\,, \end{array}\right. \tag{3.1}$$

where $U_l^{(k)}$, $V_l^{(k)}$ are defined in (1.3).

PROPOSITION 3.1. *For every pair (k,l) of integers $k\geqq 0$ and $l\geqq 0$, we have*

$$\frac{\partial}{\partial x} U_l^{(k)}(x,t)=\tilde{U}_l^{(k)}(x,t),\quad \frac{\partial}{\partial x} V_l^{(k)}(x,t)=\tilde{V}_l^{(k)}(x,t)\,. \tag{3.2}$$

PROOF. Let us prove this by an induction process on k. From (1.3) and (3.1) we see that (3.2) hold for $l\geqq 0$ if $k=0$. Next, we suppose that (3.2) hold for $l\geqq 0$ when $k=m$. Then from (1.3) and (3.1) we have

$$\begin{aligned} \frac{\partial}{\partial x} U_l^{(m+1)} &= \sum_{i=0}^{l}\tilde{U}_i^{(m)}u_{l-i}+\sum_{i=0}^{l} U_i^{(m)}\tilde{u}_{l-i} \\ &= m\sum_{i=0}^{l}\sum_{s=0}^{i}\tilde{u}_s\{U_{i-s}^{(m-1)}u_{l-i}\}+\sum_{i=0}^{l}\tilde{u}_i U_{l-i}^{(m)} \\ &= m\sum_{s=0}^{l}\tilde{u}_s\{\sum_{j=0}^{l-s} U_j^{(m-1)}u_{l-s-j}\}+\sum_{i=0}^{l}\tilde{u}_i U_{l-i}^{(m)} \\ &= m\sum_{s=0}^{l}\tilde{u}_s U_{l-s}^{(m)}+\sum_{i=0}^{l}\tilde{u}_i U_{l-i}^{(m)}=(m+1)\sum_{i=0}^{l}\tilde{u}_i U_{l-i}^{(m)}=\tilde{U}_l^{(m+1)}\,. \end{aligned}$$

Similarily we have

$$\frac{\partial}{\partial x} V_l^{(m+1)}=\tilde{V}_l^{(m+1)}\,.$$

Therefore by an induction process the proposition now follows at once. Q. E. D.

Now, differentiating (1.2) and using (3.1) and (3.2) we obtain the following system of partial differential equations for $\tilde{u}_l$ and $\tilde{v}_l$ $(l\geqq 1)$:

$$(3.3)\quad \begin{cases} \dfrac{\partial\tilde{u}_l}{\partial t}+\lambda\dfrac{\partial\tilde{u}_l}{\partial x}=(-v_0)\tilde{u}_l+(-u_0)\tilde{v}_l-\sum_{i=0}^{l-1}\tilde{u}_iv_{l-i}-\sum_{i=0}^{l-1}\tilde{v}_iu_{l-i}+\sum_{k=1}^{\infty}a_k\tilde{U}_{l-1}^{(k)}, \\ \dfrac{\partial\tilde{v}_l}{\partial t}+\mu\dfrac{\partial\tilde{v}_l}{\partial x}=v_0\tilde{u}_l+u_0\tilde{v}_l+\sum_{i=0}^{l-1}\tilde{u}_iv_{l-i}+\sum_{i=0}^{l-1}\tilde{v}_iu_{l-i}+\sum_{k=1}^{\infty}b_k\tilde{V}_{l-1}^{(k)},\ (x,t)\in\Omega_T, \\ \tilde{u}_l(x,0)=0,\ \tilde{v}_l(x,0)=0,\ x\in R. \end{cases}$$

Here, from (1.7), (1.8), (1.17) and (1.20) we see that both $\sum_{k=1}^{\infty}ka_ku_0^{k-1}$ and $\sum_{k=1}^{\infty}kb_kv_0^{k-1}$ converge uniformly over Ω_T. Therefore both $\sum_{k=1}^{\infty}a_k\tilde{U}_0^{(k)}$ and $\sum_{k=1}^{\infty}b_k\tilde{V}_0^{(k)}$ converge uniformly over Ω_T. Hence for $l=1$ the right-hand sides of (3.3) are well-defined. In order to show that the right-hand sides of (3.3) are well-defined for $l\geqq 2$, we must show that both of the series

$$(3.4)\quad \sum_{k=1}^{\infty}a_k\tilde{U}_{l-1}^{(k)}(x,t),\ \sum_{k=1}^{\infty}b_k\tilde{V}_{l-1}^{(k)}(x,t)\ (l\geqq 2)$$

converge uniformly over Ω_T, which will be mentioned in REMARK 3.5.

Now, we define $\tilde{r}_l$ $(l\geqq 1)$ and $\tilde{K}_l^{(k)}$ $(l\geqq 0,\ k\geqq 0)$ inductively by means of the following relations:

$$(3.5)\quad \begin{cases} \tilde{r}_l=\sum_{k=1}^{\infty}q_k\tilde{K}_{l-1}^{(k)}+2L_T(\sum_{i=0}^{l-1}\tilde{r}_ir_{l-i})\ (l\geqq 1), \\ \tilde{K}_l^{(0)}=0\ (l\geqq 0), \\ \tilde{K}_l^{(k)}=k\sum_{i=0}^{l}\tilde{r}_iK_{l-i}^{(k-1)}\ (l\geqq 0,\ k\geqq 1), \end{cases}$$

where $r_0,\tilde{r}_0$ are defined in (1.8) and (1.15) respectively and r_l and $K_l^{(k)}$ are defined in (2.1). The validity of the definition of $\tilde{r}_1$ follows from (1.20) since $\tilde{r}_1=\sum_{k=1}^{\infty}kq_kr_0^{k-1}\tilde{r}_0$, where from (2.1) and (3.5) we see that $\tilde{K}_0^{(k)}=kr_0^{k-1}\tilde{r}_0$. The convergency of $\sum_{k=1}^{\infty}q_k\tilde{K}_{l-1}^{(k)}$ $(l\geqq 2)$ will be examined in (3.15).

PROPOSITION 3.2. *The implicit function $\tilde{w}(z)$, determined by*

$$(3.6)\quad \tilde{F}(z,\tilde{w})=z\tilde{w}\sum_{k=1}^{\infty}kq_kw(z)^{k-1}+2L_T(w(z)-r_0)\tilde{w}-(\tilde{w}-\tilde{r}_0)=0,$$

has the expansion

$$(3.7)\quad \tilde{w}(z)=\sum_{l=0}^{\infty}\tilde{r}_lz^l,\ |z|<\tilde{\rho},$$

where $\tilde{\rho}$ is some positive number independent of $\tilde{r}_0$, and $w(z)$ is given in (2.3).

PROOF. By PROPOSITION 2.1 if $|z|<\rho_0$ then $|w(z)|<\rho_1$. Hence by PROPOSITION 1.4 we see that $\sum_{k=1}^{\infty}kq_kw(z)^{k-1}$ converges for $|z|<\rho_0$. Therefore the right-hand side of (3.6) is well-defined on $\{(z,\tilde{w})|\ |z|<\rho_0,\ |\tilde{w}|<+\infty\}$.

Now, since $\tilde{F}(z, \tilde{w})$ is linear for $\tilde{w}$ and since $\tilde{F}(0, \tilde{r}_0)=0$, the function $\tilde{w}(z)$ determined by (3.6) has the expansion of the form

$$\tilde{w}(z)=\sum_{n=0}^{\infty}\tilde{c}_n z^n, \ |z|<\tilde{\rho}, \tag{3.8}$$

where $\tilde{c}_0=\tilde{r}_0$ and $\tilde{\rho}$ is some positive number. We define $\tilde{\rho}_0$ by

$$\tilde{\rho}_0=\min\{\rho_0, \tilde{\rho}\}, \tag{3.9}$$

where ρ_0 is defined in (2.3). Then, by the Weierstrass' double series theorem we define $\tilde{E}_l^{(k)}$ $(k\geqq 1, l\geqq 0)$ by means of the following relations:

$$k(\sum_{l=0}^{\infty}\tilde{c}_l z^l)\cdot(\sum_{l=0}^{\infty}r_l z^l)^{k-1}=\sum_{l=0}^{\infty}\tilde{E}_l^{(k)}z^l, \ |z|<\tilde{\rho}_0. \tag{3.10}$$

In view of (2.1) we have

$$(\sum_{l=0}^{\infty}r_l z^l)^k=\sum_{l=0}^{\infty}K_l^{(k)}z^l, \ |z|<\tilde{\rho}_0. \tag{3.11}$$

Therefore from (2.1), (3.9) and (3.10) we have

$$\tilde{E}_l^{(k)}=k\sum_{i=0}^{l}\tilde{c}_i K_{l-i}^{(k-1)} \ (k\geqq 1, l\geqq 0), \tag{3.12}$$

where we define $\tilde{E}_l^{(0)}=0$ $(l\geqq 0)$. Substituting (3.8) into (3.6) and considering (3.9) we have

$$\sum_{l=1}^{\infty}(\sum_{k=1}^{\infty}q_k\tilde{E}_{l-1}^{(k)})z^l+2L_T\sum_{l=1}^{\infty}(\sum_{i=1}^{l-1}\tilde{c}_i r_{l-i})z^l=\sum_{l=1}^{\infty}\tilde{c}_l z^l, \ |z|<\tilde{\rho}_0. \tag{3.13}$$

Comparing each coefficient of z^l of (3.13) we obtain

$$\tilde{c}_l=\sum_{k=1}^{\infty}q_k\tilde{E}_{l-1}^{(k)}+2L_T\sum_{i=0}^{l-1}\tilde{c}_i r_{l-i} \ (l\geqq 1). \tag{3.14}$$

Therefore, from (3.5), (3.12) and (3.14) we easily see that

$$\tilde{c}_l=\tilde{r}_l \ (l\geqq 0), \ \tilde{E}_l^{(k)}=\tilde{K}_l^{(k)} \ (l\geqq 0, k\geqq 0).$$

Hence the proposition follows at once. Furthermore we have

$$\sum_{k=1}^{\infty}q_k\tilde{K}_l^{(k)}<+\infty \ (l\geqq 0). \tag{3.15}$$

Q.E.D.

Under these preparations we will prove the main proposition of this section.

PROPOSITION 3.3. *For* $\tilde{U}_l^{(k)}(x,t)$ *and* $\tilde{V}_l^{(k)}(x,t)$ $(l\geqq 0, k\geqq 0)$ *defined in* (3.1) *we have*

$$|\tilde{U}_l^{(k)}(x,t)|, \ |\tilde{V}_l^{(k)}(x,t)|\leqq\tilde{K}_l^{(k)}L_T{}^l, \ (x,t)\in\Omega_T. \tag{3.16}$$

PROOF. In the same manner of the proof of PROPOSITION 2.2, we will prove this through four steps.

(i) It is obvious from (3.1) and (3.5) that (3.16) hold for $k=0$ and $l\geqq 0$. From (2.1), (3.1) and (3.5) we see that $\tilde{U}_0^{(1)}=\tilde{u}_0$, $\tilde{V}_0^{(1)}=\tilde{v}_0$ and $\tilde{K}_0^{(1)}=\tilde{r}_0$ hold. Hence we see from (1.13) and (1.15) that (3.16) hold for $k=1$ and $l=0$.

(ii) We suppose that (3.16) hold for $1\leqq k\leqq s$ and $0\leqq l\leqq n$. Then from (3.1), (3.5) and (2.11) we have

$$|\tilde{U}_l^{(s+1)}|\leqq(s+1)\sum_{i=0}^{l}|\tilde{u}_i|\,|U_{l-i}^{(s)}|\leqq((s+1)\sum_{i=0}^{l}\tilde{r}_iK_{l-i}^{(s)})L_T{}^l=\tilde{K}_l^{(s+1)}L_T{}^l\ (0\leqq l\leqq n)\,.$$

Similarly we have

$$|\tilde{V}_l^{(s+1)}|\leqq\tilde{K}_l^{(s+1)}L_T{}^l\ (0\leqq l\leqq n)\,.$$

Hence by an induction process on k, the estimates (3.16) hold for $k\geqq 0$ and $0\leqq l\leqq n$.

(iii) From (1.17), (3.15) and (ii) we see that the system (3.3) has meaning for $l=n+1$. Therefore applying Haar's inequality (1.10) to (3.3) for $l=n+1$ and using (ii) and (3.5) we have

$$\begin{aligned}|\tilde{U}_{n+1}^{(1)}|=|\tilde{u}_{n+1}|&\leqq L_T\{2(\textstyle\sum_{i=0}^{n}\tilde{r}_ir_{n+1-i})L_T{}^{n+1}+(\sum_{k=1}^{\infty}q_k\tilde{K}_n^{(k)})L_T{}^n\}\\&=\{\textstyle\sum_{k=1}^{\infty}q_k\tilde{K}_n^{(k)}+2L_T\sum_{i=0}^{n}\tilde{r}_ir_{n+1-i}\}L_T{}^{n+1}=\tilde{r}_{n+1}L_T{}^{n+1}=\tilde{K}_{n+1}^{(1)}L_T{}^{n+1}\,.\end{aligned}$$

Similarly we have $|\tilde{V}_{n+1}^{(1)}|\leqq\tilde{K}_{n+1}^{(1)}L_T{}^{n+1}$. Therefore (3.16) hold for $k=1$ and $l=n+1$.

(iv) From (i), (ii) and (iii) we easily see that (3.16) hold for $k\geqq 0$ and $l\geqq 0$.

Q. E. D.

REMARK 3.4. From (1.3) and (3.1) we see that $\tilde{U}_l^{(1)}=\tilde{u}_l$ and $\tilde{V}_l^{(1)}=\tilde{v}_l$, and from (2.1) and (3.5) we see that $\tilde{K}_l^{(1)}=\tilde{r}_l$. Therefore putting $k=1$ in (3.16) we have

$$|\tilde{u}_l(x,t)|,\ |\tilde{v}_l(x,t)|\leqq\tilde{r}_lL_T{}^l,\ (x,t)\in\Omega_T\,, \tag{3.17}$$

which will be used in proving PROPOSITION 3.6 and LEMMA 4.3.

REMARK 3.5. In view of (3.15) and (3.16) we easily see that both $\sum_{k=0}^{\infty}a_k\tilde{U}_{l-1}^{(k)}(x,t)$ and $\sum_{k=0}^{\infty}b_k\tilde{V}_{l-1}^{(k)}(x,t)$ $(l\geqq 1)$ appearing in (3.3) converge uniformly over Ω_T.

PROPOSITION 3.6. *For $\bar{u}_l$ and $\bar{v}_l$, which are introduced in* (1.9), *we have*

$$|\bar{u}_l(x,t)|,\ |\bar{v}_l(x,t)|\leqq\eta\tilde{r}_lL_T{}^l+\left(2r_0+\frac{1}{L_T}\right)r_lL_T{}^l,\ (x,t)\in\Omega_T\,, \tag{3.18}$$

where η is defined in (1.14).

PROOF. In view of (1.9) the first expression in (1.2) can be rewritten by

$$\bar{u}_l=-\lambda\tilde{u}_l-\textstyle\sum_{i=0}^{l}u_{l-i}v_i+\sum_{k=0}^{\infty}a_kU_{l-1}^{(k)}\,.$$

From (2.1), (2.11), (2.12) and (3.17) we have

$$\begin{aligned}|\bar{u}_l|&\leqq|\lambda|\,|\tilde{u}_l|+\textstyle\sum_{i=0}^{l}|u_{l-1}|\,|v_i|+\sum_{k=0}^{\infty}|a_k|\,|U_{l-1}^{(k)}|\\&\leqq\eta\tilde{r}_lL_T{}^l+2r_0r_lL_T{}^l+\{(\textstyle\sum_{i=1}^{l-1}r_{l-i}r_i)L_T+\sum_{k=0}^{\infty}q_kK_{l-1}^{(k)}\}L_T{}^{l-1}\end{aligned}$$

$$=\eta\tilde{r}_l L_T{}^l+\left(2r_0+\frac{1}{L_T}\right)r_l L_T{}^l .$$

Similarly we have

$$|\tilde{v}_l|\leqq\eta\tilde{r}_l L_T{}^l+\left(2r_0+\frac{1}{L_T}\right)r_l L_T{}^l . \qquad \text{Q. E. D.}$$

§4. The proof of main theorem

First, we will define ε_T' by

$$\varepsilon_T'=\frac{\tilde{\rho}_0}{L_T}, \tag{4.1}$$

where $\tilde{\rho}_0$ is defined in (3.9).

LEMMA 4.1. *For any $T>0$, if $|\varepsilon|<\varepsilon_T'$ then both of the series in* (5) *converge uniformly over Ω_T.*

PROOF. In view of (2.12) we have

$$|\sum_{l=0}^{\infty}u_l\varepsilon^l|,\ |\sum_{l=0}^{\infty}v_l\varepsilon^l|\leqq\sum_{l=0}^{\infty}r_l(L_T|\varepsilon|)^l,\ (x,t)\in\Omega_T . \tag{4.2}$$

If $|\varepsilon|<\varepsilon_T'$, then from (3.9) and (4.1) we have $L_T|\varepsilon|<\rho_0$. Therefore, from (2.3) we have $\sum_{l=0}^{\infty}r_l(L_T|\varepsilon|)^l<+\infty$. Hence the lemma now follows at once. Q. E. D.

We remark here that for any $T>0$ we can define ε_T as both of the relations

$$\begin{cases} \varepsilon_T\leqq\varepsilon_T' ,\\ \sum_{l=1}^{\infty}r_l(L_T|\varepsilon_T|)^l\leqq\dfrac{\rho_1}{2} , \end{cases} \tag{4.3}$$

are satisfied. Then, from (1.18) we have

$$\sum_{l=0}^{\infty}r_l(L_T|\varepsilon|)^l<\rho_1,\ |\varepsilon|<\varepsilon_T . \tag{4.4}$$

Hence we have the following lemma.

LEMMA 4.2. *For any $T>0$, if $|\varepsilon|<\varepsilon_T$ then we have*

$$|\sum_{l=0}^{\infty}u_l(x,t)\varepsilon^l|,\ |\sum_{l=0}^{\infty}v_l(x,t)\varepsilon^l|<\rho_1,\ (x,t)\in\Omega_T . \tag{4.5}$$

PROOF. It is obvious from (2.12) and (4.4). Q. E. D.

LEMMA 4.3. *For any $T>0$, if $|\varepsilon|<\varepsilon_T$ then the series $\sum_{l=0}^{\infty}\tilde{u}_l(x,t)\varepsilon^l$, $\sum_{l=0}^{\infty}\tilde{v}_l(x,t)\varepsilon^l$, $\sum_{l=0}^{\infty}\bar{u}_l(x,t)\varepsilon^l$ and $\sum_{l=0}^{\infty}\bar{v}_l(x,t)\varepsilon^l$ converge uniformly over Ω_T.*

PROOF. In view of (3.17) and (3.18) we have

$$|\sum_{l=0}^{\infty}\tilde{u}_l\varepsilon^l|,\ |\sum_{l=0}^{\infty}\tilde{v}_l\varepsilon^l|\leqq\sum_{l=0}^{\infty}\tilde{r}_l(L_T|\varepsilon|)^l,$$

$$|\sum_{l=0}^{\infty}\bar{u}_l\varepsilon^l|,\ |\sum_{l=0}^{\infty}\bar{v}_l\varepsilon^l|\leqq\eta\sum_{l=0}^{\infty}\tilde{r}_l(L_T|\varepsilon|)^l+\left(2r_0+\frac{1}{L_T}\right)\sum_{l=0}^{\infty}r_l(L_T|\varepsilon|)^l.$$

If $|\varepsilon|<\varepsilon_T$ then from (4.1) and (4.3) we have $L_T|\varepsilon|<\tilde{\rho}_0$. Therefore from (3.7) and (3.9) we see that $\sum_{l=0}^{\infty}\tilde{r}_l(L_T|\varepsilon|)^l<+\infty$. We already saw in the proof of LEMMA 4.1 that $\sum_{l=0}^{\infty}r_l(L_T|\varepsilon|)^l<+\infty$. Hence the lemma now follows at once. Q.E.D.

Under these preparations we obtain the following theorem.

THEOREM 4.4. *For any* $T>0$, *if* $\phi, \psi\in\mathfrak{B}^1(R)$, $0\leqq\phi(x)$, $\psi(x)\leqq M_T$ *and* $|\varepsilon|<\varepsilon_T$ *then the Cauchy problem* (1)-(2) *has solutions* u *and* v *which are unique and belong to* $C^1(\Omega_T)$. *The solutions* u *and* v *are analytic with respect to* ε *and are expressed in the form of* (5). *The right-hand sides of* (5) *converge uniformly over* Ω_T. *Here* M_T *is arbitrary positive constant such that* $M_T\leqq\dfrac{1}{T}\log\dfrac{1+\sqrt{1+2\rho_1T}}{2}$ (*see* (1.19)), *and* ε_T *is arbitrary positive constant which satisfy* (4.3), *where* r_l ($l\geqq0$), L_T *and* $\tilde{\rho}_0$ *are defined when we put* $M=M_T$ *in* (1.8) and in (1.15).

PROOF. In view of LEMMA 4.3, both of the series in (5) are diffentiable term by term with respect to x and t on Ω_T. And from the manner of the constructions of u_l and v_l ($l\geqq0$) and from LEMMA 4.1 and 4.2, if $|\varepsilon|<\varepsilon_T$ then u and v given in (5) are solutions of the Cauchy problem (1)-(2). The uniform convergency follows from LEMMA 4.1 and the first inequality in (4.3) at once. The uniqueness of solutions is obvious from the general theory (see Nagumo [4]). Q.E.D.

REMARK 4.5. If g and h appearing in (1) are entire functions (i.e. $\rho_1=+\infty$), then THEOREM 4.4 can be rewrited as follows.

For any positive constants T and M, if $0\leqq\phi(x)$, $\psi(x)\leqq M$ and $|\varepsilon|<\varepsilon(T, M)$ then the Cauchy problem (1)-(2) has solutions u and v which are unique and belong to $C^1(\Omega_T)$. Here we put $\varepsilon(T, M)=\rho_0/L_T$, and $\tilde{\rho}_0$ and L_T are defined when we put $r_0=Me^{MT}$ by using M given above.

In this case, of course, the analyticity with respect to ε and the uniform convergency over Ω_T of solutions hold also.

§ 5. The existence of traveling wave-like solutions

In this section, as an application of THEOREM 4.4 we shall show that the

Cauchy problem (1) with some initial conditions has traveling wave-like solutions. For this purpose we need THEOREM 5.1 which states the existence of traveling wave solutions $u_0(x,t)=u_0(x-\xi t)$ and $v_0(x,t)=v_0(x-\xi t)$ (for the traveling wave solution, see Aronson and Weinberger [1]).

THEOREM 5.1. *The Cauchy problem*

$$(5.1)\qquad \begin{cases} \dfrac{\partial u_0}{\partial t}+\lambda\dfrac{\partial u_0}{\partial x}=-u_0v_0\,, \\[2ex] \dfrac{\partial v_0}{\partial t}+\mu\dfrac{\partial v_0}{\partial x}=u_0v_0, \ (x,t)\in R\times R_+\,, \end{cases}$$

with initial data

$$(5.2)\qquad \begin{cases} u_0(x,0)=\phi(x)=\dfrac{a\{a(\lambda-\xi)+b(\mu-\xi)\}}{a(\lambda-\xi)+b(\mu-\xi)e^{\gamma x}} \\[2ex] v_0(x,0)=\psi(x)=\dfrac{b\{a(\lambda-\xi)+b(\mu-\xi)\}}{a(\lambda-\xi)e^{-\gamma x}+b(\mu-\xi)}\,, \end{cases}$$

has traveling wave solutions of the form

$$(5.3)\qquad \begin{cases} u_0(x,t)=u_0(x-\xi t)=\dfrac{a\{a(\lambda-\xi)+b(\mu-\xi)\}}{a(\lambda-\xi)+b(\mu-\xi)e^{\gamma(x-\xi t)}}\,, \\[2ex] v_0(x,t)=v_0(x-\xi t)=\dfrac{b\{a(\lambda-\xi)+b(\mu-\xi)\}}{a(\lambda-\xi)e^{-\gamma(x-\xi t)}+b(\mu-\xi)}\,, \end{cases}$$

where $a>0$, $b>0$, $(\lambda-\xi)(\mu-\xi)>0$ *and* $\gamma=\dfrac{a}{\mu-\xi}+\dfrac{b}{\lambda-\xi}$.

PROOF. Putting $s=x-\xi t$, $u_0(x,t)=u_0(s)$ and $v_0(x,t)=v_0(s)$ in (5.1) and putting $x=0$ in (5.2), then the Cauchy problem (5.1)-(5.2) will be reduced to the Cauchy problem of ordinary differential equations for u_0 and v_0:

$$(5.4)\qquad \begin{cases} (\lambda-\xi)\dfrac{du_0}{ds}=-u_0v_0\,, \\[2ex] (\mu-\xi)\dfrac{dv_0}{ds}=u_0v_0\,, \\[2ex] u_0(0)=a,\ v_0(0)=b\,. \end{cases}$$

Adding the first expression to the second one in (5.4) we have

$$\frac{d}{ds}\{(\lambda-\xi)u_0+(\mu-\xi)v_0\}=0\,.$$

Solving this equation with initial data in (5.4) we obtain

$$(5.5)\qquad (\lambda-\xi)u_0+(\mu-\xi)v_0=a(\lambda-\xi)+b(\mu-\xi)\,.$$

Eliminating v_0 from (5.5) and the first expression in (5.4) we have

$$\frac{du_0}{ds}=\left(\frac{1}{\mu-\xi}u_0-\frac{a}{\mu-\xi}-\frac{b}{\lambda-\xi}\right)u_0 .$$

Solving this equation with $u_0(0)=a$, we have

$$u_0(s)=\frac{a\{a(\lambda-\xi)+b(\mu-\xi)\}}{a(\lambda-\xi)+b(\mu-\xi)e^{\gamma s}} . \tag{5.6}$$

Substituting (5.6) into (5.5) we have

$$v_0(s)=\frac{b\{a(\lambda-\xi)+b(\mu-\xi)\}}{a(\lambda-\xi)e^{-\gamma s}+b(\mu-\xi)} . \tag{5.7}$$

Therefore, putting $s=x-\xi t$ in (5.6) and (5.7) we obtain the traveling wave solutions (5.3). Q. E. D.

REMARK 5.2. It is easily seen from (5.3) that if $\gamma<0$ we have

$$u_0(-\infty,0)=0,\quad u_0(+\infty,0)=p ,$$
$$v_0(-\infty,0)=q,\quad v_0(+\infty,0)=0 ,$$

and if $\gamma>0$ we have

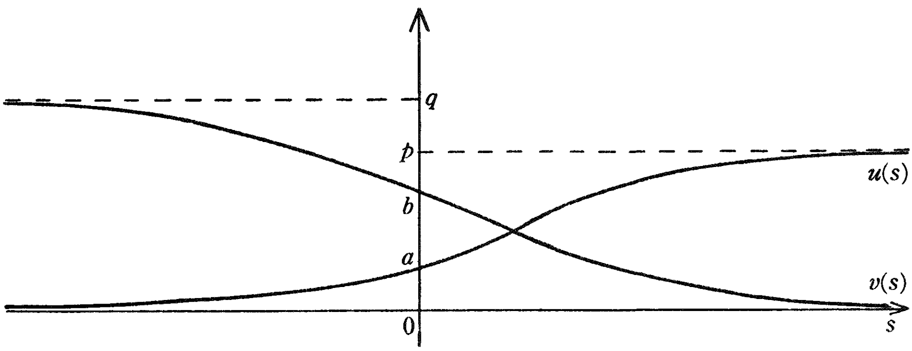
$$u_0(-\infty,0)=p,\quad u_0(+\infty,0)=0 ,$$
$$v_0(-\infty,0)=0,\quad v_0(+\infty,0)=q ,$$

where $p=a+b\dfrac{\mu-\xi}{\lambda-\xi}$ and $q=a\dfrac{\lambda-\xi}{\mu-\xi}+b$. Hence we have the relations

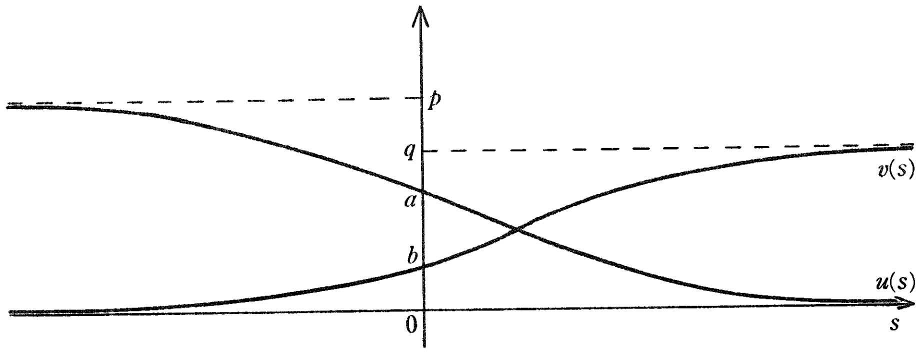
$$pq=aq+bp,\ \gamma=\frac{p}{\mu-\xi}=\frac{q}{\lambda-\xi}=\frac{p-q}{\mu-\lambda},\ \xi=\frac{\lambda p-\mu q}{p-q} .$$

Here, we will sketch the graphs of the solutions $u_0(s)$ and $v_0(s)$ according to the case of $\gamma<0$ or $\gamma>0$. Since $\lambda\neq\mu$, without loss of generality we may assume that $\lambda<\mu$.

In case of $\gamma<0$ $(0<p<q)$



in case of $\gamma>0$ $(0<q<p)$



We call the solutions u and v of the Cauchy problem (1)-(2) traveling wave-like solution when u and v are written by means of some functions f_0, f_1, g_0 and g_1 in the following form

$$(5.8)\qquad \begin{cases} u(x,t)=f_0(x-\xi t)+\varepsilon f_1(x,t)\,, \\ v(x,t)=g_0(x-\xi t)+\varepsilon g_1(x,t)\,, \end{cases}$$

where ξ is some constant, f_1 and g_1 are bounded over Ω_T and the absolute value of ε is sufficiently small.

Now, the initial data (5.2) satisfy the conditions in (4) since $a>0$, $b>0$ and $(\lambda-\xi)(\mu-\xi)>0$. If put

$$M(a,b)=\max\left\{a+b\frac{\mu-\xi}{\lambda-\xi},\ a\frac{\lambda-\xi}{\mu-\xi}+b\right\},$$

then we have

$$0\leqq\phi(x),\ \psi(x)\leqq M(a,b)\,.$$

Thus, in view of THEOREM 4.4 and 5.1 we establish the main theorem of this section.

THEOREM 5.3. *For any $T>0$, let us take positive numbers a and b so that the inequality $M(a,b)<\rho_1/2$ holds (see* (1.18)) *and take ε such that the inequality $|\varepsilon|<\varepsilon_T$ is satisfied, where ε_T is defined in* (4.3). *Then the Cauchy problem* (1) *and* (5.2) *has traveling wave-like solutions. Here, ε_T is defined in* (4.3), *and r_l $(l\geqq 0)$ are defined when we put $M=M(a,b)$ in* (1.8) and in (1.15).

PROOF. In view of THEOREM 4.4, the solutions of the Cauchy problem (1) and (5.2) can be expressed by

$$
(5.9) \qquad \begin{cases} u(x,t)=u_0(x-\xi t)+\sum_{l=1}^{\infty} u_l(x,t)\varepsilon^l, \\ v(x,t)=v_0(x-\xi t)+\sum_{l=1}^{\infty} v_l(x,t)\varepsilon^l, \ (x,t)\in\Omega_T, \end{cases}
$$

where u_0 and v_0 are traveling wave solutions (5.3) of the Cauchy problem (5.1) and (5.2). In view of LEMMA 4.1, if we take $|\varepsilon|$ sufficiently small then we can make the absolute value of the second term of the right-hand sides in (5.9) as small as possible over Ω_T. Therefore, the solutions (5.9) certainly have the form of (5.8). Hence the Cauchy problem (1) and (5.2) has traveling wave-like solutions.

Q. E. D.

REMARK 5.4. In THEOREM 5.3, if g and h are entire functions (i.e. $\rho_1=+\infty$) then we can arbitrarily take positive numbers a and b independent of T (cf. REMARK 4.5).

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