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DEGREE OF THE STANDARD ISOMETRIC MINIMAL IMMERSIONS OF THE SYMMETRIC SPACES OF RANK ONE INTO SPHERES

Dedicated to Professor Isamu Mogi on his 60th birthday

By

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Introduction

Let $(M=G/K, g)$ be an irreducible symmetric space of compact type. We denote by Δ the Laplace-Beltrami operator of (M, g) acting on the space of C^{∞} functions on M. We denote by λ_k the k-th eigen-value of Δ and by V^k the corresponding eigen-space. For each integer k, $k\geqq 1$, there exists an isometric minimal immersion x_k of M into the unit sphere in $R^{m(k)+1}$ defined by an orthonormal base of V^k, which we call the k-th standard isometric minimal immersion of M.

do Carmo and Wallach [2] showed that the k-th standard isometric minimal immersion of a sphere S^n has degree k. We showed that the k-th standard minimal immersion of a complex projective space CP^n, $n\geqq 2$, has degree $2k$ [5]. In these cases the degree of the immersion coincides with the algebraic degree of the homogeneous polynomials which define the immersion. In this note we determine the degree of the standard isometric minimal immersions of the other symmetric spaces of rank one into spheres.

THEOREM A. *Let x_k be the k-th standard isometric minimal immersion of a quaternion projective space QP^n, $n\geqq 2$. Then x_k has degree $2k$.*

Let $\pi: S^{4n+3}\to QP^n$ be the Hopf fibration, where we consider S^{4n+3} as the unit sphere in $Q^{n+1}=C^{2n+2}$. Then for each eigen-function f on QP^n which belongs to V^k, there exists a homogeneous harmonic polynomial P_f of type (k, k) on C^{2n+2} which induces f through π. So the degree of the immersion coincides with the algebraic degree of the homogeneous polynomials which define the immersion.

THEOREM B. *Let x_k be the k-th standard isometric minimal immersion of the Cayley projective plane $CayP^2$. Then x_k has degree $2k$.*

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1. Preliminalies

Let (M^m, g) be an irreducible symmetric space of compact type. We denote by Δ the Laplace-Beltrami operator of (M, g). Let λ_k be the k-th eigen-value of Δ, $0=\lambda_0<\lambda_1<\lambda_2<\cdots$, and V^k be the corresponding eigen-space. We define an inner product $(\,,)$ in V^k as follows:

$$(f, h)=\int_M f\cdot h d\mu,\ f, h\in V^k$$

where $d\mu$ is the canonical measure of (M, g) normalized in such a way that

$$\int_M d\mu=\dim.\ V^k=m(k)+1.$$

Let $f_0, f_1, \cdots, f_{m(k)}$ be an orthonormal base of V^k. Then

$$x_k: M\rightarrow R^{m(k)+1}\ ;\ p\rightarrow(f_0(p), f_1(p), \cdots, f_{m(k)}(p))$$

realizes an isometric minimal immersion of $(M, (\lambda_k/m)g)$ into the unit sphere in $R^{m(k)+1}$. We call this isometric minimal immersion the k-th *standard minimal immersion* of M.

Let (G, K) be a symmetric pair corresponding to M. Then G acts on V^k by

$$(\sigma\cdot f)(p)=f(\sigma^{-1}\cdot p),\ \sigma\in G,\ p\in M.$$

Under this action of G on V^k $v_0=\sum_{i=0}^{m(k)} f_i(eK)f_i$ is a K-fixed unit vector. Then x_k is equivalent to the following isometric minimal immersion of M into the unit sphere $S^{m(k)}$ in V^k under the identification $V^k\rightarrow R^{m(k)}$; $f\rightarrow(x_i)$ where $f=\sum_{i=0}^{m(k)} x_i f_i$;

$$x_k: M\rightarrow S^{m(k)}\ ;\ \sigma K\rightarrow\sigma\cdot v_0,\ \sigma\in G.$$

Thus x_k is an equivariant immersion, i.e., there exists a Lie group homomorphism ρ of G into the isometry group $I(S^{m(k)})$ of the unit sphere $S^{m(k)}$ in $R^{m(k)+1}$ such that

$$x_k(\sigma\cdot p)=\rho(\sigma)\cdot x_k(p),\ \sigma\in G,\ p\in M.$$

Let $B_{1|p}=(x_k)_*$ and let $B_{2|p}$ be the second fundamental form of x_k at p. Inductively we define the higher fundamental forms. Here we denote by $O_p^j(M)$, $j\geqq 2$, the linear span of Image $B_{j|p}$, and by $N_{j|p}$ the orthogonal projection of the tangent space $T_p(S^{m(k)})$ to the orthogonal complement of $(T_pM+O_p^2(M)+\cdots+O_p^j(M))$:

$$N_{j|p}: T_p(S^{m(k)})\rightarrow(T_pM+O_p^2(M)+\cdots+O_p^j(M))^{\perp}.$$

Then the $(j+1)$-th fundamental form $B_{j+1|p}$ is defined by

$$B_{j+1|p}(u_0, u_1, \cdots, u_j)=[(\nabla_{u_0}B_j(U_1, U_2, \cdots, U_j))]^{N_{j|p}},$$
$$u_0, u_1, \cdots, u_j \in T_pM,$$

where ∇ is the covariant derivative in $S^{m(k)}$ and U_i, $1\leqq i\leqq j$, are local extensions of u_i. The smallest positive integer d such that $B_{d|p}\neq 0$ and $B_{d+1|p}=0$ is called the *degree* of x_k. By the following Lemma 1 the definition of the degree is independent of the choice of p.

LEMMA 1. (1) *B_j is G-invariant,* i.e.,

$$B_{j|\sigma\cdot p}(\sigma\cdot u_1, \cdots, \sigma\cdot u_j)=\rho(\sigma)\cdot B_{j|p}(u_1, \cdots, u_j),$$
$$\sigma\in G,\ u_1, u_2, \cdots, u_j\in T_pM.$$

So we get

$$\rho(\sigma)\cdot O_p^j(M)=O_{\sigma\cdot p}^j(M),\ \sigma\in G.$$

(2) *$B_{j|p}$ is a symmetric j-multilinear mapping of T_pM onto $O_p^j(M)$.*

Since $B_{j|p}$ is symmetric we regard it as a linear mapping of the j-th symmetric power $S^j(T_pM)$ onto $O_p^j(M)$ as follows:

$$B_{j|p}(u_1\cdots u_j)=B_{j|p}(u_1, \cdots, u_j)$$

where $u_1\cdots u_j$ is the symmetric product of $u_1, \cdots, u_j\in T_pM$. Extend the isotropy action of K on $T_{eK}M$ to $S^j(T_{eK}M)$ in a natural manner. Hereafter we only consider the fundamental forms at the origin $eK\in M$. So we omit the subscript eK. We have the following

LEMMA 2. (1) *B_j is a K-homomorphism of $S^j(T_{eK}M)$ to $O_{eK}^j(M)$.*
(2) *V^k admits the following orthogonal direct sum decomposition as K-module*

$$V_k=Rv_0+T_{eK}M+O_{eK}^2(M)+\cdots+O_{eK}^d(M), \tag{1.1}$$

where d is the degree of x_k.

For the proof of Lemma 1 and Lemma 2 we refer, for instance, to [5].

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively, and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. We identify $\mathfrak{p}$ with the tangent space $T_{eK}M$ in a natural manner. Then the isotropy action of K on $\mathfrak{p}$ is the adjoint action of K on $\mathfrak{p}$. We identify the j-th symmetric power $S^j(T_{eK}M)=S^j(\mathfrak{p})$ with the space of real valued homogeneous polynomials on $\mathfrak{p}$ of degree j in a natural manner. Take an orthonormal base $e_1, \cdots, e_m$ of $\mathfrak{p}$ and put

$r=\sum_{i=1}^{m} e_i\cdot e_i \in S^j(\mathfrak{p})$. Since the adjoint action of K on $\mathfrak{p}$ leaves invariant the inner product $B_{|\mathfrak{p}\times\mathfrak{p}}$ induced by the Killing form B of $\mathfrak{g}$ we have the following decomposition of $S^j(\mathfrak{p})$ as K-module ([8], p. 255)

$$S^j(\mathfrak{p})=\begin{cases} H_j & , \ j=0,1, \\ H_j+r\cdot S^{j-2}(\mathfrak{p}) & \text{(direct sum)}, \ j\geqq 2. \end{cases}$$

where H_j is the space of homogeneous harmonic polynomials of degree j on $\mathfrak{p}$. We have the following Lemma due to do Carmo and Wallach [2].

LEMMA 3. $\operatorname{Ker} B_j \supset r\cdot S^{j-2}(p)$, $j\geqq 2$.

2. Proof of Theorem B

Let $\mathfrak{g}$ be the compact real form of the complex simple Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{f}_4$. Let G be the connected Lie group with the Lie algebra $\mathfrak{g}$. Then G contains a Lie subgroup K which is isomorphic to Spin(9) and (G, K) is a symmetric pair corresponding to the Cayley projective plane Cay P^2 [6]. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. Let $\tilde{\mathfrak{k}}$ [resp. $\tilde{\mathfrak{p}}$] be the complexification of $\mathfrak{k}$ [resp. $\mathfrak{p}$] in $\tilde{\mathfrak{g}}$ and θ be the automorphism of $\tilde{\mathfrak{g}}$ such that $\theta|\tilde{\mathfrak{k}}=\mathrm{id.}$ and $\theta|\tilde{\mathfrak{p}}=-\mathrm{id.}$

Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{h}$ is also a maximal abelian subalgebra of $\mathfrak{g}$ and the complexification $\tilde{\mathfrak{h}}$ of $\mathfrak{h}$ in $\tilde{\mathfrak{g}}$ is a common Cartan subalgebra of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$. We regard roots and weights as elements of $(-1)^{1/2}\mathfrak{h}$ via B. We denote by $\mathfrak{r}$ the set of non-zero roots of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$. We fix a lexicographic order in $(-1)^{1/2}\mathfrak{h}$ and take simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $\mathfrak{r}$ such that

$$(2B(\alpha_i, \alpha_j)/B(\alpha_j, \alpha_j))=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Take fundamental weight system $\mu_1, \mu_2, \mu_3, \mu_4$ of $\tilde{\mathfrak{g}}$. Then it is easily seen that $\theta=\exp(\mathrm{ad}\, 2\pi(-1)^{1/2}\mu_4)$ and that

$$\beta_1=2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\ \beta_2=-\alpha_1,\ \beta_3=-\alpha_2,\ \beta=-\alpha_3, \tag{2.1}$$

is a fundamental root system of $\tilde{\mathfrak{k}}$ ([4], p. 121), i.e., each root of $\tilde{\mathfrak{k}}$ is written uniquely in the form $\beta=\sum_{i=1}^{4} m_i\beta_i$ with all non-positive or all non-negative integers m_i. Let $\nu_1, \nu_2, \nu_3, \nu_4$ be the fundamental weight system of $\tilde{\mathfrak{k}}$. From (2.1) and [3] (p. 69) we get

$$\mu_4=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4=\beta_1+\beta_2+\beta_3+\beta_4, \tag{2.2}$$

$$\nu_1=\beta_1+\beta_2+\beta_3+\beta_4\,, \tag{2.3}$$
$$\nu_4=\beta_1/2+\beta_2+3\beta_3/2+2\beta_4\,.$$

Since Cay P^2 is a symmetric space of rank one, each eigen-space V^k of Δ acting on real valued C^∞ functions is a class one orthogonal representation of (G, K), i.e., V^k is G-irreducible and contains a non-zero K-fixed vector [1]. It is known that $(V^k)^C$ is also G-irreducible. So $(V^k)^C$ is a class one unitary representation. We know that the highest weight of irreducible G-module $(V^k)^C$ is $k\mu_4$ [8]. To prove Theorem B we need the following

LEMMA 4 (Smith [7]). *Let $e_1, \cdots, e_{16}$ be an orthonormal base of $\mathfrak{p}$ with respect to $B_{|\mathfrak{p}\times\mathfrak{p}}$. Let $r=\sum_{i=1}^{16} e_i\cdot e_i\in S^2(\mathfrak{p})$. Then $(S^j(\mathfrak{p}))^C$ is decomposed under adjoint representation as follows:*

$$(S^j(\mathfrak{p}))^C=\begin{cases} V(\nu_4) & ,\ j=1\,, \\ \sum_{2p+q=j} V(p\nu_1+q\nu_4)+r\cdot(S^{j-2}(\mathfrak{p}))^C & \text{(direct sum)},\ j\geqq 2\,, \end{cases} \tag{2.4}$$

where $V(\nu)$ is the irreducible K-module with highest weight ν.

Extend the j-th fundamental form $B_j: S^j(\mathfrak{p})\to O^j_{eK}(\mathrm{Cay}\,P^2)$ to a complex linear mapping $B_j: (S^j(\mathfrak{p}))^C\to (O^j_{eK}(\mathrm{Cay}\,P^2))^C$. Then by Lemma 3 and Lemma 4 we get

$$(O^j_{eK}(\mathrm{Cay}\,P^2))^C=B_j(\textstyle\sum_{2p+q=j} V(p\nu_1+q\nu_4))\,.$$

By Schur's Lemma $B_j|V(p\nu_1+q\nu_4)$ is zero or an isomorphism. So we denote by I the set of indices (p, q) such that $B_j|V(p\nu_1+q\nu_4)$ is an isomorphism. Then we get the following decomposition of $(V^k)^C$ from (1.1)

$$(V^k)^C= V(0)+V(\nu_4)+\textstyle\sum_{j=2}^{d}\sum_{2p+q=j,\,(p,q)\in I} V(p\nu_1+q\nu_4)\,, \tag{2.5}$$

where d is the degree of the k-th standard minimal immersion. From (2.5) we get

$$d=\mathrm{Max}_{(p,q)\in I}(2p+q)\,. \tag{2.6}$$

Let $(V^k)^C=\sum_{\mu\in M} V_\mu$ be the weight space decomposition of $(V^k)^C$ as a G-module. Then it is also the weight space decomposition as a K-module since K has the same rank as G.

Let (p, q) be a pair contained in I. Then by the above remark $p\nu_1+q\nu_4$ is a weight of $(V^k)^C$ as a G-module. Each weight of $(V^k)^C$ as a G-module is written in the following form

$$k\mu_4-\textstyle\sum_{i=1}^{4} m_i\alpha_i\,,$$

with non-negative integers m_i. Choose m_i such that

(2.7) $$p\nu_1+q\nu_4=k\mu_4-\sum_{i=1}^4 m_i\alpha_i\,.$$

By (2.2) and (2.3) we get

$$p\nu_1+q\nu_4=(p+q/2)\beta_1+(p+q)\beta_2+(p+3q/2)\beta_3+(p+2q)\beta_4$$

$$k\mu_4-\sum_{i=1}^4 m_i\alpha_i$$

$$=(k-m_4/2)\beta_1+(k+m_1-m_4)\beta_2+(k+m_2-3m_4/2)\beta_3+(k+m_3-2m_4)\beta_4\,.$$

Comparing the coefficient of β_1 on both sides of (2.7) we get

$$p+q/2=k-m_4/2, \quad \text{for any} \quad (p, q)\in I\,.$$

So we get $d\leqq 2k$ from (2.6).

Let $\pi_{p,q}:(V^k)^C\to V(p\nu_1+q\nu_4)$, $(p, q)\in I$, be the orthogonal projection with respect to the decomposition (2.5). Let v_0 be the weight vector corresponding to $k\mu_4$. Since $v_0=\sum_{(p,q)\in I}\pi_{p,q}(v_0)$ there exists a pair $(p, q)\in I$ such that $\pi_{p,q}(v_0)\neq 0$. Since $\pi_{p,q}$ is a K-homomorphism $k\mu_4$ is a weight of $V(p\nu_1+q\nu_4)$. Choose non-negative integers n_i such that

(2.8) $$k\mu_4=p\nu_1+q\nu_4-\sum_{i=1}^4 n_i\beta_i\,.$$

By (2.2) and (2.3) we get

$$k\mu_4=k(\beta_1+\beta_2+\beta_3+\beta_4)$$

$$p\nu_1+q\nu_4-\sum_{i=1}^4 n_i\beta_i$$

$$=(p+q/2-n_1)\beta_1+(p+q-n_2)\beta_2+(p+3q/2-n_3)\beta_3+(p+2q-n_4)\beta_4\,.$$

Comparing the coefficient of β_1 on both sides of (2.8) we get

$$p+q/2-n_1=k\,.$$

So we get $d\geqq 2k$ from (2.6). So the Theorem B is proved.

We can prove Theorem A in the same manner as the proof of Theorem B. Necessary facts are found in [7].

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