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CODIVISIBLE MODULES, WEAKLY-CODIVISIBLE MODULES AND STRONGLY η-PROJECTIVE MODULES

(Dedicated to Prof. G. Azumaya for his sixtieth birthday)

By

Masahisa SATO

1. Introduction.

In [1], P. E. Bland has studied the strongly M-projective module and the strongly M-projective cover. As their general notions, we define the strongly η-projective module and the strongly η-projective cover for any class $\eta\subset\text{Mod-}R$, (for the definitions, refer to section 2) and by considering the pre-torsion theory associated with the radical t_η, ($t_\eta(K_R)=\cap\{\text{Ker}(f)\mid f\in\text{Hom}_R(K_R, M_R), M_R\in\eta\}$ for any right R-module K_R), we shall show that the above notions can be translated into the new notions weakly codivisible module and weakly codivisible cover with respect to $(\mathcal{T}, \mathcal{F})$ associated with the radical t_η and that new or generalized results are obtained. Through all the sections, we shall generalize the results of P. E. Bland [1], M. L. Teply [7] and K. M. Rangaswarmy [4].

In [1, Proposition 5] and [1, Proposition 6], it is proved that if $\text{Cog}(M_R)$ is closed under factors, then

(1) *B_R has a strongly M-projective cover iff $B/B\cdot\text{Ann}(M_R)$ has a projective cover as an $R/\text{Ann}(M_R)$-module.*

(2) *Every R-module has a strongly M-projective cover iff $R/\text{Ann}(M_R)$ is a right perfect ring.*

But we shall show in Corollaries 8, 9 that these statements are valid without the above assumption on $\text{Cog}(M_R)$.

By [1, Proposition 7], if M_R is an injective module, then any strongly M-projective module is codivisible with respect to the hereditary torsion theory cogenerated by M_R. So we shall characterize under what conditions about the pre-torsion theory $(\mathcal{T}_t, \mathcal{T}_t)$ associated with the radical t a strongly η-projective module is codivisible.

We have equivalent conditions in Theorem 12 that

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(1) *If* $B/B\cdot t(R)$ *is a projective* $R/t(R)$*-module, then* B_R *is a codivisible module.*

(2) *Any weakly codivisible module (resp. strongly* η*-projective module) is a codivisible module.*

(3) $A\cdot t(R)\cap K_R=0$ *for any weakly-codivisible module* A_R *and its submodule* K_R *such that* $K_R\in\mathcal{F}_t$. *(i.e.* $t(A_R)$ *has no non-zero torsion-free submodule.)*

(4) $t(B_R)=B_R\cap t(A_R)$ *for any codivisible module* A_R *and its submodule* B_R.

(5) $M\cdot t(R)$ *has no non-zero torsion-free submodule for any (cyclic) module* M_R.

These conditions are deep related to a pseudo-hereditary pre-torsion theory. In fact, conditions in Theorem 12 hold iff $(\mathcal{T}_t, \mathcal{F}_t)$ is a pseudo-hereditary pre-torsion theory (Theorem 14). Furthermore this implies it holds the converse of [4, Theorem 8] which asserts if $(\mathcal{T}, \mathcal{F})$ is a pseudo-hereditary torsion theory, then $B/B\cdot t(R)$ is a projective $R/t(R)$-module iff B_R is a codivisible module. Hence our result Theorem 12 proves that [1, Proposition 7] and [4, Theorem 8] are essentially the same contents.

As an immediate consequence, we have the following generalization of [4, Corollary 15] that any R-module is codivisible iff $R/t(R)$ is a semi-simple Artinian ring and $(\mathcal{T}_t, \mathcal{F}_t)$ is pseudo-hereditary. We shall also generalize the result [4, Theorem 8] on the pseudo-hereditaryness in a torsion theory to those in a pre-torsion theory associated with a radical (Theorem 13).

In the final section, we study a module M_R such that $M_R\cdot t(R)=M_R$. It is proved in [7, Lemma 3] and [4, Corollary 9] that if $(\mathcal{T}, \mathcal{F})$ is a pseudo-hereditary torsion theory, $M_R\cdot t(R)=M_R$ implies that M_R is codivisible. We shall, however, show that $M_R\cdot t(R)=M_R$ for a torsion theory $(\mathcal{T}, \mathcal{F})$ iff M_R is torsion and has a colocalization with respect to $(\mathcal{T}, \mathcal{F})$. In fact, this result is valid under more weaker situation that $(\mathcal{T}, \mathcal{F})$ is a pre-torsion theory such that $A/t(B_R)$ is codivisible for any codivisible module A_R and $B_R\subset A_R$ (Theorem 17). As an application, we obtain the equivalent conditions which are a generalization of [7, Corollary 1] and [7, Proposition 1];

(1) $R/t(R)$ *is a semi-perfect ring.*

(2) *Every simple R-module has a codivisible cover.*

(3) *Every simple* $R/t(R)$*-module has a codivisible cover as an R-module.* (Corollary 18).

We shall, at the same time, another proof of Theorem of K. Ohtake that every module has the colocalization iff the torsion-free class $\mathcal{F}$ is closed under factors and extensions (Corollary 19).

2. Definitions.

Let R be a ring with unit and Mod-R the category of unital right R-modules. For $\eta \subset$ Mod-R, we denote "**Cog** (η)" $= \{M_R \mid M_R \subset \prod L_i$ for some $L_i \in \eta\}$, i.e. the class cogenerated by η, and "**Ann** (η)" $= \cap \{\mathrm{Ann}(M_R) \mid M_R \in \eta\}$. $M_R \in$ Mod-R is called "**strongly η-projective**" if $\mathrm{Hom}_R(M_R, -)$ preserves the exactness of every short exact sequence $0 \to K_R \to L_R \to H_R \to 0$ such that $L_R = \prod L_i$ for some $L_i \in \eta$. A "**strongly η-projective cover**" of N_R means a strongly η-projective module P_R with an epimorphism $P_R \to N_R \to 0$ whose kernel is small and η-independent in P_R. Here a submodule $K_R \subset L_R$ is called "**η-independent**" in L_R if, for any non-zero $K_R^* \subset K_R$, the canonical map

$$\mathrm{Hom}_R(L_R, M_R) \longrightarrow \mathrm{Hom}_R(K_R^*, M_R)$$

is non-zero for some $M_R \in \mathrm{Cog}(\eta)$. In the case that η consists of a single element M_R, the above definitions coincide with the original definitions of strongly M-projective module and strongly M-projective covers in [1].

For a subfunctor t of the identity functor on Mod-R which is called a "**pre-radical**", we denote

$$\mathcal{T}_t = \{M_R \in \text{Mod-}R \mid t(M_R) = M_R\} \qquad \text{and}$$

$$\mathcal{F}_t = \{M_R \in \text{Mod-}R \mid t(M_R) = O\}$$

whose elements are said to be "**torsion**" and "**torsion-free**" respectively. A pre-radical t is called a "**radical**" if $t(M/t(M_R)_R) = O$ for any $M_R \in$ Mod-R and is "**idempotent**" if $t(t(M_R)) = t(M_R)$ for any $M_R \in$ Mod-R. We call the pair $(\mathcal{T}_t, \mathcal{F}_t)$ a "**pre-torsion theory**" (resp. "**torsion theory**") if t is a radical (resp. idempotent radical). For a detail, refer to [6]. For $\eta \subset$ Mod-R, we define a pre-radical "$\boldsymbol{t_\eta}$" by

$$t_\eta(K_R) = \cap \{\mathrm{Ker}(f) \mid f \in \mathrm{Hom}_R(K_R, M_R),\ M_R \in \eta\}$$

for any $K_R \in$ Mod-R. Clearly it is a radical. In [2] H. Katayama has remarked that any radical t is represented as $t = t_\eta$ for some $\eta \subset$ Mod-R.

A module M_R is "**codivisible**" (resp. "**weakly-codivisible**") if $\mathrm{Hom}_R(M_R, -)$ preserves the exactness of every short exact sequence $O \to K_R \to L_R \to H_R \to O$ such that $K_R \in \mathcal{F}_t$ (resp. $L_R \in \mathcal{F}_t$). Clearly a codivisible module is a weakly codivisible module. A "**codivisible cover**" of M_R means a codivisible P_R with an epimorphism $P_R \to M_R \to O$ whose kernel is small in P_R and torsion-free. An epimorphism $f: P_R \to M_R \to O$ is called a "**weakly codivisible cover**" of M_R if P_R is weakly codivisible, $\mathrm{Ker}(f)$ is small in P_R and $t(P_R) \cap \mathrm{Ker}(f) = O$. In the case that t is a radical, if a module M_R has a codivisible cover, then it has a weakly codivisible cover (for the proof, see Lemma 11). A "**colocalization**" of M_R is an R-homomorphism $f: P_R \to M_R$ such that P_R is torsion codivisible and $\mathrm{Ker}(f)$ & $\mathrm{Cok}(f) \in \mathcal{F}_t$. For a detail, see [3] and [5]. A codivisible cover, a weakly codivisible cover and a colocalization of M_R are unique up to the isomorphism if they exist (for the proof, see Lemma 6).

A pre-torsion theory $(\mathcal{T}, \mathcal{F})$ is called "**pseudo-hereditary**" (resp. "**hereditary**") if any submodule of $t(R_R)$ (resp. $M_R \in \mathcal{T}$) is torsion.

3. Basic Property for Radicals.

PROPOSITION 1. *Let t be a pre-radical. t is a radical iff $t=t_\eta$ for some $\eta \subset \text{Mod-}R$.*

In this case, $\mathcal{F}_t = \mathrm{Cog}(\eta)$ and $t = t_{\mathcal{F}_t}$.

PROOF. "If part" is clear. So we assume t is a radical and we put $\eta = \mathcal{F}_t$. Let $M_R \in \text{Mod-}R$, $K_R \in \mathcal{F}_t$ and $f \in \mathrm{Hom}_R(M_R, K_R)$. f induces $t(f): t(M_R) \to t(K_R)$ and $t(K_R)=O$ since $K_R \in \mathcal{F}_t$, so $f(t(M_R))=O$, hence $t(M_R) \subset t_\eta(M_R)$. We consider the canonical map $i: M_R \to M_R/t(M_R)_R$, then $i(t_\eta(M_R))=O$ since $M/t(M_R) \in \mathcal{F}_t$. So $t_\eta(M_R) \subset t(M_R)$. Thus $t=t_\eta$. Next we prove $\mathcal{F}_t = \mathrm{Cog}(\eta)$. Let $M_R \in \mathrm{Cog}(\eta)$. Then there are $L_i \in \eta$, $i \in I$ such that $M_R \hookrightarrow \Pi_{i \in I} L_i$, hence $t_\eta(M_R)=O$, thus $M_R \in \mathcal{F}_t$. Assume $M_R \in \mathcal{F}_t$. For every $O \neq x \in M_R$, there is $L_x \in \eta$ and $f_x: M_R \to L_x$ such that $f_x(x) \neq 0$, which means $\Pi f_x: M_R \to \Pi_{0 \neq x \in M_R} L_x$ is a monomorphism, thus $M_R \in \mathrm{Cog}(\eta)$.

COROLLARY 2. *Let $(\mathcal{T}, \mathcal{F})$ be the pre-torsion theory associated with a radical t and η a subclass of* Mod-R *such that $t=t_\eta$. Then the following properties hold.*

(1) $t(R_R)=\mathrm{Ann}(\eta)=\mathrm{Ann}(\mathcal{F})$.

(2) $M_R \cdot t(R_R) \subset t(M_R)$ *for any* $M_R \in \text{Mod-}R$.

(3) *$\mathcal{T}$ is closed under factors, direct sums and extensions.*

(4) *For any $M_R \in \mathrm{Mod}R$ and $K_R \subset M_R$ such that $K_R \in \mathcal{F}$, if $t(M_R) \cap K_R$ is a direct summand of M_R, then $t(M_R) \cap K_R = 0$.*

PROOF. Proof of (1). By Proposition 1, $t(R_R)=t_{\mathcal{F}}(R_R)$. So

$$\begin{aligned} t_\eta(R) &= \cap \{\mathrm{Ker}(f) \mid f \in \mathrm{Hom}_R(R_R, M_R),\ M_R \in \eta\} \\ &= \cap \{\mathrm{Ann}(m) \mid m \in M_R,\ M_R \in \eta\} \\ &= \cap \{\mathrm{Ann}(M_R) \mid M_R \in \eta\}\ . \end{aligned}$$

Proof of (2). For any $x \in M_R$, we define $f: R_R \to M_R$ by $f(r)=x \cdot r$ for every $r \in R$. Since $x \cdot t(R_R)=f(t(R_R))$ and $f(t(R_R)) \subset t(M_R)$, $M \cdot t(R_R) \subset t(M_R)$.

Proof of (3). Since $t=t_{\mathcal{F}}$, $t(M_R)=M_R$ iff $\mathrm{Hom}_R(M_R, K_R)=O$ for any $K_R \in \mathcal{F}$.

$\mathrm{Hom}_R(-, K_R)$ is a right exact functor, so (3) holds.

Proof of (4). We put $M_R=(t(M_R)\cap K_R)\oplus M_R^*$ for some $M_R^*\subset M_R$. $t(M_R)=t(t(M_R)\cap K_R)\oplus t(M_R^*)=t(M_R^*)\subset M_R^*$. Hence $t(M_R)\cap K_R=t(M_R)\cap K_R\cap M_R^*=O$.

REMARK: The proof of (2) is valid under the assumption that t is a pre-radical. The above proofs of (2) and (4) are suggested by the refree.

4. Weakly Codivisible Modules and Strongly η-Projective Modules.

In this section, we study basic properties of weakly codivisible modules and strongly η-projective modules.

PROPOSITION 3. *Let $(\mathcal{T}, \mathcal{F})$ be the pre-torsion theory associated with a radical t. Then it holds that*

(1) *If A_R is weakly codivisible with respect to $(\mathcal{T}, \mathcal{F})$, then $A/A\cdot t(R)$ is a projective $R/t(R)$-module.*

(2) *If A_R is weakly codivisible, then $t(A_R)=A_R\cdot t(R)$.*

(3) *Let $O\to A_R\to B_R\to C_R\to O$ be an exact sequence. If C_R is codivisible, then $t(A_R)=t(B_R)\cap A_R$.*

PROOF. Proof of (1). We put f an epimorphism $\sum\oplus(R/t(R))_R\to A/A\cdot t(R)_R\to O$ and j the canonical map $A_R\to A/A\cdot t(R)_R\to O$. We consider the next diagram with exact rows;

$$\begin{array}{ccccccccc} O & \longrightarrow & A\cdot t(R_R)_R & \longrightarrow & A_R & \xrightarrow{j} & A/A\cdot t(R)_R & \longrightarrow & O \\ & & & & & & \| & & \\ O & \longrightarrow & \mathrm{Ker}(f)_R & \longrightarrow & \sum\oplus(R/t(R)) & \xrightarrow{f} & A/A\cdot t(R)_R & \longrightarrow & O. \end{array}$$

By assumption, there is $g: A_R\to\sum\oplus(R/t(R))_R$ such that $j=fg$. By Corallary 2, $g(A\cdot t(R))=O$. So there is $\bar{g}: A/A\cdot t(R)_R\to\sum\oplus(R/t(R))_R$ such that $g=\bar{g}j$. Since $j=fg=f\bar{g}j$ and j is an epimorphism, $1=f\bar{g}$, which means $A/A\cdot t(R)_R$ is a direct summand of $\sum\oplus(R/t(R))_R$, hence $A/A\cdot t(R)$ is a projective $R/t(R)$-module.

Proof of (2). $A\cdot t(R)\subset t(A_R)$ by Corollary 2, so $t(A/A\cdot t(R))=t(A_R)/A\cdot t(R)$, but $t(A/A\cdot t(R))=O$ by (1), hence $t(A_R)=A_R\cdot t(R_R)$.

Proof of (3). Since $t(A_R)\subset t(B_R)$, $t(B/t(A_R))=t(B_R)/t(A_R)$. On the other hand, the exact sequence $O\to A/t(A_R)_R\to B/t(A_R)_R\to C_R\to O$ splits since C_R is codivisible and $A_R/t(A_R)\in\mathcal{F}$, so we put $B/t(A_R)=A/t(A)\oplus\bar{C}$ for $\bar{C}_R\subset B/t(A_R)_R$. Since a radical commutes with the direct sums, we have

$$\begin{aligned} t(B_R)/t(A_R) &= t(B_R/t(A_R)) \\ &= t(A_R/t(A_R))\oplus t(\bar{C}_R) \end{aligned}$$

$$=t(\bar{C}_R)\subset\bar{C}_R\,.$$

Thus $(A_R\cap t(B_R))/t(A_R)=(A_R/t(A_R))\cap(t(B_R)/t(A_R))=O$, i.e.

$$A_R\cap t(B_R)=t(A_R)\,.$$

REMARK: (1) in the above proposition is a generalization of [4, Corollary 7].

THEOREM 4. *Let t be a radical, $(\mathcal{T}, \mathcal{F})$ the corresponding pre-torsion theory and η a subclass of* Mod-R *such that $t=t_\eta$. Then the following statements are equivalent for $M_R\in$* Mod-R.

(1) *$M/M\cdot t(R)$ is a projective $R/t(R)$-module.*

(2) *$M/M\cdot\mathrm{Ann}(\eta)$ is a projective $R/\mathrm{Ann}(\eta)$-module.*

(3) *M_R is weakly codivisible with respect to $(\mathcal{T}, \mathcal{F})$.*

(4) *M_R is a strongly η-projective module.*

PROOF. Clearly (1) and (2) are equivalent by Corollary 2. (3) implies (1) is proved by Proposition 3.

(1) implies (3). Let $O\rightarrow A_R\rightarrow B_R\xrightarrow{i}C_R\rightarrow O$ be an exact sequence such that $B_R\in\mathcal{F}$ and $f: M_R\rightarrow C_R$. $B_R\cdot t(R)=O$ by Corollary 2, so $C_R\cdot t(R)=O$, hence $f(M_R\cdot t(R))=f(M_R)\cdot t(R)=O$. It induces $\bar{f}: M/M\cdot t(R)_R\rightarrow C_R$ such that $f=\bar{f}j$ where $j: M_R\rightarrow M/M\cdot t(R)_R$ is the canonical map. Clearly $\bar{f}$ and i are $R/t(R)$-homomorphisms, so there is an R-homomorphism $h: M/M\cdot t(R)_R\rightarrow B_R$ such that $\bar{f}=ih$ since $M/M\cdot t(R)$ is a projective $R/t(R)$-module. Thus $f=i(hj)$, so (3) holds.

(3) implies (4). It holds since $\eta\subset\mathcal{F}_t=\mathrm{Cog}(\eta)$ by Proposition 1.

(4) implies (3). Let $O\rightarrow A_R\rightarrow B_R\xrightarrow{i}C_R\rightarrow O$ be an exact sequence such that $B_R\in\mathcal{F}$ and $f: M_R\rightarrow C_R$ any R-homomorphism. By Proposition 1, $\mathcal{F}=\mathrm{Cog}(\eta)$, hence there are $L_i\in\eta$ $(i\in I)$ for some index set I such that $B_R\subset\prod_{i\in I}L_i$. We consider the following commutative diagram with exact rows;

$$\begin{tikzcd}
 & & & M_R \arrow[d, "f"] & \\
O \arrow[r] & A_R \arrow[r] \arrow[d, equal] & B_R \arrow[r, "i"] \arrow[d, hook] & C_R \arrow[r] \arrow[d, "k"] & O \\
O \arrow[r] & A_R \arrow[r] & \prod L_i \arrow[r] & \prod L_i/A_R \arrow[r] & O\,.
\end{tikzcd}$$

By assumption, there is an R-homomorphism $g: M_R\rightarrow\prod L_i$ such that $kf=jg$. Since B_R is a fibre product (i.e. pull back) of (k, j), there is an R-homomorphism $\bar{f}: M_R\rightarrow B_R$ such that $f=i\bar{f}$. Thus M_R is weakly codivisible.

COROLLARY 5. *For a pre-torsion theory* $(\mathcal{T}, \mathcal{F})$, *the following statements are equivalent.*

(1) *Every R-module is a weakly codivisible module.*

(2) $R/t(R)$ *is a semi-simple Artinian ring.*

PROOF. This is a direct consequence of Theorem 4.

Theorem 7 generalizes [1, Proposition 5] and shows that it is proved without the assumption in [1] that Cog $(\{M\})$ is closed under factors. Before proving the theorem, we prove the following lemma.

LEMMA 6. *Let* η *be a subclass of* Mod-R *and* $M_R \in$ Mod-R. *Then it holds that*

(1) *A submodule* L_R *of* M_R *is* η*-independent in* M_R *iff* $t_\eta(M_R) \cap L_R = O$.

(2) *An epimorphism* $P_R \to M_R$ *is a strongly* η*-projective cover of* M_R *iff it is a weakly codivisible cover.*

(3) *A strongly* η*-projective cover of* M_R *is unique up to the isomorphism if it exists.*

PROOF. (1) and (2) are clear by definitions and Theorem 4.

Proof of (3). Let $O \to K_R \xrightarrow{k} A_R \xrightarrow{f} M_R \to O$ and $O \to L_R \xrightarrow{l} B_R \xrightarrow{g} M_R \to O$ be strongly η-projective covers of M_R. Since L_R is η-independent in B_R, there exists an R-homomorphism $h: B_R \to \prod_{i \ni I} M_i$ for some $M_i \in \eta$ and an index set I such that $h \cdot l$ is a monomorphism. So we have a commutative diagram with exact rows;

$$\begin{array}{ccccccccc}
O & \longrightarrow & K_R & \xrightarrow{k} & A_R & \xrightarrow{f} & M_R & \longrightarrow & O \\
 & & & & & & \| & & \\
O & \longrightarrow & L_R & \xrightarrow{l} & B_R & \xrightarrow{g} & M_R & \longrightarrow & O \\
 & & \wr\downarrow h^* & & \downarrow h & & \downarrow \bar{h} & & \\
O & \longrightarrow & hl(L_R) & \longrightarrow & \prod M_i & \xrightarrow{j} & (\prod M_i)/hl(L_R) & \longrightarrow & O
\end{array}$$

where j is the canonical map and h^* and $\bar{h}$ are induced maps of h. Since A_R is strongly η-projective, there exists an R-homomorphism $p: A_R \to \prod M_i$ sunh that $jp = \bar{h}f$. By the fact that h^* is an isomorphism, B_R is a fibre product of $(j, \bar{h})$. So there is an R-homomorphism $s: A_R \to B_R$ such that $f = gs$. Since g is a minimal epimorphism, s is an epimorphism. Clearly Ker$(s) \subset K_R$, so Ker(s) is small and η-independent in A_R. Repeating the same discussion as above, we can show that s is a splitting epimorphism. Hence s is an isomorphism since Ker(s) is small in A_R.

THEOREM 7. *Let* η *be a subclass of* Mod-R *and* $t=t_\eta$ *a radical. The following assertions are equivalent for a given* $B_R \in$ Mod-R.

(1) B_R *has a strongly* η*-projective cover.*

(2) B_R *has a weakly codivisible cover.*

(3) $B/B\cdot\mathrm{Ann}(\eta)$ *has a projective cover as an* $R/\mathrm{Ann}(\eta)$*-module.*

(4) $B/B\cdot t(R)$ *has a projective cover as an* $R/t(R)$*-module.*

PROOF. (1) and (2) are equivalent by Lemma 6. Also (3) and (4) are equivalent by Corollary 2.

(2) implies (4). Let $O\to K_R\to A_R\to B_R\to O$ be a weakly codivisible cover of B_R. By Proposition 3, $t(A_R)=A_R\cdot t(R)$, hence $A\cdot t(R)_R\cap K_R=O$. So we have a commutative diagram with exact rows;

$$\begin{tikzcd}
O \arrow[r] & K_R \arrow[r] \arrow[d, equal] & A_R \arrow[r] \arrow[d] & B_R \arrow[r] \arrow[d] & O \\
O \arrow[r] & K_R \arrow[r] & A/A\cdot t(R)_R \arrow[r] & B/B\cdot t(R)_R \arrow[r] & O
\end{tikzcd}$$

By Theorem 4, $A/A\cdot t(R)$ is a projective $R/t(R)$-module. Since an epimorphic image of a small submodule is small, K_R is small in $A/A\cdot t(R)$. Hence the lower sequence of the above diagram is a projective cover of $B/B\cdot t(R)$ as an $R/t(R)$-module.

(4) implies (2). Let $O\to K\to Q\to B/B\cdot t(R)\to O$ be a projective cover of $B/B\cdot t(R)$ as an $R/t(R)$-module. We consider these modules as R-modules and put (A_R, g, f) a fibre product of $Q_R\to B/B\cdot t(R)_R$ and $B_R\to B/B\cdot t(R)_R$. We have a commutative diagram with exact rows and columns;

$$\begin{tikzcd}
 & & O \arrow[d] & O \arrow[d] & \\
 & & \mathrm{Ker}(f)_R \arrow[r, "\bar{g}", "\sim"'] \arrow[d] & B\cdot t(R)_R \arrow[r] \arrow[d] & O \\
O \arrow[r] & K_R^* \arrow[r] \arrow[d, "\bar{f}"] & A_R \arrow[r, "g"] \arrow[d, "f"] & B_R \arrow[r] \arrow[d] & O \\
O \arrow[r] & K_R \arrow[r] & Q_R \arrow[r] \arrow[d] & B/B\cdot t(R)_R \arrow[r] \arrow[d] & O \\
 & & O & O &
\end{tikzcd}$$

We first show $\mathrm{Ker}(f)=A_R\cdot t(R)$.

$$\begin{aligned} f(A\cdot t(R)) &= f(A_R)\cdot t(R) \\ &= Q_R\cdot t(R) \\ &= O, \end{aligned}$$

so
$$A_R\cdot t(R_R)_R \subset \mathrm{Ker}(f)\,.$$
Thus
$$\begin{aligned}\bar{g}(A\cdot t(R)) &= g(A\cdot t(R))\\ &= g(A_R)\cdot t(R)\\ &= B_R\cdot t(R)\,.\end{aligned}$$

Since $\bar{g}$ is an isomorphism, $A\cdot t(R)=\mathrm{Ker}(f)$. By this fact, Q and $A/A\cdot t(R)$ are isomorphic as R-modules, hence as $R/t(R)$-modules. Here Q is a projective $R/t(R)$-module, so A_R is a weakly codivisible module by Theorem 4. Next we show K_R^* $(=\mathrm{Ker}(g))$ is small in A_R. Assume $K_R^*+L_R=A_R$ for $L_R\subset A_R$. Then $K_R^*\cdot t(R)+L\cdot t(R)=A\cdot t(R)$. But $K\cdot t(R)=O$, so $K^*\cdot t(R)=O$, thus $L\cdot t(R)=A\cdot t(R)$, hence $\mathrm{Ker}(f)=A\cdot t(R)=L\cdot t(R)\subset L_R$. On the other hand, $f(K_R^*)+f(L_R)=\bar{f}(A_R)=Q_R$, which means $f(L_R)=Q_R$ since K_R is small in Q_R and $f(K_R^*)=K_R$. Since f is an epimorphism, $L_R+\mathrm{Ker}(f)_R=A_R$, thus $L_R=A_R$. Last we show $t(A_R)\cap K_R^*=O$. $t(A_R)=A_R\cdot t(R)$ by Proposition 3, so $O=\mathrm{Ker}(\bar{f})=K_R^*\cap \mathrm{Ker}(f)_R=K_R^*\cap(A_R\cdot t(R))_R=K_R^*\cap t(A_R)_R$. This completes the proof of the theorem.

By Theorem 7, we get following corollaries.

COROLLARY 8. *The following statements are equivalent for* $M_R\in\mathrm{Mod}\text{-}R$ *and* $B_R\in\mathrm{Mod}\text{-}R$.

(1) B_R *has a strongly M-projective cover.*

(2) $B/B\cdot t(R)$ *has a projective cover as an* $R/\mathrm{Ann}(M_R)$*-module.*

REMARK: This fairly generalizes both [1, Proposition 5] and [4, Theorem 10] as we state before.

COROLLARY 9. *Let* η *be a subclass of* Mod-R. *Then we have next equivalent conditions.*

(1) *Every R-module in* Mod-R *has a strongly* η*-projective cover.*

(2) $R/\mathrm{Ann}(\eta)$ *is a right perfect ring.*

REMARK: This is also a generalization of [1, Theorem 6] and [4, Theorem 11].

By applying Theorem 7 only to finitely generated modules, we have (c. f. [4, Theorem 12])

COROLLARY 10. *The following statements are equivalent.*

(1) *Every finitely generated (resp. cyclic) R-module has a strongly* η*-projec-*

tive cover.

(2) *Every finitely generated (resp. cyclic) R-module has a weakly codivisible cover.*

(3) $R/\mathrm{Ann}(\eta)$ *is a semi-perfect ring.*

5. A Pseudo-Hereditary Pre-Torsion Theory.

From Proposition 3, it is easily seen that when B_R is codivisible with respect to $(\mathcal{T}_t, \mathcal{F}_t)$, then $B/B\cdot t(R)$ is a projective $R/t(R)$-module. On the other hand, [4, Theorem 8] has shown that the converse of the above result holds if $(\mathcal{T}_t, \mathcal{F}_t)$ is a pseudo-hereditary torsion theory.

Under the assumption that t is a radical, we shall first study equivalent conditions for which the converse of the above result holds. In fact, we shall prove that the converse holds iff $(\mathcal{T}_t, \mathcal{F}_t)$ is a pseudo-hereditary pre-torsion theory. This result means that the equivalent conditions of [1, Proposition 7] are nothing but a paraphrase of our result in the special case that $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory.

LEMMA 11. *Let $(\mathcal{T}, \mathcal{F})$ be a pre-torsion theory with the radical t. Then it holds that*

(1) *If A_R is weakly codivisible with respect to $(\mathcal{T}, \mathcal{F})$ and $B_R \subset t(A_R)$, then A/B_R is weakly codivisible.*

(2) *For any $M_R \in \mathrm{Mod}\text{-}R$, there is an exact sequence $O \to K_R \to A_R \to M_R \to O$ such that A_R is weakly codivisible and $K_R \in \mathcal{F}$.*

(3) *For any $M_R \in \mathrm{Mod}\text{-}R$, there is an exact sequence $O \to K_R \to A_R \to M_R \to O$ such that A_R is weakly codivisible and $t(A_R) \cap K_R = O$.*

PROOF. Proof of (1). Let $s: K_R \to L_R \to O$ be an epimorphism such that $K_R \in \mathcal{F}$. Assume $f: A/B_R \to L_R$ is an R-homomorphism. Since A_R is weakly codivisible, there is $\bar{f}: A_R \to K_R$ such that $s\bar{f} = fp$ where p is the canonical map $A_R \to A/B_R$. $\bar{f}(B_R) \subset \bar{f}(t(A_R)) = O$ since $t(A_R) = A_R \cdot t(R_R)$ by Proposition 3, so $\bar{f}(B_R) = O$, thus there is an R-homomorphism $g: A/B_R \to K_R$ such that $\bar{f} = gp$. $fp = s\bar{f} = sgp$ and p is an epimorphism. This shows $f = sg$, as was to be shown.

Proof of (2) and (3). We consider an exact sequence $O \to \mathrm{Ker}(f)_R \to P_R \xrightarrow{f} M_R \to O$ such that P_R is projective. The exact sequence $O \to \mathrm{Ker}(f)/t(\mathrm{Ker}(f))_R \to P/t(\mathrm{Ker}(f))_R \to M_R \to O$ satisfies (2) by (1). The exact sequence

$$O \longrightarrow \mathrm{Ker}(f)/(\mathrm{Ker}(f) \cap t(P_R))_R \longrightarrow P/(\mathrm{Ker}(f) \cap t(P_R))_R \longrightarrow M_R \longrightarrow O$$

satisfies (3) by (1).

REMARK: (3) in Lemma 11 is a generalization of [1, Lemma 1].

THEOREM 12. *Let* $(\mathcal{T}, \mathcal{F})$ *be a pre-torsion theory with the radical* t *and* η *a subclass of* Mod-R *such that* $t=t_\eta$. *The following statements are equivalent.*

(1) *If* $M/M\cdot t(R)$ *is a projective* $R/t(R)$*-module, then* M_R *is a codivisible module with respect to* $(\mathcal{T}, \mathcal{F})$.

(2) *Every weakly codivisible module is codivisible.*

(2)* *Every strongly* η*-projective module is codivisible.*

(3) *For every weakly codivisible module* A_R, $A_R\cdot t(R)_R\cap K_R=O$ *for any torsion-free submodule* K_R *of* A_R.

(3)* *For every codivisible module* A_R,

(a) $(A_R\cdot t(R))_R\cap K_R=O$ *for any torsion-free submodule* K_R *of* A_R.

(b) $A_R/t(B_R)$ *is codivisible for any* $B_R\subset A_R$.

(4) *For every weakly codivisible module* A_R, $t(A_R)_R$ *has no non-zero torsion-free submodule.*

(4)* *For every codivisible module* A_R,

(*a*) $t(A_R)_R$ *has no non-zero torsion-free submodule.*

(b) $A_R/t(B_R)$ *is codivisible for any* $B_R\subset A_R$.

(5) *For every weakly codivisible module* A_R, $t(B_R)_R=t(A_R)_R\cap B_R$ *for every submodule* $B_R\subset A_R$.

(5)* *For every codivisible module* A_R, $t(B_R)_R=t(A_R)_R\cap B_R$ *for every submodule* $B_R\subset A_R$.

(6) *For any* $M_R\in$ Mod-R, $M_R\cdot t(R)_R$ *has no non-zero torsion-free submodule.*

(7) *For any cyclic module* C_R, $C_R\cdot t(R)_R$ *has no non-zero torsion-free submodule.*

(*) *In these cases* (1)—(7), *for any* $M_R\in$ Mod-R, *there is an exact sequence* $O\to K_R\to A_R\to M_R\to O$ *such that* A_R *is codivisible and* $K_R\in\mathcal{F}$.

Furthermore the property (b) *of* (3)* *or* (4)* *is equivalent that* $t(t(B_R))=t(B_R)$ *by Proposition* 3, (3).

PROOF. The equivalences of (1), (2) and (2)* hold by Theorem 4.

(2) implies (3). The exact sequence

$$O \longrightarrow A\cdot t(R)_R\cap K_R \longrightarrow A_R \longrightarrow A_R/(A\cdot t(R)_R\cap K_R)_R \longrightarrow O$$

splits since $A/(A\cdot t(R)\cap K)_R$ is codivisible by Lemma 11, (1) and the assumption. Thus $A\cdot t(R)_R\cap K_R=O$ by Corollary 2.

The equivalence of (3) and (4) holds since $t(A_R)=A_R\cdot t(R)$ by Proposition 3.

(6) implies (4). It is clear.

(4) implies (6). By Lemma 11, there exists an exact sequence $O\to K_R\to A_R$

$\to M_R \to O$ such that A_R is weakly codivisible and $t(A_R)_R \cap K_R = O$. So we have a commutative diagram with exact rows;

$$\begin{array}{ccccccccc} O & \longrightarrow & A_R\cdot t(R)_R\cap K_R & \longrightarrow & A_R\cdot t(R)_R & \longrightarrow & M_R\cdot t(R)_R & \longrightarrow & O \\ & & \cap & & \cap & & \cap & & \\ O & \longrightarrow & K_R & \longrightarrow & A_R & \longrightarrow & M_R & \longrightarrow & O \end{array}$$

Since $t(A_R) = A_R\cdot t(R)$ by Proposition 3, $A_R\cdot t(R)_R \cong M_R\cdot t(R)_R$, so $M_R\cdot t(R)_R$ has no non-zero torsion-free submodule.

(5) implies (5)*. It is clear.

(5)* implies (1). Let A_R be a codivisible module and a submodule B_R of A_R. Then by assumption, it holds

$$t(t(B_R)) = t(A_R)_R \cap t(B_R)_R$$

$$= t(B_R)_R$$

since

$$t(A_R) \supset t(B_R),$$

so

$$t(B_R) \in \mathcal{T}.$$

Using this fact, there is an exact sequence $O \to K_R \to P_R \to M_R \to O$ such that P_R is codivisible and $K_R \in \mathcal{T}$ by similar way in Lemma 11. $t(P_R)_R \cap K_R = t(K_R) = O$ by assumption, so we have a commutative diagram with exact rows;

$$\begin{array}{ccccccccc} O & \longrightarrow & K_R & \longrightarrow & P_R & \longrightarrow & M_R & \longrightarrow & O \\ & & \| & & \downarrow & & \downarrow & & \\ O & \longrightarrow & K_R & \longrightarrow & P_R/P\cdot t(R)_R & \longrightarrow & M_R/M\cdot t(R)_R & \longrightarrow & O. \end{array}$$

Since $M_R/M\cdot t(R)$ is a projective $R/t(R)$-module, the lower row sequence splits as R-modules, hence so does the upper row sequence. Thus M_R is codivisible.

(4) implies (5). We remark $A/t(B_R)$ is weakly codivisible by Lemma 11, (1).

$$(t(A_R)_R \cap B_R)/t(B_R) \subset t(A_R)/t(B_R) = t(A/t(B_R))$$

and

$$(t(A_R)_R \cap B_R)/t(B_R) \subset B_R/t(B_R).$$

By assumption, $(t(A_R) \cap B_R)/t(B_R) = O$ since it is torsion-free. Hence $t(A_R)_R \cap B_R = t(B_R)$.

The equivalence of (3)* and (4)* is clear.

(1) and (3) imply (3)* is also clear.

(4)* implies (5)*. It is proved similarly as (4) implies (5).

(6) implies (7). It is clear.

(7) implies (6). We first show that if M_R is finitely generated, then M_R has the property (6) by induction on the number of generators of M_R. By assumption, it holds in case $n=1$. Assume $n \geqq 1$, $M_R = m_1R + \cdots + m_nR + m_{n+1}R$ and K_R is a

torsion-free submodule of $M_R \cdot t(R)$.

$$m_{n+1}t(R)_R \cap K_R \in \mathcal{F}$$

and

$$m_{n+1}t(R)_R \cap K_R \subset (m_{n+1}R) \cdot t(R_R) ,$$

hence

$$m_{n+1}t(R)_R \cap K_R = O .$$

So

$$K_R \oplus (m_{n+1}t(R_R)) \subset M_R \cdot t(R)$$

and

$$\begin{aligned} K_R &\cong (K_R \oplus (m_{n+1}t(R)))/m_{n+1}t(R_R) \\ &\subset M_R \cdot t(R_R)/m_{n+1}t(R_R) \\ &= (\bar{m}_1 \cdot R + \cdots + \bar{m}_n \cdot R) \cdot t(R_R) \end{aligned}$$

where

$$\bar{m}_i = m_i + m_{n+1}t(R) , \qquad i=1, \cdots , n .$$

By induction hypothesis, $K_R = O$. Let $M_R \in \text{Mod-}R$ and K_R a torsion-free submodule of $M_R \cdot t(R)$. For any $k \in K_R$, it has an expansion $k = m_1t_1 + \cdots + m_nt_n$ for some $m_i \in M_R$ and $t_i \in t(R)$, $i=1, \cdots , n$. So $kR \subset (m_1R + \cdots + m_nR) \cdot t(R)$ and $kR \in \mathcal{F}$, thus $kR_R = O$. Hence $K_R = O$. This completes the proof of the theorem

REMARK: In the proof of this theorem, the simplification of the proof that (2) implies (3), (4) implies (5) are suggested by the refree.

We recall the definition of a pseudo-hereditary pre-torsion theory that any submodule of $t(R_R)_R$ is torsion. We have the following theorem (c. f. Theorem 12).

THEOREM 13. *Let $(\mathcal{T}, \mathcal{F})$ be a pre-torsion theory with the radical t. Then the following assertions are equivalent.*

(1) *$(\mathcal{T}, \mathcal{F})$ is the pseudo-hereditary pre-torsion theory.*

(2) *For every $M_R \in \text{Mod-}R$, any submodule of $M_R \cdot t(R_R)$ is torsion.*

(3) *For every weakly codivisible module A_R, any submodule of $t(A_R)_R$ is torsion.*

(3)* *For every codivisible module A_R, any submodule of $t(A_R)_R$ is torsion.*

(4) *For a module M_R such that $t(M_R) = M_R \cdot t(R_R)$, any submodule of $t(M_R)_R$ is torsion.*

(5) *For a module M_R such that $t(M_R) = M_R \cdot t(R)$, $t(N_R) = t(M_R) \cap N_R$ for every $N_R \subset M_R$.*

PROOF. (4) implies (3), (3) implies (3)*, (2) implies (1) are clear.

(3) implies (2). By a similar way in Lemma 11, (3) using the assumption, there is an exact sequence $O \to K_R \to A_R \xrightarrow{f} M_R \to O$ such that A_R is codivisible and

$K_R \cap t(A_R)=O$. Thus $t(A_R)=A_R \cdot t(R) \cong M_R \cdot t(R)$. Hence (2) holds.

(5) implies (4). Let $N_R \subset t(M_R)=M_R \cdot t(R)$. By assumption, $t(N_R)=t(M_R)_R \cap N_R=N_R$.

(1) implies (5). $t(N_R) \subset t(M_R)_R \cap N_R$ is clear. Assume $x \in t(M_R)_R \cap N_R$ and decompose

$$x=\sum_{i=1}^{k_x} m_i^{(x)} t_i^{(x)}$$

for $m_i^{(x)} \in M_R$ and $t_i^{(x)} \in t(R)$ since $x \in M_R \cdot t(R)$. We put

$$P_x=(t_1^{(x)}, \cdots, t_{k_x}^{(x)})R \subset \sum_{i=1}^{k_x} \oplus t(R)$$

via

$$f: \sum_x \oplus P_x \longrightarrow t(M_R)_R \cap N_R$$

$$f(\sum_x (t_1^{(x)}, \cdots, t_{k_x}^{(x)}) r_x) = \sum_x (\sum_{i=1}^{k_x} m_i^{(x)} t_i^{(x)} r_x) = \sum_x x r_x$$

for $r_x \in R_R$. Clearly f is an epimorphism, so it is sufficient to show any submodule of $\sum_{i=1}^{n} \oplus t(R)$ is torsion since $P_x \subset \sum_{i=1}^{k_x} \oplus t(R)$ and $\mathcal{T}$ is closed under factors and direct sums. As we proved in Corollary 2, $\mathcal{T}$ is closed under extensions. So a similar proof in [7, Lemma 3] gives this fact by induction.

The next is a generalization of a result [4, Theorem 8].

THEOREM 14. *The properties in Theorem* 12 *and Theorem* 13 *are equivalent.*

PROOF. (3) in Theorem 13 implies (4) in Theorem 12 is clear, so we shall prove that (7) in Theorem 12 implies (1) in Theorem 13. Let $L_R \subset t(R)_R$. $L/t(L_R) \in \mathcal{F}$ and $L/t(L_R) \subset (R/t(L_R)) \cdot t(R)$. By assumption (7), $L/t(L_R)=O$, thus $L_R=t(L_R)$.

Next corollary is a generalization of [4, Corollary 15].

COROLLARY 15. *Let* $(\mathcal{T}, \mathcal{F})$ *be a pre-torsion theory with the radical* t. *Then the following assertions are equivalent.*

(1) *Every R-module is codivisible with respect to* $(\mathcal{T}, \mathcal{E})$.

(2) (a) $R/t(R)$ *is a semi-simple Artinian ring, and*

(b) $(\mathcal{T}, \mathcal{T})$ *is pseudo-hereditary.*

PROOF. (1) implies (2). (1) satisfies the property Theorem 12, (1). Hence

$(\mathcal{T}, \mathcal{F})$ is pseudo-hereditary by Theorem 14. (a) is clear from Corollary 5.

(2) implies (1). It is clear from Corollary 5, Theorem 12, Theorem 13 and Theorem 14.

Next we give an example, which shows (a) and (b) in Corollary 15 are independent.

EXAMPLE: *We put* Z *a ring of integers,* $\eta=\{Z/pZ\}$ *and* $t=t_\eta$ *where* p *is a prime number. Then*

(1) $(\mathcal{T}_t, \mathcal{F}_t)$ *is not pseudo-hereditary.*

(2) $Z/t(Z)$ *is a semi-simple Artinian ring.*

(3) *Every Z-module is a weakly codivisible module.*

(4) Z/pZ *has not codivisible cover.*

Since $t(Z)=pZ$, $t(pZ)=p^2Z\neq t(Z)$ and $Z/t(Z)$ is a field. Hence (1), (2) and (3) hold by Theorem 7 and Corollary 5. If Z/pZ has a codivisibl cover, then it must be of the form $O\to pZ/p^2Z\to Z/p^2Z\to Z/pZ\to O$. But Z/p^2Z is not codivisible by Proposition 3 since $t(p^2Z)=p^3Z\neq p^2Z$.

By Corollary 9, if every R-module has a codivisible cover, then $R/t(R)$ is a right perfect ring. So on the analogy of Corollary 15, we propose the next conjecture.

CONJECTURE. (*) *If every right R-module has a codivisible cover with respect to a pre-torsion theory* $(\mathcal{T}, \mathcal{F})$, *then* $(\mathcal{T}, \mathcal{F})$ *is pseudo-hereditary.*

6. The modules $M_R\cdot t(R_R)=M_R$.

M. L. Teply in [7] has proved that for a pseudo-hereditary torsion theory $(\mathcal{T}, \mathcal{F})$, M_R is codivisible if $M_R\cdot t(R_R)=M_R$. In this section, we shall characterize those modules M_R such that $M_R\cdot t(R_R)=M_R$ by the notion of the colocalization of a module.

LEMMA 16. *Let* $(\mathcal{T}, \mathcal{F})$ *be a pre-torsion theory with the radical* t. *The following assertions are equivalent for a given R-module* M_R.

(1) $\mathrm{Hom}_R(t(M_R)_R, L_R/T_R)=O$ *for any* $T_R\subset L_R\in\mathcal{F}$.

(2) $t(M_R)\in\mathcal{T}$ *and* $t(M_R)$ *is weakly codivisible.*

(3) $t(M_R)\cdot t(R_R)=t(M_R)$.

PROOF. (1) implies (2). By Proposition 1, $t=t_{\mathcal{F}}$. Hence $t(M_R)\in\mathcal{T}$ since $\mathrm{Hom}_R(t(M_R)_R, L_R)=O$ for any $L_R\in\mathcal{F}$. The weakly codivisibility of $t(M_R)$ is clear.

(2) implies (3). By Proposition 3, $t(t(M_R))=t(M_R)\cdot t(R_R)$. But $t(M_R)\in\mathcal{T}$, $t(t(M_R))=t(M_R)$. So $t(M_R)=t(M_R)\cdot t(R_R)$.

(3) implies (1). Assume $f\in\operatorname{Hom}_R(t(M_R)_R, L_R/T_R)$. $t(M_R)=t(M_R)\cdot t(R_R)$ implies $f(t(M_R))=f(t(M_R))\cdot t(R_R)$. But $f(t(M_R))\subset L_R/T_R$ and $(L_R/T_R)\cdot t(R_R)=(L_R\cdot t(R_R)+T_R)/T_R=O$. Thus $f=O$.

THEOREM 17. *Let $(\mathcal{T}, \mathcal{F})$ be a pre-torsion theory with the radical t such that if A_R is codivisible, then $A/t(B_R)_R$ is codivisible for any $B_R\subset A_R$. Then the following statements are equivalent.*

(1) *M_R has the colocalization.*

(2) *$\operatorname{Hom}_R(t(M_R)_R, L_R/T_R)=O$ for any $T_R\subset L_R\in\mathcal{F}$.*

(3) *$t(M_R)\in\mathcal{T}$ and $t(M_R)$ is weakly codivisible.*

(4) *$t(M_R)\cdot t(R_R)=t(M_R)$.*

PROOF. The equivalences of (2), (3) and (4) are proved by Lemma 16.

(1) implies (4). Let $f: C(M_R)\to M_R$ be a colocalization of M_R. $C(M_R)\in\mathcal{T}$ and is codivisible, hence $C(M_R)=t(C(M_R))=C(M_R)\cdot t(R_R)$ by Proposition 3, hence $f(C(M_R))=f(C(M_R))\cdot t(R_R)\subset M_R\cdot t(R_R)\subset t(M_R)$. On the other hand, $M_R/f(C(M_R))\in\mathcal{F}$, hence $O=t(M_R/f(C(M_R)))=t(M_R)/f(C(M_R))$. Thus $t(M_R)\subset f(C(M_R))$, so $t(M_R)=f(C(M_R))$ and $t(M_R)=t(M_R)\cdot t(R_R)$.

(2) implies (1). We consider an exact sequence $O\to K_R\to P_R\to t(M_R)_R\to O$ such that P_R is projective. Since $t(P_R/t(K_R))=t(P_R)/t(K_R)$, we have a commutative diagram with exact rows and columns;

$$\begin{tikzcd}
 & O \arrow[d] & O \arrow[d] & O \arrow[d] & \\
O \arrow[r] & (K_R\cap t(P_R))/t(K_R)_R \arrow[r] \arrow[d] & t(P_R)/t(K_R)_R \arrow[r, "\bar{f}"] \arrow[d, "i"] & \operatorname{Im}(fi)_R \arrow[r] \arrow[d, "j"] & O \\
O \arrow[r] & K_R/t(K_R)_R \arrow[r] \arrow[d] & P_R/t(K_R)_R \arrow[r, "f"] \arrow[d] & t(M_R)_R \arrow[r] \arrow[d, "k"] & O \\
O \arrow[r] & (K_R+t(P_R))/t(P_R)_R \arrow[r] \arrow[d] & P_R/t(P_R)_R \arrow[r] \arrow[d] & \operatorname{Cok}(j)_R \arrow[r] \arrow[d] & O \\
 & O & O & O &
\end{tikzcd}$$

By assumption, $k=O$. Hence $\operatorname{Im}(fi)=t(M_R)$. So $(K_R+t(P_R))/t(P_R)=P_R/t(P_R)$. Here $P_R/t(P_R)$ is weakly codivisible by Lemma 11, so the left column sequence splits, hence so does the middle column sequence. Thus $t(P_R)/t(K_R)$ is a direct summand of $P_R/t(K_R)$. But $P_R/t(K_R)$ is codivisible by assumption, hence $t(P_R)/t(K_R)$ is codivisible. Furthermore

$$t(P_R/t(K_R)) \cong t((t(P_R)/t(K_R)) \oplus (P_R/t(P_R)))$$
$$= t(t(P_R)/t(K_R)) \oplus t(P_R/t(P_R))$$
$$= t(t(P_R/t(K_R)))\,.$$

Clearly this isomorphism is an injection $t(i)$, hence $t(P_R/t(K_R)) = t(t(P_R/t(K_R)))$. Thus it is torsion. This shows $\bar{f}: t(P_R)/t(K_R)_R \to M_R$ is a colocalization of M_R.

The next corollary is a generalization of [7, Proposition 1] and [7, Corollary 1].

COROLLARY 18. *Under same assumption as in Theorem* 17, *the following assertions are equivalent.*

(1) *$R/t(R)$ is a semi-perfect ring.*

(2) *Every simple R-module has a codivisible cover.*

(3) *Every simple $R/t(R)$-module has a codivisible cover as an R-module.*

PROOF. (1) implies (2). Assume S_R to be a simple R-module. If $S_R \cdot t(R_R) = S_R$, then $S_R = t(S_R) = S_R \cdot t(R_R)$. Hence S_R has a colocalization by Theorem 17. This is a codivisible cover of S_R. If $S_R \cdot t(R_R) = O$, then S is a simple $R/t(R)$-module. By assumption (1), S has a projective cover as an $R/t(R)$-module, say $O \to K \to P \to S \to O$. Since P is a direct summand of a direct sum of $R/t(R)$ as an $R/t(R)$--module, so is as an R-module. Thus P_R is a codivisible R-module since a direct sum of $R/t(R)$ is codivisible by assumption. So the above exact sequence is a codivisible cover of S_R as an R-module.

(2) implies (3). It is clear.

(3) implies (1). Let S be a simple $R/t(R)$-module and $O \to K_R \to P_R \xrightarrow{f} S_R \to O$ a codivisible cover of S_R as an R-module. Since $t(P_R) = P_R \cdot t(R)$ by Proposition 3, $f(t(P_R)) = f(P_R) \cdot t(R) = S_R \cdot t(R) = O$, hence $O \to (K_R + t(P_R))/t(P_R) \to P_R/t(P_R) \to S \to O$ is an exact sequence as an $R/t(R)$-module. By Theorem 4, $P/t(P_R)$ is a projective $R/t(R)$-module, hence the above sequence is a projective cover of S as an $R/t(R)$-module. Thus $R/t(R)$ is a semi-perfect ring.

COROLLARY 19. (K. Ohtake)

Let $(\mathcal{T}, \mathcal{F})$ be a pre-torsion theory with the radical t. Then the following statements are equivalent.

(1) *Every R-module has a colocalization.*

(2) *$\mathcal{F}$ is closed under factors and extensions.*

PROOF. (2)implies (1). In this case, t must be an idempotent radical, so it is clear from Theorem 17.

(1) implies (2). By Theorem 17, $t(M_R)\in\mathcal{T}$ for any $M_R\in$Mod-R because in the proof that (1) implies (4) the codivisibility of A_R is not necessary. Hence t is an idempotent radical, so $\mathcal{T}$ is closed under extensions. Thus the assumption of Theorem 17 is satisfied. Hence (2) holds by Theorem 17 since $\mathrm{Hom}_R(t(L_R/T_R)_R, L_R/T_R)=O$ for any $T_R\subset L_R\in\mathcal{T}$.

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Institute of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun,
Ibaraki, 305 Japan.