

A LIMIT THEOREM FOR CONDITIONAL RANDOM WALK

By

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§1. Introduction

Let X, X_1, X_2, \dots , be a sequence of real valued independent, identically distributed random variables defined on a probability space (Ω, β, P) such that

$$(I) \quad EX=0 \quad \text{and} \quad 0 < \sigma^2 := E(X^2) < \infty.$$

Let $\{S_n; n \geq 0\}$ be a random walk defined by

$$S_0=0 \quad \text{and} \quad S_n := X_1 + \dots + X_n \quad \text{for } n \geq 1.$$

The main purpose of this paper is to show the following

THEOREM. *In addition to (I) suppose that*

$$(II) \quad E(X^2[\log(1+|X|)]^\alpha) > \infty$$

for some constant $\alpha > 1$. Then

$$P(S_n/\sigma\sqrt{n} \leq x | S_k > 0 \text{ for every } k, 1 \leq k \leq n) \xrightarrow{d} 1 - \exp(-x^2/2)$$

(convergence in distribution of distribution functions on the semi-infinite interval $[0, \infty)$).

The result of Theorem (without the condition (II)) was announced by Spitzer [19], page 162, in a footnote "Added in proof". But the proof was not published. The rigorous proof was given by Iglehart [10], Proposition 2.1 under the condition that random walk has finite third absolute moment and in addition the maximal span one when it is integer-valued. It has been open whether his condition is necessary or not (see Iglehart [11], page 177). Our Theorem asserts that his condition is not necessary.

In §2 we discuss asymptotic property of probability distributions of random walk. In §3 we prepare several lemmas which play important role in our proof of Theorem. In §4 we prove our Theorem.

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§ 2. Asymptotic property of distributions of random walk

1. First of all we introduce a lemma, which is simple but plays a fundamental role in our discussion. Define

$$\Phi_n(x) := P(S_n/\sigma\sqrt{n} \leq x) \text{ and } \Phi(x) := (2\pi)^{-1/2} \int_{-\infty}^x e^{-(x^2/2)} dx \text{ the}$$

standard normal distribution. Then we have

LEMMA 1. *If (I) holds, then for each continuous function $\phi(x)$ on $(-\infty, \infty)$ such that $\phi(x)/x^2 \rightarrow 0$ ($|x| \rightarrow \infty$) we have*

$$(1) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \phi(x) d\Phi_n(x) = \int_{-\infty}^{\infty} \phi(x) d\Phi(x).$$

PROOF ([6], Problem 14 in chapter 8). For arbitrary $\varepsilon > 0$ choose $K_\varepsilon > 0$ such that $\sup\{|\phi(x)|/x^2; |x| \geq K_\varepsilon\} \leq \varepsilon/2$. Note that

$$\begin{aligned} I_n := & \left| \int_{-\infty}^{\infty} \phi(x) d\Phi_n(x) - \int_{-\infty}^{\infty} \phi(x) d\Phi(x) \right| \leq \left| \int_{|x| < K_\varepsilon} \phi(x) d\Phi_n(x) \right. \\ & \left. - \int_{|x| < K_\varepsilon} \phi(x) d\Phi(x) \right| + \int_{|x| \geq K_\varepsilon} |\phi(x)| d\Phi_n(x) + \int_{|x| \geq K_\varepsilon} |\phi(x)| d\Phi(x). \end{aligned}$$

Since $\Phi_n \xrightarrow{d} \Phi$ by (I) and $\Phi(x)$ is continuous for all x , the first term of the right hand side $\rightarrow 0$ ($n \rightarrow \infty$). For the second term

$$\int_{|x| \geq K_\varepsilon} |\phi(x)| d\Phi_n(x) = \int_{|x| \geq K_\varepsilon} \frac{|\phi(x)|}{x^2} x^2 d\Phi_n(x) \leq (\varepsilon/2) \int_{-\infty}^{\infty} x^2 d\Phi_n(x) = \varepsilon/2.$$

Similarly

$$\int_{|x| \geq K_\varepsilon} |\phi(x)| d\Phi(x) \leq \varepsilon/2.$$

Hence we have $\overline{\lim}_{n \rightarrow \infty} I_n \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n \rightarrow \infty} I_n = 0$. This completes the proof.

2. In this paragraph we introduce the following two lemmas.

LEMMA 2. *If (I) holds, then for all $\varepsilon \geq 0$, $n \geq 1$ and $-\infty < x < \infty$*

$$(1) \quad P(x \leq S_n \leq x + \varepsilon) \leq A_\varepsilon / \sqrt{n},$$

where positive constant A_ε depends only on ε .

LEMMA 3. *If (I) holds, then there exist positive constants B and d such that*

$$(2) \quad P(-d < S_n < 0) \geq B/\sqrt{n} \text{ for every } n \geq 1.$$

Lemmas 2 and 3 relate to local limit theorem of sums of independent random variables. Refer, e.g., to [9], [20], [6] and [21], where more refined results are obtained under the stronger condition that X is centered nonlattice or centered lattice in addition to (I). See for the definition of centered nonlattice and centered lattice, [6], 10.4.

Lemma 2 is a simple modification of [9] or [21], and we omit the proof. For Lemma 3, when X is centered nonlattice or centered lattice, it is a consequence of [6], Theorem 10.17. Hence we prove it for the remaining case in the following stronger form (refer to [20], E2 in § 7). We may assume without losing generality X has span one, that is, $P(X \in \{x_0 + z; z = 0, \pm 1, \pm 2, \dots\}) = 1$, where x_0 is a constant in $[0, 1)$. Let $f(t) := E[\exp(itX)] = \sum_{z=-\infty}^{\infty} p_z \exp[it(x_0 + z)] = \exp(itx_0) \tilde{f}(t)$, where $p_z = P(X = x_0 + z)$ and $\tilde{f}(t) := \sum_{z=-\infty}^{\infty} p_z \exp(itz)$.

LEMMA 4. *In addition to (I) suppose that X has a lattice distribution with a span one. Then we have the following results.*

(i) *There exists a $t_0 > 0$ such that $M = 2\pi/t_0$ a positive integer and*

$$\{t; -\infty < t < \infty \text{ with } |f(t)| = 1\} = \{lt_0; l = 0, \pm 1, \pm 2, \dots\}.$$

(ii) *Let $d > M$ and N be large enough. Then for each $n \geq N$ there exists at least one $x \in (-d, 0)$ such that $P(S_n = x) > 0$. Moreover*

$$\lim_{n \rightarrow \infty} \max\{\sigma \sqrt{2\pi n} P(S_n = x); -d < x < 0\} = \lim_{n \rightarrow \infty} \min\{\sigma \sqrt{2\pi n} P(S_n = x); -d < x < 0 \text{ with } P(S_n = x) > 0\} = M.$$

PROOF. Let us prove (i). Let $t_0 := \inf\{t > 0; |f(t)| = 1\}$. Then by the continuity of characteristic function $|f(t_0)| = 1$, and by [9], Theorem 2 in § 14 $t_0 > 0$. Put $\tilde{f}(t_0) = \exp(i\alpha)$, $0 \leq \alpha < 2\pi$. Then $\sum_{z=-\infty}^{\infty} p_z \exp[i(t_0 z - \alpha)] = 1$, and this implies

$$(3) \quad "p_z > 0 \text{ only if } t_0 z - \alpha = 2\pi r \quad r = 0, \pm 1, \pm 2, \dots".$$

Using (3), we have

$$\tilde{f}(t) = \sum_{r=-\infty}^{\infty} p_{(2\pi r + \alpha)/t_0} \exp[i(2\pi r + \alpha)t/t_0] = \exp(i\alpha t/t_0) \times \sum_{r=-\infty}^{\infty} p_{(2\pi r + \alpha)/t_0} \exp(2\pi i r t/t_0).$$

Then for each $l = 0, \pm 1, \pm 2, \dots$,

$$(4) \quad \tilde{f}(t + lt_0) = \exp(il\alpha) \exp(i\alpha t/t_0) \sum_{r=-\infty}^{\infty} (p_{(2\pi r + \alpha)/t_0} \exp(2\pi i r t/t_0) \times \exp(2\pi i l r)) = \exp(il\alpha) \tilde{f}(t).$$

Note that $|\tilde{f}(t)| < 1$ for $0 < t < t_0$. Combining this and (4), we have (i).

Let us prove (ii). Since $|\tilde{f}(t)| < 1$ for $t \neq lt_0$, $l = 0, \pm 1, \pm 2, \dots$, and since $\tilde{f}(2\pi) = 1$, there exists a positive integer M such that $Mt_0 = 2\pi$. Since $1 = \tilde{f}(2\pi) = \tilde{f}(Mt_0) =$

$\exp(iM\alpha)$ by (4), there exists a positive integer r_0 such that

$$(5) \quad M\alpha = 2\pi r_0.$$

Now rewrite $\tilde{f}(t)$, using (5). Then

$$(6) \quad \tilde{f}(t) = \exp(ir_0 t) \sum_{r=-\infty}^{\infty} p_{Mr+r_0} \exp(Mirt).$$

Let $\tilde{X}_n := X_n - x_0$ and $\tilde{S}_n := \tilde{X}_1 + \cdots + \tilde{X}_n$ for $n \geq 1$. Since \tilde{S}_n has the characteristic function $\tilde{f}^n(t)$, we have by the inversion formula of Fourier series and by (4)

$$(7) \quad \begin{aligned} 2\pi P(\tilde{S}_n = k) &= 2\pi P(S_n = k + nx_0) = \int_{-t_0/2}^{2\pi - t_0/2} e^{-ikt} \tilde{f}^n(t) dt \\ &= \sum_{l=0}^{M-1} \int_{-t_0/2}^{t_0/2} \exp[-ik(lt_0 + t)] \tilde{f}^n(lt_0 + t) dt = \sum_{l=0}^{M-1} \exp[2\pi il(nr_0 - k)/M] \\ &\quad \times \int_{-t_0/2}^{t_0/2} e^{-ikt} \tilde{f}^n(t) dt \\ &= \begin{cases} M \int_{-t_0/2}^{t_0/2} e^{-ikt} \tilde{f}^n(t) dt & \text{if } nr_0 - k = \text{integral multiple of } M \dots (*) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $|\tilde{f}(t)| < 1$ for $0 < |t| \leq t_0/2$, we can apply similar technique as in [9], Theorem in § 49 to obtain the asymptotic behaviour of $\int_{-t_0/2}^{t_0/2} e^{-ikt} \tilde{f}^n(t) dt$ as $n \rightarrow \infty$. Then we have for arbitrary fixed $d > 0$

$$(8) \quad \int_{-t_0/2}^{t_0/2} e^{-ikt} \tilde{f}^n(t) dt \sim \sqrt{2\pi}/(\sigma\sqrt{n}) \quad (n \rightarrow \infty),$$

when k varies with n such that

$$(9) \quad -d < k + nx_0 < 0.$$

By (7), (8) and (9)

$$(10) \quad \sigma\sqrt{2\pi n} P(S_n = k + nx_0) \sim M \quad (n \rightarrow \infty)$$

when k varies with n such that (*) and (9). Hence in order to complete the proof we check that we can choose for each $n \geq 1$ integer k which satisfies (*) and (9). When $d > M$, we can really do this by choosing an integer m such that $n(r_0 + x_0) < mM < n(r_0 + x_0) + d$ and by taking $k = nr_0 - mM$. This completes the proof.

3. In this paragraph we show

THEOREM 5. *In addition to (I) suppose (II) holds for some $\alpha > 1$. Then for all $\epsilon \geq 0$, $n \geq 1$ and $x \geq 0$*

$$(1) \quad P(x \leq \max\{S_k; 0 \leq k \leq n\} \leq x + \epsilon) \leq C_\epsilon / \sqrt{n},$$

where positive constant C_ε depends only on ε .

REMARK. When X is integer-valued and has the maximal span one, a stronger result than our Theorem 5 is announced in [16] without proof. Refer also to [1] and [2].

Before proceeding to our proof of Theorem 5, we introduce two lemmas. Let S and S^{-1} be the Fourier transformation and its inverse, that is, for $v, w \in L^1(-\infty, \infty)$

$$(Sv)(t) := \int_{-\infty}^{\infty} e^{itx} v(x) dx, (S^{-1}w)(x) := (1/2\pi) \int_{-\infty}^{\infty} e^{-itx} w(t) dt.$$

LEMMA 6. (i) Let $H(x) := (3/8\pi) [\sin(x/4)/(x/4)]^4$. Then $H(x)$ is a probability density, and the characteristic function $h(t) := (SH)(t)$ is given by

$$h(t) = \begin{cases} 1 - 6t^2 + 6|t|^3 & 0 \leq |t| < 1/2 \\ 2(1 - |t|)^3 & 1/2 \leq |t| < 1 \\ 0 & |t| \geq 1 \end{cases}$$

(ii) For each $\varepsilon > 0$, let $H_\varepsilon(x) := \varepsilon H(\varepsilon x)$ and $h_\varepsilon(t) := h(t/\varepsilon)$. Then $H_\varepsilon(x)$ is a probability density with the property; $h_\varepsilon(t) = (SH_\varepsilon)(t)$ and $H_\varepsilon(x) = (S^{-1}h_\varepsilon)(x)$.

The functions $H(x)$ and $h(t)$ was first introduced and used in [8] to estimate the difference of two distribution functions.

Let $f(t) := E[\exp(itX)]$. Then we have

LEMMA 7. In addition to (I) suppose (II) holds for some $\alpha > 0$. Then

(i)
$$f(t) = \exp(-\sigma^2 t^2 [1 + \gamma(t)]/2),$$

where $\gamma(t)$ is a continuous function such that $\gamma(t) = O(1/|\log|t||^\alpha)$ ($t \rightarrow 0$),

(ii)
$$|f^n(t) - \exp(-\sigma^2 n t^2/2)| \leq \sigma^2 n t^2 |\gamma(t)| \exp(-\sigma^2 n t^2/4)$$
 for $|t| \leq \varepsilon_1$,

where ε_1 is a constant such that $0 < \varepsilon_1 < 1$ and $\max\{|\gamma(t)|; |t| \leq \varepsilon_1\} \leq 1/2$.

Under the condition of the lemma

(2)
$$\operatorname{Re}[f(t)] = 1 - \sigma^2 t^2/2 + O(t^2/|\log|t||^\alpha)$$

and

(3)
$$\operatorname{Im}[f(t)] = O(t^2/|\log|t||^\alpha) \quad (t \rightarrow 0).$$

The first expansion is given in [12], Theorem 11.3.4. In order to prove the second estimate rewrite $\operatorname{Im}[f(t)]$ as

$$\operatorname{Im}[f(t)] = [f(t) - f(-t)]/(2i) = t^2 [f''(\theta t) - f''(-\theta t)]/(4i),$$

where $0 < \theta < 1$. Note that

$$|f''(\theta t) - f''(-\theta t)| \leq \int_{-\infty}^{\infty} |e^{i\theta t x} - e^{-i\theta t x}| x^2 dF(x)$$

$$= 2 \int_{-\infty}^{\infty} |\sin(\theta t x)| x^2 dF(x),$$

where $F(x) := P(X < x)$. For an estimate of the last term we apply the technique in [12], Theorem 11.3.4, and obtain

$$|f''(\theta t) - f''(-\theta t)| = O(1/|\log |t||^\alpha) \quad (t \rightarrow 0).$$

This proves (3). Combining (2) and (3), we obtain (i). (ii) is obvious from (i).

PROOF OF THEOREM 5. To simplify the notations we may take $\sigma = 1$. In our proof we adopt the techniques shown in [14] and [15]. Let

$$(4) \quad \tilde{F}_n(x) := P(\max\{S_k; 0 \leq k \leq n\} < x) \quad \text{and} \quad \rho_n(t) := \int_{-\infty}^{\infty} e^{itx} d\tilde{F}_n(x)$$

for $n \geq 0$. By [14] we have the identity;

$$(5) \quad \rho_n(t) = \sum_{k=0}^n f^k(t) \bar{\rho}_{n-k}(t),$$

where $\bar{\rho}_0(t) = 1$ and $\bar{\rho}_k(t) = \rho_k(t) - f(t)\rho_{k-1}(t)$ for $k \geq 1$. Put $\delta = \varepsilon_1$. Then

$$(6) \quad S^{-1}(\rho_n h_\delta)(x) = \int_{-0}^{\infty} H_\delta(x-y) dF_n(y) \geq \int_{x-0}^{x+\delta+0} H_\delta(x-y) \\ \times d\tilde{F}_n(y) \geq CP(x \leq \max\{S_k; 0 \leq k \leq n\} \leq x + \delta),$$

where $C = H_\delta(\delta) > 0$. Let

$$(7) \quad 2\pi S^{-1}(\rho_n h_\delta)(x) = \int_{-\delta}^{\delta} e^{-itx} \rho_n(t) dt + \int_{-\delta}^{\delta} e^{-itx} \rho_n(t) [h_\delta(t) - 1] dt = : I + J. \\ I = \sum_{k=0}^n \int_{-\delta}^{\delta} e^{-itx} [f^k(t) - e^{-(kt^2/2)}] \bar{\rho}_{n-k}(t) dt + \sum_{k=0}^n \int_{-\delta}^{\delta} e^{-(itx + kt^2/2)} \bar{\rho}_{n-k}(t) dt = : I_1 + I_2.$$

Note that

$$(8) \quad \bar{\rho}_n(t) = \int_{-\infty}^0 (1 - e^{itx}) d\bar{F}_n(x),$$

where $\bar{S}_n := \max\{S_k; 1 \leq k \leq n\}$ and $\bar{F}_n(x) := P(\bar{S}_n < x)$. By [14], Lemma 1

$$\int_{-\infty}^0 x d\bar{F}_n(x) = -(1/n) \int_0^{\infty} x dF_n(x),$$

where $F_n(x) := P(S_n < x)$. By Lemma 1 (take $\phi(x) = \chi_{(0, \infty)}(x)x$, where $\chi_{(0, \infty)}(x)$ is the indicator function on the set $(0, \infty)$)

$$\int_0^{\infty} x dF_n(x) \sim \sigma \sqrt{n/(2\pi)},$$

so that we have

$$(9) \quad - \int_{-\infty}^0 x d\bar{F}_n(x) \sim \sigma / \sqrt{2\pi n} \quad (n \rightarrow \infty).$$

By (8)

$$|\bar{\rho}_n(t)| \leq -|t| \int_{-\infty}^0 x d\bar{F}_n(x),$$

and by (9) we have for all $n \geq 1$ and $-\infty < t < \infty$

$$(10) \quad |\bar{\rho}_n(t)| \leq K|t|/\sqrt{n},$$

where K is a positive constant independent of n and t . Applying Lemma 7 and (10), we have

$$\begin{aligned} I_1 &= O\left(\sum_{0 \leq k \leq \delta-4} \int_0^\delta (n-k)t^2 \exp[-(n-k)t^2/4] dt\right) + O\left(\sum_{\delta-4 < k < n} \frac{k}{\sqrt{n-k}}\right. \\ &\quad \times \left. \left[\int_0^{k^{-1/4}} + \int_k^{\delta} \right] \frac{t^3}{|\log t|^\alpha} e^{-(kt^2/4)} dt\right) = O(1/\sqrt{n}) + O\left(\sum_{k=1}^{n-1} [k(1 + \right. \\ &\quad \left. + \log k)^\alpha \sqrt{n-k}]^{-1}\right) + O\left(\sum_{k=1}^{n-1} k e^{-(\sqrt{k}/4)} / \sqrt{n-k}\right) = O(1/\sqrt{n}). \end{aligned}$$

Note that we used $\alpha > 1$ in the last estimate. Estimate of I_2 is as follows.

$$\begin{aligned} (11) \quad I_2 &= \int_{-\delta}^\delta e^{-itx} \bar{\rho}_n(t) dt + \int_{-\delta}^\delta \exp[-(itx + nt^2/2)] dt + \\ &\quad + \sum_{k=1}^{n-1} \int_{-\infty}^\infty \exp[-(itx + kt^2/2)] \bar{\rho}_{n-k}(t) dt - \sum_{k=1}^{n-1} \int_{|t| > \delta} \exp[-(itx + kt^2/2)] \times \\ &\quad \times \bar{\rho}_{n-k}(t) dt =: \sum_{j=1}^4 i_j. \end{aligned}$$

Obviously $i_1, i_2 = O(1/\sqrt{n})$. $i_4 = O(1/\sqrt{n})$, because

$$i_4 = O\left(\sum_{k=1}^{n-1} (n-k)^{-1/2} \int_\delta^\infty t e^{-(kt^2/2)} dt\right) = O\left(\sum_{k=1}^{n-1} e^{-(k\delta^2/2)} k^{-1} \times (n-k)^{-1/2}\right) = O(1/\sqrt{n}).$$

Let us estimate i_3 , using a technique in [15]. Since

$$\begin{aligned} \bar{\rho}_n(t) &= \int_{-\infty}^0 \int_x^0 ite^{itz} dz d\bar{F}_n(x) = it \int_{-\infty}^0 e^{itz} \bar{F}_n(z) dz, \\ i_3 &= \sum_{k=1}^{n-1} \int_{-\infty}^\infty \exp[-(itx + kt^2/2)] it \int_{-\infty}^0 e^{ity} \bar{F}_{n-k}(y) dy dt = \\ &= \sqrt{2\pi} \sum_{k=1}^{n-1} k^{-3/2} \int_{-\infty}^0 (x-y) e^{-[(x-y)^2/2k]} \bar{F}_{n-k}(y) dy = \\ &= O\left(\sum_{k=1}^{n-1} k^{-3/2} x e^{-(x^2/2k)} \int_{-\infty}^0 \bar{F}_{n-k}(y) dy\right) + O\left(\sum_{k=1}^{n-1} k^{-3/2} \int_{-\infty}^0 |y| e^{-(y^2/2k)} \times \right. \\ &\quad \left. \times \bar{F}_{n-k}(y) dy\right) =: i' + i''. \end{aligned}$$

For i' , since

$$(12) \quad \int_{-\infty}^0 \bar{F}_{n-k}(y) dy = - \int_{-\infty}^0 y d\bar{F}_{n-k}(y),$$

we have by (9)

$$i' = O\left(\sum_{k=1}^{n-1} (n-k)^{-1/2} k^{-3/2} x e^{-(x^2/2k)}\right).$$

Applying the estimate (84) in [14], we have $i' = O(1/\sqrt{n})$. For i'' divide i'' by

$$i'' = O\left(\left[\sum_{1 \leq k \leq n/2} + \sum_{n/2 < k < n}\right] k^{-3/2} \int_{-\infty}^0 |y| e^{-(y^2/2k)} \bar{F}_{n-k}(y) dy\right) =: i_1'' + i_2''.$$

For $1 \leq k \leq n/2$ $\bar{F}_{n-k}(y) \leq \bar{F}_{[n/2]_G}(y)$, where $[a]_G := \max\{n; \text{integer such that } n \leq a\}$,

$$\begin{aligned} i_1'' &= O\left(\int_{-\infty}^0 \left[\sum_{1 \leq k \leq n/2} k^{-3/2} |y| e^{-(y^2/2k)}\right] \bar{F}_{[n/2]_G}(y) dy\right) = \\ &= O\left(\int_{-\infty}^0 \left[\sqrt{n} \sum_{1 \leq k \leq n/2} (n-k)^{-1/2} k^{-3/2} |y| e^{-(y^2/2k)}\right] \bar{F}_{[n/2]_G}(y) dy\right). \end{aligned}$$

Again applying (84) in [14], we have

$$i_1'' = O\left(\int_{-\infty}^0 \bar{F}_{[n/2]_G}(y) dy\right) = O(1/\sqrt{n}).$$

For i_2''

$$\begin{aligned} i_2'' &= O\left(\sum_{n/2 < k < n} k^{-1} \int_{-\infty}^0 (|y|/\sqrt{k}) e^{-(y^2/2k)} \bar{F}_{n-k}(y) dy\right) = \\ &= O\left(\sum_{n/2 < k < n} k^{-1} \int_{-\infty}^0 \bar{F}_{n-k}(y) dy\right) = O\left(\sum_{n/2 < k < n} k^{-1} (n-k)^{-1/2}\right) = O(1/\sqrt{n}). \end{aligned}$$

Hence we have $i'' = O(1/\sqrt{n})$, and $i_3 = O(1/\sqrt{n})$. Summarizing the above estimates, we have $I_2 = O(1/\sqrt{n})$ and then $I = O(1/\sqrt{n})$.

Let us estimate J .

$$\begin{aligned} J &= \sum_{k=0}^n \int_{-\delta}^{\delta} e^{-itx} [f^k(t) - e^{-(kt^2/2)}] \bar{\rho}_{n-k}(t) (h_{\delta}(t) - 1) dt + \\ &+ \sum_{k=0}^n \int_{-\delta}^{\delta} \exp(-itx - kt^2/2) \bar{\rho}_{n-k}(t) (h_{\delta}(t) - 1) dt =: J_1 + J_2. \end{aligned}$$

Applying Lemma 7, (10) and $h_{\delta}(t) - 1 = O(t^2)$ ($t \rightarrow 0$), we have

$$\begin{aligned} J_1 &= O\left(\int_0^{\delta} nt^4 e^{-(nt^2/4)} dt\right) + O\left(\sum_{k=1}^{n-1} \left[\frac{k}{\sqrt{n-k}} \int_0^{\delta} t^5 e^{-(kt^2/4)} dt\right]\right) \\ &= O(n^{-3/2}) + O\left(\sum_{k=1}^{n-1} (k^2 \sqrt{n-k})^{-1}\right) = O(1/\sqrt{n}). \end{aligned}$$

Similarly we have $J_2 = O(1/\sqrt{n})$, and hence $J = O(1/\sqrt{n})$.

Summing up the estimates on I and J , we have

$$(13) \quad S^{-1}(\rho_n h_{\delta})(x) \leq C'/\sqrt{n},$$

where C' is a positive constant independent of $n \geq 1$ and $x \geq 0$. By (6) and (13)

$$(14) \quad P(x \leq \max\{S_k; 0 \leq k \leq n\} \leq x + \delta) \leq C' / \sqrt{n},$$

where $C' = C'/C$ a positive constant independent of $n \geq 1$ and $x \geq 0$. Obviously (1) holds for $0 \leq \varepsilon \leq \delta$ by taking $C_\varepsilon = C'$. For $\varepsilon > \delta$ (1) holds by taking $C = (1 + \varepsilon/\delta)C'$, in fact because by (14)

$$P(x \leq \max\{S_k; 0 \leq k \leq n\} \leq x + \varepsilon) \leq \sum_{0 \leq l \leq \varepsilon/\delta} P(x + \delta l \leq \max\{S_k; 0 \leq k \leq n\} \leq x + \delta(l+1)) \leq (1 + \varepsilon/\delta)C' / \sqrt{n}.$$

This completes the proof.

4. In this paragraph we introduce some results from the theory of factorization developed, e. g., in [20] and [5]. Define random functions \underline{S}_n and M_n by

$$\underline{S}_n := \min\{S_k; 1 \leq k \leq n\}, \quad M_n := \max\{0 \leq k \leq n; S_k = \min\{0, \underline{S}_n\}\}.$$

Then we have

LEMMA 8 ([19]). *If (I) holds, then*

$$(1) \quad P(\underline{S}_n > 0) = P(S_1 > 0, \dots, S_n > 0) \sim e^\alpha (\pi n)^{-1/2} \quad (n \rightarrow \infty),$$

where $\alpha = \sum_{k=1}^{\infty} (1/k) [P(S_k > 0) - 1/2]$ is a convergent series.

REMARK. By [17] $\sum_{k=1}^{\infty} (1/k) |P(S_k > 0) - 1/2| < \infty$.

PROPOSITION 9. *If (I) holds, then there exists non-negative, nondecreasing function $C(x)$ such that, for all $n \geq 1$, $0 \leq k \leq n$ and $x > 0$*

$$(2) \quad P(M_n = k, -x < S_k \leq 0) \leq C(x) (k+1)^{-3/2} (n-k+1)^{-1/2}.$$

In order to prove Proposition 9 we need

LEMMA 10. *For all $x > 0$ and $n \geq 0$*

$$P(M_n = n, -x < S_n \leq 0) = \sum_{k=1}^n (1/k!) \sum_{\substack{j_1, \dots, j_k \geq 1 \\ j_1 + \dots + j_k = n}} (j_1 \cdots j_k)^{-1} \\ \times \int_{-\infty}^{+0} \cdots \int_{-\infty}^{+0} \chi_{(-x, 0]}(y_1 + \dots + y_k) dP(S_{j_1} < y_1) \cdots dP(S_{j_k} < y_k).$$

PROOF. By a simple change of [5], Theorem 2 in § 16,

$$(3) \quad \sum_{n=0}^{\infty} u^n E[M_n = n; e^{i\lambda S_n}] = \exp\left[\sum_{n=1}^{\infty} u^n (1/n) \int_{-\infty}^{+0} e^{i\lambda x} dP(S_n < x)\right].$$

Expand the right hand side in the Taylor series. After the rearrangement of summation, compare the coefficients of each term on both sides. Then we have

$$(4) \quad E[M_n = n; e^{i\lambda S_n}] = \sum_{k=1}^n (1/k!) \sum_{\substack{j_1, \dots, j_k \geq 1 \\ j_1 + \dots + j_k = n}} (j_1 \cdots j_k)^{-1}$$

$$\times \int_{-\infty}^{+0} \cdots \int_{-\infty}^{+0} \exp[i\lambda(y_1 + \cdots + y_k)] dP(S_{j_1} < y_1) \cdots dP(S_{j_k} < y_k)$$

By the linearity of summation and integration, (4) holds for $c_1 e^{i\lambda_1 x} + \cdots + c_m e^{i\lambda_m x}$ for $-\infty < c_j, \lambda_j < \infty$ ($1 \leq j \leq m, m \geq 1$) in place of $e^{i\lambda x}$, and this completes the proof.

PROOF OF PROPOSITION 9. By $P(M_n = k, -x < S_k \leq 0) = P(M_k = k, -x < S_k \leq 0) \times P(\underline{S}_{n-k} > 0)$ for $0 \leq k \leq n$, and by Lemma 8, it is enough to prove (2) for $k = n$. Applying Lemma 10, we have

$$(5) \quad P(M_n = n, -x < S_n \leq 0) \leq \sum_{k=1}^n [d(x)^k / k!] \times \sum_{\substack{j_1, \dots, j_k \geq 1 \\ j_1 + \dots + j_k = n}} (j_1 \cdots j_k)^{-3/2} = : I$$

where $d(x)$ is a positive, nondecreasing function defined below. In the above estimate we used Lemma 2, which implies

$$P(-x < S_n \leq 0) \leq \sum_{0 \leq k \leq x} P(-k-1 < S_n \leq -k) \leq A_1(x+1)/\sqrt{n} = : d(x)/\sqrt{n}.$$

For $1 \leq k \leq n$

$$\sum_{\substack{j_1, \dots, j_k \geq 1 \\ j_1 + \dots + j_k = n}} (j_1 \cdots j_k)^{-3/2} \leq k(n/k)^{-3/2} \left(\sum_{j=1}^n j^{-3/2} \right)^{k-1} \leq n^{-3/2} k^{5/2} A^{k-1},$$

where $A = \sum_{j=1}^{\infty} j^{-3/2} < \infty$. Hence we have

$$(6) \quad I \leq A^{-1} \left[\sum_{k=1}^n k^{5/2} (Ad(x))^k / k! \right] n^{-3/2} \leq C(x) n^{-3/2},$$

where $C(x) := \sum_{k=1}^{\infty} k^{5/2} (Ad(x))^k / k!$ a positive nondecreasing function. This completes the proof.

§ 3. Auxiliary lemmas

First we consider the moment of the conditional distributions.

LEMMA 11 ([19], Corollary 3.6). *If (I) holds, then*

$$\lim_{n \rightarrow \infty} E[S_n / (\sigma \sqrt{n}) | \underline{S}_n > 0] = \sqrt{\pi/2} = \int_0^{\infty} x d(1 - e^{-(x^2/2)}).$$

In [19] the conditioning is not " $\underline{S}_n > 0$ " but " $\underline{S}_n \geq 0$ ". But this difference is not the matter.

REMARK. In addition to (I) suppose that (II) holds for $\alpha = 1$. Then we can prove

$$\lim_{n \rightarrow \infty} E[(S_n / \sigma \sqrt{n})^2 | \underline{S}_n > 0] = 2 = \int_0^{\infty} x^2 d(1 - e^{-(x^2/2)}).$$

Let $A_n := \{\underline{S}_n > 0\}$ and ${}_m \underline{S}_n := \min\{S_k; m < k \leq n\}$ for $1 < m < n$. Now we take $\sigma = 1$ for simplicity without losing generality.

LEMMA 12. Suppose that (I) holds. Let $\phi \in C_b^1([0, \infty))$ the class of bounded functions with bounded continuous derivative. Then for each $\epsilon > 0$ there exists $\delta_\epsilon \in (0, 1)$ and positive integer n_ϵ such that

$$|E[\phi(S_{n-m}/\sqrt{n-m})|A_{n-m}] - E[\phi(S_n/\sqrt{n})|A_n]| < \epsilon$$

for all $n \geq n_\epsilon$ and $m \leq \delta_\epsilon n$.

PROOF. To simplify the notations we may assume $\|\phi\| \leq 1$ and $\|\phi'\| \leq 1$, where $\|\phi\| := \sup\{|\phi(x)|; 0 \leq x < \infty\}$ for $\phi \in C_b([0, \infty))$ the class of bounded continuous functions. Let $f_n := E[\phi(S_n/\sqrt{n})|A_n]$. Since for $0 \leq m < n$ $A_n = A_{n-m} \cap \{S_{n-m} > 0\}$ and

$$\phi(S_n/\sqrt{n}) = \phi\left(\frac{S_{n-m}}{\sqrt{n}}\right) + (S_n - S_{n-m})\phi'([S_{n-m} + \theta(S_n - S_{n-m})]/\sqrt{n})/\sqrt{n},$$

for some $\theta \in (0, 1)$, we have

$$(1) \quad P(A_n)f_n = E[A_n; \phi(S_n/\sqrt{n})] = E[A_{n-m}; P(S_m > -x)_{|x=S_{n-m}}\phi(S_{n-m}/\sqrt{n})] \\ + E[A_{n-m}; E[S_m > -x; S_m\phi'([x + \theta S_m]/\sqrt{n})/\sqrt{n}]_{|x=S_{n-m}}] =: I + J.$$

Since

$$\phi\left(\frac{S_{n-m}}{\sqrt{n}}\right) = \phi\left(\frac{S_{n-m}}{\sqrt{n-m}}\right) + \left(\sqrt{1 - \frac{m}{n}} - 1\right) \frac{S_{n-m}}{\sqrt{n-m}} \phi'\left(\frac{S_{n-m}}{\sqrt{n-m}} + \theta'\left(\sqrt{1 - \frac{m}{n}} - 1\right) \frac{S_{n-m}}{\sqrt{n-m}}\right)$$

for some $\theta' \in (0, 1)$, we have

$$(2) \quad I = E[A_{n-m}; \phi(S_{n-m}/\sqrt{n})] - E[A_{n-m}; P(S_m \leq -x)_{x=S_{n-m}}\phi(S_{n-m}/\sqrt{n})] \\ = E\left[A_{n-m}; \phi\left(\frac{S_{n-m}}{\sqrt{n-m}}\right)\right] + \left(\sqrt{1 - \frac{m}{n}} - 1\right) E\left[A_{n-m}; \phi'(*) \frac{S_{n-m}}{\sqrt{n-m}}\right] - E\left[A_{n-m}; P(S_m \leq -x)_{x=S_{n-m}}\phi\left(\frac{S_{n-m}}{\sqrt{n-m}}\right)\right] \\ - \left(\sqrt{1 - \frac{m}{n}} - 1\right) E\left[A_{n-m}; P(S_m \leq -x)_{x=S_{n-m}}\right] \\ \times \phi'(*) \frac{S_{n-m}}{\sqrt{n-m}}] = P(A_n)f_{n-m} + [P(A_{n-m}) - P(A_n)]f_{n-m} + \left(\sqrt{1 - \frac{m}{n}} - 1\right) P(A_{n-m}) \\ E\left[(1 - P(S_m \leq -x)_{x=S_{n-m}})\phi'(*) \frac{S_{n-m}}{\sqrt{n-m}} | A_{n-m}\right] - E\left[A_{n-m}; \phi\left(\frac{S_{n-m}}{\sqrt{n-m}}\right) P(S_m \leq -x)_{x=S_{n-m}}\right] \\ =: P(A_n)f_{n-m} + \sum_{j=1}^3 I_j,$$

where $\phi'(*) = \phi'(S_{n-m}/\sqrt{n-m} + \theta'(\sqrt{1 - m/n} - 1)S_{n-m}/\sqrt{n-m})$. For I_3

$$(3) \quad |I_3| \leq E\left[A_{n-m}; P\left(\frac{S_m}{\sqrt{m}} \leq -x\sqrt{1/\delta - 1}\right)_{x=S_{n-m}/\sqrt{n-m}}\right] \leq P\left(A_{n-m}, \frac{S_{n-m}}{\sqrt{n-m}} \geq d\right) \\ \times P(S_m/\sqrt{m} \leq -d\sqrt{1/\delta - 1}) + P\left(A_{n-m}, \frac{S_{n-m}}{\sqrt{n-m}} < d\right) =: I_3' + I_3'',$$

where $d > 0$. Let $p = [(n-m)/2]_G$. Then for $0 \leq m \leq \delta n$

$$I_3'' \leq E[A_p; P(-\sqrt{p}x \leq S_{n-m-p} < -\sqrt{p}x + \sqrt{n-md})_{|x=S_p/\sqrt{p}}].$$

By Lemma 2

$$\begin{aligned} P(-\sqrt{p}x \leq S_{n-m-p} < -\sqrt{p}x + \sqrt{n-md}) &\leq \sum_{0 \leq l \leq \sqrt{n-md}} P(-\sqrt{p}x + l \leq S_{n-m-p} < -\sqrt{p}x \\ &+ l + 1) \leq A_1(\sqrt{n-md} + 1)/\sqrt{n-m-p} \leq K_1 d + K_2/\sqrt{n}, \end{aligned}$$

where $K_1 = \sqrt{2/(1-\delta)}A_1$ and $K_2 = \sqrt{2/(1-\delta)}$. Hence we have

$$I_3'' \leq (K_1 d + K_2/\sqrt{n})P(A_p).$$

For $0 \leq m \leq \delta n$, J is estimated as

$$(4) \quad |J| \leq \sqrt{\delta} P(A_{n-m}) E[|S_m|/\sqrt{m}].$$

By the estimates from (1) to (4)

$$(5) \quad f_n = f_{n-m} + R,$$

where R is estimated as

$$\begin{aligned} (6) \quad |R| &\leq \left[\frac{P(A_{n-m})}{P(A_n)} - 1 \right] + \frac{P(A_{n-m})}{P(A_n)} \left(\delta E \left[\frac{S_{n-m}}{\sqrt{n-m}} \middle| A_{n-m} \right] + \sqrt{\delta} E[|S_m|/\sqrt{m}] \right. \\ &\quad \left. + P(S_m/\sqrt{m} \leq -d\sqrt{1/\delta-1}) + (K_1 d + K_2/\sqrt{n}) \frac{P(A_p)}{P(A_n)} \right) = : \sum_{i=1}^5 R_i. \end{aligned}$$

For $0 < \delta < 1/2$

$$0 \leq \overline{\lim}_{n \rightarrow \infty} \max\{R_1; 0 \leq m \leq \delta n\} \leq 1/\sqrt{1-\delta} - 1 < 2\delta.$$

By Lemma 1

$$\lim_{m \rightarrow \infty} E[|S_m|/\sqrt{m}] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} |u| e^{-(u^2/2)} du = (2/\pi)^{1/2},$$

and then $K_3 = \sup\{E[|S_m|/\sqrt{m}]; m \geq 1\} < \infty$. By Lemma 11,

$$\lim_{n \rightarrow \infty} \max\{E[S_{n-m}/\sqrt{n-m} | A_{n-m}]; 0 \leq m \leq \delta n\} = (\pi/2)^{1/2}.$$

Then we have for $0 < \delta < 1/2$

$$0 \leq \overline{\lim}_{n \rightarrow \infty} \max\{R_2 + R_3; 0 \leq m \leq \delta n\} \leq [(\pi/2)^{1/2} + K_3] \sqrt{\delta}.$$

By (I) $-S_m/\sqrt{m}$ converges in distribution to the one sided normal distribution (refer, e.g., [4], 10), so that we have

$$\sup\{P(S_m/\sqrt{m} \leq -d\sqrt{1/\delta-1}); m \geq 1\} \rightarrow 0 \text{ as } \delta d^{-2} \rightarrow 0.$$

Combining (5), (6) and the succeeding estimates, we have $R \rightarrow 0$ as $n \rightarrow \infty$, $d \rightarrow +0$

and $\delta \rightarrow +0$ such that $\delta d^{-2} \rightarrow 0$. This completes the proof.

For $0 < k < n$ let

$$(7) \quad I_{n,k} = \begin{cases} \frac{E[S_{k-1} < 0 < S_k; (n-k)^{-1/2} S_k P(x + \underline{S}_{n-k} > 0) | x = S_k]}{P(S_{k-1} < 0 < S_k \text{ and } \underline{S}_{n-k} > 0)} & \text{if the denominator } \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

LEMMA 13. *In addition to (I) suppose (II) holds for $\alpha > 1$. Then*

$$\lim_{\delta \rightarrow +0} \overline{\lim}_{n \rightarrow \infty} \max\{I_{n,k}; 0 < k < \delta n\} = 0.$$

PROOF. Since by Theorem 5

$$\begin{aligned} P(x + \underline{S}_{n-k} > 0) &= P(\max\{-S_j; 0 \leq j \leq n-k\} < x) \leq \sum_{0 \leq l \leq x} P(l \leq \max\{-S_j; 0 \leq j \\ &\leq n-k\} < l+1) \leq C_1(x+1)/\sqrt{n-k}, \end{aligned}$$

and since $S_k < X_k$ if $S_{k-1} < 0 < S_k$, we have

$$\text{the numerator of } I_{n,k} \leq C_1 E[S_{k-1} < 0 < S_k; X_k^2 + X_k] / \sqrt{n-k}.$$

Obviously

$$\text{the denominator of } I_{n,k} \geq P(S_{k-1} < 0 < S_k) P(\underline{S}_{n-k} > 0).$$

Then

$$(8) \quad I_{n,k} \leq C_1 \frac{E[S_{k-1} < 0 < S_k; X_k^2 + X_k]}{\sqrt{n-k} P(S_{k-1} < 0 < S_k) \sqrt{n-k} P(\underline{S}_{n-k})}.$$

We estimate above by dividing it into two cases. In case $P(X > d_0) = 0$ for some $d_0 > 0$

$$0 \leq I_{n,k} \leq (d_0^2 + d_0) [\sqrt{n-k} P(\underline{S}_{n-k})]^{-1} / \sqrt{n-k}$$

Hence we have the assertion of the lemma by Lemma 8. In case $P(X > d) > 0$ for all $d > 0$, choose $d > 0$ large enough. Then we have by Lemma 3

$$(9) \quad \begin{aligned} P(S_{k-1} < 0 < S_k) &\geq P(-d < S_{k-1} < 0, X_k > d) = P(X > d) P(-d < S_{k-1} < 0) \\ &\geq B / \sqrt{k-1}. \end{aligned}$$

and then

$$\overline{\lim}_{n \rightarrow \infty} \max\{I_{n,k}; 0 < k < \delta n\} \leq (\text{posi. const.}) \sqrt{\delta / (1-\delta)} \rightarrow 0 \quad (\delta \rightarrow 0).$$

This completes the proof.

For $0 < k < n$ and $M > 0$, let

$$J_{n,k,M} := \begin{cases} \frac{P(S_{k-1} < 0, S_k > M \text{ and } \underline{k}S_n > 0)}{P(S_{k-1} < 0 < S_k \text{ and } \underline{k}S_n > 0)} & \text{if the denominator } \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

LEMMA 14. *Under the same condition as Lemma 13, for each $\delta \in (0, 1)$*

$$\lim_{M \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \max\{J_{n,k,M}; 0 < k < \delta n\} = 0.$$

PROOF. By a similar estimate as in (8), we have

$$(10) \quad J_{n,k,M} \leq \frac{E[S_{k-1} < 0, S_k \geq M; P(\underline{S}_{n-k} / \sqrt{n-k} > -x / \sqrt{n-k})_{|x=S_k}]}{P(S_{k-1} < 0 < S_k)P(A_{n-k})} \\ \leq C_1 \frac{E[S_{k-1} < 0, S_k \geq M; X_k + 1]}{P(S_{k-1} < 0 < S_k)\sqrt{n-k}P(A_{n-k})}$$

We estimate above by dividing it into two cases.

In case $P(X > d_0) = 0$ for some $d_0 > 0$, $J_{n,k,M} = 0$ for all n, k ($0 < k < n$) and $M > d_0$ by the definition. This implies the assertion of the lemma.

In case $P(X > d) > 0$ for all $d > 0$, choose $d > 0$, large enough so that we have (9) in the proof of Lemma 13. Let us estimate the numerator. For $\nu = 0$ or 1

$$E[S_{k-1} < 0, X_k > M - S_{k-1}; X_k^\nu] \leq \sum_{i=0}^{\infty} E[-i-1 \leq S_{k-1} < -i, X_k \geq M+i; X_k^\nu] \\ = \sum_{i=0}^{\infty} P(-i-1 \leq S_{k-1} < -i) E[X \geq M+i; X^\nu] \leq A_1 E\left[\sum_{i=0}^{\infty} \chi_{[M+i, \infty)}(X_i) X_i^\nu\right] / \sqrt{k-1},$$

In the last estimate we used Lemma 2. Hence we have

$$(11) \quad J_{n,k,M} \leq A_1 E\left[\sum_{i=0}^{\infty} \chi_{[M+i, \infty)}(X_i) X_i^\nu\right] / [\sqrt{k} P(S_{k-1} < 0 < S_k) \sqrt{n-k} P(A_{n-k})].$$

Note that for $\nu = 0$ or 1, and $M \geq 0$

$$0 \leq \sum_{i=0}^{\infty} \chi_{[M+i, \infty)}(X_i) X_i^\nu \leq \sum_{i=0}^{\infty} \chi_{[i, \infty)}(X_i) X_i^\nu < \infty$$

almost surely, because

$$E\left[\sum_{i=0}^{\infty} \chi_{[i, \infty)}(X_i) X_i^\nu\right] = \sum_{i=0}^{\infty} E[X > i; X^\nu] \leq K \sum_{i=2}^{\infty} [i(\log i)^\alpha]^{-1} < \infty,$$

where K is a positive constant. The last estimate follows from the condition that (II) holds for $\alpha > 1$, because for $\nu = 0$ or 1, and $i \geq 2$

$$\infty > E[X > 1; X^2(\log X)^\alpha] \geq E[X > i; X^\nu X^{2-\nu}(\log X)^\alpha] \geq i(\log i)^\alpha E[X > i; X^\nu],$$

that is, $E[X > i-2; X^\nu] \leq K[i(\log i)^\alpha]^{-1}$. Now applying the dominated convergence

theorem twice, we have for $\nu=0$ or 1

$$(12) \quad \lim_{M \rightarrow \infty} E \left[\sum_{i=0}^{\infty} \chi_{[M+i, \infty)}(X_i) X_i^\nu \right] = 0.$$

Combining (9), (11), (12) and Lemma 8, we have the assertion of the lemma. This completes the proof.

§ 4. Proof of Theorem

1. Our idea of proof is suggested by the method of identification of the limit process given in [3] (refer also to [10]). It really works when we use a limit theorem for the fluctuation of a random walk given in [4]. Let $E_i := \{S_i = 0\} \cup \{S_{i-1} < 0 < S_i\} \cup \{S_{i-1} < 0 < S_i\}$ the event of zero crossing at i , and $T_n(\omega) := \max\{0 \leq k \leq n; \omega \in E_k\}$ the last zero crossing step up to the n -th step. Let B_t ($0 \leq t \leq 1, B_0 = 0$) be the standard Brownian motion, and $T := \sup\{t \in [0, 1]; B_t = 0\}$. Then we have

PROPOSITION 15. *If (I) holds, then for each $\delta \in (0, 1)$ and $\phi \in C_b([0, \infty))$*

$$(1) \quad \lim_{n \rightarrow \infty} E \left[\phi \left(\frac{1}{\sqrt{1 - T_n/n}} \frac{S_n}{\sigma \sqrt{n}} \right) \mid 0 < T_n/n < \delta, S_n > 0 \right] = \int_0^\infty \phi(x) d(1 - e^{-(x^2/2)}).$$

PROOF. Since by 11 in [4] $(T_n/n, S_n/\sigma\sqrt{n}) \xrightarrow{d} (T, B_1)$ and $P(T \in dt, B_1 \in dx) = tg(t, x) dt dx$, where for $0 < t < 1$ and $-\infty < x < \infty$

$$g(t, x) = (2\pi)^{-1} x [t(1-t)]^{-3/2} \exp(-x^2/[2(1-t)])$$

we have

$$\begin{aligned} \text{the left hand side of (1)} &= \lim_{n \rightarrow \infty} \left(E \left[0 < T_n/n < \delta, S_n > 0; \right. \right. \\ &\left. \left. \phi \left(\frac{1}{\sqrt{1 - T_n/n}} \frac{S_n}{\sigma \sqrt{n}} \right) \right] / P(0 < T_n/n < \delta, S_n > 0) \right) = E \left[0 < T < \delta, B_1 > 0; \right. \\ &\left. \phi \left(\frac{B_1}{\sqrt{1 - T}} \right) \right] / P(0 < T < \delta, B_1 > 0) = \int_0^\infty \phi(x) d(1 - e^{-(x^2/2)}). \end{aligned}$$

This completes the proof.

2. Again we take $\sigma=1$ for simplicity. Let us rewrite the left hand side of (1) for $\phi \in C_b^1([0, \infty))$. By a simple consideration on the last zero crossing step T_n , we have

$$(2) \quad \begin{aligned} E \left[0 < T_n/n < \delta, S_n > 0; \phi \left(\frac{1}{\sqrt{1 - T_n/n}} \frac{S_n}{\sqrt{n}} \right) \right] &= \sum_{0 < k < \delta n} E \left[S_k = 0, k S_n > 0; \right. \\ &\left. \phi \left(\frac{S_n}{\sqrt{n - k}} \right) \right] + \sum_{0 < k < \delta n} E \left[S_{k-1} < 0 < S_k, k S_n > 0; \phi \left(\frac{S_n}{\sqrt{n - k}} \right) \right] =: I + J. \end{aligned}$$

By the stationality and the independence of increments of a random walk, we have

$$I = \sum_{0 < k < \delta n} E \left[S_k = 0; E \left[\underline{S}_{n-k} > 0; \phi \left(\frac{S_n}{\sqrt{n-k}} \right) \right] \right] = \sum_{0 < k < \delta n} P(S_k = 0) P(\underline{S}_{n-k} > 0) \\ \times E \left[\phi \left(\frac{S_{n-k}}{\sqrt{n-k}} \right) | \underline{S}_{n-k} > 0 \right] = \sum_{0 < k < \delta n} P(S_k = 0, \underline{S}_{n-k} > 0) f_{n-k},$$

where $f_n := E[\phi(S_n/\sqrt{n}) | A_n]$. Similarly we have

$$J = \sum_{0 < k < \delta n} E \left[S_{k-1} < 0 < S_k; E \left[x + \underline{S}_{n-k} > 0; \phi \left(\frac{x + S_{n-k}}{\sqrt{n-k}} \right) \right]_{|x=S_k} \right].$$

Using the Taylor expansion of

$$\phi \left(\frac{x + S_{n-k}}{\sqrt{n-k}} \right) = \phi \left(\frac{S_{n-k}}{\sqrt{n-k}} \right) + \phi' \left(\frac{S_{n-k} + \theta x}{\sqrt{n-k}} \right) \frac{x}{\sqrt{n-k}}$$

where $0 < \theta < 1$, we have

$$(3) \quad J = \sum_{0 < k < \delta n} E \left[S_{k-1} < 0 < S_k \leq M; E \left[x + \underline{S}_{n-k} > 0; \phi \left(\frac{S_{n-k}}{\sqrt{n-k}} \right) \right]_{|x=S_k} \right] \\ + \sum_{0 < k < \delta n} E \left[S_{k-1} < 0, S_k > M; E \left[x + \underline{S}_{n-k} > 0; \phi \left(\frac{S_{n-k}}{\sqrt{n-k}} \right) \right]_{|x=S_k} \right] + R. T. \\ =: J_1 + R^{(1)} + R^{(2)},$$

where M is a large positive number which will be suitably chosen later.

Let us rewrite J_1 . Since for $x > 0$

$$E[x + \underline{S}_n > 0; \phi(S_n/\sqrt{n})] = \sum_{k=0}^{n-1} E \left[M_k = k, -x < S_k \leq 0; \right. \\ \left. E \left[\underline{S}_{n-k} > 0; \phi \left(\frac{y + S_{n-k}}{\sqrt{n}} \right) \right]_{|y=S_k} \right] + E[M_n = n, -x < S_n \leq 0; \phi(S_n/\sqrt{n})],$$

and since

$$\phi \left(\frac{y + S_{n-k}}{\sqrt{n}} \right) = \phi \left(\frac{S_{n-k}}{\sqrt{n-k}} \right) + \phi' \left(\frac{S_{n-k}}{\sqrt{n-k}} + \theta \left[\left(\sqrt{\frac{n-k}{n}} - 1 \right) \frac{S_{n-k}}{\sqrt{n-k}} \right. \right. \\ \left. \left. + \frac{y}{\sqrt{n}} \right] \right) \left[\left(\sqrt{\frac{n-k}{n}} - 1 \right) \frac{S_{n-k}}{\sqrt{n-k}} + \frac{y}{\sqrt{n}} \right]$$

for some $\theta \in (0, 1)$, we have

$$(4) \quad E[x + \underline{S}_n > 0; \phi(S_n/\sqrt{n})] = \sum_{k=0}^{n-1} E \left[M_k = k, -x < S_k \leq 0; E \left[\underline{S}_{n-k} > 0; \phi \left(\frac{S_{n-k}}{\sqrt{n-k}} \right) \right] \right] \\ + E[M_n = n, -x < S_n \leq 0; \phi(S_n/\sqrt{n})] + \left(\sum_{0 \leq k < \delta n} + \sum_{\delta n \leq k < n} \right) \left(\sqrt{\frac{n-k}{n}} - 1 \right) E \left[M_k = k,$$

$$\begin{aligned}
 & -x < S_k \leq 0; E \left[\mathbb{1}_{S_{n-k} > 0; \phi'(*) \frac{S_{n-k}}{\sqrt{n-k}}} \right]_{|y=S_k} + \sum_{k=0}^{n-1} E[M_k = k, -x < S_k \leq 0; \\
 & (S_k/\sqrt{n})E[\mathbb{1}_{S_{n-k} > 0; \phi'(*)}]_{|y=S_k}] = : \sum_{k=0}^{n-1} P(M_n = k, -x < S_k \leq 0) f_{n-k} \\
 & + E[M_n = n, -x < S_n \leq 0; \phi(S_n/\sqrt{n})] + R_{\delta n}^{(3,1)} + R_{\delta n}^{(3,2)} + R_n^{(3,3)},
 \end{aligned}$$

where

$$\phi'(*) = \phi' \left(\frac{S_{n-k}}{\sqrt{n-k}} + \theta \left[\left(\sqrt{\frac{n-k}{n}} - 1 \right) \frac{S_{n-k}}{\sqrt{n-k}} + \frac{y}{\sqrt{n}} \right] \right).$$

Using (4), we have

$$\begin{aligned}
 J_1 &= \sum_{0 < k < \delta n} E[S_{k-1} < 0 < S_k \leq M; \sum_{j=0}^{n-k-1} P(M_{n-k} = j, -x < S_j \leq 0)_{|x=S_k} f_{n-k-j}] \\
 &+ R.T. = \sum_{0 < k < \delta n} E[S_{k-1} < 0 < S_k \leq M; \sum_{0 \leq j < \delta(n-k)} P(M_{n-k} = j, -x < S_j \leq 0)_{|x=S_k} f_{n-k-j}] \\
 &+ (R.T.)' = : J_2 + R^{(4)},
 \end{aligned}$$

where $R^{(4)}$ is given as

$$\begin{aligned}
 R^{(4)} &= \sum_{0 < k < \delta n} E[S_{k-1} < 0 < S_k \leq M; \sum_{\delta(n-k) \leq j < n-k} P(M_{n-k} = j, -x < S_j \leq 0)_{|x=S_k} f_{n-k-j}] \\
 &+ \sum_{0 < k < \delta n} E \left[S_{k-1} < 0 < S_k \leq M; E \left[M_{n-k} = n-k, -x < S_{n-k} \leq 0; \phi \left(\frac{S_{n-k}}{\sqrt{n-k}} \right) \right]_{|x=S_k} \right] \\
 &+ \sum_{0 < k < \delta n} E[S_{k-1} < 0 < S_k \leq M; R_{\delta(n-k)}^{(3,1)}] + \sum_{0 < k < \delta n} E[S_{k-1} < 0 < S_k \leq M; R_{\delta(n-k)}^{(3,2)}] \\
 &+ \sum_{0 < k < \delta n} E[S_{k-1} < 0 < S_k \leq M; R_n^{(3,3)}] = : \sum_{i=1}^5 R^{(4,i)}.
 \end{aligned}$$

Summarizing these calculations, we have

$$(5) \quad E \left[0 < T_n/n < \delta, S_n > 0; \phi \left(\frac{1}{\sqrt{1-T_n/n}} \frac{S_n}{\sqrt{n}} \right) \right] = I + J_2 + R^{(1)} + R^{(2)} + \sum_{i=1}^5 R^{(4,i)}.$$

3. In this paragraph we estimate the remainder terms. To simplify the notations we may assume $\|\phi\| \leq 1$ and $\|\phi'\| \leq 1$. For $R^{(1)}$

$$(6) \quad |R^{(1)}| \leq \sum_{0 < k < \delta n} P(S_{k-1} < 0, S_k > M \text{ and } \underline{k} S_n > 0) \leq P(0 < \exists k < \delta n \text{ such} \\
 \text{that } S_{k-1} < 0 < S_k \text{ and } \underline{k} S_n > 0) \max\{J_{n,k,M}; 0 < k < \delta n\},$$

where $J_{n,k,M}$ is the one in Lemma 14. In the above estimate we used the following wellknown inequality; for real numbers a_1, \dots, a_n and positive numbers b_1, \dots, b_n

$$\frac{a_1 + \dots + a_n}{b_1 + \dots + b_n} \leq \max\{a_k/b_k; 1 \leq k \leq n\}.$$

Similarly for $R^{(2)}$

$$(7) \quad |R^{(2)}| \leq \sum_{0 < k < \delta n} E \left[S_{k-1} < 0 < S_k; \frac{S_k}{\sqrt{n-k}} P(x + \underline{S}_{n-k} > 0) \Big|_{x=S_k} \right] \leq P(0 < \exists k < \delta n) \\ \text{such that } S_{k-1} < 0 < S_k \text{ and } \underline{k} \underline{S}_n > 0) \max\{I_{n,k}; 0 < k < \delta n\},$$

where $I_{n,k}$ is the one in Lemma 13. For $R_{\delta n}^{(3,1)}$

$$(8) \quad |R_{\delta n}^{(3,1)}| \leq \delta \sum_{0 < k < \delta n} P(M_k = k, -x < S_k \leq 0) E \left[\underline{S}_{n-k} > 0; \frac{S_{n-k}}{\sqrt{n-k}} \right] \\ = \delta \sum_{0 < k < \delta n} P(M_n = k, -x < S_k \leq 0) E \left[\frac{S_{n-k}}{\sqrt{n-k}} \Big| \underline{S}_{n-k} > 0 \right],$$

for $R_{\delta n}^{(3,2)}$

$$(9) \quad |R_{\delta n}^{(3,2)}| \leq \sum_{\delta n \leq k < n} P(M_k = k, -x < S_k \leq 0) E \left[\underline{S}_{n-k} > 0; \frac{S_{n-k}}{\sqrt{n-k}} \right] \\ = \sum_{\delta n \leq k < n} P(M_n = k, -x < S_k \leq 0) E \left[\frac{S_{n-k}}{\sqrt{n-k}} \Big| \underline{S}_{n-k} > 0 \right],$$

and for $R_n^{(3,3)}$

$$(10) \quad |R_n^{(3,3)}| \leq (x/\sqrt{n}) \sum_{k=1}^{n-1} P(M_n = k, -x < S_k \leq 0) \leq (x/\sqrt{n}) P(x + \underline{S}_n > 0).$$

$$|R^{(4,1)}| + |R^{(4,2)}| \leq \sum_{0 < k < \delta n} E \left[S_{k-1} < 0 < S_k \leq M; \left(\sum_{j=0}^{n-k} P(M_{n-k} = j, -x < S_j \leq 0) \Big|_{x=S_k} \right) \right. \\ \left. < S_j \leq 0 \Big|_{x=S_k} \right] \left(\frac{\sum_{\delta(n-k) \leq j \leq n-k} P(M_{n-k} = j, -x < S_j \leq 0) \Big|_{x=S_k}}{\sum_{j=0}^{n-k} P(M_{n-k} = j, -x < S_j \leq 0) \Big|_{x=S_k}} \right),$$

Since $P(M_n = 0) = P(\underline{S}_n > 0) \sim e^\alpha (\pi n)^{-1/2}$ by Lemma 8, we have from Proposition 9

$$(11) \quad \sup \left\{ \frac{\sum_{\delta n \leq k \leq n} P(M_n = k, -x < S_k \leq 0)}{\sum_{0 \leq k \leq n} P(M_n = k, -x < S_k \leq 0)}; 0 < x \leq M \right\} \leq \sup \left\{ \sum_{\delta n \leq k \leq n} P(M_n = k, \right. \\ \left. -x < S_k \leq 0) / P(M_n = 0); 0 < x \leq M \right\} \leq K_4 C(M) \delta^{-3/2} n^{-1/2},$$

where K_4 is a positive constant independent of δ , n and M . Hence we have

$$(12) \quad |R^{(4,1)}| + |R^{(4,2)}| \leq K_4 \delta^{-3/2} (1-\delta)^{-1/2} C(M) n^{-1/2} P(0 < \exists k < \delta n) \\ \text{such that } S_{k-1} < 0 < S_k \leq M \text{ and } \underline{k} \underline{S}_n > 0).$$

In order to estimate $R^{(4,3)}$ to $R^{(4,5)}$ we use (8), (9) and (10). For $R^{(4,3)}$

$$(13) \quad |R^{(4,3)}| \leq \sum_{0 < k < \delta n} E [S_{k-1} < 0 < S_k \leq M; K_5 \delta \sum_{0 < j < \delta(n-k)} P(M_{n-k} = j, -x$$

$$\langle S_j \leq 0 \rangle_{|x=S_k} \leq K_5 \delta P(0 < \exists k < \delta n \text{ such that } S_{k-1} < 0 < S_k \leq M \text{ and } \underline{k} S_n > 0),$$

where $K_5 = \sup\{E[S_n/\sqrt{n}|A_n]; n \geq 1\} < \infty$ by Lemma 11. For $R^{(4,4)}$ applying (11) as in the estimate of $|R^{(4,1)}| + |R^{(4,2)}|$, we have

$$(14) \quad |R^{(4,4)}| \leq \sum_{0 < k < \delta n} E[S_{k-1} < 0 < S_k \leq M; K_5 \sum_{\delta(n-k) \leq j \leq n-k} P(M_{n-k} = j, -x < S_j \leq 0)_{|x=S_k}] \leq K_4 K_5 \delta^{-3/2} (1-\delta)^{-1/2} C(M) n^{-1/2} P(0 < \exists k < \delta n \text{ such that } S_{k-1} < 0 < S_k \leq M \text{ and } \underline{k} S_n > 0)$$

For $R^{(4,5)}$

$$(15) \quad |R^{(4,5)}| \leq \sum_{0 < k < \delta n} E \left[S_{k-1} < 0 < S_k \leq M; \frac{S_k}{\sqrt{n-k}} P(x + \underline{S}_{n-k} > 0)_{|x=S_k} \right] \leq M [(1-\delta)n]^{-1/2} P(0 < \exists k < \delta n \text{ such that } S_{k-1} < 0 < S_k \text{ and } \underline{k} S_n > 0).$$

4. By Lemma 12, $f_{n-k} \cong f_n \pm \varepsilon$ for all $n \geq n_\varepsilon$ and $k \leq 2\delta n$, $2\delta \leq \delta_\varepsilon$. Here and hereafter we take the compound notations \pm, \cong in signs and inequalities in order. Then we have

$$(16) \quad I \cong (f_n \pm \varepsilon) P(0 < \exists k < \delta n \text{ such that } S_k = 0 \text{ and } \underline{k} S_n > 0),$$

and

$$(17) \quad J_2 \cong (f_n \pm \varepsilon) \sum_{0 < k < \delta n} E[S_{k-1} < 0 < S_k \leq M; \sum_{0 \leq j < \delta(n-k)} P(M_{n-k} = j, -x < S_j \leq 0)_{|x=S_k}] = (f_n \pm \varepsilon) (P(0 < \exists k < \delta n \text{ such that } S_{k-1} < 0 < S_k \text{ and } \underline{k} S_n > 0) - P(0 < \exists k < \delta n \text{ such that } S_{k-1} < 0, S_k > M \text{ and } \underline{k} S_n > 0) - \sum_{0 < k < \delta n} E \left[S_{k-1} < 0 < S_k \leq M; \left(\sum_{0 \leq j \leq n-k} P(M_{n-k} = j, -x < S_j \leq 0)_{|x=S_k} \right) \frac{\left(\sum_{\delta(n-k) \leq j \leq n-k} P(M_{n-k} = j, -x < S_j \leq 0)_{|x=S_k} \right)}{\sum_{0 \leq j \leq n-k} P(M_{n-k} = j, -x < S_j \leq 0)} \right]) = (f_n \pm \varepsilon) (j_1 - j_2 - j_3).$$

Combining (5), (16) and (17), we have

$$(18) \quad E \left[\phi \left(\frac{1}{\sqrt{1-T_n/n}} \frac{S_n}{\sqrt{n}} \right) | 0 < T_n/n < \delta, S_n > 0 \right] \cong f_n \pm \varepsilon + [-(f_n \pm \varepsilon) (j_2 + j_3) + R^{(1)} + R^{(2)} + \sum_{i=1}^5 R^{(4,i)}] / P(0 < T_n/n < \delta, S_n > 0).$$

By (6) and (17) $0 \leq \{|R^{(1)}|, j_2\} / P(0 < T_n/n < \delta, S_n > 0) \leq \max\{j_{n,k,M}; 0 < k < \delta n\}$, so that we have by Lemma 14

$$(19) \quad \lim_{M \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \{|R^{(1)}|, j_2\} / P(0 < T_n/n < \delta, S_n > 0) = 0.$$

By (7) and Lemma 13

$$(20) \quad \lim_{\delta \rightarrow +0} \overline{\lim}_{n \rightarrow \infty} |R^{(2)}|/P(0 < T_n/n < \delta, S_n > 0) \leq \lim_{\delta \rightarrow +0} \overline{\lim}_{n \rightarrow \infty} \max\{I_{n,k}; 0 < k < \delta n\} = 0.$$

By (12), (14) and (17) we have for each fixed $\delta \in (0, 1)$ and $M > 0$

$$(21) \quad 0 \leq \{|R^{(4,1)}| + |R^{(4,2)}|, |R^{(4,4)}|, j_3\}/P(0 < T_n/n < \delta, S_n > 0) \leq K/\sqrt{n} \rightarrow 0$$

($n \rightarrow \infty$), where K is a positive constant. By (13) $|R^{(4,3)}|/P(0 < T_n/n < \delta, S_n > 0) \leq K_\varepsilon \delta$, then

$$(22) \quad \lim_{\delta \rightarrow +0} \overline{\lim}_{n \rightarrow \infty} |R^{(4,3)}|/P(0 < T_n/n < \delta, S_n > 0) = 0.$$

By (15)

$$(23) \quad \lim_{n \rightarrow \infty} |R^{(4,5)}|/P(0 < T_n/n < \delta, S_n > 0) = 0.$$

Taking $\varepsilon > 0$ arbitrary small, we have from (1), (18) to (23)

$$(24) \quad \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int_0^\infty \phi(x) dP(S_n/\sqrt{n} \leq x | S_n > 0) = \int_0^\infty \phi(x) d(1 - e^{-(x^2/2)})$$

for $\phi \in C_b^1([0, \infty))$.

For the final step of the proof, we show that (24) holds for every $\phi(x) = \chi_{[a,b)}(x)$ $0 \leq a < b < \infty$. We can do this, approximating $\chi_{[a,b)}(x)$ by two sequences of functions in $C_b^1([0, \infty))$ from above and below. Hence we have $P(S_n/\sqrt{n} \leq x | S_n > 0) \xrightarrow{d} 1 - \exp(-x^2/2)$. This completes the proof.

Added in proof. After this paper was received the author was informed by Professor F. Spitzer that our Théorem had been proved under only condition (I) by Erwin Bolthausen: *On a functional central limit theorem for random walks conditioned to stay positive*, Ann. Prob. 4 (1976), 480-485.

Our proof is different from that of Bolthausen, and is based on the fact that random walk excursion after the last zero crossing step can be approximated by the conditional random walk (see for the detail, § 4).

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