TSUKUBA J. MATH.
Vol. 3, No. 2 (1979), 59–80

CONSTRUCTIONS OF MODULAR FORMS BY MEANS OF TRANSFORMATION FORMULAS FOR THETA SERIES

By

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Let F be a positive integral symmetric matrix of degree m, and Z a variable on the Siegel space H_n of degree n. Let Φ be a spherical function of order ν with respect to F which is of the form

$$\Phi(G)=\begin{cases} 1 & (\nu=0) \\ |{}^tGF^{1/2}\eta|^{\nu} & (\nu>0) \end{cases} \qquad \text{for } m\times n \text{ complex matrices } G$$

with an $m\times n$ matrix η such that ${}^t\eta\eta=0$ if $\nu>1$.

We define a theta series associated with F by setting

$$\theta_{F,U,V}(Z;\Phi)=\sum_G \Phi(G+V)\exp(\mathrm{tr}(Z{}^t(G+V)F(G+V)+2{}^t(G+V)U)),$$

where U, V are $m\times n$ real matrices, tr denotes the trace of a corresponding square matrix and G runs through all $m\times n$ integral matrices. We write simply $\theta_{F,U,V}(Z)$ for the theta series $\theta_{F,U,V}(Z;\Phi)$ when Φ is of order 0.

For congruence subgroups of $SL_2(\boldsymbol{Z})$ the transformation formulas for theta series of degree 1 associated with F are well known. For example, we can find transformation formulas for theta series of degree 1 in [7], [8], in which multipliers are explicitly determined. Transformation formulas for the theta series $\theta_{F,U,V}(Z;\Phi)$ of degree $n\geq 1$ are also established in [1] in the case where F is even and U, V are zero (the condition on U, V is not necessary if Φ is of order 0 [9]). Using these results we can get many examples of Siegel modular forms for congruence subgroups.

In this paper we determine a transformation formula for the theta series $\theta_{F,U,V}(Z;\Phi)$ associated with a positive integral symmetric matrix F and any real matrices U, V and using this, we get some examples of cusp forms for some congruence subgroups Γ' of $Sp_n(\boldsymbol{Z})$. Cusp forms of weight $n+1$ for Γ' induce differential forms of the first kind on the nonsingular model of the modular function field with respect to Γ'. Our result shows that the geometric genus of the nonsingular model of the modular function field with respect to Γ' is positive.

Received November 30, 1978. Revised August 23, 1979.

For example, this is the case where (i) $\Gamma'=\Gamma(4)$ if $n>1$, (ii) $\Gamma'=\Gamma(2N^2)$ for $N>1$ if $n\equiv 0$ (2), (iii) $\Gamma'=Sp_n(\boldsymbol{Z})$ if $n=24$ (cf. H. Maass [5]), (iv) $\Gamma'=\Gamma(N)$ for $N\geq 2$ if $n\equiv 0$ (8), (v) $\Gamma'=\Gamma(2,4)$ or $\Gamma(N^2)$ for $N>1$ if $n\equiv 7$ (8).

Notation.

We denote by $\boldsymbol{Z}_+, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{C}$, the set of all positive rational integers, the ring of rational integers, the rational number field, the real number field and the complex number field. Let K be a subset of $\boldsymbol{C}$. We denote by $M_{m,n}(K)$ the set of all $m\times n$ matrices with entries in K; simply K^m denotes $M_{m,1}(K)$ and $SM_m(K)$ denotes the set of all symmetric matrices of degree m with entries in K. We denote by 1_n the identity matrix of degree n. For $X\in M_{m,m}(\boldsymbol{C})$ and $Y\in M_{m,n}(\boldsymbol{C})$, we set $X[Y]={}^tYXY$.

We denote the modular group $Sp_n(\boldsymbol{Z})$ simply by Γ. Γ acts on the Siegel space H_n by the usual modular transformations

$$Z\longmapsto MZ=(AZ+B)(CZ+D)^{-1} \quad \text{for } M=\begin{pmatrix} AB \\ CD \end{pmatrix}\in\Gamma.$$

Let Γ' be a congruence subgroup of Γ, and χ a map of Γ' to $\boldsymbol{C}^*=\{c\in\boldsymbol{C}\,|\,c\neq 0\}$. A holomorphic function f on H_n is called a *modular form* of weight $k\left(\in\frac{1}{2}\boldsymbol{Z}_+\right)$ for Γ' with a multiplier χ if f satisfies $f(MZ)=\chi(M)|CZ+D|^k f(Z)$ for any $M\in\Gamma'$. Here the factor of automorphy $|CZ+D|^{1/2}$ is always determined by the condition that $-\pi/2<\arg(|\sqrt{-1}C+D|^{1/2})\leq\pi/2$ and $|CZ+D|^k$ is determined by $|CZ+D|^k=(|CZ+D|^{1/2})^{2k}$. Such f is called a *cusp form* of weight k for Γ' with a multiplier χ if in the Fourier expansion

$$|CZ+D|^{-k}f(MZ)=\sum_S a(S)\varepsilon(\mathrm{tr}(ZS)) \quad \text{for all } M\in\Gamma,$$

$a(S)$ vanishes for S with $|S|=0$, where $\varepsilon(*)=\exp(\sqrt{-1}\pi *)$.

We introduce several congruence subgroups of Γ. Let Θ denote the *theta group* $\left\{M=\begin{pmatrix} AB \\ CD \end{pmatrix}\in\Gamma\,|\,({}^tAC)_\Delta\equiv({}^tBD)_\Delta\equiv 0\ (2)\right\}$ where for a square matrix (x_{ij}) of degree n, $(x_{ij})_\Delta$ denotes ${}^t(x_{11},\cdots,x_{nn})$. Let N be a positive integer. Then we set $\Gamma_0(N)=\{M\in\Gamma\,|\,C\equiv 0\ (N)\}$, $\Gamma(N)=\{M\in\Gamma\,|\,A\equiv D\equiv 1_n\ (N),\ B\equiv C\equiv 0\ (N)\}$ and $\Theta_0(N)=\{M\in\Gamma_0(N)\,|\,({}^tBD)_\Delta\equiv 1/N({}^tAC)_\Delta\equiv(B^tA)_\Delta\equiv 1/N(D^tC)_\Delta\equiv 0\ (2)\}$. For two positive integers N_1, N_2 we put $\Gamma_0(N_1, N_2)=\{M\in\Gamma\,|\,B\equiv 0\ (N_1),\ C\equiv 0\ (N_2)\}$. For a positive even integer N we put $\Gamma(N, 2N)=\{M\in\Gamma(N)\,|\,({}^tAC)_\Delta\equiv({}^tBD)_\Delta\equiv 0\ (2N)$, $\Theta_1(N)=\{M\in\Gamma_0(N)\,|\,1/N({}^tAC)_\Delta\equiv 1/N(D^tC)_\Delta\equiv 0\ (2)\}$ and $\Theta_2(N)=\{M\in\Gamma_0(N)\,|\,({}^tBD)_\Delta\equiv(B^tA)_\Delta\equiv 0\ (2)\}$.

We denote by $(-)$ the *generalized Legendre symbol* to which we add the following significance; (i) $\left(\frac{0}{1}\right)=1$ and (ii) if a is an odd integer congruent to 1 mod 4 and b is a positive even integer, then $\left(\frac{a}{b}\right)=\left(\frac{b}{a}\right)$. (cf. [2])

1. Transformation formulas

For u, v, x and $y \in \boldsymbol{C}^n$ we define a theta series by setting

$$\vartheta_{u,v}(Z; x, y)=\sum_{g\equiv v \bmod Z} \varepsilon(Z[g+y]+2^t g(x+u)+{}^t yx),$$

where the summation is taken over all $g \in \boldsymbol{C}^n$ such that $g-v \in \boldsymbol{Z}^n$. From Satz 8 in [10] we get easily the following

LEMMA 1. *Let u, v, x and $y \in \boldsymbol{C}^n$, and $M=\begin{pmatrix} AB \\ CD \end{pmatrix} \in \Gamma$. Setting*

$$u_M={}^tDu+{}^tBv+\frac{1}{2}({}^tBD)_\Delta,\ v_M={}^tCu+{}^tAv+\frac{1}{2}({}^tAC)_\Delta \text{ and}$$

$$E(u, v, M)=\varepsilon(-{}^t({}^tCu+{}^tAv)({}^tDu+{}^tBv+({}^tBD)_\Delta)+{}^tvu),$$

we have

$$\vartheta_{u,v}(MZ; Ax-By, -Cx+Dy)$$
$$=\chi(M)E(u, v, M)|CZ+D|^{1/2}\vartheta_{u_M,v_M}(Z; x, y)$$

where $\chi(M)$ is the 8-th root of 1 depending only on M.

Let F be a positive real symmetric matrix of degree $m>0$. For U, V, X and $Y \in M_{m,n}(\boldsymbol{C})$, we set

$$\theta_{F,U,V}(Z; X, Y)=\sum_{G\equiv V \bmod Z} \varepsilon(\mathrm{tr}(ZF[G+Y]+2^tG(X+U)+{}^tYX),$$

where the summation is taken over all the matrices $G \in M_{m,n}(\boldsymbol{C})$ such that $G-V \in M_{m,n}(\boldsymbol{Z})$.

The idea of the proof of the next theorem is due to A. N. Andrianov and G. N. Maloletkin [1], whose idea is based on the interpretation of the theta series $\theta_{F,U,V}(Z; X, Y)$ of degree n associated with positive quadratic forms F of degree m as specializations of the standard theta series $\vartheta_{u,v}(Z; x, y)$ of degree mn.

For square matrices A and $B=(b_{ij})$ respectively of degree m and n, we define a tensor product by

$$A\otimes B=\begin{pmatrix} Bb_{11}\cdots\cdots Ab_{1n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Ab_{n1}\cdots\cdots Ab_{nn} \end{pmatrix}.$$

Let F be a positive real symmetric matrix of degree m. We define three maps which we shall denote by the same sign $\sim$, in the following way:

$$\sim : H_n \longrightarrow H_{mn} \quad \text{defined by} \quad Z \longmapsto \tilde{Z} = F \otimes Z$$

$$\sim : Sp_n(\boldsymbol{R}) \longrightarrow Sp_{mn}(\boldsymbol{R}) \text{ defined by } M = \begin{pmatrix} AB \\ CD \end{pmatrix} \longmapsto \tilde{M} = \begin{pmatrix} \tilde{A}\tilde{B} \\ \tilde{C}\tilde{D} \end{pmatrix} = \begin{pmatrix} 1_m \otimes A & F \otimes B \\ F^{-1} \otimes C & 1_m \otimes D \end{pmatrix}$$

$$\sim : M_{m,n}(\boldsymbol{C}) \longrightarrow \boldsymbol{C}^{mn} \quad \text{defined by} \quad X = (x_1, \cdots, x_n) \longmapsto \tilde{X} = {}^t({}^t x_1, \cdots, {}^t x_n).$$

Then under the above notation we have $\tilde{M}\tilde{Z} = \widetilde{MZ}$, $|\tilde{C}\tilde{Z} + \tilde{D}| = |CZ + D|^m$, $\tilde{Z}[\tilde{G}] = \mathrm{tr}(ZF[G])$, ${}^t\tilde{A}\tilde{X} = \widetilde{XA}$, ${}^t\tilde{B}\tilde{X} = \widetilde{FBX}$, ${}^t\tilde{G}\tilde{X} = \widetilde{F^{-1}XC}$, ${}^t\tilde{D}\tilde{X} = \widetilde{XD}$, $({}^t\tilde{B}\tilde{D})_\Delta = \widetilde{F_\Delta {}^t({}^tBD)_\Delta}$, $({}^t\tilde{A}\tilde{C})_\Delta = \widetilde{(F^{-1})_\Delta {}^t({}^tAC)_\Delta}$ and ${}^t\tilde{Y}\tilde{X} = \mathrm{tr}({}^tYX)$. If both F and NF^{-1} ($N \in \boldsymbol{Z}_+$) are integral, then we have $\widetilde{\Gamma_0(N)} \subset Sp_n(\boldsymbol{Z})$. Moreover, if both F and NF^{-1} are even, then $\widetilde{\Gamma_0(N)}$ is contained in the theta group of degree mn.

We obtain $\theta_{F,U,V}(Z; X, Y) = \vartheta_{\tilde{U},\tilde{V}}(\tilde{Z}; \tilde{X}, \tilde{Y})$, and hence by Lemma 1 we get the following

THEOREM 1. *Let F be a positive real symmetric matrix of degree $m > 0$. Let $M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in Sp_n(\boldsymbol{R})$ with $\tilde{M} \in Sp_{mn}(\boldsymbol{Z})$. For $U, V \in M_{m,n}(\boldsymbol{C})$, set*

$$U_M = UD + FVB + \frac{1}{2} F_\Delta {}^t({}^tBD)_\Delta. \quad V_M = F^{-1}UC + VA + \frac{1}{2}(F^{-1})_\Delta {}^t({}^tAC)_\Delta \text{ and}$$

$$E_F(U, V, M) = \varepsilon(\mathrm{tr}(-{}^t(F^{-1}UC + VA)(UD + FVB + F_\Delta {}^t({}^tBD)_\Delta) + {}^tVU).$$

Then we have

$$\begin{aligned} &\theta_{F,U,V}(MZ; X{}^tA - FY{}^tB, -F^{-1}X{}^tC + Y{}^tD) \\ &\quad = \chi_F(M) E_F(U, V, M) |CZ + D|^{m/2} \theta_{F,U_M,V_M}(Z; X, Y) \end{aligned}$$

where $\chi_F(M) = \chi_F^{(n)}(M)$ is the 8-th root of 1 depending only on n, F and M.

Suppose that $m = \deg(F)$ is $\geq n$. Let l be any integer such that $n \leq l \leq m$, and L any subset of $\{1, \cdots, m\}$ with l elements. Put $L = \{j_1, \cdots, j_l\}$ with $j_1 < \cdots < j_l$. We denote by η_L the matrix in $M_{m,l}(Z)$ whose

(i) j-th row $= e_i$ if $j = j_i \in L$

(ii) j-th row $= 0$ if $j \notin L$,

e_i being the i-th row of the identity matrix 1_l of degree l. Take a pair (η, ν) in $M_{l,n}(\boldsymbol{C}) \times \boldsymbol{Z}_+$ which satisfies both of the conditions that (i) ${}^t\eta\eta = 0$ if $\nu > 1$ and that (ii) $\nu = 1$ if $l = n$. For $G \in M_{m,n}(\boldsymbol{C})$ we set $\Phi(G) = |{}^tGF^{1/2}\eta_L\eta|^\nu$. We define a theta series with Φ by setting

$$\theta_{F,U,V}(Z;\Phi;X,Y)=\sum_{G\equiv V \bmod Z}\Phi(G)\varepsilon(\mathrm{tr}(ZF[G+Y]+2{}^tG(X+U)+{}^tYX)),$$

the summation being taken over all the matrices $G\in M_{m,n}(\boldsymbol{C})$ such that $G-V\in M_{m,n}(\boldsymbol{Z})$.

Let $\xi=(\xi_{ij})$ be an $l\times n$ variable matrix and $\partial=\left(\dfrac{\partial}{\partial\xi_{ij}}\right)$ the corresponding matrix of differential operators. We introduce the differential operator $\det^{\nu}({}^t\eta\partial)$. In Lemma 3 of [1], the following equation is proved. For $P\in SM_n(\boldsymbol{C})$ and $Q\in M_{l,n}(\boldsymbol{C})$ and for $c\in\boldsymbol{C}$, we have

$$\begin{aligned}&\det{}^{\nu}({}^t\eta\partial)\ (\mathrm{tr}(P{}^t\xi\xi+2{}^tQ\xi)+c)\\&=|2\sqrt{-1}\pi(P{}^t\xi+{}^tQ)\eta|^{\nu}\varepsilon(\mathrm{tr}(P{}^t\xi\xi+2{}^tQ\xi)+c).\end{aligned}$$

THEOREM 2. *Suppose* $n\leq m=\deg(F)$. *Let* l *be any integer with* $n\leq l\leq m$ *and* L *a subset of* $\{1,\cdots,m\}$ *with* l *elements. Let* $\eta\in M_{l,n}(\boldsymbol{C})$ *and put* $\Phi(G)=|{}^tGF^{1/2}\eta_L\eta|^{\nu}$ $(\nu\in\boldsymbol{Z}_+)$ *for* $G\in M_{m,n}(\boldsymbol{C})$. *Then we have*

$$\begin{aligned}&\theta_{F,U,V}(MZ;\Phi;X{}^tA-FY{}^tB,\ -F^{-1}X{}^tC+Y{}^tD)\\&=\chi_F(M)E_F(U,V,M)|CZ+D|^{(m/2)+\nu}\theta_{F,U_M,V_M}(Z;\Phi;X,Y),\end{aligned}$$

in either case that (i) $\nu>1$, $l>n$ *and* ${}^t\eta\eta=0$, *or that* (ii) $\nu=1$ *and* $l\geq n$, *where* $M=\begin{pmatrix}AB\\CD\end{pmatrix}$ *is as in Theorem* 1 *and* X,Y *are matrices in* $M_{m,n}(\boldsymbol{C})$ *such that* ${}^tXF^{-1/2}\eta_L={}^tYF^{1/2}\eta_L=0$.

Proof. Take an $m\times n$ matrix ξ' such that entries of its i-th rows ($i\in L$) are independent variables and its j-th rows ($j\notin L$) are 0. Then we have ${}^tXF^{-1/2}\xi'={}^tYF^{1/2}\xi'=0$. Setting $\xi={}^t\eta_L\xi'$ and substituting X for $F^{1/2}\xi'+X$ in the formula of Theorem 1, we obtain

$$\begin{aligned}&\sum_{G\equiv V \bmod Z}\varepsilon(\mathrm{tr}(-(CZ+D)^{-1}C{}^t\xi\xi+2(CZ+D)^{-1}{}^tGF^{1/2}\eta_L\xi+MZF[G-F^{-1}X{}^tC+Y{}^tD]\\&\qquad+2{}^tG(U+X{}^tA-FY{}^tB)+{}^t(-F^{-1}X{}^tC+Y{}^tD)\,(X{}^tA-FY{}^tB)))\\&=\chi_F(M)E_F|CZ+D|^{m/2}\sum_{G\equiv V_M \bmod Z}\varepsilon(\mathrm{tr}(2{}^tGF^{1/2}\eta_L\xi+ZF[G+Y]+2{}^tG(U_M+X)+{}^tYX).\end{aligned}$$

Applying the differential operator $\det^{\nu}({}^t\eta\partial)$ at $\xi=0$, we get the desired result.

In the similar way as in the proof of Theorem 2, we get the following corollary.

Let $k\in\boldsymbol{Z}_+$. Let L_i $(1\leq i\leq k)$ be subsets of $\{1,\cdots,m\}$ with $l_i(\geq n)$ elements such that $L_i\cap L_j=\phi$ if $i\neq j$. For $i=1,\cdots,k$ take pairs (η_i,ν_i) in $M_{l_i,n}(\boldsymbol{C})\times\boldsymbol{Z}_+$ which satisfy both conditions that (i) ${}^t\eta_i\eta_i=0$ if $\nu_i>0$ and that (ii) $\nu_i=1$ if $l_i=n$. For

$G\in M_{m,n}(\boldsymbol{C})$ we set $\Phi(G)=|{}^tGF^{1/2}\eta_{L_1}\eta_1|^{\nu_1}\cdots|{}^tGF^{1/2}\eta_{L_k}\eta_k|^{\nu_k}$. We define a theta series with Φ by

$$\theta_{F,U,V}(Z;\Phi;X,Y)=\sum_{G\equiv V \bmod Z}\Phi(G)e(\mathrm{tr}(ZF[G+Y]+2{}^tG(X+U)+{}^tYX)),$$

for U, V, X and $Y\in M_{m,n}(\boldsymbol{C})$.

COROLLARY. Let L_i, η_i, ν_i $(1\leq i\leq k)$ and Φ be stated as above. Then we have

$$\theta_{F,U,V}(MZ;\Phi;X{}^tA-FY{}^tB,\ -F^{-1}X{}^tC+Y{}^tD)$$

$$=\chi_F(M)E_F(U,V,M)|CZ+D|^{(m/2)+\Sigma\nu_i}\theta_{F,U_M,V_M}(Z;\Phi;X,Y),$$

where $M=\begin{pmatrix}AB\\CD\end{pmatrix}$ *is as in Theorem* 1 *and* X, Y *are matrices in* $M_{m,n}(\boldsymbol{C})$ *such that* ${}^tXF^{-1/2}\eta_{L_i}={}^tYF^{1/2}\eta_{L_i}=0$ *for* $i=1,\cdots,k$.

2. Computation of χ_F I

We shall compute χ_F (cf. Theorem 1) in the following four cases (up to ±1 when $\deg(F)$ is odd). Let F be a positive integral symmetric matrix of degree $m>0$. Let N be a positive integer such that NF^{-1} is integral.

① $M\in\Theta_0(N)$.

② F is even. $M\in\Gamma_0(2N)$, or $M\in\Theta_0(N)$, or $M\in\Theta_1(N)$ for an even N.

③ NF^{-1} is even. $M\in\Gamma_0(2,N)$, or $M\in\Theta_0(N)$, or $M\in\Theta_2(N)$ for an even N.

④ Both F and NF^{-1} are even. $M\in\Gamma_0(N)$.

First we must generalize Lemma 5 in [1]. We put

$$P_U=\begin{pmatrix}{}^tU^{-1} & \\ & U\end{pmatrix},\quad Q_S=\begin{pmatrix}1_n & S\\ 0 & 1_n\end{pmatrix},\quad R_S=\begin{pmatrix}1_n & 0\\ S & 1_n\end{pmatrix}$$

with $U\in SL_n(\boldsymbol{Z})$ and $S\in SM_n(\boldsymbol{Z})$.

LEMMA 2. *Let K be the group generated by the elements of $\Gamma_0(N_1,N_2)$ (resp. $\Theta_0(N)$, resp. $\Theta_1(N)$, resp. $\Theta_2(N)$) of the form P_U, Q_S and R_S. Then for any* $M=\begin{pmatrix}AB\\CD\end{pmatrix}\in\Gamma_0(N_1,N_2)$ *(resp. $\Theta_0(N)$, $\Theta_1(N)$, $\Theta_2(N)$), there exist matrices M_1 and $M_2\in K$ such that*

$$M_1MM_2=\left(\begin{array}{cccc|cccc} a & & & & b & & & \\ & 1 & 0 & & & 0 & 0 & \\ & & \ddots & & & & \ddots & \\ & 0 & & 1 & & 0 & & 0 \\ \hline c & & & & d & & & \\ & 0 & 0 & & & 1 & 0 & \\ & & \ddots & & & & \ddots & \\ & 0 & & 0 & & 0 & & 1 \end{array}\right).$$

Moreover $|D|\equiv d \bmod N_1N_2$ (resp. $\bmod N$).

Proof. We treat only the case of $\Theta_0(N)$. Then K is generated by P_U, Q_S and R_T with $U\in SL_n(\boldsymbol{Z})$, even $S\in SM_n(\boldsymbol{Z})$ and $T\in SM_n(N\boldsymbol{Z})$ such that $\frac{1}{N}T$ is even.

We shall prove the assertion by induction on n. When $n=1$, the assertion is trivial. Let us suppose $n>1$. By the elementary divisor theorem there exist $U, V\in SL_n(\boldsymbol{Z})$ such that UDV is diagonal. Hence we may assume $D=\mathrm{diag}(d_1,\cdots,d_n)$.

Step I. We may assume $d_n=1$.

Putting $C=(c_{ij})$ we have $g.c.d\ (c_{n1},\cdots,c_{nn},d_n)=1$. First we assume that d_n is an odd integer. There are even integers $s_1,\cdots,s_n$ such that $s_1c_{n1}+\cdots+s_nc_{nn}=2\ g.c.d\ (c_{n1},\cdots,c_{nn})$. Let us put

$$S=\begin{pmatrix} & s_1 & \\ 0 & \vdots & 0 \\ s_1\cdots\cdots s_{n-1} & & s_n \\ 0 & s_n & 0 \end{pmatrix},\qquad MQ_S=\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \text{ and } D'=(d'_{ij}).$$

Then we have $d'_{n,n-1}=2\ g.c.d(c_{n1},\cdots,c_{nn})$ and $d'_{nn}=d_n+c_{n,n-1}s_n$, and hence $g.c.d\ (d'_{n,n-1},d'_{nn})=1$. Now again by the elementary divisor theorem we may assume that D' is of the form $D'=\mathrm{diag}(d_1',\cdots,d'_{n'},1)$. Secondly we assume that d_n is an even integer. Then for some i, c_{ni} is an odd integer. Take an integer j different from i with $1\leq j\leq n$. There are integers $s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_n$ and an even integer s_j such that $s_1c_{n1}+\cdots+s_nc_{nn}=g.c.d\ (c_{n1},\cdots,c_{nn})$. Let us put

$$S=\begin{pmatrix} & s_1 & \\ 0 & \vdots & 0 \\ s_1\cdots\cdots & s_j & \cdots\cdots s_n \\ 0 & \vdots & 0 \\ & s_n & \end{pmatrix},\qquad MQ_S=\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \text{ and } D'=(d'_{ij}).$$

Then we have $d'_{nj}=g.c.d\ (c_{n1},\cdots,c_{nn})$, $d'_{nn}=d_n+c_{nj}s_n$ and hence $g.c.d\ (d'_{nj},d'_{nn})=1$. Again by the elementary divisor theorem we may assume that D' is of the form $D'=\mathrm{diag}(d_1',\cdots,d'_{n-1},1)$.

Step II. The assertion is true.

Let us put $Q_SMR_T=\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$. Then since $D=\mathrm{diag}\ (d_1,\cdots,d_{n-1},1)$, we can now select Q_S and R_T such that the last row of C and the last column of B are zero. The symplectic condition yields that A', B' and C' have the form

$$A'=\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix},\quad B'=\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix},\quad C'=\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

By the induction hypothesis this proves the lemma.

In the case of $\Gamma_0(N_1, N_2)$, $\Theta_1(N)$ and $\Theta_2(N)$ the similar proof is applivable.

Applying Theorem 1 to the case ①, ②, ③ and ④ with $U=V=X=Y=0$, we have

$$\theta_{F,0,0}(MZ)=\chi_F^{(n)}(M)|CZ+D|^{m/2}\theta_{F,0,0}(Z).$$

Hence $\chi_F^{(n)}$ is a character if m is even. Let us denote by $\chi_F^{(n)}/\{\pm 1\}$ the composition map of $\chi_F^{(n)}$ and the quatient map: $\boldsymbol{C}^* \longrightarrow \boldsymbol{C}^*/\{\pm 1\}$. $\chi_F^{(n)}/\{\pm 1\}$ is a homomorphism whether m is even or odd. As we shall see in the next section, $\chi_F^{(n)}$ (resp. $\chi_F^{(n)}/\{\pm 1\}$) is trivial on K (see Lemma 2 for the notation) if m is even (resp. odd).

Assume that $M=\begin{pmatrix} AB \\ CD \end{pmatrix}$ satisfies at least one of the four conditions ①, ②, ③ and ④, and $\begin{pmatrix} ab \\ cd \end{pmatrix}$ is the matrix in $SL_2(\boldsymbol{Z})$ corresponding to M in Lemma 2. Then using Siegel's Φ-operator we obtain

$$\chi_F^{(n)}(M)=\chi_F^{(1)}\begin{pmatrix} ab \\ cd \end{pmatrix}=\mathrm{sgn}(d)^{m/2}\left(\frac{(-1)^{m/2}|F|}{d}\right) \text{ if } m \text{ is even,}$$

and

$$\chi_F^{(n)}(M)=\pm\varepsilon\left(\frac{d-1}{4}\right) \text{ if } m \text{ is odd.}$$

(see also Appendix).

Through easy calculation we get the following

THEOREM 3. *Let F be a positive integral symmetric matrix of degree m, and N a positive integer such that NF^{-1} is integral. Put $|F|=2^sK$ with g.c.d $(2, K)=1$.*

(1) *In any one of the following four cases, we have for any even positive integer m*

$$\chi_F^{(n)}(M)=\mathrm{sgn}(|D|)^{m/2}\left(\frac{(-1)^{m/2}|F|}{abs(D)}\right).$$

① $8|N$ *and* $M\in\Theta_0(N)$, $4|N$ *and* $M\in\Theta_0(2N)$, $2|N$ *and* $M\in\Gamma_0(2, 2N)$, $2|s$ *and* $4|N$ *and* $M\in\Theta_0(N)$, $2|s$ *and* $2|N$ *and* $M\in\Theta_0(2N)$, *or* $2|s$ *and* $M\in\Gamma_0(2, 2N)$,

② (*F is even.*) $8|N$ *and* $M\in\Theta_1(N)$, $4|N$ *and* $M\in\Theta_0(2N)$, $2|s$ *and* $4|N$ *and* $M\in\Theta_1(N)$, $2|s$ *and* $2|N$ *and* $M\in\Theta_1(2N)$, *or* $M\in\Gamma_0(2N)$,

③ (NF^{-1} is even.) $8|N$ *and* $M\in\Theta_2(N)$, $2|s$ *and* $4|N$ *and* $M\in\Theta_2(N)$, *or* $M\in\Gamma_0(2, N)$

④ (*Both F and NF^{-1} are even.*) $M\in\Gamma_0(N)$ *with* $N>1$.

In case ④ with $N=1$ we have $\chi_F^{(n)}(M)=1$ for all M.

(2) *In any one of the following four cases, we have for any odd integer m*

$$\chi_F^{(n)}(M)=\pm\varepsilon\left(\frac{d-1}{4}\right).$$

① $4|N$ *and* $M\in\Theta_0(N)$, $2|N$ *and* $M\in\Theta_0(2N)$, *or* $M\in\Gamma_0(2, 2N)$,
② $4|N$ *and* $M\in\Theta_1(N)$, *or* $2|N$ *and* $M\in\Theta_1(2N)$,
③ $4|N$ *and* $M\in\Theta_2(N)$, *or* $2|N$ *and* $M\in\Gamma_0(2, N)$,
④ $M\in\Gamma_0(N)$.

REMARK. For even m the case ④ with $N=1$ is investigated in [11].

COROLLARY. *Let F and N be as in Theorem* 3. *Then we have*

$$\chi_F^{(n)}(M)=\mathrm{sgn}(|D|)^{m/2}\left(\frac{(-1)^{m/2}|F|}{abs(D)}\right)\quad \textit{if } m=\deg(F) \textit{ is even,}$$

$$\chi_F^{(n)}(M)=\pm\varepsilon\left(\frac{|D|-1}{4}\right)\quad \textit{if } m \textit{ is odd,}$$

in the following four cases ① $M\in\Gamma_0(2, 2N)$, ② (F *is even.*) $M\in\Gamma_0(2N)$, ③ (NF^{-1} *is even.*) $M\in\Gamma_0(2, N)$ *and* ④ (*Both* F *and* NF^{-1} *are even.*) $M\in\Gamma_0(N)$.

3. Computation of χ_F II

LEMMA 3. (*The inversion formula*) *Let F be a positive real symmetric matrix of degree m. Then for* U, V, X *and* $Y\in M_{m,n}(\boldsymbol{C})$ *we have*

$$\theta_{F,U,V}(Z;X,Y)=|F|^{-n/2}|-\sqrt{-1}Z|^{-m/2}\theta_{F^{-1},V,U}(-Z^{-1};Y,-X),$$

where $|-\sqrt{-1}Z|^{1/2}$ *is determined to be positive for purely imaginary Z in* H_n.

Proof. We have the inversion formula for the standard theta series

$$\vartheta_{u,v}(Z;x,y)=|-\sqrt{-1}Z|^{-1/2}\vartheta_{v,u}(-Z^{-1};y,-x),$$

where $|-\sqrt{-1}Z|^{-1/2}$ is positive for purely imaginary $Z\in H_n$. From this we get the inversion formula for θ_F in the same argument as in the proof of Theorem 1.

COROLLARY. *Let F be as in Lemma* 3. *Assume that there is a positive real number h such that hF is integral. Put* $\boldsymbol{G}=M_{m,n}(\boldsymbol{Z})$. *Then we have*

$$\begin{aligned}&\theta_{F,U,V}(-Z^{-1};X,Y)\\&=|F|^{-n/2}|-\sqrt{-1}Z|^{m/2}\sum_{H:h^{-1}F^{-1}G/G}\theta_{h^2F,hFV,-h^{-1}F^{-1}U+H}(Z;hFY;-h^{-1}F^{-1}X),\end{aligned}$$

where $|-\sqrt{-1}Z|^{1/2}$ is positive for purely imaginary Z in H_n.

Hereafter we assume that F and $M=\begin{pmatrix}AB\\CD\end{pmatrix}$ satisfy the condition ①, ②, ③ or ④ with $N>1$. Let $H\in F^{-1}\boldsymbol{G}$. We have the following two formulas:

$$(*)\quad\left\{\ \theta_{F,0,H}(-Z^{-1})=|F|^{-1/2}|-\sqrt{-1}Z|^{m/2}\sum_{K:F^{-1}G/G}\varepsilon(\mathrm{tr}(2^tHFK))\theta_{F,0,K}(Z),\right.$$

$$\theta_{F,0,H}(Z)=\sum_{K:(dF)^{-1}G/G,\ K{}^tD\equiv\bmod G}\theta_{dF,0,K}\left(\frac{1}{d}Z[D]\right)$$

for $D\in M_{n,n}(\boldsymbol{Z})$ such that $|D|\neq 0$ and for $d\in\boldsymbol{Z}_+$ such that dD^{-1} is integral.

Let us put $M'=\begin{pmatrix}-B & A\\ -D & C\end{pmatrix}=M\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\in Sp_n(\boldsymbol{Z})$. Let d be a positive integer such that dD^{-1} is integral. Then we have

$$\theta_{F,0,0}(M'Z)=\sum_{G:G{}^tD^{-1}/G}\theta_{dF,0,G}\left(\frac{1}{d}M'Z[D]\right)\quad\text{(by the second formula of (*))}$$

$$=\sum_{G:G{}^tD^{-1}/G}\theta_{dF,0,G}\left(\frac{1}{d}{}^tBD-(dZ-dD^{-1}C)^{-1}\right)$$

$$=\sum_{G:G{}^tD^{-1}/G}\varepsilon(\mathrm{tr}({}^tBD{}^tGFG))\theta_{dF,0,G}(-(dZ-dD^{-1}C)^{-1})$$

$$=\sum_{G:G{}^tD^{-1}/G}\varepsilon(\mathrm{tr}({}^tBD{}^tGFG)|dF|^{-n/2}|-\sqrt{-1}\,(dZ-dD^{-1}C)|^{m/2}$$

$$\times\sum_{K:(dF)^{-1}G/G}\varepsilon(\mathrm{tr}(2d{}^tGFK))\theta_{dF,0,K}(dZ-dD^{-1}C)$$

(by the first formula of (*))

$$=|dF|^{-n/2}|-\sqrt{-1}\,(dZ-dD^{-1}C)|^{m/2}$$

$$\times\sum_{G:G{}^tD^{-1}/G}\ \sum_{K:(dF)^{-1}G/G}\varepsilon(\mathrm{tr}({}^tBD{}^tGFG+2d{}^tGFK-d^2D^{-1}C{}^tKFK))\theta_{dF,0,K}(dZ)$$

Now

$$\sum_{G:G{}^tD^{-1}/G}\varepsilon(\mathrm{tr}({}^tBD{}^tGFG+2d{}^tGFK-d^2D^{-1}C{}^tKFK))$$

$$=\sum_{G:G{}^tD^{-1}/G}\varepsilon(\mathrm{tr}({}^tBD{}^t(G-dKD^{-1}C)F(G-dKD^{-1}C)+2d{}^tAD{}^tGFK-d^{2t}AC{}^tKFK))$$

$$=\sum_{0:G{}^tD^{-1}/G}\varepsilon(\mathrm{tr}({}^tBD{}^tGFK)).$$

Using the second formula of (*) for $D=d1_n$, we get

$$\theta_{F,0,0}(M'Z)$$

$$=|dF|^{-n/2}|-\sqrt{-1}\,(dZ-dD^{-1}C)|^{m/2}\sum_{G:G{}^tD^{-1}/G}(\mathrm{tr}({}^tBD{}^tGFG))\sum_{K:F^{-1}G/G}\theta_{F,0,K}(Z).$$

Substituting $-Z^{-1}$ for Z and using the first formula of (*), we get

$$\theta_{F,0,0}(MZ)$$

$$=|dF|^{-n/2}|\sqrt{-1}\,dD^{-1}(CZ+D)Z^{-1}|^{m/2}\sum_{G:G{}^tD^{-1}/G}\varepsilon(\mathrm{tr}({}^tBD{}^tGFG))$$

$$\times\sum_{K:F^{-1}G/G}|F|^{-n/2}|-\sqrt{-1}\,Z|^{m/2}\sum_{L:F^{-1}G/G}\varepsilon(\mathrm{tr}(2{}^tLFK))\theta_{F,0,L}(Z).$$

Observing that

$$\sum_{K:F^{-1}G/G} \varepsilon(\mathrm{tr}(2{}^tLFK)) = \begin{cases} 0 & \text{if } L \not\equiv 0 \bmod \boldsymbol{G} \\ |F|^n & \text{if } L \equiv 0 \bmod \boldsymbol{G}, \end{cases}$$

we obtain

$$\theta_{F,0,0}(MZ)$$
$$= |-\sqrt{-1}\,Z|^{m/2} |\sqrt{-1}\,D^{-1}(CZ+D)Z^{-1}|^{m/2} \sum_{G:G{}^tD^{-1}/G} \varepsilon(\mathrm{tr}({}^tBD{}^tGFG))\theta_{F,0,0}(Z).$$

The above computation is well known for $n=1$. (cf. [4], [7], [8] the section 2). Thus we obtain;

LEMMA 4. *Let* $|\sqrt{-1}X+1_n|^{1/2}$ *be a functiou on* $SM_n(\boldsymbol{R})$ *which is the branch taking the value* 1 *at* $X=0$. *Suppose that* F *and* $M=\begin{pmatrix} AB \\ CD \end{pmatrix}$ *satisfy one of the four conditions* ①, ②, ③ *and* ④ *with* $N>1$. *Let us denote by* $\varepsilon(C, D)$ *the complex number given by*

$$\varepsilon(C, D)\, abs(D)^{-1/2} |\sqrt{-1}\,C+D|^{1/2} = |\sqrt{-1}\,D^{-1}C+1_n|^{1/2}.$$

Then we have

$$\chi_F^{(n)}(M) = \varepsilon(C, D)^m\, abs(D)^{-m/2} \sum_{G:G{}^tD^{-1}/G} \varepsilon(\mathrm{tr}({}^tBD{}^tGFG)).$$

COROLLARY. *If* M *is in the form of* P_U, Q_S *or* R_S (*cf.* §2), *then we have*

$$\chi_F^{(n)}(M) = 1 \text{ if } m \text{ is even},$$

$$\chi_F^{(n)}(M) = \pm 1 \text{ if } m \text{ is odd}.$$

4. Constructions of cusp forms

Let $k \in \frac{1}{2}\boldsymbol{Z}_+$ and let χ be a map of Γ' to $\boldsymbol{C}^*$. We denote by $[\Gamma', k, \chi]$ (resp. $[\Gamma', k]$) the space of cusp forms of weight k for Γ' with a multiplier χ (resp. a trivial multiplier).

We apply a differential operator $\det^\nu({}^t\eta\partial)$ to the formula in Corollary to Lemma 3. Then we get

$$\theta_{F,U,V}(-Z^{-1}; X, Y)$$
$$= (\sqrt{-1})^{mn/2} h^{n\nu} |F|^{-n/2} |-Z|^{(m/2)+\nu}$$
$$\times \sum_{H:h^{-1}F^{-1}G/G} \theta_{h^2F,hFV,-h^{-1}F^{-1}U+H}(Z; \Phi; hFY, -h^{-1}F^{-1}X),$$

where Φ and ν are as in Theorem 2. Any $M \in \Gamma$ can be written in the form of a product of P_U, Q_S and $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$ with $U\in GL_n(\boldsymbol{Z})$ and $S\in SM_n(\boldsymbol{Z})$ (cf. § 2 for the notation). Hence in the Fourier expansion

$$|CZ+D|^{-(m/2)-\nu}\theta_{F,U,V}(MZ;\Phi;X,Y)=\sum_{S\geq 0}a(S)\varepsilon(\mathrm{tr}(ZS)) \quad \text{for all } M\in\Gamma,$$

the coefficient $a(S)$ vanishes for S with $|S|=0$, since $\Phi(G)$ vanishes if rank $({}^tGFG)<n$. Thus $\theta_{F,U,V}(Z;\Phi)$ will be a cusp form so long as it is a modular form.

(1) Cusp forms of weight $\frac{n}{2}+1$

PROPOSITION 1. a) *We have*

$$\dim\left[\Gamma(2), \frac{n}{2}+1, \chi\right]>0$$

with $\chi(M)=\chi_{1_n}(M)\varepsilon\left(\mathrm{tr}\left(\frac{1}{2}B+\frac{1}{2}(D-1_n)-\frac{1}{4}C{}^tD-\frac{1}{4}B{}^tA\right)\right)$. *Especialy we have*

$$\dim\left[\Gamma(4,8), \frac{n}{2}+1, \chi_{1_n}\right]>0.$$

b) *Let F be a positive even symmetric matrix and N a positive integer such that NF^{-1} is even. Then we have*

$$\dim\left[\Gamma(hN), \frac{n}{2}+1, \chi_{hF}\right]>0 \quad \textit{for } h\geq 3$$

and

$$\dim\left[\Gamma(2N), \frac{n}{2}+1, \chi\right]>0$$

with $\chi(M)=\chi_F(M)\varepsilon\left(\mathrm{tr}\left(\frac{1}{2}(D-1_n)-\frac{1}{4}F^{-1}C{}^tD-\frac{1}{4}FA{}^tB\right)\right)$.

c) *If N is divisible by a square of some odd prime, then we have*

$$\dim\left[\Gamma(N), \frac{n}{2}+1, \chi_F\right]>0.$$

Proof. a) We apply Theorem 2 with $n=l=m$, $F=1_n$, $\Phi(G)=|G|$, $X=Y=0$, $U=V=\frac{1}{2}1_n$ and $M=\begin{pmatrix} AB \\ CD \end{pmatrix}\in\Gamma(2)$. Then we have

$$\theta_{1_n,(1/2)1_n,(1/2)1_n}(MZ;\Phi)=\chi(M)|CZ+D|^{(n/2)+1}\theta_{1_n,(1/2)1_n,(1/2)1_n}(Z;\Phi)$$

with $\chi(M)=\chi_{1_n}(M)\varepsilon\left(\mathrm{tr}\left(\frac{1}{2}B+\frac{1}{2}(D-1_n)-\frac{1}{4}C{}^tD-\frac{1}{4}B{}^tA\right)\right)$. Hence $\theta_{1_n,(1/2)1_n,(1/2)1_n}(Z;\Phi)$ is a cusp form for $\Gamma(2)$ with a multiplier χ.

Let us denote its Fourier expansion by $\sum_{S>0} a(S)\varepsilon(\mathrm{tr}(ZS))$. $a(S)$ is given by $a(S)=\varepsilon\left(\frac{n}{2}\right)\sum_{G\equiv(1/2)1_n \bmod Z,\, {}^tGG=S}\varepsilon(\mathrm{tr}(G))|G|$. We must show that $\theta_{1_n,(1/2)1_n,(1/2)1_n}(Z;\Phi)$ is a non-zero function. To do this, it sufficies to show that there is $S>0$ such that $a(S)\neq 0$. The Fourier coefficient for $\frac{1}{4}1_n$ is

$$a\left(\frac{1}{4}1_n\right)=\varepsilon\left(\frac{n}{2}\right)\sum_{G\equiv(1/2)1_n \bmod Z,\, {}^tGG=(1/4)1_n}\varepsilon(\mathrm{tr}(G))|G|$$
$$=2^{-n}\varepsilon\left(\frac{n}{2}\right)\sum_{G\equiv 1_n \bmod 2Z,\, {}^tGG=1_n}\varepsilon\left(\mathrm{tr}\left(\frac{1}{2}G\right)\right)|G|.$$

Since $G\equiv 1_n \bmod 2\boldsymbol{Z}$, we have $|G|=|(g_{ij})|\equiv g_{11}\cdots g_{nn} \bmod 4$. If $n\equiv 0 \bmod 4$, then we have $g_{11}\cdots g_{nn}=1$ or -1 according as $\mathrm{tr}(G)\equiv 0$ or $2 \bmod 4$; hence $\varepsilon\left(-\frac{n}{2}\right)a\left(\frac{1}{4}1_n\right)>0$. Similarly we have $\varepsilon\left(-\frac{n}{2}\right)a\left(\frac{1}{4}1_n\right)<0$ if $n\equiv 2 \bmod 4$, $\sqrt{-1}\,\varepsilon\left(-\frac{n}{2}\right)a\left(\frac{1}{4}1_n\right)<0$ if $n\equiv 1 \bmod 4$ and $\sqrt{-1}\,\varepsilon\left(-\frac{n}{2}\right)a\left(\frac{1}{4}1_n\right)>0$ if $n\equiv 3 \bmod 4$.

b) Let F and N be as in the proposition. Let us put $\Phi(G)=|G|$. It is shown in [5] that for an integer $h\geq 3$, $\theta_{hF,0,(1/h)1_n}(Z;\Phi)$ is a non-zero cusp form of weight $\frac{n}{2}+1$ for $\Gamma(hN)$ with a multiplier χ_{hF}. It remains to show that $\theta_{F,(1/2)1_n,(1/2)1_n}(Z;\Phi)$ is a non-zero cusp form for $\Gamma(2N)$ with a multiplier $\chi(M)=\chi_F(M)\varepsilon\left(\mathrm{tr}\left(\frac{1}{2}(D-1_n)-\frac{1}{4}F^{-1}C^tD-\frac{1}{4}A^tB\right)\right)$. By Theorem 2 we have a formula for $M=\begin{pmatrix}AB\\CD\end{pmatrix}\in\Gamma(2N)$.

$$\theta_{F,(1/2)1_n,(1/2)1_n}(MZ;\Phi)=\chi(M)|CZ+D|^{(n/2)+1}\theta_{F,(1/2)1_n,(1/2)1_n}(Z;\Phi).$$

If $\sum_{S>0} a(S)\varepsilon(\mathrm{tr}(ZS))$ is its Fourier expansion, then we have

$$a\left(\frac{1}{4}F\right)=\varepsilon\left(\frac{n}{2}\right)\sum_{G\equiv(1/2)1_n \bmod Z,\, {}^tGFG=(1/4)F}\varepsilon(\mathrm{tr}(G))|G|$$
$$=2^{-n}\varepsilon\left(\frac{n}{2}\right)\sum_{G\equiv 1_n \bmod 2Z,\, {}^tGFG=F}\varepsilon\left(\mathrm{tr}\left(\frac{1}{2}G\right)\right)|G|.$$

Using the same argument as in a), we get $a\left(\frac{1}{4}F\right)\neq 0$. Thus we get the desired result.

c) For an odd prime $h>1$ with $h^2|N$, it is easily checked that $\theta_{F,0,(1/h)1_n}(Z;\Phi)$ is in $\left[\Gamma(N),\frac{n}{2}+1,\chi_F\right]$. If $a\left(\frac{1}{h^2}1_n\right)$ is the Fourier coefficient for $\frac{1}{h^2}1_n$, then we have

$$a\left(\frac{1}{h^2}1_n\right)=\sum_{G\equiv(1/h)1_n \bmod Z,\ {}^tGFG=(1/h^2)F}|G|$$

$$=h^{-n}\sum_{G\equiv 1_n \bmod hZ,\ {}^tGFG=F}|G|>0.$$

Hence $\theta_{F,0,(1/h)1_n}(Z;\Phi)$ is a non-zero cusp form.

(2) Cusp forms of weight $\geq n$

Let F be a positive real symmetric matrix of degree $m>0$. Let V be an $m\times n$ matrix with entries in $\boldsymbol{Q}$, and h the least common multiple of the denominators of the entries of V. Suppose that there exists a prime p with $p|h$ such that $\overline{hV}\in M_{m,n}(\boldsymbol{Z}/p\boldsymbol{Z})$ is of rank n, where $\overline{hV}$ denotes the reduction of hV mod p. Then for all $G\in M_{m,n}(\boldsymbol{Q})$ with $G\equiv V$ mod $\boldsymbol{Z}$, $F[G]$ is a nonsingular matrix; hence in the Fourier expansion $\theta_{F,U,V}(Z)=\sum_{S\geq 0}a(S)\varepsilon(\mathrm{tr}(ZS))$ $(U\in M_{m,n}(\boldsymbol{R}))$, $a(S)$ vanishes for S with $|S|=0$.

(i) Let F be a positive even symmetric matrix of degree $m\geq 2n$. Let N be a positive integer such that NF^{-1} is even. For $U, V\in M_{m,n}(\boldsymbol{Q})$ and $M=\begin{pmatrix}AB\\CD\end{pmatrix}\in \Gamma_0(N)$, we have $(U, FV)\begin{pmatrix}DC\\BA\end{pmatrix}=(U_M, FV_M)$ mod $\boldsymbol{Z}$. Let p be a prime with $(p, N)=1$ (hence $(p, |F|)=1$) and take $U, V\in M_{m,n}\left(\frac{1}{p}\boldsymbol{Z}\right)$ so that $\overline{p(U, FV)}\in M_{m,n}(\boldsymbol{Z}/p\boldsymbol{Z})$ is of rank $2n$. Then $\overline{p(U_M, V_M)}$ is also of rank $2n$ for all $M\in\Gamma_0(N)$. Using the notation in Corollary to Lemma 3, we have $(U, FV)\begin{pmatrix} & 1_n\\ -1_n & \end{pmatrix}\equiv(FV, F(-F^{-1}U+H))$ $\equiv(FV, -U)$ mod $\boldsymbol{Z}$; hence $\overline{(U, FV)\begin{pmatrix} & 1_n\\ -1_n & \end{pmatrix}}$ is also of rank $2n$. Since $\Gamma_0(N)$ and $\begin{pmatrix} & 1_n\\ -1_n & \end{pmatrix}$ generate Γ, in the Fourier expansion

$$|CZ+D|^{-m/2}\theta_{F,U,V}(MZ)=\sum_{S\geq 0}a(S)\varepsilon(\mathrm{tr}(ZS))\quad\text{for all } M=\begin{pmatrix}AB\\CD\end{pmatrix}\in\Gamma,$$

$a(S)$ vanishes for S with $|S|=0$. For $M\in\Gamma(pN)$ we have $U_M\equiv U$, $V_M\equiv V$ mod $\boldsymbol{Z}$ and hence $\theta_{F,U,V}(Z)\in\left[\Gamma(pN), \frac{m}{2}, \chi\right]$ for some multiplier χ.

(ii) For $F=1_m$ we get $2(U, V)\begin{pmatrix}DC\\BA\end{pmatrix}\equiv 2(U_M, V_M)$ mod $\boldsymbol{Z}$ for $U, V\in M_{m,n}(\boldsymbol{R})$ and $M=\begin{pmatrix}AB\\CD\end{pmatrix}\in\Gamma$. Hence for an odd prime p if we take $U, V\in M_{mn}\left(\frac{1}{p}\boldsymbol{Z}\right)$ so that $\overline{2p(U, V)} \in M_{m,n}(\boldsymbol{Z}/p\boldsymbol{Z})$ is of rank $2n$, then $\theta_{F,U,V}(\boldsymbol{Z})$ is in $\left[\Gamma(2p), \frac{m}{2}, \chi\right]$ for some χ.

(iii) Suppose $m \geq 2n+1$ and set $F = 1_m$. Take $T \in M_{m,2n}\left(\frac{1}{2}\boldsymbol{Z}\right)$ so that $\overline{2\left(T + \frac{1}{2}\begin{pmatrix} 0 \\ {}^t u \end{pmatrix}\right)} \in M_{m,2n}(\boldsymbol{Z}/2\boldsymbol{Z})$ is of rank $2n$ for any $u \in \boldsymbol{Z}^{2n}$. Then for any M in $GL_{2n}(\boldsymbol{Z})$, TM also has this property. Set

$$W = \begin{pmatrix} 1 & & -1 \\ & \ddots & \vdots \\ & & 1-1 \\ & & 1 \end{pmatrix} \in M_{m,m}(\boldsymbol{Z}).$$

Then we have $W(U_M, V_M) = W(U, V)\begin{pmatrix} DC \\ BA \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ {}^t u \end{pmatrix}$ for $M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in \Gamma$ and for some $u \in \boldsymbol{Z}^{2n}$. Thus if $W(U, V)$ has the property stated above, so does $W(U_M, V_M)$. Especially $\overline{2V_M} \in M_{m,n}(\boldsymbol{Z}/2\boldsymbol{Z})$ is of rank $2n$ for any $M \in \Gamma$. Hence we get $\theta_{F,U,V}(Z) \in [\Gamma(2), m/2, \chi]$ for some χ.

Examples of non-zero cusp forms

(i)′ Let F be a positive even symmetric matrix of degree $m \geq 2n$ which is of the form $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$ with $\deg(F_1), \deg(F_2) \geq n$. Let N be a positive integer such that NF^{-1} is even and let p be a prime such that $(p, N) = 1$. It is easily checked that for

$$U = \begin{pmatrix} \frac{1}{p} 1_n \\ 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 \\ \frac{1}{p} 1_n \end{pmatrix} \in M_{m,n}\left(\frac{1}{p}\boldsymbol{Z}\right)$$

$\overline{p(U, FV)} \in M_{m,2n}(\boldsymbol{Z}/p\boldsymbol{Z})$ is of rank $2n$, and $\theta_{F,U,V}(\boldsymbol{Z})$ is in $[\Gamma(pN), m/2, \chi]$ with $\chi(M) = \varepsilon(\mathrm{tr}(2^t VFVB - {}^t C^t UF^{-1}UD - {}^t A^t VFVB))$. $\theta_{F,U,V}(Z)$ is a non-zero function. In fact, we have $\theta_{F,U,V}(Z) = \theta_{F_1,U',0}(Z)\theta_{F_2,0,V'}(Z)$ with

$$U' = \begin{pmatrix} \frac{1}{p} 1_n \\ 0 \end{pmatrix} \in M_{\deg(F_1),n}\left(\frac{1}{p}\boldsymbol{Z}\right), \quad V' = \begin{pmatrix} 0 \\ \frac{1}{p} 1_n \end{pmatrix} \in M_{\deg(F_2),n}\left(\frac{1}{p}\boldsymbol{Z}\right).$$

Here $\theta_{F_2,0,V'}(Z)$ is obviously non-zero and so is $\theta_{F_1,U',0}(Z)$ (for example, use the inversion formula).

(ii)′ Set $F = 1_m$ with $m \geq 2n$. Let p be an odd prime, and U, V the same matrices as in (i)′. Then we have a non-zero cusp form $\theta_{1_m,U,V}(Z)$ of weight $m/2$ for $\Gamma(2p)$ with the multiplier $\chi(M) = \chi_{1_m}(M)\varepsilon\left(\mathrm{tr}\left(\frac{2}{p^2}B - \frac{1}{p^2}C^t D - \frac{1}{p^2}A^t B\right)\right)$.

(iii)′ Set $F=1_m$ with $m\geq 2n+1$ and let U, V be as above with $p=2$. Then $\overline{2W(U, V)+\begin{pmatrix}0\\ {}^t u\end{pmatrix}}\in M_{m,2n}(\boldsymbol{Z}/2\boldsymbol{Z})$ is of rank $2n$ for any $u\in\boldsymbol{Z}^{2n}$. Hence we have a non-zero cusp form $\theta_{1_m,U,V}(Z)\in[\Gamma(2), m/2, \chi]$ with $\chi(M)=\chi_{1_m}(M)\varepsilon\left(\mathrm{tr}\left(\frac{1}{2}B-\frac{1}{4}C^tD-\frac{1}{4}A^tB\right)\right)$.

(3) Cusp forms of weight $n+1$ with a trivial multiplier

THEOREM 4. a) *We have*

$$\dim[\Gamma(4), n+1]>0 \quad \textit{for } n>1.$$

Let $F=\begin{pmatrix}F_1 & \\ & F_2\end{pmatrix}$ *be a positive even symmetric matrix of degree* $2n+2$ *with* $\deg(F_1)$, $\deg(F_2)>n$, *and* N *a positive integer such that* NF^{-1} *is even. Then we have*

$$\dim[\Gamma(h^2N), n+1]>0 \quad \textit{for an odd } h>1$$

and

$$\dim[\Gamma(2N, 4N), n+1]>0 \quad \textit{if } N \textit{ is odd}.$$

b) *Let* n *be even. Then we have*

$$\dim[\Gamma(2h^2), n+1]>0 \quad \textit{for an odd } h>1.$$

Let F *be a positive even symmetric matrix of degree* n, *and* N *a positive integer such that* NF^{-1} *is even. Then we have*

$$\dim[\Gamma(hN), n+1]>0 \quad \textit{for } h\geq 2$$

and

$$\dim[\Gamma(N), n+1]>0 \quad \textit{if } N \textit{ is divisible by a square of some odd integer } >1.$$

For $n=24$ *we have*

$$\dim[\Gamma, 25]>0.$$

Proof. a) Suppose $n>1$. From (2)

$$(**) \qquad \theta_{1_{2n+2},U,V}(Z)$$

is a non-zero cusp form for $\Gamma(2)$ with the multiplier $\chi(M)=\chi_{1_{2n+2}}(M)\varepsilon(\mathrm{tr}(2^tVVB-{}^tUUD^tC-{}^tVVB^tA))$ where we put

$$U=\frac{1}{2}\begin{pmatrix}1 & & 0\\ 0 & \ddots & \\ 1 & \cdots\cdots & 1\\ & 0 & \end{pmatrix},\quad V=\frac{1}{2}\begin{pmatrix} & 0 & \\ 1 & & \\ 0 & \ddots & \\ 1 & \cdots\cdots & 1\end{pmatrix}\in M_{2n+2,n}\left(\frac{1}{2}\boldsymbol{Z}\right).$$

Since $\chi_{1_{2n+2}}(M)$ is trivial on $\Gamma(4)$ (cf. Corollary to Theorem 3) and since both 4^tUU and 4^tVV are even, χ is trivial on $\Gamma(4)$. Thus we get $\dim[\Gamma(4), n+1]>0$ for $n>1$.

The remaining cases have already investigated in (2).

b) Let n be an even integer. Throughout the proof $\Phi(G)$ denotes the determinant of G.

For an odd $h>1$, we have $\theta_{1_n,0,(1/h)1_n}(Z)\in[\Gamma(2h), n/2, \chi]$ and $\theta_{1_n,0,(1/h)1_n}(Z;\Phi)\in[\Gamma(2h), n/2+1, \chi']$ with $\chi'(M)=\chi_{1_n}(M)\varepsilon(\mathrm{tr}(1/h^2(2\,1_n-A)^tB))$. Hence we have $\theta_{1_n,0,(1/h)1_n}(Z)\theta_{1_n,0,(1/h)1_n}(Z;\Phi)\in[\Gamma(2h), n+1, \chi]$ with $\chi(M)=\varepsilon(\mathrm{tr}(1/h^2(2\,1_n-A)^tB))$. Since χ is trivial on $\Gamma(2h^2)$, $\theta_{1_n,0,(1/h)1_n}(Z)\theta_{1_n,0,(1/h)1_n}(Z;\Phi)$ is a cusp form for $\Gamma(2h^2)$ with a trivial multiplier. It remains to shows that both $\theta_{1_n,0,(1/h)1_n}(Z)$ and $\theta_{1_n,0,(1/h)1_n}(Z;\Phi)$ are non-zero functions. Obviously the former is non-zero, and it is easy to check that the latter is non-zero, using the same method as in the proof of Proposition 1 c).

Let F and N be as in the theorem. For $h\geq 3$, $\theta_{hF,0,0}(Z)\times\theta_{hF,0,(1/h)1_n}(Z;\Phi)$ is a non-zero cusp form of weight $n+1$ for $\Gamma(hN)$ by Proposition 1 b). Hence we get $\dim[\Gamma(hN), n+1]>0$ for $h\geq 3$.

If N is odd, then $\theta_{F,(1/2)1_n,(1/2)1_n}(Z)$ is non-zero modular form, since we have $\theta_{F,0,(1/2)1_n}(MZ)=\chi_F(M)E_F(0,(1/2)1_n, M)\theta_{F,(N/2)1_n(1/2)1_n}(Z)=\chi_F(M)E_F(0,(1/2)1_n, M)\theta_{F,(1/2)1_n,(1/2)1_n}(Z)$ for $M=\begin{pmatrix}1_n & NF^{-1}\\ 0 & 1_n\end{pmatrix}$. Hence $\theta_{F,(1/2)1_n,(1/2)1_n}(Z)\theta_{F,(1/2)1_n,(1/2)1_n}(Z;\Phi)$ is a non-zero cusp form by Proposition 1 b). Hence we get $\dim[\Gamma(2N), n+1]>0$ for an odd N. If N is even, then obviously $\dim[\Gamma(2N), n+1]$ is positive since $[\Gamma(4), n+1]$ is contained in $[\Gamma(2N), n+1]$.

If N is divisible by a square of some odd integer $h>1$, then $\theta_{F,0,0}(Z)\theta_{F,0,(1/2)1_n}(Z;\Phi)$ is a non-zero cusp form for $\Gamma(N)$ with a trivial multiplier by Proposition 1 c). Hence we have $\dim[\Gamma(N), n+1]>0$.

For $n=24$ H. Maass has shown an existence of an even matrix of degree 24 with the determinant 1, for which $\theta_{F,0,0}(Z;\Phi)$ is a non-zero cusp form of weight 13 for Γ with a trivial multiplier. Hence $\theta_{F,0,0}(Z)\theta_{F,0,0}(Z;\Phi)$ is a non-zero cusp form of weight 25 for Γ with a trivial multiplier and we get $\dim[\Gamma, 25]>0$.

REMARK 1. A cusp form of weight $n+1$ for $\Gamma(4)$ corresponds to a differential form of the first kind on the nonsingular model $\overline{H_n/\Gamma(4)}$ of the modular function field with respect to $\Gamma(4)$. Our result shows that the geometric genus of $\overline{H_n/\Gamma(4)}$ is positive if $n>1$. On the other hand we know that for $n=1$, $\overline{H_1/\Gamma(4)}$ is a rational curve.

REMARK 2. When $n=2$, the cusp form (**) is just the example of a cusp form of weight 3 found by S. Raghavan in [6]. In fact we get

$$(**)=\prod_i \vartheta_{u_i,v_i}(Z,0,0)$$

where (u_i, v_i) varies over the set

$$\left\{\begin{pmatrix}0 & 1/2\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1/2\end{pmatrix}, \begin{pmatrix}0 & 1/2\\ 0 & 1/2\end{pmatrix}, \begin{pmatrix}1/2 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1/2 & 0\end{pmatrix}, \begin{pmatrix}1/2 & 0\\ 1/2 & 0\end{pmatrix}\right\}.$$

(4) Examples of cusp forms of degree 2 and weight 3

Let F be a positive even symmetric matrix of degree $m\in 2\boldsymbol{Z}, >n$, and N a positive integer such that NF^{-1} is even. We have a transformation formula

$$(***)\quad \begin{aligned}&\theta_{F,U,V}(Z;\Phi)\\ &=\varepsilon(\mathrm{tr}(A^tB^tVFV+2(D-1_n)^tVU-C^tUFU))|CZ+D|^{(m/2)+\nu}\theta_{F,U,V}(Z;\Phi)\end{aligned}$$

for $M=\begin{pmatrix}AB\\ CD\end{pmatrix}\in\Gamma(N)$ and $U, V\in M_{m,n}\left(\frac{1}{N}\boldsymbol{Z}\right)$ with $NF^{-1}U\in M_{m,n}(\boldsymbol{Z})$, where Φ and ν are as in Theorem 2. Let us denote its Fourier expansion by $\sum_{S\geq 0} a(S)\varepsilon(\mathrm{tr}(ZS))$. Then $a(S)$ is given by

$$a(S)=\varepsilon(2\,\mathrm{tr}({}^tVU))\sum_{G\in M_{m,n}(\boldsymbol{Z}),\,F[G+V]=S}\varepsilon(2\,\mathrm{tr}({}^tGU))\Phi(G+V).$$

Using this formula, we give some examples of non-zero cusp forms of degree 2 and weight 3 for principal congruence subgroups with a trivial multiplier. It seems that we answer a question in [3] concerning "konkrete Beispiele von Spitzenformen".

$\theta_{F,U,V}(Z;\Phi)$ becomes such a cusp form for $\Gamma(N)$ in the following cases. Let us set

$$G=\begin{pmatrix}g_1 & g_5\\ g_2 & g_6\\ g_3 & g_7\\ g_4 & g_8\end{pmatrix}\in M_{4,2}(\boldsymbol{Z}),\quad G_1=\begin{pmatrix}g_3 & g_7\\ g_4 & g_8\end{pmatrix},\quad G_2=\begin{pmatrix}g_1 & g_5\\ g_4 & g_8\end{pmatrix},\quad G_3=\begin{pmatrix}g_2 & g_6\\ g_4 & g_8\end{pmatrix}.$$

(i) $$N=5;\ F=\begin{pmatrix}2&1&&\\1&2&1&\\&1&2&1\\&&1&2\end{pmatrix},\quad \Phi(G)=|G_2|,\quad U=\frac{1}{5}\begin{pmatrix}1&4\\2&3\\3&2\\4&1\end{pmatrix},\quad V=\frac{1}{5}\begin{pmatrix}4&-1\\-3&2\\2&-3\\-1&4\end{pmatrix}$$

(ii) $$N=13;\ F=\begin{pmatrix}2&1&&\\1&2&1&\\&1&2&1\\&&1&4\end{pmatrix},\quad \Phi(G)=|G_1|,\quad U=\frac{1}{13}\begin{pmatrix}5&0\\3&0\\1&0\\7&0\end{pmatrix},\quad V=\frac{1}{13}\begin{pmatrix}4&1\\-8&2\\12&-3\\-3&4\end{pmatrix}$$

(iii) $$N=17;\ F=\begin{pmatrix}2&1&&\\1&2&1&\\&1&4&1\\&&1&2\end{pmatrix},\quad \Phi(G)=|G_1|,\quad U=\frac{1}{17}\begin{pmatrix}0&4\\0&-2\\0&-3\\0&5\end{pmatrix},\quad V=\frac{1}{17}\begin{pmatrix}2&-1\\-4&2\\6&-3\\3&10\end{pmatrix}$$

(iv) $N=29\,;\ F=\begin{pmatrix}2&1&&\\1&2&1&\\&1&6&1\\&&1&2\end{pmatrix},\quad \Phi(G)=|G_1|,\quad U=\frac{1}{29}\begin{pmatrix}0&6\\0&7\\0&11\\0&7\end{pmatrix},\quad V=\frac{1}{29}\begin{pmatrix}2&-1\\-4&2\\6&-3\\-3&16\end{pmatrix}$

(v) $N=4h-1\ (h\geq 2)\,;\ F=\begin{pmatrix}2&1&&\\1&2h&&\\&&2&1\\&&1&2h\end{pmatrix},\quad \Phi(G)=|G_3|,\quad U=0,\quad V=\begin{pmatrix}-1&0\\2&0\\0&-1\\0&2\end{pmatrix}$

(vi) $N=20h-7\ (h\geq 2)\,;\ F=\begin{pmatrix}4&1&&\\1&2&1&\\&1&2&1\\&&1&2h\end{pmatrix},\quad \Phi(G)=|G_2|,\quad U=0,\quad V=F^{-1}\begin{pmatrix}1&0\\0&0\\0&0\\0&1\end{pmatrix}$

(vii) $N=20h-3\ (h\geq 2)\,;\ F=\begin{pmatrix}2&1&&\\1&2&1&\\&1&4&1\\&&1&2h\end{pmatrix},\quad \Phi(G)=|G_1|,\quad U=0,\quad V=F^{-1}\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}$

(viii) $N=24h-11\ (h\geq 2)\,;\ F=\begin{pmatrix}2&1&&\\1&2&1&\\&1&2h&1\\&&1&4\end{pmatrix},\quad \Phi(G)=|G_1|,\quad U=0,\quad V=F^{-1}\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}$

(ix) $N=24h-7\ (h\geq 2)\,;\ F=\begin{pmatrix}2&1&&\\1&4&1&\\&1&2&1\\&&1&2h\end{pmatrix},\quad \Phi(G)=|G_3|,\quad U=0,\quad V=F^{-1}\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}$

REMARK. Let p be a prime integer with $3<p<100$. Then p is one of the following: 5, 13, 17, 29, $4h$-1, $20h$-3, $20h$-7, $24h$-11, $24h$-7 for some $h\geq 2$. Hence noting cusp forms which appear in the proof of Theorem 4, we can easily obtain a non-zero cusp forms of weight 3 for $\Gamma(N)$ with a trivial multiplier where N is any integer with $3<N\leq 100$.

Now we shall prove the above $\theta_{F,U,V}(Z\,;\Phi)$ are non-zero cusp forms of weight 3 with a trivial multiplier. We treat only the cases (i) and (v). To the remaining cases almost the same argument is applicable.

Case (i). We get ${}^tVFV=\frac{1}{5}\begin{pmatrix}4&-1\\-1&4\end{pmatrix}$, ${}^tVU=\frac{2}{5}\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $5F^{-1}U\in M_{4,2}(\boldsymbol{Z})$ and ${}^tUFU=\frac{1}{5}\begin{pmatrix}2&1\\1&2\end{pmatrix}$. Then it is easy to check that $\theta_{F,U,V}(Z\,;\Phi)$ is a cusp form of weight 3 with a trivial multiplier, using the formula (***). We must show that it is a non-zero function. Put $S_0=\frac{1}{5}\begin{pmatrix}4&-1\\-1&4\end{pmatrix}$. Then we have

$$a(S_0)=\sum_G \varepsilon(2/5(g_1+2g_2+3g_3+4g_4+4g_5+3g_6+2g_7+g_8))|G_2+S_0|,$$

where G runs over the set of all 4×2 integral matrices such that ${}^tG_2+G_2+{}^tGFG=0$. The equation ${}^tG_2+G_2+{}^tGFG=0$ has the following twenty integral solutions. Let us put $a_1={}^t(-1,0,0,0)$, $a_2={}^t(-1,1,0,0)$, $a_3={}^t(-1,1,-1,0)$, $a_4={}^t(-1,1,-1,1)$,

$b_1=a_3-a_4$, $b_2=a_2-a_4$, $b_3=a_1-a_4$, $b_4=-a_4$ and $0={}^t(0,0,0,0)$. Then all the integral solutions are

$$G=(0,0),(0,b_1),(0,b_2),(0,b_3),(a_1,0),(a_1,b_1),(a_1,b_2),(a_1,b_4),$$
$$(a_2,0),(a_2,b_1),(a_2,b_3),(a_2,b_4),(a_3,0),(a_3,b_2),(a_3,b_3),(a_3,b_4)$$
$$(a_4,b_1),(a_4,b_2),(a_4,b_3),(a_4,b_4).$$

Then we have

$$a(S_0)=1+\varepsilon\left(\frac{3}{5}\right).$$

Thus $\theta_{F,U,V}(Z;\Phi)$ is a non-zero function.

Case (v). Obviously $\theta_{F,U,V}(Z;\Phi)$ is a cusp form of weight 3 for $\Gamma(N)$ with a trivial multiplier. We shall show that it is a non-zero function. Put $S_0=\frac{1}{N}\begin{pmatrix}2&0\\0&2\end{pmatrix}$. Then we have

$$a(S_0)=\sum_G |G_3+S_0|,$$

where G runs over the set of all 4×2 integral matrices such that ${}^tG_3+G_3+{}^tGFG=0$. The integral solution of the equation ${}^tG_3+G_3+{}^tGFG=0$ is only $G=0$. Hence we have

$$a(S_0)=|S_0|=\frac{4}{N^2}.$$

Thus $\theta_{F,U,V}(Z;\Phi)$ is a non-zero function.

5. Appendix

Let F be a positive integral symmetric matrix of degree $m>0$ and $M\in\Gamma$ satisfy one of the four conditions ①, ②, ③ and ④ in §2. If $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\boldsymbol{Z})$ is the matrix corresponding to M in Lemma 2, then it satisfies one of the four conditions ①, ②, ③ and ④ below;

① $b\equiv 0\ (2)$, $c\equiv 0\ (2N)$,

② (F is even.) $b\equiv 0\ (2)$, $c\equiv 0\ (N)$,

③ (NF^{-1} is even.) $b\equiv 0\ (2)$, $c\equiv 0\ (N)$,

④ (Both F and NF^{-1} are even.) $c\equiv 0\ (N)$.

In these cases $\chi_F\begin{pmatrix}a&b\\c&d\end{pmatrix}=\varepsilon(c,d)^m|d|\sum_{G:d^{-1}Z^m/Z^m}\varepsilon(\mathrm{tr}(bd^tGFG))$ can be computed as in [8]. Moreover the invariance of $\chi_F\begin{pmatrix}a&b\\c&d\end{pmatrix}$ by $\begin{pmatrix}1&m\\0&1\end{pmatrix}$ with $m\in\boldsymbol{Z}$ (resp. $m\in 2\boldsymbol{Z}$) for an even

F (resp. an integral F) gives some informations on F and N.

PROPOSITION. (i)
② *Suppose that F is even and NF^{-1} is integral. If m is odd, then $4|N$, or $2|N$ and $|F|=2^{2r+1}K$ with $r\geq 0$ and an odd K. If m is even, then $4|N$, or $2|N$ and $|F|=2^{2r}K$ with $r>0$ and an odd K, or $|F|\equiv m+1$ (4).*
③ *Suppose that F is integral and NF^{-1} is even. If m is odd, then $4|N$, or $2|N$ and $|F|=2^{2r}K$ with $r\geq 0$ and an odd K. If m is even, then $4|N$, or $2|N$ and $|F|=2^{2r}K$ with $r\geq 0$ and an odd K, or $|F|\equiv m+1$ (4).*
④ *Suppose that both F and NF^{-1} are even. If m is odd, then $8|N$, or $4|N$ and $|F|=2^{2r+1}K$ with $r\geq 0$ and an odd K. If m is even, then $8|N$, or $4|N$ and $|F|=2^{2r}K$ with $r>0$ and an odd K, or $2|N$ and $|F|=2^{2r}K$ with $r>0$ and $K\equiv m+1$ (4), or $|F|\equiv m+1$ (4).*
It is known that $m\equiv 0$ (8) if $|F|=1$.

(ii) *Suppose that $M=\begin{pmatrix} ab \\ cd \end{pmatrix}$ and F satisfy one of the four conditions ①, ②, ③ and ④ mentioned above. In case ④ with $N=1$, we have*

$$\chi_F^{(n)}(M)=1 \quad \textit{for all } M\in SL_2(\boldsymbol{Z}).$$

In the remaining cases d is always non-zero. If m is odd, then we have

$$\chi_F^{(n)}(M)=sgn(c)^{m(sgn(d)-1)/2}\varepsilon\left(\frac{m(d-1)}{4}\right)\left(\frac{c}{d}\right)^m\left(\frac{|F|}{d}\right).$$

If m is even, then we have

$$\chi_F^{(n)}(M)=sgn(d)^{m/2}\left(\frac{(-1)^{m/2}|F|}{|d|}\right).$$

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