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ON THE DEFINITION OF THE WAVE FRONT SET OF A DISTRIBUTION

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§ 1. Introduction.

Gabor [2] has indicated an invariant definition of the wave front set of a distribution. His definition in a slightly modified form is as follows. Let X be a manifold, $(x_0, \xi_0)\in T^*X\backslash 0$ and $u\in\mathscr{D}'(X)$. Then we say that (x_0, ξ_0) is in the complement of the wave front set $WF(u)$ of u, if and only if there exists a conic open neighborhood $\Gamma\subset T^*X\backslash 0$ of (x_0, ξ_0) such that for every compact set F of real-valued functions $f\in C^\infty(X)$ and every $g\in C_0^\infty(X)$ with $(x, df(x))\in\Gamma$ for $x\in\mathrm{supp}\, g$, we have for every integer $k\geq 0$, when $\tau\to\infty$,

$$\langle e^{-i\tau f}g, u\rangle = O(\tau^{-k})$$

uniformly in $f\in F$.

In terms of local coordinates an equivalent definition of $WF(u)$ has been given using the Fourier transform [1], [3]. Let X be identified with an open subset of R^n and T^*X with $X\times R^n$. Then $(x_0, \xi_0)\notin WF(u)$ if and only if there exists a function $\chi\in C_0^\infty(X)$ with $\chi(x_0)\neq 0$ and a conic neighborhood $\Xi\subset R^n\backslash 0$ of ξ_0 such that for every integer $k\geq 0$ we have

$$\langle e^{-i\langle x, \xi\rangle}\chi, u\rangle = O(|\xi|^{-k}),$$

when $|\xi|\to\infty$ in Ξ.

In the first definition we test the distribution u by oscillatory test functions $e^{-i\tau f}g$ with arbitrary phase functions $f\in F$, while in the second we need only test u by a single oscillating function $e^{-i\langle x, \xi\rangle}\chi$ with the linear phase $\langle x, \xi\rangle$ depending on a parameter ξ. This suggests that testing the distribution u by a single oscillatory test function of the form $e^{-i\tau\psi(x, \sigma)}\chi(x)$ containing a parameter σ will suffice under a suitable condition on the dependence of the phase function $\psi(x, \sigma)$ on the parameter σ. The purpose of this paper is to give a sufficient condition. The phase function $\psi(x, \sigma)$ is allowed to be nonlinear and this will probably be useful in the calculus of wave front sets. It is possible to prove the result in this paper using the theory of Fourier integral operators. However we give here an elementary

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proof using a nonlinear analog of the Fourier transform.

§ 2. Statement of the result.

Let X be an open subset of R^n and (x_0, ξ_0) an element of $T^*X\backslash 0$. We consider an oscillatory test function $e^{-i\tau\psi(x,\sigma)}\chi(x)$ depending on a parameter σ. Here the testing phase $\psi(x, \sigma)$ is a real-valued C^∞ function defined on $X\times\Sigma$, Σ an open subset of R^N, such that $\psi'_x(x_0, \sigma_0)=\xi_0$ at $(x_0, \sigma_0)\in X\times\Sigma$, and $\chi(x)$ is a function in $C_0^\infty(X)$ such that $\chi(x_0)\neq 0$. From Gabor's definition of wave front set it follows that, when $(x_0, \xi_0)\notin WF(u)$,

$$\int e^{-i\tau\psi(x,\sigma)}\chi(x)u(x)dx=O(\tau^{-k}),\quad \tau\rightarrow\infty,$$

uniformly in σ near σ_0, for every integer $k\geq 0$. Conversely, if the phase function $\psi(x, \sigma)$ satisfies a kind of transversality condition about the dependence on the parameter σ, then testing a given distribution $u\in\mathscr{D}'(X)$ by the oscillating function $e^{-i\tau\psi}\chi$ gives us information about the wave front set $WF(u)$ of u near (x_0, ξ_0).

THEOREM 2.1. Let $\psi(x, \sigma)$ be a real-valued C^∞ function defined on $X\times\Sigma$ and assume that

(2.1) $$\psi'_x(x_0, \sigma_0)=\xi_0 \text{ and}$$

(2.2) the rank of the $n\times(N+1)$ matrix $(\xi_0, \psi''_{x\sigma}(x_0, \sigma_0))$ is equal to n.

Let $\chi(x)$ be a function in $C_0^\infty(X)$ such that $\chi(x_0)\neq 0$. Given $u\in\mathscr{D}'(X)$, if there exists a neighborhood V of σ_0 such that for every integer $k\geq 0$

$$\int e^{-i\tau\psi(x,\sigma)}\chi(x)u(x)dx=O(\tau^{-k}),\quad \tau\rightarrow\infty,$$

uniformly in $\sigma\in V$, then $(x_0, \xi_0)\notin WF(u)$.

If we introduce a homogeneous variable $\theta=(\tau, \tau\sigma)$, $\tau\in R^+$, in place of σ, then the above theorem is reformulated as follows. Let Θ be a conic open subset of $R^{N+1}\backslash 0$. The phase function $\psi(x, \theta)$ is now a real-valued C^∞ function defined in $X\times\Theta$, positively homogeneous of degree 1 with respect to θ and such that $\psi'_x(x_0, \theta_0)=\xi_0$ at $(x_0, \theta_0)\in X\times\Theta$.

THEOREM 2.2. Assume that

(2.3) the rank of the $n\times(N+1)$ matrix $\psi''_{x\theta}(x_0, \theta_0)$ is equal to n.

Let $u\in\mathscr{D}'(X)$. If there exists a conic neighborhood V of θ_0 such that for every integer $k\geq 0$

$$\int e^{-i\psi(x,\theta)}\chi(x)u(x)dx=O(|\theta|^{-k}),\ |\theta|\to\infty,$$

uniformly in $\theta\in V$, then $(x_0, \xi_0)\notin WF(u)$.

In Theorem 2.1, renumbering the σ variables, we may assume that

$$\det(\xi_0, \phi''_{x\sigma'}(x_0, \sigma_0))\neq 0. \tag{2.2'}$$

Here $\sigma=(\sigma', \sigma'')$, $\sigma'=(\sigma_1, \cdots,\sigma_{n-1})$, $\sigma''=(\sigma_n, \cdots, \sigma_N)$. If we set $\Sigma'=\{\sigma'\in R^{n-1};\ (\sigma', \sigma''_0)\in\Sigma\}$ and $\phi_1(x, \sigma')=\phi(x, \sigma', \sigma''_0)$ for $(x, \sigma')\in X\times\Sigma'$, then

$$\int e^{-i\tau\psi_1(x,\sigma')}\chi(x)u(x)dx=O(\tau^{-k}),\ \tau\to\infty,$$

uniformly in $\sigma'\in V'=V\cap\Sigma'$. Therefore we need only prove the theorems when $N=n-1$. In §4 we prove Theorem 2.2 with $N=n-1$ under the assumption that

$$\det\phi''_{x\theta}(x_0, \theta_0)\neq 0. \tag{2.3'}$$

§ 3. An analog of Fourier transform.

In this section we shall define an analog of Fourier transform with nonlinear phase function.

Let X be an open subset of R^n, Θ a conic open subset of $R^n\backslash 0$. We denote by $\mathcal{O}(\Theta)$ the space of $w\in C^\infty(\Theta)$ such that for some constant M depending only on w and for every compact set $K\subset\Theta$ and every multiindex α, the estimate

$$|D_\theta{}^\alpha w(\theta)|\leq C_{\alpha,K}|\theta|^M,\ \theta\in K^c,$$

is valid for some constant $C_{\alpha,K}$. Here we set $K^c=\{\tau\theta;\ \tau\geq 1, \theta\in K\}$.

Let $\phi(x, \theta)$ be a real-valued C^∞ function defined in $X\times\Theta$ and positively homogeneous of degree 1 with respect to θ. For every $v\in\mathcal{E}'(X)$ we set

$$\mathcal{F}v(\theta)=\int e^{-i\psi(x,\theta)}v(x)dx,\ \theta\in\Theta,$$

Then $\mathcal{F}v$ belongs to $\mathcal{O}(\Theta)$ and $\mathcal{F}$ is a continuous linear operator from $\mathcal{E}'(X)$ to $\mathcal{O}(\Theta)$, if $\mathcal{O}(\Theta)$ is equipped with a suitable inductive limit topology.

By S^m we denote the space of classical symbols of order m. Let $a\in S^0(X\times\Theta)$ be such that $\operatorname{supp} a\subset X\times K^c$ for some compact set $K\subset\Theta$. If $\phi'_x(x, \theta)\neq 0$ in $X\times K^c$, then for every $w\in\mathcal{O}(\Theta)$ the oscillatory integral

$$\tilde{\mathcal{F}}_a w(x)=\int e^{i\psi(x,\theta)}a(x, \theta)w(\theta)d\theta$$

defines a distribution $\tilde{\mathcal{F}}_a w\in\mathcal{D}'(X)$ and $\tilde{\mathcal{F}}_a$ is a continuous linear operator from $\mathcal{O}(\Theta)$ to $\mathcal{D}'(X)$.

Now we shall show that under the condition (2.3)', we can find a symbol

$a\in S^0$ such that $\tilde{\mathscr{F}}a\mathscr{F}$ is micro-locally equal to the identity operator in a conic neighborhood of $(x_0, \xi_0)\in T^*X\backslash 0$. More precisely we have the following theorem.

THEOREM 3.1. Let $\psi(x, \theta)$ be a real-valued C^∞ function defined in $X\times\Theta$ and positively homogeneous of degree 1 with respect to θ. We assume that

$$\psi'_x(x_0, \theta_0)=\xi_0\neq 0, \tag{3.1}$$

$$\det \psi''_{x\theta}(x_0, \theta_0)\neq 0. \tag{3.2}$$

Then there exists a symbol $a(x, \theta)\in S^0(X\times\Theta)$ and a compact set $K\subset\Theta$ such that $\operatorname{supp} a\subset X\times K^c$ and $\psi'_x(x, \theta)\neq 0$ in $X\times K^c$. Morever we can find an open neighborhood X_1 of x_0 and a conic open neighborhood $\Xi_1\subset R^n\backslash 0$ of ξ_0 so that for every $v\in\mathscr{E}'(X)$ we have

$$\langle e^{-i\tau f}g, \tilde{\mathscr{F}}a\mathscr{F}v\rangle\sim\langle e^{-i\tau f}g, v\rangle, \quad \tau\rightarrow\infty,$$

when $f\in C^\infty(X)$ is real-valued and $g\in C_0^\infty(X)$ has support in X_1 and $f'(x)\in\Xi_1$ for $x\in\operatorname{supp} g$. If f and g depend on a parameter, then the above asymptotic relation is valid uniformly in the parameter.

In the theorem we have used the notation

$$a(\tau)\sim b(\tau), \quad \tau\rightarrow\infty,$$

when we have, for every integer $k\geq 0$,

$$a(\tau)-b(\tau)=O(\tau^{-k}), \quad \tau\rightarrow\infty.$$

PROOF. There is a map ξ: $X\times X\times\Theta\rightarrow R^n$ which is C^∞ and positively homogeneous of degree 1 with respect to θ and such that

$$\psi(x, \theta)-\psi(y, \theta)=\langle x-y, \xi(x, y, \theta)\rangle \text{ in } X\times X\times\Theta.$$

This implies that

$$\xi(x, x, \theta)=\psi'_x(x, \theta),$$

$$\xi'_\theta(x, x, \theta)=\psi''_{x\theta}(x, \theta).$$

From (3.1), (3.2) we have $\xi(x_0, x_0, \theta_0)=\xi_0$ and $\det \xi'_\theta(x_0, x_0, \theta_0)\neq 0$. Therefore the map $(x, y, \theta)\longrightarrow(x, y, \xi(x, y, \theta))$ is a diffeomorphism of a conic open neighborhood $X_0\times X_0\times\Theta_0$ of (x_0, x_0, θ_0) onto a conic open neighborhood $W\subset X_0\times X_0\times(R^n\backslash 0)$ of (x_0, x_0, ξ_0).

Choose a C^∞ function $\zeta(x, y, \xi)$ defined in R^{3n} with $\operatorname{supp}\zeta\subset L^c$ for some compact subset L of W. Furthermore assume that ζ is positively homogeneous of degree 0 with respect to ξ when ξ is large, and $\zeta=1$ in a conic neighborhood $X_1\times X_1\times\Xi_1\subset W$ of (x_0, x_0, ξ_0) for large ξ. If we set

$$b_0(x, y, \theta)=(2\pi)^{-n}\zeta(x, y, \xi(x, y, \theta))|\det \xi'_\theta(x, y, \theta)|$$

then $b_0\in S^0(X\times X\times \Theta)$ and $\operatorname{supp} b_0\subset H\times H\times K^c$ for some compact sets $H\subset X_0$ and $K\subset\Theta_0$. The oscillatory integral

$$B(x, y)=(2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}\zeta(x, y, \xi)d\xi$$

$$=\int e^{i[\psi(x,\theta)-\psi(y,\theta)]}b_0(x, y, \theta)d\theta$$

defines a distribution kernel $B\in\mathscr{D}'(X\times X)$. It is easily proved by the method of stationary phase that for every $v\in\mathscr{E}'(X)$

$$\langle e^{-i\tau f}g, Bv\rangle\sim\langle e^{-i\tau f}g, v\rangle,\quad \tau\to\infty,$$

if $f\in C^\infty(X)$ is real-valued and $g\in C_0^\infty(X)$ has support in X_1 and $f'(x)\in\Xi_1$ for $x\in\operatorname{supp} g$.

Now we wish to replace $b_0(x, y, \theta)$ by a simple symbol $a(x, \theta)\in S^0(X\times\Theta)$. We set $a_0(x, \theta)=b_0(x, x, \theta)$. Then $a_0\in S^0(X\times\Theta)$ with $\operatorname{supp} a_0\subset H\times K^c$ and positively homogeneous of degree 0 when θ is large. Since $\psi(x, \theta)-\psi(y, \theta)=\langle x-y, \xi(x, y, \theta)\rangle$ and $\det\xi'_\theta(x, y, \theta)\neq 0$ in $X_0\times X_0\times\Theta_0$, $(\partial/\partial\theta_j)[\psi(x, \theta)-\psi(y, \theta)]=0$, $j=1, \cdots, n$, if and only if $x=y$. Since $\det\psi''_{x\theta}(x, \theta)=\det\xi'_\theta(x, x, \theta)\neq 0$ when $(x, \theta)\in X_0\times\Theta_0$, the differentials of $(\partial/\partial\theta_j)[\psi(x, \theta)-\psi(y, \theta)]$ are linearly independent in $X_0\times X_0\times\Theta_0$. So we can find $c_{0j}\in S^0(X\times X\times\Theta)$ with $\operatorname{supp} c_{0j}\subset H\times H'\times K^c$, positively homogeneous of degree 0 with respect to θ when θ is large, and such that

$$b_0(x, y, \theta)-h(y)a_0(x, \theta)=\sum_{j=1}^n c_{0j}(x, y, \theta)\frac{\partial}{\partial\theta_j}[\psi(x, \theta)-\psi(y, \theta)],$$

where h is a cutoff function $\in C_0^\infty(X)$ with $H'=\operatorname{supp} h\subset X_0$ and such that $h(x)=1$ for $x\in H$. Next we set

$$b_1(x, y, \theta)=i\sum_{j=1}^n\frac{\partial}{\partial\theta_j}c_{0j}(x, y, \theta).$$

It is possible to choose successively $a_k\in S^{-k}(X\times\Theta)$, b_k, $c_{kj}\in S^{-k}(X\times X\times\Theta)$ with $\operatorname{supp} a_k\subset H\times K^c$, $\operatorname{supp} b_k$, $\operatorname{supp} c_{kj}\subset H\times H'\times K^c$ and positively homogeneous of degree $-k$ with respect to θ when θ is large and such that

$$a_k(x, \theta)=b_k(x, x, \theta),$$

$$b_k(x, y, \theta)-h(y)a_k(x, \theta)=\sum_{j=1}^n c_{kj}(x, y, \theta)\frac{\partial}{\partial\theta_j}[\psi(x, \theta)-\psi(y, \theta)],$$

$$b_{k+1}(x, y, \theta)=i\sum_{j=1}^n\frac{\partial}{\partial\theta_j}c_{kj}(x, y, \theta).$$

Then the asymptotic sum $a(x, \theta)$ of $a_k(x, \theta)$,

$$a(x, \theta) \sim \sum_{k=0}^{\infty} a_k(x, \theta),$$

belongs to $S^0(X \times \Theta)$ and $\operatorname{supp} a$ is contained in $H \times K^c$.

Let $A \in \mathscr{D}'(X \times X)$ be the distribution kernel defined by the oscillatory integral

$$A(x, y) = \int e^{i[\psi(x,\theta) - \psi(y,\theta)]} a(x, \theta) d\theta.$$

It remains to prove that for every $v \in \mathscr{E}'(X)$

$$\langle e^{-i\tau f} g, Av \rangle \sim \langle e^{-i\tau f} g, Bv \rangle, \quad \tau \to \infty,$$

when $f \in C^\infty(X)$ is real-valued and $g \in C_0^\infty(X)$ has support in X_1 and $f'(x) \in \Xi_1$ for $x \in \operatorname{supp} g$. By repeated integrations by parts

$$\langle e^{-i\tau f} g, Bv - A(hv) \rangle$$

$$= \iiint e^{-i\tau f(x)} g(x) e^{i[\psi(x,\theta) - \psi(y,\theta)]} [b_0(x, y, \theta) - h(y) a(x, \theta)] v(y) dx dy d\theta$$

$$= \iiint e^{-i\tau f(x)} g(x) e^{i[\psi(x,\theta) - \psi(y,\theta)]} [b_k(x, y, \theta) - h(y) R_k(x, \theta)] v(y) dx dy d\theta$$

where $R_k = a - a_0 - \cdots - a_{k-1} \in S^{-k}$. Using the substitution $\theta \to \tau\theta$ we have

$$\langle e^{-i\tau f} g, Bv - A(hv) \rangle$$

$$= \iiint e^{i\tau[-f(x) + \psi(x,\theta) - \psi(y,\theta)]} g(x) [b_k(x, y, \tau\theta) - h(y) R_k(x, \tau\theta)] v(y) \tau^n dx dy d\theta.$$

Hence by the method of stationary phase

$$\langle e^{-i\tau f} g, Bv - A(hv) \rangle = O(\tau^{-k}), \quad \tau \to \infty,$$

for every k. Finally we use again the methcd of stationary phase to show that

$$\langle e^{-i\tau f} g, A(hv) \rangle \sim \langle e^{-i\tau f} g, Av \rangle, \quad \tau \to \infty.$$

This concludes the proof of Theorem 3.1.

§ 4. Proof of Theorem 2.2.

We keep the notation in the proof of Theorem 3.1 and shrinking X_1 if necessary we may assume that $\chi(x) \neq 0$ when $x \in X_1$. From Theorem 3.1 it follows that

$$\langle e^{-i\tau f} g, u \rangle = \langle e^{-i\tau f} \chi^{-1} g, \chi u \rangle$$

$$\sim \langle e^{-i\tau f} \chi^{-1} g, \tilde{\mathscr{F}}_a \mathscr{F}(\chi u) \rangle, \quad \tau \to \infty,$$

if $f \in C^\infty(X)$ is real-valued and $g \in C_0^\infty(X)$ has support in X_1 and $f'(x) \in \Xi_1$ for $x \in \operatorname{supp} g$. Hence it is sufficient to prove that

$$\langle e^{-i\tau f}g, \mathscr{F}_a w\rangle \sim 0, \quad \tau\to\infty, \tag{3.1}$$

when $w\in\mathscr{O}(\Theta)$ satisfies the estimate

$$|w(\theta)|\leq C_k|\theta|^{-k}, \quad \theta\in V\cap K^c, \tag{3.2}$$

where V is a conic neighborhood of θ_0. If we set

$$I(\theta, \tau)=\int e^{i[\psi(x,\theta)-\tau f(x)]}g(x)a(x, \theta)dx,$$

then we have

$$\langle e^{-i\tau f}g, \mathscr{F}_a w\rangle=\int I(\theta, \tau)w(\theta)d\theta. \tag{3.3}$$

From the proof of Theorem 3.1, it follows that the map $(x, \theta)\longrightarrow(x, \psi'_x(x, \theta))$ is a diffeomorphism of $X_0\times\Theta_0$ onto a conic open neighborhood W' of (x_0, ξ_0). The image of $X_0\times(\Theta_0\cap V)$ under this map is a conic neighborhood of (x_0, ξ_0). Hence there exists $\varepsilon>0$, a neighborhood X_2 of x_0 and a conic neighborhood Ξ_2 of ξ_0 such that

$$|\psi'_x(x, \theta)-\xi|\geq\varepsilon(|\theta|+|\xi|)$$

if $x\in X_2$, $\theta\in\Theta_0\backslash V$ and $\xi\in\Xi_2$.

Now put $X_3=X_1\cap X_2$, $\Xi_3=\Xi_1\cap\Xi_2$ and assume that $\operatorname{supp} g\subset X_3$ and $f'(x)\in\Xi_3$ for $x\in\operatorname{supp} g$. Then $\psi'_x(x, \theta)-\tau f'(x)\neq0$ when $x\in\operatorname{supp} g$, $\theta\in\Theta_0\backslash V$ and $\tau\in R^+$. We set

$$L=|\psi'_x(x, \theta)-\tau f'(x)|^{-2}\langle\psi'_x(x, \theta)-\tau f'(x), D_x\rangle.$$

The coefficients of L are positively homogeneous of degree -1 with respect to (θ, τ) and we have

$$Le^{i[\psi(x,\theta)-\tau f(x)]}=e^{i[\psi(x,\theta)-\tau f(x)]}.$$

By repeated integrations by parts, if $\theta\in\Theta_0\backslash V$, $\tau\in R^+$,

$$I(\theta, \tau)=\int e^{i[\psi(x,\theta)-\tau f(x)]}({}^tL)^k(g(x)a(x, \theta))dx,$$

where ${}^tL$ is the adjoint of L. Since the coefficients of $({}^tL)^k$ are of degree $-k$ and $a\in S^0(X\times\Theta)$ has support in $X\times K^c$, K a compact subset of Θ_0, we have

$$|I(\theta, \tau)|\leq C_k'(|\theta|+\tau)^{-k} \tag{3.4}$$

when $\theta\in\Theta\backslash V$ and $\tau\in R^+$.

If we choose $\delta>0$ sufficiently small, then $\psi'_x(x, \theta)-\tau f'(x)\neq0$ when $x\in\operatorname{supp} g$, $\theta\in K^c$ and $|\theta|<\delta\tau$. Hence the estimate (3.4) is also valid when $\theta\in K^c$ and $|\theta|<\delta\tau$. Note that $I(\theta, \tau)$ is bounded, (3.5) $$|I(\theta, \tau)| \leq C, \quad \theta \in \Theta, \quad \tau \in R^+,$$

and since $\operatorname{supp} a \subset X \times K^c$, $I(\theta, \tau) = 0$ when $\theta \notin K^c$. From the definition of the space $\mathcal{O}(\Theta)$ we have

(3.6) $$|w(\theta)| \leq C_{0,K} |\theta|^M, \quad \theta \in K^c.$$

We divide the integral in (3.3) into two parts J_1 and J_2, J_1 over V and J_2 over $\Theta \backslash V$. From (3.2), (3.4)~(3.6), it follows that

$$|J_1| \leq C C_k \int_{|\theta| \geq \delta\tau} |\theta|^{-k} d\theta + C'_k C_0 \int_{|\theta| < \delta\tau} (|\theta| + \tau)^{-k} d\theta = \text{const}\ \tau^{n-k},$$

$$|J_2| \leq C'_k C_{0,K} \int (|\theta| + \tau)^{-k} |\theta|^M d\theta = \text{const}\ \tau^{n+M-k}.$$

Combining these two estimates we obtain (3.1), which completes the proof.

References

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