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AN ALTERNATIVE PROOF OF BIRMAN-HILDEN-VIRO'S THEOREM

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In this paper we shall give an alternative proof of the thorem of Birman-Hilden-Viro (the following theorem (i), (iii), (iv), c.f. [2], [3], [6]).

THEOREM. (i) *Every closed orientable 3-manifold M of Heegaard genus* ≤ 2 *is homeomorphic to the 2-fold branched covering space of* S^3 *with a 3-bridge link L as its branch line.*

(ii) *In particular, if M is a homology sphere, then L is a knot. (More generally L is a knot iff the order of the 1-dimensional homology group of M is odd.)*

(iii) *There is an algorithm to construct L from (a Heegaard diagram of) M.*

(iv) *M is homeomorphic to* S^3 *iff L is the trivial knot. (And hence there is an algorithm to decide whether M is* S^3 *or not.)*

(v) *Each equivalence class of Heegaard splittings determines a unique knot type.*

(vi) *L is not uniquely determined by M. (L depends on a Heegaard splitting of M.)*

REMARK 1. (iv) is proved as follows: If M is homeomorphic to S^3, then by (i) S^3 is the 2-fold branched covering of S^3 with L as its branch line. This gives an involution of S^3 with fixed points L. By the result of [8], L must be a trivial knot. Whether L is a trivial knot or not is decided by the algorithm of Haken in [9].

REMARK 2. (v) is proved in Theorem 8 in [3] and (vi) is proved in [5] and [10].

In the following we shall prove (i).

Suppose that M is a closed orientable 3-manifold of Heegaard genus ≤ 2. Then M has a Heegaard splitting of genus 2. Hence we may suppose that M is obtained by pasting suitably surfaces of two handle-bodies of genus 2. (See the figure 1.)

In this pasting let the loops d', e', f' on M_1 correspond to the loops d, e, f on M_2. In this case we may suppose that the loops a, b, c and the loops d', e', f' intersect transversally in only a finite number of points.

Moreover we may suppose that there is no section of the loops which bounds

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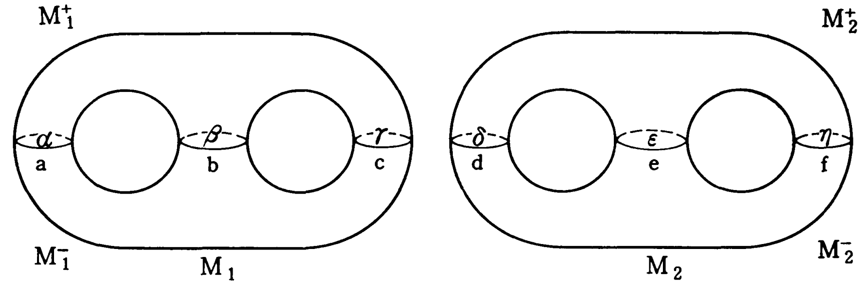

Fig. 1

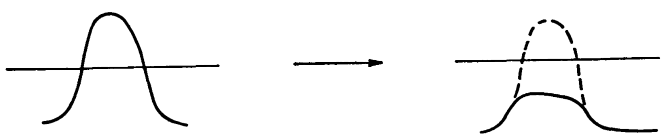

Fig. 2.

2-disk, since it can be eliminated by the isotopy deformation of the figure 2.

Now let us divide M_1 into two parts (M_1^+ and M_1^-) by the meridian disks α, β, γ, whose boundaries are a, b, c, respectively. M_1^+ and M_1^- are 3-disks. On their surfaces, the sections cut by α, β, γ, and the arcs cut by a, b, c are drawn.

Now the number of intersection points of d', e', f' and each of a, b, c on M_1 is even. For, each loop a, b or c is partitioned by three intersections into the parts corresponding to M_2^+ and the parts corresponding to M_2^- alternatively. (M_2^+ and M_2^- are the parts corresponding to M_2^- alternatively. (M_2^+ and M_2^- are the upper and lower parts of M_2 respectively.)

Let the number of intersection points of d', e', f' and each of a, b, c be $2\bar{a}$, $2\bar{b}$, $2\bar{c}$ respectively.

Consider the charts drawn on M_1^+ and M_1^-. Two kinds of these charts are possible, as illustrated in the figure 3-I and 3-II.

Here x in M_1^+ in figure 3-I shows the number af arcs connecting β and γ, etc. Also, for example, α^+ and α^- are two sections cut by α.

As in the figure, x, y, z, x', y', z' are expressed by $\bar{a}$, $\bar{b}$, $\bar{c}$ and in either case we conclude $x=x'$, $y=y'$, $z=z'$.

REMARK. We may exclude the case where M_1^+ is I-type and M_1^- is II-type. For, in this case, from $x=\bar{b}+\bar{c}-\bar{a}\geqq 0$ and $x'=\bar{a}-\bar{b}-\bar{c}\geqq 0$ follows that $x=x'=0$.

Now since $x=x'$, $y=y'$, $z=z'$ in either case the charts drawn on M_1^+ and on M_1^- may be considered as the same. For convenience we may assume that each of α, β, γ is a circle and the intersection points divide these circles equally.

Let $\sigma: M_1^+ \longrightarrow M_1^-$ be the orientation preserving motion by which the chart

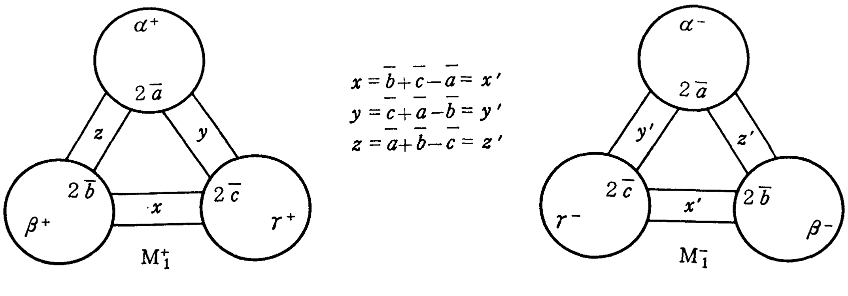

Fig. 3-I.

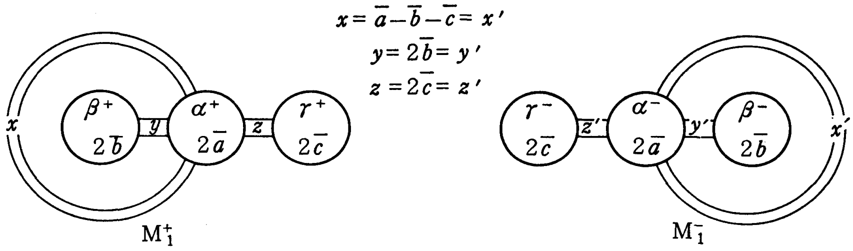

Fig. 3-II.

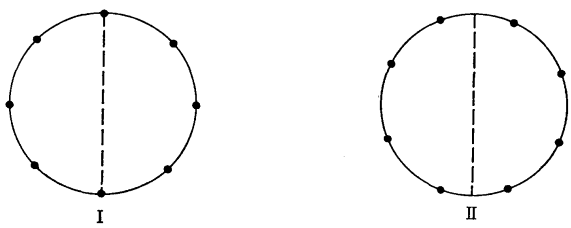

Fig. 4.

drawn on M_1^+ is transformed into the one drawn on M_1^-.

Moreover, the cuts of α, β, γ into α^+, β^+, γ^+ and α^-, β^-, γ^- determine a correspondence τ of α^+, β^+, γ^+ on M^+ and α^-, β^-, γ^- on M_1^-. We may suppose that the correspondence τ is a motion and orientation reversing.

Then the correspondences

$$\alpha^+ \xrightarrow{\sigma} \alpha^- \xrightarrow{\tau^{-1}} \alpha^+$$
$$\beta^+ \xrightarrow{\sigma} \beta^- \xrightarrow{\tau^{-1}} \beta^+$$
$$\gamma^+ \xrightarrow{\sigma} \gamma^- \xrightarrow{\tau^{-1}} \gamma^+$$

are the motions which map α^+, β^+, γ^+ into themselves and orientation reversing. Hence they are symmetries with respect to proper symmetric axes.

Now we show that these symmetric axes are diameters passing through the intersection points.

Only the following two cases I and II in figure 4 are possible, since the numbers of intersection points are even. But we must show that the case II is impossible.

In the following we consider the following example of the figure 5 for illustration.

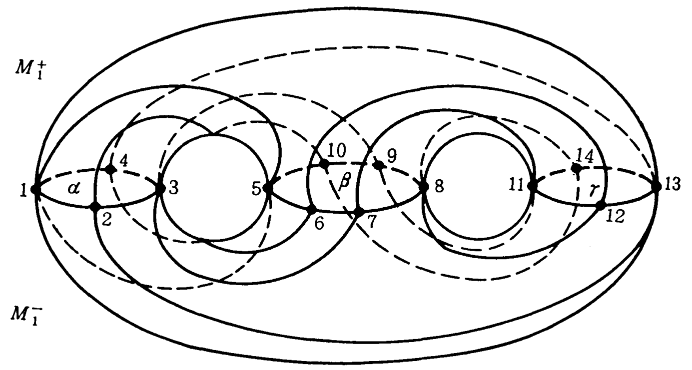

Fig. 5.

Suppose that the loops d', e', f' on M_1 are as shown in this chart. If we cut M_1 through α, β, γ, we have the figure 6.

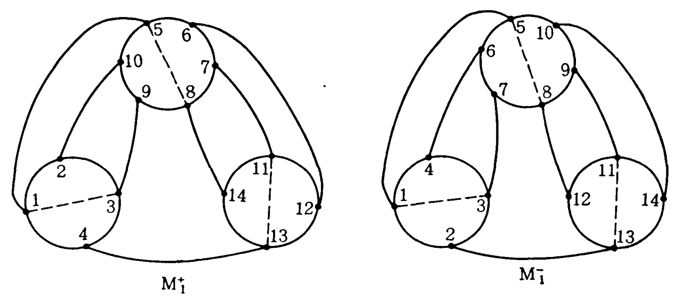

Fig. 6.

If we consider only the surface of M_1, this is the 2-fold branched covering space of S^2 with the branch points 1, 3, 5, 8, 11, and 13. (See the figure 7.)

We show that on the base space S^2, these three loops become simple arcs connecting six branch points. Then our claim will have been proved.

Suppose that this is not the case. Then on the base space S^2, these three loops become one loop and one simple arc. (Only the two cases are possible since we are considering three loops on the torus of genus 2.)

But if so, this loops divides S^2 into two parts and also on the torus of genus 2, two loops on this loop divide this torus of genus 2 into two parts. But returning back to the first (figure 1) neither two loops among d, e, f on M_2 divide the sur-

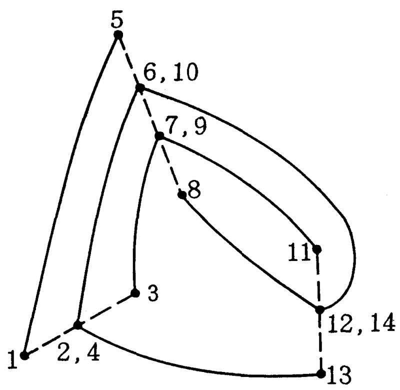

Fig. 7.

face of M_2 into parts, and we have reached the contradiction. Hence we have proved our claim. Hence six branch points are intersection points.

Now let us construct the link (knot in this case) as shown in the figure 9. This link L constructed is of three bridge type *i.e.* a link with 3 upper paths and 3 lower paths.

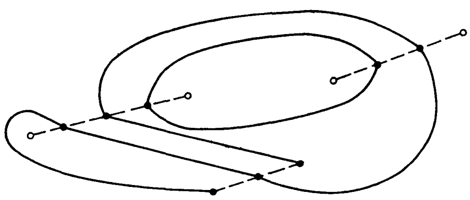

Fig. 8.

Next we shall show that the given manifold M is the 2-fold branched covering space of S^3 with this link L as branch line.

For this end we first study 3-bridge links in general. Given a chart of 3-bridge

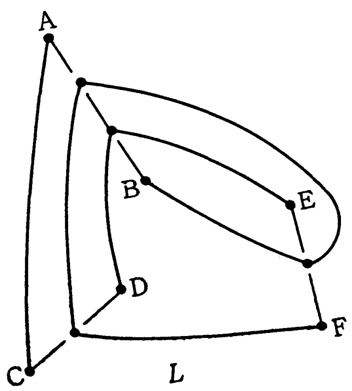

Fig. 9.

link L as above, consider the curve obtained by connecting A, B, C, D, E, F on a plane with upper paths slightly above the plane and lower paths slightly below the plane after the chart. This curve is a realization of L.

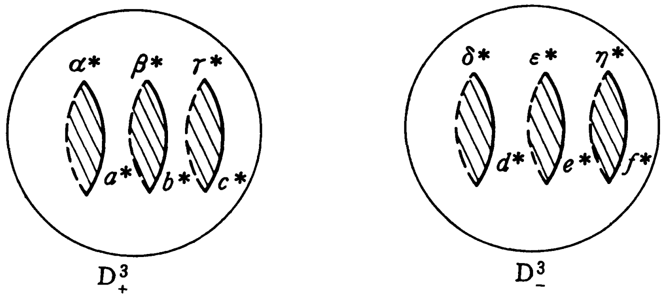

Fig. 10.

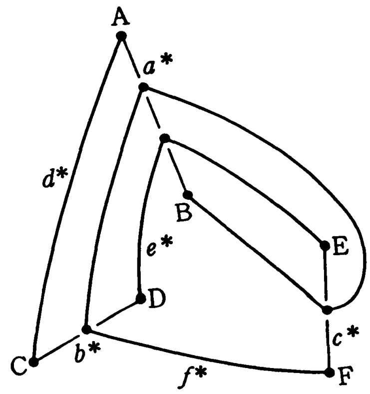

Fig. 11.

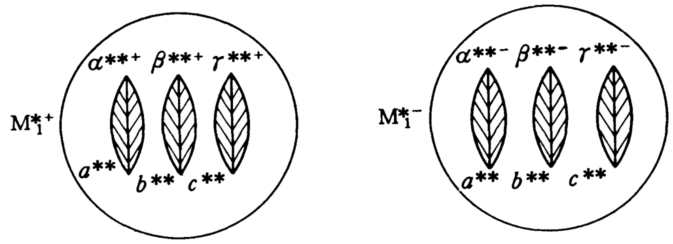

Fig. 12.

The above explanation, which was considered in 3 dimensional Euclidean space, can be translated into the one which is considered in S^3 as follows.

Let S^3 be divided by S^2 into disks D^3_+ and D^3_- and let six points A, B, C, D, E, F be on this S^2. Moreover, suppose that three simple arcs p, q, r connecting two of A, B, C, D, E, F are drawn in D^3_+ and that these are unknotted and unlinked to each other. In other words this means that 3 disjoint simple arcs connecting A, B, C, D, E, F are drawn on the S^2 and then 3 simple arcs passing near these arcs are drawn in D^3_+. Moreover suppose that 3 simple arcs s, t, u are drawn in D^3_- similarly.

The link L is the union of these six simple arcs p, q, r, s, t, u.

Now let M^* be the 2-fold branched covering space of S^3 with L as its branch line, and let us construct a Heegaard splitting of M^*.

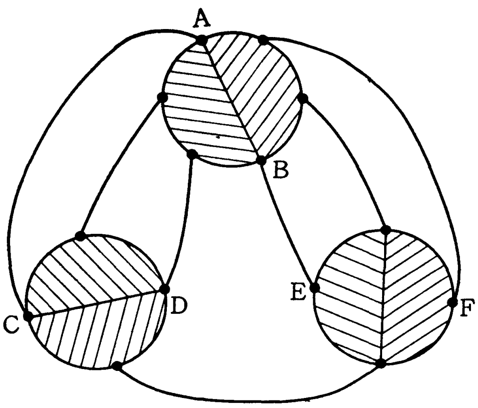

Fig. 13.

Since L consists of six arcs p, q, r, s, t, u and p, q, r are in D^3_+, and s, t, u are in D^3_-, M^* is the union of the 2-fold branched covering space M_1^* of D^3_+ with p, q, r as its branch line and the 2-fold branched covering space M_2^* of D^3_- with s, t, u as its branch line. This gives a Heegaard splitting of M^* of genus 2.

Moreover a^*, b^*, c^* and d^*, e^*, f^* of the above figure correspond to the meridian loops a, b, c of figure 1. Let these be a^{**}, b^{**}, c^{**}, d^{**}, e^{**}, f^{**}. Let $d^{**\prime}$, $e^{**\prime}$, $f^{**\prime}$ be loops on M_1^* corresponding to d^{**}, e^{**}, f^{**} by sewing.

For example, in the case of the link of the figure 9, the chart is as in figure 11:

Moreover α^*, β^*, γ^*, δ^*, ε^*, η^* become meridian disks α^{**}, β^{**}, γ^{**}, δ^{**}, ε^{**}, η^{**} on the branched covering space.

If we cut M_1^* along α^{**}, β^{**}, γ^{**} (c.f. the figure 10) we obtain the figure 12. (The oblique lines show cut ends.)

These cut ends are corresponding on M_1^{*+} and M_1^{*-} symmetrically to each other.

In the case of the link of figure 9, we obtain the figure 13. But this is the same as the figure 6. This shows that the given Heegaard diagram is the same as the one obtained as above and hence that M and M^* are homeomorphic. Q.E.D.

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