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A CHARACTERIZATION OF COMPLEX PROJECTIVE SPACES BY LINEAR SUBSPACE SECTIONS

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1. Introduction

It is conjectured in [2] that a complex projective space will be characterized from the standpoint of the positivity of sectional curvature. This conjecture is partially supported. Namely, a compact Kähler manifold (M, g) with positive curvature is biholomorphically homeomorphic to a complex projective space, if, for examples, one of the following conditions is satisfied;

i) $\dim_{\mathbf{C}} M=2$ ([2]),
ii) the Kähler metric g is Einstein ([1]),
iii) the group of holomorphic transformations acts on M transitively ([6]) and
iv) $\dim_{\mathbf{C}} M=3$ or 4 and $H^*(M; \boldsymbol{Z})\cong H^*(\mathbf{P}_n(\mathbf{C}); \boldsymbol{Z})$, $n=\dim_{\mathbf{C}} M$ ([4]).

These conditions play essential role in each result.

In this connection, we are in a position to consider the following assertion.

ASSERTION If a compact complex manifold M admits a closed complex submanifold, in particular, a closed complex hypersurface which is biholomorphically homeomorphic to a complex projective space, then M itself is biholomorphically homeomorphic to a complex projective space.

If this assertion is verified, the conjecture due to Frankel can be reduced to the following conjecture.

CONJECTURE A compact Kähler manifold with positive curvature will admit a closed complex submanifold endowed with a Kähler metric of positive curvature.

Of course, the submanifold of positive curvature may not be a Kähler submanifold of the ambient manifold.

In general, the assertion is false. For example, a product manifold $\mathbf{P}_n(\mathbf{C})\times M$, where M is a compact complex manifold, has $\mathbf{P}_n(\mathbf{C})$ as a closed complex submanifold, but the total manifold can never be biholomorphically homeomorphic to a complex projective space. Hence, the submanifold in the assertion must satisfy further as-

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sumptions.

A compact Kähler manifold with positive curvature has the positive definite Ricci tensor, hence its first Chern class c_1 is positive. It is, then, an algebraic variety of a complex projective space by the aid of Kodaira's imbedding theorem. Therefore, the compact complex manifold stated in the assertion is furthermore assumed to be a closed submanifold of a complex projective space.

The assertion is held under the conditions that the submanifold is given as a section by a linear subspace in an ambient projective space and that it is biholomorphically homeomorphic to a complex projective space. This fact is precisely stated in Theorem 1.

The main purpose of this paper is to give a proof of Theorem 1. It is shown by the aid of a generalized Lefschetz's theorem ([5]) together with a characterization theorem of a complex projective space in terms of Chern classes ([7]).

2. Theorem and Corollaries

The following theorem characterizes a complex projective space by a linear subspace section in an ambient complex projective space.

THEOREM 1. *Let M be an n-dimensional closed complex submanifold in an N-dimensional complex projective space $\mathbf{P}_N(\mathbf{C})$.*

Assume that there is a linear subspace V in $\mathbf{P}_N(\mathbf{C})$ of codimension r $(\leqq n-2)$ such that a section $M\cap V$ of M by V is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbf{C})$. Then, M itself is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$.

Note that $r\leqq n-2$ is necessary in proving Theorem 1, since the surjectivity of $\iota_*: H_2(M\cap V; \boldsymbol{Z})\to H_2(M; \boldsymbol{Z})$ is guaranteed under the requirement of r.

The following is an immediate conclusion from Theorem 1.

COROLLARY 2. *Let M be as in Theorem 1. If there is a sequence of linear subspaces $\{V^1, \cdots, V^k\}$ of $\mathbf{P}_N(\mathbf{C})$, $r=\sum_{i=1}^{n} r_i\leqq n-2$, $r_i=\mathrm{codim}_{\mathbf{C}} V^i$ such that*

i) *$M^{(i)}$ is a closed complex submanifold of $M^{(i-1)}, i=1, \cdots, k$,*

ii) *$M^{(k)}$ is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbf{C})$,*

where $M^{(i)}=M\cap V^1\cap\cdots\cap V^i, i=1, \cdots, k$ and $M^{(0)}=M$, then M is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$.

Since $M^{(k)}$ is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbf{C})$, $M^{(k-1)}$ is also biholomorphically homeomorphic to a complex projective space by the result of Theorem 1. Hence an inductive argument verifies Corollary 2.

COROLLARY 3. *Let M be as in Theorem* 1. *If there is a closed complex hypersurface S in $\mathbf{P}_N(\mathbf{C})$ such that a section $M\cap S$ is biholomorphically homeomorphic to $\mathbf{P}_{n-1}(\mathbf{C})$, then M is also biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$.*

If, moreover, there is a sequence of closed complex hypersurfaces $\{S^1, \cdots, S^k\}$, $k\leqq n-2$ such that

i) *$M^{(i)}$ is a hypersurface of $M^{(i-1)}, i=1, \cdots, k$ and*

ii) *$M^{(k)}$ is biholomorphically homeomorphic to $\mathbf{P}_{n-k}(\mathbf{C})$,*

where $M^{(i)}=M\cap S^1\cap\cdots\cap S^i, i=1, \cdots, k$ and $M^{(0)}=M$, then M itself is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$.

Corollary 3 is shown by the aid of Veronese mapping. Veronese mapping v_m: $\mathbf{P}_N(\mathbf{C})\rightarrow\mathbf{P}_{N'}(\mathbf{C}), N'=\binom{N+m}{m}-1$, is defined as follows ([8]). Let $u_{i_0 i_1\cdots i_N}$'s be homogeneous coordinates in $\mathbf{P}_{N'}(\mathbf{C})$ where $i_0, i_1, \cdots, i_N$ are nonnegative integers such that $i_0+i_1+\cdots+i_N=m$. v_m is defined by $u_{i_0 i_1\cdots i_N}\circ v_m=z_0^{i_0}\cdot z_1^{i_1}\cdots z_N^{i_N}$, where $z_0, z_1, \cdots, z_N$ are the homogeneous coordinates in $\mathbf{P}_N(\mathbf{C})$. It follows from the definition that the Veronese mapping is an imbedding.

Since the hypersurface S of $\mathbf{P}_N(\mathbf{C})$ in Corollary 3 is given as zero points of a homogeneous polynomial of degree m, $\sum_{i_0+i_1+\cdots+i_N=m} a_{i_0 i_1\cdots i_N} z_0^{i_0}\cdot z_1^{i_1}\cdots z_N^{i_N}$, S is imbedded onto $v_m(S)=H\cap v_m(\mathbf{P}_N(\mathbf{C}))$, where H is a hyperplane in $\mathbf{P}_{N'}(\mathbf{C})$ defined by $\sum a_{i_0 i_1\cdots i_N} u_{i_0 i_1\cdots i_N}=0$. Thus, $M\cap S$ is imbedded onto $v_m(M)\cap v_m(S)=v_m(M)\cap H$ which is biholomorphically homeomorphic to $\mathbf{P}_{n-1}(\mathbf{C})$ by the assumption. From Theorem 1, $v_m(M)$, hence, M is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$. Hence we have the first part of Corollary 3. The second part of the corollary is easily obtained.

3. Proof of Theorem 1

Let ι: $M'\rightarrow M$ and j: $M\rightarrow\mathbf{P}_N(\mathbf{C})$ be the imbeddings, where $M'=M\cap V$. Let τ_M, $\tau_{M'}$ and ν be the tangent bundle of M, the tangent bundle of M' and the normal bundle of M' in M, respectively.

If we denote by $[V]$ the vector bundle over $\mathbf{P}_N(\mathbf{C})$ defined by V, then the normal bundle of V in $\mathbf{P}_N(\mathbf{C})$ is the pullback of $[V]$. Moreover, it is well-known that ν is isomorphic to the pullback of the normal bundle of V in $\mathbf{P}_N(\mathbf{C})$. Therefore we have

$$\iota^*\tau_M=\tau_{M'}\oplus\iota^*j^*[V].$$

Since V is a linear subspace of codimension r, there is a hyperplane H in $\mathbf{P}_N(\mathbf{C})$ such that $[V]=r[H]$, where $[H]$ is the line bundle over $\mathbf{P}_N(\mathbf{C})$ defined by H. Hence we have

$$(1)\qquad \iota^*c_1(M)=c_1(M')+r\iota^*j^*c_1([H]),$$

where c_1's denote the first Chern classes.

Since $[V]$ is positive in the sense of Griffiths ([5]) and $M'=M\cap V$ is a non-singular zero locus of a non-trivial global section of $\mathcal{O}(j^*[V])$, by the aid of a generalized Lefschetz's theorem (see Theorem H in [5]), we obtain the following two exact sequences under the condition $r\leqq n-2$;

$$H_2(M';\ \boldsymbol{Z}) \xrightarrow{\iota_*} H_2(M;\ \boldsymbol{Z}) \longrightarrow 0$$

and

$$0 \longrightarrow H_1(M';\ \boldsymbol{Z}) \xrightarrow{\iota_*} H_1(M;\ \boldsymbol{Z}) \longrightarrow 0\,.$$

Since M' is homeomorphic to a complex projective space, we have $H_2(M';\ \boldsymbol{Z})\cong \boldsymbol{Z}$ and $H_1(M';\ \boldsymbol{Z})=0$. Hence we obtain the following exact sequence;

$$0 \longrightarrow H_2(M';\ \boldsymbol{Z}) \xrightarrow{\iota_*} H_2(M;\ \boldsymbol{Z}) \longrightarrow 0$$

which, together with $H_1(M';\ \boldsymbol{Z})=H_1(M;\ \boldsymbol{Z})=0$, implies that ι^*: $H^2(M;\ \boldsymbol{Z})\rightarrow H^2(M';\ \boldsymbol{Z})$ is an isomorphism.

If α is a positive generator of $H^2(M;\ \boldsymbol{Z})\cong\boldsymbol{Z}$, then $\iota^*\alpha$ is also a positive generator of $H^2(M';\ \boldsymbol{Z})$. Thus we have $\iota^*j^*c_1([H])\geqq\iota^*\alpha$, and hence, $\iota^*c_1(M)\geqq c_1(M')+r\iota^*\alpha$. Since $c_1(M')=(n-r+1)\iota^*\alpha$, which is derived from the fact that M' is biholomorphically homeomorphic to an $(n-r)$-dimensional complex projective space, we have $c_1(M)\geqq(n-r+1)\alpha+r\alpha=(n+1)\alpha$ by the injectivity of ι^*: $\boldsymbol{H}^2(M;\ \boldsymbol{Z})\rightarrow H^2(M';\ \boldsymbol{Z})$.

Therefore, Theorem 1 follows from a result of [7].

4. Further Remarks

1) A linear subspace of a complex projective space is also a complex projective space. And its section by another linear subspace gives a linear subspace again. This is a trivial example which supports Theorem 1. We have a non-trivial example for Theorem 1 as follows. Recall the Veronese mapping v_m: $\mathbf{P}_n(\mathbf{C})\rightarrow\mathbf{P}_N(\mathbf{C})$, $N=\binom{n+m}{m}-1$. The section of $v_m(\mathbf{P}_n(\mathbf{C}))$ by the hyperplane of a form; $u_{m0\cdots0}=0$, in $\mathbf{P}_N(\mathbf{C})$ gives a hyperplane $z_0=0$ in $\mathbf{P}_n(\mathbf{C})$. On the contrary, the section by the hyperplane of a form; $u_{m0\cdots0}+u_{0m0\cdots0}+\cdots+u_{0\cdots0m}=0$, gives the hypersurface of degree m; $\sum_{j=0}^{n} z_j{}^m=0$ in $\mathbf{P}_n(\mathbf{C})$.

2) In [3], a pair (V, L) of a compact variety V and a line bundle L is called a polarized variety, if L is ample. If a compact complex manifold M is imbedded in a complex projective space, then a hyperplane section $M\cap H$ of M induces a polarized variety $(M, [M\cap H])$, since $[M\cap H]$ is very ample. Theorem 1 is, if es-

pecially $r=1$, implicated with Theorem 6.1 in [3].

3) With respect to Conjecture stated in Introduction, we have the following consideration.

Let (M, g) be a compact Kähler manifold with positive curvature. Then, M is a closed complex submanifold of a complex projective space. A hyperplane section M' of M gives a hypersurface which is defined by a certain holomorphic function, which we denote by f, locally. A relation between the holomorphic bisectional curvature $H'\sigma, \tau$ of M' with respect to the induced metric and $H\sigma, \tau$ of (M, g) is given as follows;

$$H'\sigma, \tau = H\sigma, \tau - \frac{|H_f(Z, W)|^2}{||df||^2 ||Z||^2 ||W||^2} . \tag{2}$$

Here σ and τ are holomorphic planes tangent to $M', \sigma = X \wedge IX, \tau = Y \wedge IY$ and $Z = X - \sqrt{-1}\, IX, W = Y - \sqrt{-1}\, IY$. H_f denotes the complex Hessian of f, i.e., $H_f = (\nabla_i \nabla_j f)$ and $||df||^2 = \Sigma g^{\bar{j}i} \frac{\partial f}{\partial z^i} \overline{\frac{\partial f}{\partial z^j}}$.

(2) is obtained by the similar argument as that in [9].

From (2), we have the following statement which locally supports Conjecture with respect to the holomorphic bisectional curvature.

For any point p of M and an arbitrary positive number ε, there are a neighborhood U of p and a holomorphic function f defined on U which satisfy the following;

i) $\{q \in U;\ f(q) = 0\}$ is a hypersurface of M which contains p

and

ii) on the hypersurface endowed with the induced metric,

$|H'\sigma, \tau - H\sigma, \tau| < \varepsilon$ for any pair of holomorphic planes σ and τ tangent to the hypersurface.

This statement is observed as follows. Among all charts around p we can choose a certain normal chart (U', x^i), $x^i(p) = 0$, with respect to which the components $g_{i\bar{j}}$'s of the metric g satify

$$g_{i\bar{j}}(x^i) = \delta_{ij} + \sum_{s,t} R_{i\bar{j}s\bar{t}}(p) x^s x^{\bar{t}} + o(r^3) , \tag{3}$$

where $r = (\sum_t |x^t|^2)^{1/2}$, and $R_{i\bar{j}s\bar{t}}$'s are the components of the curvature tensor R.

Assume that a holomorphic function f on U' is of the form, $f(x^i) = \sum_i a^i x^i + o(r^4), (a^i) \neq 0$, of course, such an f exists indeed. Then, $\nabla_i \nabla_j f(p) = \partial^2 f / \partial x^i \partial x^j (p) - \sum_k \Gamma^k_{ij}(p) \partial f / \partial x^k (p) = 0$, where Γ^k_{ij}'s are the Christoffel's symbols, that is, $\Gamma^k_{ij} = \sum_u g^{\bar{u}k}\, \partial g_{i\bar{u}} / \partial x^j$. Hence, the complex Hessian $H_f = (\nabla_i \nabla_j f)$ vanishes at p. Therefore we can choose a sufficiently small neighborhood U around p such that

$$|H'\sigma,\tau-H\sigma,\tau|=\frac{|H_f(Z,W)|^2}{||df||^2||Z||^2||W||^2}<\varepsilon$$

for any pair of holomorphic planes σ and τ tangent to $\{q\in U;\ f(q)=0\}$.

Since p is arbitrary, M is covered with such a system $\{(U_p,f_p)\}_{p\in M}$ which gives local hypersurfaces. In order for the system to define a global hypersurface in M, it must satisfy the property that there is a subsystem $\{(U_\alpha,f_\alpha)\}_{\alpha\in A}$, which covers M and f_α/f_β gives a non-vanishing holomorphic function on $U_\alpha\cap U_\beta(\neq\phi)$, that is, the subsystem induces a non-singular holomorphic devisor.

It should be noticed that if this is verified, Conjecture can be supported with respect to the holomorphic bisectional curvature, since we only need to set $\varepsilon=1/2\cdot\min H\sigma,\tau$ over all pairs of holomorphic planes of M.

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