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TOPOLOGICALLY COMPLETE SPACES AND PERFECT MAPS

by

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1. Introduction. All spaces considered in this paper are assumed to be completely regular Hausdorff. A space X is called *topologically complete* if it is complete with respect to its finest uniformity. Realcompact spaces and paracompact spaces are topologically complete (cf. [2]). A continuous map f is called *perfect* if it is a closed map and each fiber is compact.

It is known that topological completeness as well as realcompactness is not preserved under perfect maps; this fact was essentially proved by Mrówka [18] and was noted in [3]. To date, the images of realcompact spaces under perfect maps were investigated by several topologists (e.g., Frolik [6], [7], Kenderov [14], Isiwata [11], [13], Blair [1], Dykes [3], [4]), however, with the exception of [3], little seems to be known about topologically complete spaces.

In this paper, we shall obtain characterizations of the images of topologically complete spaces under perfect maps and necessary and sufficient conditions for them to be topologically complete.

In section 2, for convenience, we list certain basic definitions and facts that will be used in the sequal.

In section 3, we introduce the notion of almost uniform structures. This notion is useful for dealing with the perfect images of topologically complete spaces. There are also some "tool" theorems concerning almost uniform structures.

In section 4, almost topologically complete spaces are defined in terms of an almost uniform structure, and we prove that almost topologically complete spaces characterize perfect images of topologically complete spaces. Similarly we can prove the corresponding theorem concerning Frolik's almost realcompact spaces, and consider the relationship between almost topologically complete spaces and almost realcompact spaces. Furthermore, some properties of almost topologically complete spaces are studied, in particular, it is proved that almost topological completeness is invariant under perfect maps.

In the final section 5, we consider a problem under what conditions an almost

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topologically complete space is topologically complete. To do this, we introduce the notions of b^*-spaces and weak b^*-spaces; the main theorem here is that an almost topologically complete space is topologically complete if and only if it is one of these spaces. It is shown that collectionwise normal countably paracompact spaces, M-spaces due to Morita [15] and topologically complete spaces are b^*-spaces, and that extremally disconnected spaces, M'-spaces due to Isiwata [12] and b^*-spaces are weak b^*-spaces.

Hereafter, $C(X)$ denotes the set of all real-valued continuous functions on a space X, and N is the space of positive integers. For details of topologically complete spaces, the reader is referred to [2], [8] and [16]. The terminologies and notation will be used as in [19].

2. Definitions and preliminalies.

2.1. Let X be a space. If $\mathcal{F}$ is a family of subsets of X, we write $\cap\mathcal{F}=\cap\{F|F\in\mathcal{F}\}$ and $\overline{\mathcal{F}}=\{\mathrm{cl}_X F|F\in\mathcal{F}\}$. An *almost-cover* of X is a family of subsets of X whose union is dense in X [5]. Let $\mathcal{U}$ and $\mathcal{V}$ be almost-covers of X. We say that $\mathcal{U}$ is a *refinement* of $\mathcal{V}$ in case every member of $\mathcal{U}$ is contained some member of $\mathcal{V}$. Let λ be a collection of almost-covers of X. A centered family $\mathcal{F}$ of X (i.e., with finite intersection property) is said to be *λ-Cauchy* if for any $\mathcal{U}\in\lambda$, there exist $U\in\mathcal{U}$ and $F\in\mathcal{F}$ with $F\subset U$. We say that X is *λ-complete* if $\cap\overline{\mathcal{F}}\neq\emptyset$ holds for each λ-Cauchy family $\mathcal{F}$ of X. Throughout this paper, we use the following symbols:

μ_X: the collection of all locally finite cozero-set covers of a space X.

υ_X: the collection of all countable locally finite cozero-set covers of a space X.

It is known that a space X is topologically complete (resp. realcompact) if and only if X is μ_X- (resp. υ_X-) complete.

2.2. A space X is said to be *extremally disconnected* if the closure of every open set of X is open. A map f from a space X onto a space Y is called *irreducible* if the image of each proper closed subset of X is a proper closed subset of Y. Associated with each space X, there is an extremally disconnected space $E(X)$ and a perfect irreducible map e_X from $E(X)$ onto X ([22]). The space $E(X)$ is unique up to homeomorphism and is called the *absolute* of X.

2.3. A continuous map f: $X\rightarrow Y$ is called a *quasi-perfect* map (resp. an *SZ-map*) if $f(F)$ is closed in Y for any closed (resp. zero-) set F of X and $f^{-1}(y)$ is countably compact (resp. relatively pseudocompact in X, i.e., each member of $C(X)$ is bounded on $f^{-1}(y)$) for each $y\in Y$. A space X is called an *M-* (resp. *M'-*) *space* if there is a quasi-perfect (resp. an SZ-) map from X onto a metric space. These notions were introduced by Morita [15] and Ishiwata [12] respectively. Now we shall say that a space X is (*weakly*) *CZ-expandable* if for any locally finite family $\{F_\xi|\xi\in\Xi\}$ of (regular) closed subsets of X, there exist a locally finite family $\{U_\xi|\xi\in\Xi\}$ of cozero-sets of X and a family $\{Z_\xi|\xi\in\Xi\}$ of zero-sets of X such that $F_\xi\subset Z_\xi\subset U_\xi$ for each $\xi\in\Xi$. Here a regular closed set is the closure of an open set. The notion of CZ-expandability is slightly stronger than that of cz-expandability due to Smith [21]. Clearly collectionwise normal countably paracompact spaces are CZ-expandable and extremally disconnected spaces are weakly CZ-expandable.

PROPOSITION 2.3.A. *If X is an M- (resp. M'-) space, then X is CZ- (resp. weakly CZ-) expandable.*

To prove Proposition 2.3.A, we use the following lemma.

LEMMA 2.3.B ([10, Lemma 4.2]). *Let f: $X\to T$ be an SZ-map from a space X onto a space T. If $\{H_n\}$ is a locally finite sequence of open sets of X and if $x_n\in H_n$ for each $n\in N$, then $\{f(x_n)\}$ is locally finite in T.*

PROOF OF PROPOSITION 2.3.A. Let X be an M- (resp. M'-) space. Then there is a quasi-perfect (resp. an SZ-) map f from X onto a metric space T. Let $\{F_\xi|\xi\in\Xi\}$ be a locally finite family of closed (resp. regular closed) subsets of X. Since a metric space is CZ-expandable, it suffices to show that $\{f(F_\xi)|\xi\in\Xi\}$ is locally finite in T. This is obvious in case f is a quasi-perfect map. So we assume that f is an SZ-map and each F_ξ is regular closed, that is, $F_\xi=\mathrm{cl}_X G_\xi$ for an open set G_ξ of X. If $\{f(F_\xi)|\xi\in\Xi\}$ is not locally finite at $t\in T$, then there is a countable neighborhood base $\{H_n|n\in N\}$ at t, and $f^{-1}(H_n)$ meets infinitely many G_ξ. We can take a countable subset $\{\xi(n)|n\in N\}$ of Ξ and a sequence $\{x_n|n\in N\}$ of X such that $x_n\in f^{-1}(H_n)\cap G_{\xi(n)}$. Then $\{f(x_n)|n\in N\}$ converges to t. But, since $\{G_{\xi(n)}|n\in N\}$ is locally finite, $\{f(x_n)|n\in N\}$ is locally finite in T by Lemma 2.3.B. This contradiction completes the proof.

3. Almost uniform structures and g-inverse systems.

DEFINITION 3.1. Let λ be a collection of almost-covers of a space X. We say that λ is an *almost uniform structure* of X if λ satisfies the following conditions:

(1) For every $\mathcal{U}\in\lambda$, $\mathcal{U}$ is locally finite in X.

(2) For every $\mathcal{U},\mathcal{V}\in\lambda$, there is $\mathcal{W}\in\lambda$ such that $\mathcal{W}$ is a refinement of $\mathcal{U}$ and $\mathcal{V}$.

(3) For each $x\in X$ and each neighborhood H at x, there is $\mathcal{U}\in\lambda$ such that St $(x,\bar{\mathcal{U}})\subset H$.

In this section, we show that if a space X is λ-complete for some almost uniform structure λ of X, then there exists a perfect map from a closed subspace in the product of discrete spaces onto X. To do this, it is necessary to select a λ-Cauchy family which converges to x for each $x\in X$. The key to this selection is the following observation.

DEFINITION 3.2. Let $(\Gamma, <)$ be a directed ordered set and $\{A_\gamma|\gamma\in\Gamma\}$ a collection of non-empty discrete spaces indexed by Γ. For each pair of indices γ, δ with $\gamma<\delta$, there is assigned a multi-valued map $\pi_\gamma{}^\delta$ of A_δ into A_γ satisfying the following conditions: If $\gamma<\delta<\varepsilon$, then

$$\pi_\gamma{}^\delta\circ\pi_\delta{}^\varepsilon(\alpha)\subset\pi_\gamma{}^\varepsilon(\alpha) \text{ for each } \alpha\in A_\varepsilon .$$

Then we call the collection $\{A_\gamma, \pi_\gamma{}^\delta, \Gamma\}$ a *g-inverse system.*

Let us consider the product space $A=\Pi\{A_\gamma|\gamma\in\Gamma\}$. We define a subspace Y of A as follows: $Y=\{p|p=(\alpha_\gamma|\gamma\in\Gamma)$, p satisfies the condition $p(\varDelta)$ below for any finite subset $\varDelta$ of $\Gamma\}$.

$$p(\varDelta)\begin{cases}\text{There are } \delta\in\Gamma \text{ and } \beta_0\in A_\delta \text{ such that } \gamma<\delta \text{ and}\\ \alpha_\gamma\in\pi_\gamma{}^\delta(\beta_0) \text{ for each } \gamma\in\varDelta .\end{cases}$$

Then we call Y the *g-inverse limit of* $\{A_\gamma, \pi_\gamma{}^\delta, \Gamma\}$ and denote it by $Y=g\text{-}\varprojlim\{A_\gamma, \pi_\gamma{}^\delta, \Gamma\}$ (or simply $Y=g\text{-}\varprojlim\{A_\gamma\}$).

We prove a elementary proposition about this notion.

PROPOSITION 3.3. *Let $\{A_\gamma, \pi_\gamma{}^\delta, \Gamma\}$ be a g-inverse system and $A=\Pi\{A_\gamma|\gamma\in\Gamma\}$. Then we have:*

(1) *$g\text{-}\varprojlim\{A_\gamma\}$ is a closed subspace of A.*

(2) *If there is a cofinal subset Γ_0 of Γ such that A_γ is finite for each $\gamma\in\Gamma_0$, then $g\text{-}\varprojlim\{A_\gamma\}\neq\emptyset$.*

PROOF. (1) Let $p=(\alpha_\gamma|\gamma\in\Gamma)\in A-g\text{-}\varprojlim\{A_\gamma\}$. Then there is a finite subset $\varDelta$ of Γ such that p does not satisfy the condition $p(\varDelta)$. If we put

$$U(p)=\Pi\{\{\alpha_\gamma\}|\gamma\in\varDelta\}\times\Pi\{A_\gamma|\gamma\in\Gamma-\varDelta\} ,$$

then $U(p)$ is a neighborhood at p in A disjoint from $g\text{-}\varprojlim\{A_\gamma\}$. Hence $g\text{-}\varprojlim\{A_\gamma\}$ is closed in A.

(2) In the first place, we suppose that A_γ is finite for each $\gamma\in\Gamma$. Let us set $F_\gamma=\{p\in A|p$ satisfies the condition $p(\varDelta)$ for any finite subset $\varDelta$ of $\{\beta\in\Gamma|\beta<\gamma\}\}$ for each $\gamma\in\Gamma$. Then, by the same argument as in (1), F_γ is closed in A. We shall show that each F_γ is non-empty. Let $\gamma\in\Gamma$ and let α_γ be an arbitrary fixed point of A_γ. We choose $\alpha_\beta\in\pi_\beta{}^\gamma(\alpha_\gamma)$ for each $\beta\in\Gamma$ with $\beta<\gamma$, and choose $\alpha_\delta\in A_\delta$ for each $\delta\in\Gamma$ with $\delta\nleqq\gamma$. Then the point $p=(\alpha_\gamma|\gamma\in\Gamma)$ belongs to F_γ, that is, F_γ is non-empty.

Moreover $F_\delta \subset F_\gamma$ holds whenever $\gamma < \delta$, which implies that $\{F_\gamma | \gamma \in \Gamma\}$ is a centered family of closed subsets of A. Since A is compact, $\cap\{F_\gamma | \gamma \in \Gamma\} \neq \emptyset$. On the other hand, it is easy to see that $\cap\{F_\gamma | \gamma \in \Gamma\} = g\text{-}\varprojlim\{A_\gamma\}$. Hence we have $g\text{-}\varprojlim\{A_\gamma\} \neq \emptyset$. To complete the proof, let Γ_0 be a cofinal subset of Γ such that A_γ is finite for each $\gamma \in \Gamma_0$. Since $\{A_\gamma, \pi_\gamma^\delta, \Gamma_0\}$ forms a g-inverse system, by the above argument, there is a point $q = (\beta_\gamma | \gamma \in \Gamma_0) \in g\text{-}\varprojlim\{A_\gamma, \pi_\gamma^\delta, \Gamma_0\}$. For each $\gamma \in \Gamma$, there is $\gamma' \in \Gamma_0$ with $\gamma' > \gamma$. Choosing an arbitrary point $\alpha_\gamma \in \pi_\gamma^{\gamma'}(\beta_{\gamma'})$, we put $p = (\alpha_\gamma | \gamma \in \Gamma)$. It suffices to show that $p \in g\text{-}\varprojlim\{A_\gamma, \pi_\gamma^\delta, \Gamma\}$. For any finite subset $\Delta = \{\gamma(1), \cdots, \gamma(n)\}$ of Γ, there is $\{\gamma'(1), \cdots, \gamma'(n)\} \subset \Gamma_0$ such that $\alpha_{\gamma(i)} \in \pi_{\gamma(i)}^{\gamma'(i)}(\beta_{\gamma'(i)})$ for $i = 1, \cdots, n$. Since $q \in g\text{-}\varprojlim\{A_\gamma, \pi_\gamma^\delta, \Gamma_0\}$, there are $\delta \in \Gamma_0$ and $\beta_0 \in A_\delta$ such that $\gamma'(i) < \delta$ and $\beta_{\gamma'(i)} \in \pi_{\gamma'(i)}^\delta(\beta_0)$ for each $i = 1, \cdots, n$. Then we have

$$\alpha_{\gamma(i)} \in \pi_{\gamma(i)}^{\gamma'(i)} \circ \pi_{\gamma'(i)}^{\delta}(\beta_0) \subset \pi_{\gamma(i)}^{\delta}(\beta_0) .$$

Hence p satisfies the condition $p(\Delta)$. This proves that $p \in g\text{-}\varprojlim\{A_\gamma\} \neq \emptyset$.

DEFINITION 3.4. Let $\lambda = \{\mathcal{U}_\gamma | \gamma \in \Gamma\}$, where $\mathcal{U}_\gamma = \{U(\gamma, \alpha) | \alpha \in A_\gamma\}$, be an almost uniform structure of a space X. We consider A_γ as a topological space with the discrete topology. Define an order $<$ on Γ as follows: $\gamma < \delta$ if and only if $\mathcal{U}_\delta$ is a refinement of $\mathcal{U}_\gamma$. Then $(\Gamma, <)$ is a directed ordered set. For each $\gamma, \delta \in \Gamma$ with $\gamma < \delta$, we define a multi-valued map $\pi_\gamma^\delta \colon A_\delta \to A_\gamma$ by

$$\pi_\gamma^\delta(\beta) = \{\alpha \in A_\gamma | \, U(\gamma, \alpha) \supset U(\delta, \beta)\} \text{ for } \beta \in A_\delta .$$

Then $\{A_\gamma, \pi_\gamma^\delta, \Gamma\}$ forms a g-inverse system. We call it the *g-inverse system associated with λ.*

THEOREM 3.5. *Let λ be an almost uniform structure of a space X and $\{A_\gamma. \pi_\gamma^\delta, \Gamma\}$ the g-inverse system associated with λ. If X is λ-complete, then there is a perfect map from $g\text{-}\varprojlim\{A_\gamma\}$ onto X.*

PROOF. Let us put $\lambda = \{\mathcal{U}_\gamma | \gamma \in \Gamma\}$, where $\mathcal{U}_\gamma = \{U(\gamma, \alpha) | \alpha \in A_\gamma\}$, and $A = \Pi\{A_\gamma | \gamma \in \Gamma\}$. If $p = (\alpha_\gamma | \gamma \in \Gamma) \in g\text{-}\varprojlim\{A_\gamma\}$, then $\mathcal{F}_p = \{U(\gamma, \alpha_\gamma) | \gamma \in \Gamma\}$ is a λ-Cauchy family of X. Since X is λ-complete, there is a point $x_p \in X$ such that $\cap \mathcal{F}_p = \{x_p\}$. We define a map f: $g\text{-}\varprojlim\{A_\gamma\} \to X$ by $f(p) = x_p$.

Claim 1. f is onto and $f^{-1}(x)$ is compact for each $x \in X$: Let $x \in X$. For each $\gamma \in \Gamma$, we set $B_\gamma = \{\alpha \in A_\gamma | x \in \mathrm{cl}_X U(\gamma, \alpha)\}$, then $\{B_\gamma, \pi_\gamma^\delta | B_\delta, \Gamma\}$ forms a g-inverse system. Since each $\mathcal{U}_\gamma$ is locally finite, B_γ is finite for each $\gamma \in \Gamma$. It follows from Proposition 3.3 (2) that there is $p \in g\text{-}\varprojlim\{B_\gamma\}$. Then we have $p \in g\text{-}\lim\{A_\gamma\}$ and $f(p) = x$, i.e., f is onto. If we put $B = \Pi\{B_\gamma | \gamma \in \Gamma\}$, then it is easily seen that $f^{-1}(x) = g\text{-}\varprojlim\{A_\gamma\} \cap B$. By Proposition 3.3 (1), $g\text{-}\varprojlim\{A_\gamma\}$ is closed in A. Since B is compact, it follows that

$f^{-1}(x)$ is compact.

Claim 2. f is continuous: Let $p=(\alpha_\gamma|\gamma\in\Gamma)\in g\text{-}\varprojlim\{A_\gamma\}$, and H a neighborhood at $f(p)$. Then there is $\delta\in\Gamma$ such that $\mathrm{St}(f(p), \bar{\mathcal{U}}_\delta)\subset H$. If we put

$$U(p)=(\{\alpha_\delta\}\times\Pi\{A_\gamma|\gamma\in\Gamma-\{\delta\}\})\cap g\text{-}\varprojlim\{A_\gamma\},$$

then $U(p)$ is a neighborhood at p such that $f(U(p))\subset H$.

Claim 3. f is a closed map: The proof is a modification of that of [19, VII, 2D]. Let F be a closed set of $g\text{-}\varprojlim\{A_\gamma\}$ and let $x\in X-f(F)$. Since $g\text{-}\varprojlim\{A_\gamma\}$ is closed in A, F is a closed subset of A with $F\cap f^{-1}(x)=\emptyset$. If we put $B_\gamma=\{\alpha\in A_\gamma|x\in\mathrm{cl}_X U(\gamma,\alpha)\}$ for each $\gamma\in\Gamma$ and $B=\Pi\{B_\gamma|\gamma\in\Gamma\}$, then $f^{-1}(x)=g\text{-}\varprojlim\{A_\gamma\}\cap B$. Hence we have $F\cap B=\emptyset$. Now we can assert that there is a finite subset Δ_0 of Γ such that $\cap\{\mathrm{cl}_X U(\gamma, \beta_\gamma)|\gamma\in\Delta_0\}\not\ni x$ for every $q=(\beta_\gamma|\gamma\in\Gamma)\in F$. For, if not, then we denote by $[\Gamma]$ the family of all finite subsets of Γ. We consider $[\Gamma]$ as a directed ordered set by the usual set inclusion relation. For each $\Delta\in[\Gamma]$, there is $\phi(\Delta)=(\beta_\gamma|\gamma\in\Gamma)\in F$ such that $\cap\{\mathrm{cl}_X U(\gamma, \beta_\gamma)|\gamma\in\Delta\}\ni x$. Choosing an arbitrary point $\alpha_\gamma\in B_\gamma$ for each $\gamma\in\Gamma-\Delta$, we set

$$\alpha_\gamma=\begin{cases}\beta_\gamma & \text{if } \gamma\in\Delta\,,\\ \alpha_\gamma & \text{if } \gamma\in\Gamma-\Delta\,.\end{cases}$$

If we put $\psi(\Delta)=(\alpha_\gamma|\gamma\in\Gamma)$, then $\psi(\Delta)\in B$. Since B is compact, the net $\{\psi(\Delta)|\Delta\in[\Gamma]\}$ has a cluster point $p\in B$. But it is easily seen that p is also a cluster point of $\{\phi(\Delta)|\Delta\in[\Gamma]\}$. This contradicts the fact that F is closed in A. Now we choose $\Delta_0=\{\gamma(1), \cdots, \gamma(n)\}$ satisfying the above condition. Then $\mathcal{G}=\{\mathrm{cl}_X U(\gamma(1), \alpha_{\gamma(1)})\cap\cdots\cap\mathrm{cl}_X U(\gamma(n), \alpha_{\gamma(n)})|\alpha_{\gamma(i)}\in A_{\gamma(i)}, i=1, \cdots, n\}$ is a locally finite closed cover of X satisfying $f(F)\subset\cup\{G|x\notin G\in\mathcal{G}\}$. Hence $H=X-\cup\{G|x\notin G\in\mathcal{G}\}$ is an open neighborhood at x such that $H\cap f(F)=\emptyset$. Thus $f(F)$ is closed in X. The proof is completed.

4. Images of topologically complete spaces and realcompact spaces under perfect maps. We firstly make the following definition.

DEFINITION 4.1. A space X is *almost topologically complete* if X is λ-complete for some almost uniform structure λ of X.

Since μ_X is an almost uniform structure of X, topologically complete spaces are almost topologically complete. Throughout the remainder of this paper, we use the following symbols:

ζ_X (resp. ζ_X^c): the collection of all locally finite (resp. countable locally finite) open almost-covers of a space X.

η_X (resp. η_X^c): the collection of all locally finite (resp. countable locally finite) covers of a space X.

If $\mathcal{U}$ and $\mathcal{V}$ are locally finite open almost-covers of a space X, then $\{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\}$ is also a locally finite open almost-cover of X. Therefore $\zeta_X, \zeta_X{}^c, \eta_X$ and $\eta_X{}^c$ are almost uniform structures of a space X.

As characterizations of perfect images of topologically complete spaces, we have the following theorem; (c)→(d) has been stated by Dykes [3]. For a space X, we write $\Delta(X) = \sup\{\mathrm{card}\, \mathcal{G} | \mathcal{G}$ is a locally finite family of non-empty open sets of $X\}$, where $\mathrm{card}\, \mathcal{G}$ denotes the cardinality of $\mathcal{G}$. If $\mathfrak{m}$ is a cardinal number, then we denote by $D(\mathfrak{m})$ a discrete space with cardinality $\mathfrak{m}$.

THEOREM 4.2. *For a space Y, the following conditions are equivalent:*

(a) *Y is an almost topologically complete space.*

(b) *Y is the image of a closed subspace in the product of copies of $D(\Delta(Y))$ under a perfect map.*

(c) *Y is the image of a topologically complete space under a perfect map.*

(d) *$E(Y)$ is topologically complete.*

(e) *Y is ζ_Y-complete.*

PROOF. (a)→(c). By Theorem 3.5, there exists a perfect map from a closed subspace in the product of discrete spaces onto Y. Since discrete spaces are topologically complete and topological completeness is preserved by product spaces and closed subspaces, we have (c).

(c)→(d). This implication is due to Dykes [3].

(d)→(e). To show that Y is ζ_Y-complete, let $\mathcal{F}$ be a ζ_Y-Cauchy family of Y. For each $\mathcal{U} \in \mu_{E(Y)}$, let us consider the family $\mathcal{V}(\mathcal{U}) = \{Y - e_Y(E(Y) - U) | U \in \mathcal{U}\}$. e_Y being a perfect irreducible map, we have $\mathcal{V}(\mathcal{U}) \in \zeta_Y$ (cf. [20, Proposition 3]). Then there are $U \in \mathcal{U}$ and $F \in \mathcal{F}$ such that $F \subset Y - e_Y(E(Y) - U)$, and hence $e_Y{}^{-1}(F) \subset U$. It follows that $\mathcal{G} = \{e_Y{}^{-1}(F) | F \in \mathcal{F}\}$ is a $\mu_{E(Y)}$-Cauchy family of $E(Y)$. Since $E(Y)$ is topologically complete, we have $p \in \cap \bar{\mathcal{G}}$ for some $p \in E(Y)$. Then $e_Y(p) \in \cap \bar{\mathcal{F}}$ and hence Y is ζ_Y-complete.

(e)→(b). This follows from Theorem 3.5.

(b)→(c) and (e)→(a) are obvious. Thus the proof is completed.

As a generalization of realcompact spaces, Frolik [6] introduced the notion of almost realcompact spaces. A space X is called *almost realcompact* if for every maximal open filter $\mathcal{G}$ of X with $\cap \bar{\mathcal{G}} = \emptyset$, there is a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $\cap \bar{\mathcal{G}}' = \emptyset$. Analogously to Theorem 4.2, we have the following result; (a)↔(d) and (a)↔(c) have been proved by Dykes [3] and Frolik [7] respectively.

THEOREM 4.3. *For a space Y, the following conditions are equivalent:*

(a) *Y is an almost realcompact space.*

(b) *Y is the image of a closed subspace in the product of copies of N under a perfect map.*

(c) *Y is the image of a realcompact space under a perfect map.*

(d) *$E(Y)$ is realcompact.*

(e) *Y is $\zeta_Y{}^c$-complete.*

(f) *Y is $\eta_Y{}^c$-complete.*

PROOF. The proof of the equivalence of (a), (b), (c), (d) and (e) is entirely similar to that of Theorem 4.2. (f)→(b) follows from Theorem 3.5. We shall show (a)→(f). To show (f), let $\mathcal{F}$ be a $\eta_Y{}^c$-Cauchy family of Y. We suppose to the contrary that $\cap \bar{\mathcal{F}}=\emptyset$. Set $\mathcal{G}_0=\{G|G$ is open in Y and there is $F\in\mathcal{F}$ with $F\subset G\}$. Let $\mathcal{G}$ be a maximal open filter containing $\mathcal{G}_0$. By the regularity of Y, $\cap \bar{\mathcal{G}}=\emptyset$ holds. Since Y is almost realcompact, there exists a countable subfamily $\{G_n|n\in N\}$ of $\mathcal{G}$ such that $\cap\{\mathrm{cl}_Y G_n|n\in N\}=\emptyset$. Then we may assume without loss of generality that $\{G_n|n\in N\}$ is decreasing and $G_1=Y$. For each $n\in N$, let us put $U_n=\mathrm{cl}_Y G_n-\mathrm{cl}_Y G_{n+1}$, then $\{U_n|n\in N\}\in\eta_Y{}^c$. $\mathcal{F}$ being a $\eta_Y{}^c$-Cauchy family, there are $F\in\mathcal{F}$ and $m\in N$ such that $F\subset U_m$. It follows that $Y-\mathrm{cl}_Y G_{m+1}\in\mathcal{G}_0\subset\mathcal{G}$, which contradicts the fact that $G_{m+1}\in\mathcal{G}$. Thus Y is $\eta_Y{}^c$-complete.

PROPOSITION 4.4. *An almost realcompact space is almost topologically complete. Conversely, an almost topologically complete space in which every discrete closed subset has non-measurable cardinal is almost realcompact.*

PROOF. The first assertion is an immediate consequence of Theorem 4.3. The second follows from Theorems 4.2, 4.3 and Shirota's theorem (cf. [8, 15.20]) that a topologically complete space in which every discrete closed subset has nonmeasurable cardinal is realcompact.

REMARKS 4.5. (1) Every η_Y-complete space Y is almost topologically complete. Since $\eta_Y{}^c$-completeness implies η_Y-completeness, by Proposition 4.4, an almost topologically complete space Y in which every discrete closed subset has non-measurable cardinal is η_Y-complete. However, it is open whether an almost topologically complete space Y is η_Y-complete or not in general.

(2) In [18], Mrówka constructed a non-topologically complete space Y such that Y is the union of two closed topologically complete subspaces. Then Y is also an example of an almost topologically complete space which is not topologically complete.

(3) Now we have the following relations:

realcompact	$\rightarrow$	topologically complete
$\downarrow$		$\downarrow$
almost realcompact	$\rightarrow$	almost topologically complete

Almost topologically complete spaces have nice properties as follows:

PROPOSITION 4.6. (1) *A closed subspace of an almost topologically complete space is almost topologically complete.*

(2) *The product of an arbitrary family of almost topologically complete spaces is almost topologically complete.*

PROOF. We remark that if $f_\alpha, \alpha \in A$, is a perfect map from a space X_α onto a space Y_α, then the product map f: $\Pi X_\alpha \rightarrow \Pi Y_\alpha$ (i.e., $f(p)=(f_\alpha(p_\alpha)|\alpha \in A)$, where $p=(p_\alpha|\alpha \in A) \in \Pi X_\alpha$) is perfect (cf. [19, VII, 2H]). Hence the proofs follow from Theorem 4.2.

PROPOSITION 4.7. *Let f: $X \rightarrow Y$ be an onto perfect map.*

(1) *If X is almost topologically complete, so is Y.*

(2) *If Y is almost topologically complete, so is X.*

PROOF. (1) is a consequence of Theorem 4.2. To show (2), let Y be almost topologically complete. By Proposition 4.6, $\beta X \times Y$ is almost topologically complete, where βX is the Stone-Čech compactification of X. If we put $X_1=\{(x, f(x))|x \in X\}$, then X_1 is a closed subset of $\beta X \times Y$ and is homeomorphic to X. Hence X is almost topologically complete by Proposition 4.6.

5. Almost topological completeness versus topological completeness. We shall isolate what is to be added to almost topological completeness to produce topological completeness. For the purpose, we introduce a new class of locally finite families.

NOTATION 5.1. For a space X, we use the following symbols:

$$\mu_X=\{\mathcal{U}_\gamma|\gamma \in \Gamma\}, \text{ where } \mathcal{U}_\gamma=\{U(\gamma, \alpha)|\alpha \in A_\gamma\}.$$

Let $\{A_\gamma, \pi_\gamma^\delta, \Gamma\}$ be the g-inverse system associated with μ_X. If $\mathcal{F}$ is a given family of subsets of X, then we set

$$A_\gamma(\mathcal{F})=\{\alpha \in A_\gamma | U(\gamma, \alpha) \text{ meets infinitely many members of } \mathcal{F}\} \text{ for each } \gamma \in \Gamma.$$

Then $\{A_\gamma(\mathcal{F}), \pi_\gamma^\delta | A_\delta(\mathcal{F}), \Gamma\}$ forms a g-inverse system.

DEFINITION 5.2. A family $\mathcal{F}$ of subsets of a space X is *μ-completely locally finite* if $g\text{-}\varprojlim\{A_\gamma(\mathcal{F})\}=\emptyset$.

DEFINITION 5.3. A space X is a *(weak) b^*-space* if every locally finite family of (regular) closed subsets of X is μ-completely locally finite.

Now we shall show which spaces belong to the class of (weak) b^*-spaces. We denote by μX the completion of X with respect to its finest uniformity. For the details of μX, see [16]. It is known that X is P-embedded in μX (i.e., every locally finite cozero-set cover of X has a refinement which can be extended to a locally finite cozero-set cover of μX).

LEMMA 5.4. *Let $\mathcal{F}$ be a family of subsets of a space X. The following conditions are equivalent*:

(a) *$\mathcal{F}$ is μ-completely locally finite.*

(b) *$\mathcal{F}$ is locally finite in μX.*

PROOF. We use the same symbols as in 5.1. (a)→(b). Suppose that $\mathcal{F}$ is not locally finite at $x\in\mu X$. For each $\gamma\in\Gamma$ and each $\alpha\in A_\gamma$, let us set

$$V(\gamma,\alpha)=\mu X-\mathrm{cl}_{\mu X}(X-U(\gamma,\alpha))\,,$$
$$A_\gamma(x)=\{\alpha\in A_\gamma\,|\,x\in V(\gamma,\alpha)\}\,.$$

Then $\{A_\gamma(x),\pi_\gamma^\delta|A_\delta(x),\Gamma\}$ forms a g-inverse system and $A_\gamma(x)\subset A_\gamma(\mathcal{F})$ holds for each $\gamma\in\Gamma$. Since X is P-embedded in μX, there exists a cofinal subset Γ_0 of Γ such that $A_\gamma(x)$ is finite for each $\gamma\in\Gamma_0$. Hence, by Proposition 3.3 (2), there is a point $p\in g\text{-}\varprojlim\{A_\gamma(x)\}$. Then we have $p\in g\text{-}\varprojlim\{A_\gamma(\mathcal{F})\}\neq\emptyset$, i.e., $\mathcal{F}$ is not μ-completely locally finite in X.

(b)→(a). Suppose that $\mathcal{F}$ is not μ-completely locally finite in X. Then there is $p=(\alpha_\gamma|\gamma\in\Gamma)\in g\text{-}\varprojlim\{A_\gamma(\mathcal{F})\}$. Let $\mathcal{G}=\{U(\gamma,\alpha_\gamma)|\gamma\in\Gamma\}$. Since $\mathcal{G}$ is a μ_X-Cauchy family of X, there is $x\in\mu X$ with $x\in\bigcap\{\mathrm{cl}_{\mu X}U(\gamma,\alpha_\gamma)|\gamma\in\Gamma\}$. We show that $\mathcal{F}$ is not locally finite at x. To see this, let H be a given neighborhood at x in μX. There is $f\in C(\mu X)$ such that $f(x)=0$ and $f(y)=1$ for each $y\in\mu X-H$. If we put

$$U_1=\{y\in X|\ |f(y)|<2/3\}\,,$$
$$U_2=\{y\in X|\ |f(y)|>1/3\}\,,$$

then $\{U_1,U_2\}\in\mu_X$, i.e., $\mathcal{U}_\gamma=\{U_1,U_2\}$ for some $\gamma\in\Gamma$. Since $x\notin\mathrm{cl}_{\mu X}U_2$, we have $U_1\in\mathcal{G}$. This implies that U_1 meets infinitely many members of $\mathcal{F}$, and hence so does H. It follows that $\mathcal{F}$ is not locally finite in μX. The proof is completed.

Consequently we have the following proposition.

PROPOSITION 5.5. *A topologically complete space is a b^*-space.*

Following [9], we say that a family $\mathcal{F}$ of subsets of a space X is *μ-uniformly locally* finite in X if $A_\gamma(\mathcal{F})=\emptyset$ for some $\gamma\in\Gamma$. Clearly every μ-uniformly locally finite family is μ-completely locally finite.

LEMMA 5.6. *Let* $\mathcal{F}=\{F_\xi|\xi\in\Xi\}$ *be a family of subsets of a space* X. *The following conditions are equivalent*:

(a) $\mathcal{F}$ *is* μ-*uniformly locally finite in* X.

(b) *There are a locally finite family* $\{H_\xi|\xi\in\Xi\}$ *of cozero-sets of* X *and a family* $\{Z_\xi|\xi\in\Xi\}$ *of zero-sets of* X *such that* $F_\xi\subset Z_\xi\subset H_\xi$ *for each* $\xi\in\Xi$.

(c) *As* (b), *with* "μX" *instead of* "X".

PROOF. We use the same symbols as in 5.1. (a)→(c). Let $\mathcal{F}$ be μ-uniformly locally finite in X, i.e., $A_\gamma(\mathcal{F})=\emptyset$ for some $\gamma\in\Gamma$. Since X is P-embedded in μX, there exists a locally finite cozero-set cover $\{U_\alpha|\alpha\in A\}$ of μX such that $\{U_\alpha\cap X|\alpha\in A\}$ is a refinement of $\mathcal{U}_\gamma$. Then each U_α meets at most finitely many members of $\mathcal{F}$. For each $\xi\in\Xi$, let $A_\xi=\{\alpha\in A|U_\alpha\cap F_\xi\neq\emptyset\}$, and let

$$Z_\xi=\mu X-\cup\{U_\alpha|\alpha\in A-A_\xi\} \text{ and } H_\xi=\cup\{U_\alpha|\alpha\in A_\xi\}.$$

Then $\{H_\xi|\xi\in\Xi\}$ and $\{Z_\xi|\xi\in\Xi\}$ have the desired properties.

(c)→(b) is clear.

(b)→(a). Let $\{H_\xi|\xi\in\Xi\}$ and $\{Z_\xi|\xi\in\Xi\}$ be families satisfying the stated condition. We denote by $[\Xi]$ the family of all finite subsets of Ξ. For each $\Delta\in[\Xi]$, let

$$U(\Delta)=\cap\{H_\xi|\xi\in\Delta\}-\cup\{Z_\xi|\xi\in\Xi-\Delta\}.$$

Then, by [17, Lemma 2.3], $U(\Delta)$ is a cozero-set of X, and it is easily seen that $\mathcal{U}=\{U(\Delta)|\Delta\in[\Xi]\}\in\mu_X$ and $U(\Delta)\cap F_\xi=\emptyset$ for each $\xi\in\Xi-\Delta$. It follows that $\mathcal{U}=\mathcal{U}_\gamma$ for some $\gamma\in\Gamma$, and then $A_\gamma(\mathcal{F})=\emptyset$. Thus $\mathcal{F}$ is μ-uniformly locally finite in X. The proof is completed.

We have at once from Lemma 5.6 the following result.

PROPOSITION 5.7. *A* (*weakly*) *CZ-expandable space is a* (*weak*) *b*-space.*

COROLLARY 5.8. (1) *An M-* (*M'-*) *space is a* (*weak*) *b*-space.*

(2) *A collectionwise normal countably paracompact space is a b*-space.*

(3) *An extremally disconnected space is a weak b*-space.*

PROOF. The proofs follow from Proposition 5.7 and the results in 2.3.

REMARKS 5.9. (1) A μ-completely locally finite family is not always μ-uniformly locally finite. For example, let X be the space $S\times S$, where S is the Sorgenfrey line. Since X is topologically complete, by Proposition 5.5, the locally finite family $\mathcal{F}=\{\{(-s,s)\}|s\in S\}$ is μ-completely locally finite in X. But $\mathcal{F}$ is not μ-uniformly locally finite by Lemma 5.6.

(2) There is a weak b^*-space which is not a b^*-space. Let ω_0 (resp. ω_1) be the first infinite (resp. first uncountable) ordinal. Let us set

$$X=(W(\omega_1+1)\times W(\omega_0+1))-\{(\omega_1,\omega_0)\},$$

where $W(\alpha)$ is the set of all ordinals less than α, topologized with order topology. Then X is pseudocompact and $\mu X=W(\omega_1+1)\times W(\omega_0+1)$. Since a pseudocompact space is an M'-space, by Corollary 5.8, X is a weak b^*-space. But the locally finite (in X) family $\{\{(\omega_1,\beta)\}|\beta<\omega_0\}$ is not locally finite in μX. Hence, by Lemma 5.4, X is not a b^*-space.

Now we establish the main theorem in this section.

THEOREM 5.10. *Let X be an almost topologically complete space. The following conditions are equivalent:*

(a) *X is topologically complete.*

(b) *X is a b^*-space.*

(c) *X is a weak b^*-space.*

PROOF. (a)→(b) follows from Proposition 5.5, and (b)→(c) is obvious. (c)→(a). It suffices to show that $X=\mu X$. Let $x\in\mu X$. Let us put $\zeta_X=\{\mathcal{W}_\sigma|\sigma\in\Sigma\}$, where $\mathcal{W}_\sigma=\{W(\sigma,\tau)|\tau\in T_\sigma\}$, and let $\{T_\sigma,\pi_\sigma^\rho,\Sigma\}$ be the g-inverse system associated ζ_X. Since X is a weak b^*-space, by Lemma 5.4, $\{\mathrm{cl}_{\mu X}W(\sigma,\tau)|\tau\in T_\sigma\}$ is locally finite in μX for each $\sigma\in\Sigma$, and hence it covers μX. For each $\sigma\in\Sigma$, let $T_\sigma(x)=\{\tau\in T_\sigma|x\in\mathrm{cl}_{\mu X}W(\sigma,\tau)\}$. Then each $T_\sigma(x)$ is finite and $\{T_\sigma(x),\pi_\sigma^\rho|T_\rho(x),\Sigma\}$ forms g-inverse system. By Proposition 3.3 (2), there is $p=(\tau_\sigma|\sigma\in\Sigma)\in g\text{-}\underleftarrow{\lim}\{T_\sigma(x)\}$. If we put $\mathcal{G}=\{W(\sigma,\tau_\sigma)|\sigma\in\Sigma\}$, then $\mathcal{G}$ is a ζ_X-Cauchy family such that $\{x\}=\cap\{\mathrm{cl}_{\mu X}W(\sigma,\tau_\sigma)|\sigma\in\Sigma\}$. Since X is ζ_X-complete by Theorem 4.2, we have $x\in X$, i.e., $X=\mu X$. The proof is completed.

The following corollaries are consequences of Theorems 4.2 and 5.10.

COROLLARY 5.11. *Let f be a perfect map from a topologically complete space onto a space Y. Then Y is topologically complete if and only if Y satisfies one of the conditions in Theorem* 5.10.

COROLLARY 5.12. *Let Y be the union of a locally finite family of closed topologically complete subspaces. Then Y is topologically complete if and only if Y satisfies one of the conditions in Theorem* 5.10.

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