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MONADIC SECOND ORDER LOGIC WITH AN ADDED QUANTIFIER Q

By

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Introduction

In this paper, we will show that *a monadic second order logic with an added quantifier* Q *is decidable.*

We begin with a description of some known facts concerning the decision problem for the predicate calculus. It was originally shown by L. Löwenheim (1951) that

(1) The monadic fragment of (first order) predicate calculus with equality is decidable.

Simpler proofs of (1) have been given Th. Skolem (1919) and H. Behmann (1922). Likewise there is the following result for the predicate calculus having the Chang quantifier:

(2) The monadic fragment of predicate calculus without equality containing the Chang quantifier is decidable. (A. Mostowski; 1957)

A. Slomson has extended this result further by proving, with the semantic method, that

(3) The monadic fragment of predicate calculus with the Chang quantifier and equality is decidable. (cf. [1])

On the other hand, it is also well-known that

(4) The monadic second order logic is decidable.

The sequential results mentioned above lead us in a natural way to the following "semantic" question: Is the monadic second order logic with the Chang quantifier decidable?

We extend this question to the decision problem formulated "syntactically" below; which turns to have an affirmative answer.

Let L be a monadic second order logic with an added quantifier Q, which will be defined explicitly in § 1. In addition to the usual symbols employed, L has (a) two sorts of unary predicate variables: free and bound, (b) no constants except

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logical constants (including the quantifier Q), (c) equality, (d) propositional constants: $\top$, $\bot$. For axiom sequents and rules of inference, except those for LK, L has rules of inference for second order quantifiers, a rule of inference for Q (called Yasuhara's Q-rule, cf. [2]), and the following essentially new axiom sequent for the second order quantifier $\exists$ and Q:

$$(Qv)A(v) \longrightarrow (\exists \xi)((Qv)(A(v) \wedge \xi(v)) \wedge (Qv)(A(v) \wedge \neg \xi(v))),$$

where $A(*)$ is a formula in L.

Now, our question becomes "Is L decidable?"; and we will show that the answer to this problem is "Yes".

In order to solve this problem, we shall prove a kind of a representation theorem (called "Main theorem" below), by a purely syntactic method, which is explained as follows.

Let $C_\varepsilon(*)$ be the formula

$$\bigwedge_{\alpha \in dom(\varepsilon)} \alpha^{\varepsilon(\alpha)}(*)$$

where ε is a map from a finite set of free predicate variables to $\{+, -\}$ and $\alpha^+(*)$ stands for $\alpha(*)$, $\alpha^-(*)$ stands for $\neg\alpha(*)$. Let $(\exists^i v)C_\varepsilon(v)$ be a formula expressing that "there are exactly i $*$'s which satisfy $C_\varepsilon(*)$". Then our Main theorem states:

Suppose that A is a sentence in L whose free predicate variables are all picked from among $\alpha_1, \cdots, \alpha_n$, and the numbers of whose second order and first order quantifiers are less than N, K, respectively. Then we can effectively find a Boolean combination C, which is equivalent to A in L, of sentences in

$$\left\{ \begin{matrix} (\exists^i v)C_\varepsilon(v) \\ (Qv)C_\varepsilon(v) \end{matrix} \;\middle|\; \begin{matrix} i=0, 1, \cdots, 2^N(K+1)-1; \\ dom(\varepsilon)=\{\alpha_1, \cdots, \alpha_n\} \end{matrix} \right\}.$$

Now our problem can be solved immediately as an application of this Main theorem. The reason why we use the syntactic method is because it gives us the following advantages:

(i) There are two kinds of semantic interpretations for Q (i.e. as the Chang quantifier and as the infinite quantifier), but syntactically we are able to prove the decidability of L independently of them.

(ii) Our proof of Lemma 3 in §4 gives a syntactic proof of (1) and (3).

This paper consists of four sections. After formulating a monadic second order logic L with an added quantifier Q explicitly in §1, we shall state our Main theorem in §2, which will be proved in §4. In §3, we shall give five applications of Main theorem. Our entire proof in §4 will be carried out concretely by a proof-theoretic method. This syntactic proof is made possible as a consequence of the theory underlying the work "Object logic and morphism logic" [4] (initiated by N. Motohashi). In the proof of Lemma 4, which corresponds to the induction step of the elimination of second order quantifiers, we are forced to adopt the new axiom sequent which was described previously. But this new axiom sequent can be interpreted as "an infinite set can be divided into two disjoint infinite sets", and so it is not contrary to our "mathematical commonsense". It seems, therefore, quite reasonable to adopt it as one of our axiom sequents.

I am deeply indebted to Dr. Nobuyoshi Motohashi for pointing out the significance of this "question" and for his valuable suggestions.

§ 1. Logic L

1. Symbols of L.

The symbols of L are divided into five groups as follows:

1.1) *Individual variables*:

Free variables: $a, b, \cdots, x, y, \cdots$ (with or without subscripts),

Bound variables: $u, v, \cdots$ (with or without subscripts).

1.2) *Unary predicate variables*:

Free variables: $\alpha, \beta, \cdots$ (with or without subscripts),

Bound variables: $\xi, \cdots$ (with or without subscripts).

1.3) *Logical constants*: $\neg, \wedge, \vee, \forall, \exists$, Q.

1.4) *Predicate constant*: $* = *$.

1.5) *Propositional constants*: $\top, \bot$.

So, it should be noted that L has no individual constant, function constant or predicate constant except the symbol $=$. We shall use the quantifier symbols $\forall, \exists$ both as first order quantifiers and as second order quantifiers, and the symbol Q as a first order quantifier. The symbols $\supset$ and $\equiv$ are used as abbreviations as usual.

2. Formation rules.

The rules of formation for the formulas in L are usual ones with the following added clause:

2.1) If $A(x)$ is a formula in L and v is a bound variable which does not occur in $A(x)$, then $(\mathrm{Q}v)A(v)$ is a formula in L.

Formulas in L will be denoted by $A, B, C, \cdots$ (with or without subscripts). If $A(\alpha)$ and $B(x)$ are formulas in L and v is a bound variable which does not occur in $B(x)$, then by $A(\lambda v B(v))$ we shall denote the formula obtained from $A(\alpha)$ by substituting $B(*)$ for $\alpha(*)$ in $A(\alpha)$. Notice that $A(\lambda v B(v))$ is defined so as to be a formula by avoiding any collision of the bound variables.

A sequent in L is a configuration of the form $\Gamma\rightarrow\Delta$, where Γ and Δ are finite (possible empty) sets of formulas in L. Note that this definition is not essentially different from the usual one. We shall therefore express $\{A_1, \cdots, A_m\}\rightarrow\{B_1, \cdots, B_n\}$ by $A_1, \cdots, A_m\rightarrow B_1, \cdots, B_n$ and $\Gamma_1\cup\Gamma_2\rightarrow\Delta_1\cup\Delta_2$ by $\Gamma_1, \Gamma_2\rightarrow\Delta_1, \Delta_2$ as usual.

3. Axioms and inference rules for *L*.

We use the axioms and the rules of inference for L which are divided into the following four groups.

3.1) The axioms and the rules of inference of the first order calculus LK which are formulated in Gentzen's style.

3.2) Inference rules for second order quantifiers $\forall$, $\exists$:

$$(\forall\rightarrow)\frac{A(\lambda vB(v)), \Gamma\longrightarrow\Delta}{(\forall\xi)A(\xi), \Gamma\longrightarrow\Delta}; \qquad (\rightarrow\forall)\frac{\Gamma\longrightarrow\Delta, A(\alpha)}{\Gamma\longrightarrow\Delta, (\forall\xi)A(\xi)},$$

$$(\exists\rightarrow)\frac{A(\alpha), \Gamma\longrightarrow\Delta}{(\exists\xi)A(\xi), \Gamma\longrightarrow\Delta}; \qquad (\rightarrow\exists)\frac{\Gamma\longrightarrow\Delta, A(\lambda vB(v))}{\Gamma\longrightarrow\Delta, (\exists\xi)A(\xi)},$$

where the free predicate variable α in $(\rightarrow\forall)$, $(\exists\rightarrow)$ does not occur in the lower sequent.

3.3) Axiom sequent for the quantifier Q and the second order quantifier $\exists$:

$$(\mathrm{Q}v)A(v)\longrightarrow(\exists\xi)((\mathrm{Q}v)(A(v)\wedge\xi(v))\wedge(\mathrm{Q}v)(A(v)\wedge\neg\xi(v))).$$

3.4) Inference rule for the quantifier Q (Yasuhara's Q-rule):

$$(\mathrm{Q})\ \frac{(A(a)), \Gamma\longrightarrow\Delta, B_1(a), \cdots, B_n(a)}{((\mathrm{Q}v)A(v)), \Gamma\longrightarrow\Delta, (\mathrm{Q}v)B_1(v), \cdots, (\mathrm{Q}v)B_n(v)},$$

where in the upper sequent the antecedent contains at most one formula in which a occurs, the succedent may contain no formulas in which a occurs, and a does not occur in the lower sequent.

LEMMA. The following sequents are provable in L:

(i) $(\mathrm{Q}v)A(v)\longrightarrow(\exists v)A(v)$,

(ii) $(\forall v)A(v)\longrightarrow(\mathrm{Q}v)A(v)$,

(iii) $(\mathrm{Q}v)(A(v)\vee B(v))\longrightarrow(\mathrm{Q}v)A(v)\vee(\mathrm{Q}v)B(v)$,

(iv) $(\mathrm{Q}v)A(v), (\forall v)(A(v)\supset B(v))\longrightarrow(\mathrm{Q}v)B(v)$,

(v) $(\mathrm{Q}v)(a=v)\longrightarrow$,

(vi) $(\mathrm{Q}v)A(v)\longrightarrow(\mathrm{Q}v)(A(v)\wedge v\neq a_1)$,

(vii) $(\mathrm{Q}v)A(v)\longrightarrow(\mathrm{Q}v)(A(v)\wedge v\neq a_1\wedge\cdots\wedge v\neq a_n)$,

(viii) $(\mathrm{Q}v)A(v)\longrightarrow(\exists u_1)\cdots(\exists u_n)(Iq(u_1, \cdots, u_n)\wedge A(u_1)\wedge\cdots\wedge A(u_n))$.

where $Iq(u_1, \cdots, u_n)$ is an abbreviation for $\bigwedge_{1 \leq i < j \leq n} u_i \neq u_j$.

This lemma is easily proved, so we omit it. But it should be noted that we have to use a cut-rule in the proof of (vi). Therefore, cut rules can not be eliminated in our system L.

§ 2. Main theorem

DEFINITIONS AND NOTATIONS: A *type* is a mapping from a finite set of free predicate variables to $\{+, -\}$. We shall denote types by $\varepsilon, \tau, \cdots$ (with or without subscripts), and the domain of a type ε by $dom(\varepsilon)$. A type ε will be called a type over $dom(\varepsilon)$. $C_\varepsilon(a)$ is an abbreviation for the formula

$$\bigwedge_{\alpha \in dom(\varepsilon)} \alpha^{\varepsilon(\alpha)}(a),$$

where $\alpha^{+}(a)$ is a formula $\alpha(a)$ and $\alpha^{-}(a)$ is a formula $\neg\alpha(a)$, and if dom(ε) is the empty set then $C_\varepsilon(a)$ means the propositional constant $\top$. Let $\bar{\alpha}$ be a repetition-free enumeration of dom(ε), then we may identify dom(ε) with $\bar{\alpha}$ for convenience sake. When dom(ε) is to be emphasized, $C_\varepsilon(a)$ may be expressed by $C_\varepsilon(a, \bar{\alpha})$.

If $A(x)$ is a formula in L and i is a non-negative integer, we shall use $(\exists^i v)A(v)$ as an abbreviation for

$$(\exists v_1)\cdots(\exists v_i)(\forall v)(Iq(v_1, \cdots, v_i) \wedge (A(v) \equiv (v=v_1 \vee \cdots \vee v=v_i))).$$

A *sentence* in L is a formula in L with no free individual variables. So, a sentence in L may be have free predicate variables. If A is a formula in L, by $nsq(A)$ and $nfq(A)$, we mean the number of second order quantifiers and first order quantifiers, respectively, which occur in A. Then our Main theorem is as follows:

MAIN THEOREM: *Let A be a sentence in L whose free predicate variables are all among $\bar{\alpha}$, where $\bar{\alpha}$ is a finite sequence of distinct free predicate variables. If $nsq(A) \leq N$ and $nfq(A) \leq K$ for some non-negative integers N and K, then A is equivalent in L to a Boolean combination C of sentences in*

$$\left\{ \begin{matrix} (\exists^i v)C_\varepsilon(v) \\ (Qv)C_\varepsilon(v) \end{matrix} \;\middle|\; \begin{matrix} i=0, 1, \cdots, 2^N(K+1)-1; \\ dom(\varepsilon)=\bar{\alpha} \end{matrix} \right\}.$$

Furthermore, C is obtained from A by a primitive recursive procedure.

The sentences of the form $(\exists^i v)C_\varepsilon(v), (Qv)C_\varepsilon(v)$ will be called *Basic sentences over the domain of* ε below. We shall give a proof of our Main theorem in § 4.

§ 3. Some applications of Main theorem

By $\vdash_L \Gamma \to \Delta$, we mean the sequent $\Gamma \to \Delta$ is provable in L and by $\vdash_L A$, the formula A is provable in L.

1. Decidability of L

For any sentence $A(\alpha_1, \cdots, \alpha_n)$ in L whose free predicate variables are all among $\alpha_1, \cdots, \alpha_n$, let $(\forall \xi_1)\cdots(\forall \xi_n)A(\xi_1, \cdots, \xi_n)$ be the sentence which results from $A(\alpha_1, \cdots, \alpha_n)$ on binding, by the quantifier $\forall$, all of the predicate variables that occur free in $A(\alpha_1, \cdots, \alpha_n)$. Then it is clear that

$$\vdash_L A(\alpha_1, \cdots, \alpha_n) \text{ if and only if } \vdash_L (\forall \xi_1)\cdots(\forall \xi_n)A(\xi_1, \cdots, \xi_n).$$

Now, by our Main theorem, $(\forall \xi_1)\cdots(\forall \xi_n)A(\xi_1, \cdots, \xi_n)$ is equivalent to a Boolean combination of Basic sentences over the empty set (i.e. $\top$) in L. Hence $(\forall \xi_1)\cdots(\forall \xi_n)A(\xi_1, \cdots, \xi_n)$ and so $A(\alpha_1, \cdots, \alpha_n)$ is decidable in L.

2. Completeness of L with respect to some semantics.

DEFINITIONS: An *L-structure* is a pair $(|\mathfrak{A}|, \mathfrak{A})$, where $|\mathfrak{A}|$ is an infinite set and $\mathfrak{A}$ is a map from the set consisting of all free predicate variables and the predicate constant "$=$" of L to the set of finitary relations on $|\mathfrak{A}|$ such that (i) $\mathfrak{A}(\alpha) \subseteq |\mathfrak{A}|$ for any free predicate variable α; (ii) $\mathfrak{A}(=)$ is the identity relation on $|\mathfrak{A}|$. We shall express L-structures $(|\mathfrak{A}|, \mathfrak{A}), \cdots$ by $\mathfrak{A}, \cdots$ simply.

Let $\mathfrak{A}$ be an L-structure, $A(x_1, \cdots, x_n)$ be a formula in L, and $a_1, \cdots, a_n$ be in $|\mathfrak{A}|$, then we define the relation $a_1, \cdots, a_n$ *satisfies* $A(x_1, \cdots, x_n)$ *in* $\mathfrak{A}$, which we write $\mathfrak{A} \models A[a_1, \cdots, a_n]$, as one obtained from usual definition by adding the following induction step:

Suppose that $A(x_1, \cdots, x_n)$ is of the form $(Qv)B(v, x_1, \cdots, x_n)$:

(1) $\mathfrak{A} \models (Qv)B(v)[a_1, \cdots, a_n]$ if and only if there exist exactly $\overline{\overline{\mathfrak{A}}}$ elements a's in $|\mathfrak{A}|$ such that $\mathfrak{A} \models B[a, a_1, \cdots, a_n]$; or

(2) $\mathfrak{A} \models (Qv)B(v)[a_1, \cdots, a_n]$ if and only if there exist at least ω elements a's in $|\mathfrak{A}|$ such that $\mathfrak{A} \models B[a, a_1, \cdots, a_n]$.

Q is said to be interpreted as *the Chang quantifier* (or as *the infinite quantifier*) in case (1) (or in case (2)). Whenever we make no reference to the quantifier Q particularly, one may interpret Q as either the Chang quantifier or the infinite quantifier.

Let $\mathfrak{A}$ be an L-structure, $(\Gamma \to \Delta)$ $(x_1, \cdots, x_n)$ be a sequent in L, where all free variables which occur in the elements of $\Gamma \cup \Delta$ are among $x_1, \cdots, x_n$, and $a_1, \cdots, a_n$ be in $\mathfrak{A}$, then we define the relation $a_1, \cdots, a_n$ *satisfies* $(\Gamma \to \Delta)$ $(x_1, \cdots, x_n)$ *in* $\mathfrak{A}$, which we write as $\mathfrak{A}\models(\Gamma\rightarrow\Delta)[a_1,\cdots,a_n]$, in the following manner:

$\mathfrak{A}\models(\Gamma\rightarrow\Delta)[a_1,\cdots,a_n]$ if and only if $\mathfrak{A}\not\models A[a_1,\cdots,a_n]$ for some $A(x_1,\cdots,x_n)$ in Γ, or $\mathfrak{A}\models B[a_1,\cdots,a_n]$ for some $B(x_1,\cdots,x_n)$ in Δ. A formula $A(x_1,\cdots,x_n)$ (sequent $(\Gamma\rightarrow\Delta)(x_1,\cdots,x_n)$) is *valid*, denoted by $\models A$ ($\models\Gamma\rightarrow\Delta$), if and only if it is satisfied by $a_1,\cdots,a_n$ in $\mathfrak{A}$ for any $a_1,\cdots,a_n$ in $|\mathfrak{A}|$ and any $\mathfrak{A}$.

Note that since we consider the infinite models only, the sequents which are provable in L are valid from the above definition.

THEOREM (Completeness): *A sentence is provable in L if and only if it is valid.*

PROOF. Suppose A is a sentence in L. It is obvious that if A is provable in L, then it is valid by the previous notes, and so we may only prove " if " part of the theorem. We assume that A is valid.

Let A be $A(\bar{\alpha})$, $nsq(A)\leqq N$, and $nfq(A)\leqq K$. Then by our Main theorem A is equivalent to a Boolean combination C of the Basic sentences over $\bar{\alpha}$ in L. If $\bar{\alpha}$ is the zero-sequence (i.e., the length of $\bar{\alpha}$ is equal to zero), then C is the Boolean combination of the Basic sentences over the empty set (i.e. $\top$) and so either $\vdash_L A\equiv\top$ or $\vdash_L A\equiv\bot$ holds. By the hypothesis A is valid, so $\vdash_L A\equiv\top$ must hold. Hence A is provable in L. Therefore we may assume that $\bar{\alpha}$ is a non-zero-sequence.

Suppose C is of the form

$$\bigwedge_{i=1}^{q}\bigvee_{j=1}^{r_i} D_{ij}$$

and $\{D_{i1},\cdots,D_{ir_i}\}=\{\neg A_{i1},\cdots,\neg A_{in_i},B_{i1},\cdots,B_{im_i}\}$, where $A_{i1},\cdots,A_{in_i}$ are mutually distinct Basic sentences over $\bar{\alpha}$, $B_{i1},\cdots,B_{im_i}$ are also mutually distinct Basic sentences over $\bar{\alpha}$, and $m_i+n_i\leqq r_i$. Then the following equivalence relations hold:

$$\begin{aligned}\vdash_L A \text{ iff } &\vdash_L \bigvee_{j=1}^{r_i} D_{ij} \text{ for any } i\in\{1,\cdots,q\}\\ \text{iff } &\vdash_L B_{i1}\vee\cdots\vee B_{im_i}\vee\neg A_{i1}\vee\cdots\vee\neg A_{in_i} \text{ for any } i\in\{1,\cdots,q\}\\ \text{iff } &\vdash_L A_{i1},\cdots,A_{in_i}\longrightarrow B_{i1},\cdots,B_{im_i} \text{ for any } i\in\{1,\cdots,q\}\,.\end{aligned}$$

Since such equivalence relations also hold for validity by the previous notes, we need only prove that for any $i\in\{1,\cdots,q\}$,

$$\models A_{i1},\cdots,A_{in_i}\longrightarrow B_{i1},\cdots,B_{im_i} \text{ implies } \vdash_L A_{i1},\cdots,A_{in_i}\longrightarrow B_{i1},\cdots,B_{im_i}\,.$$

To show this, it is sufficient to prove that for any distinct Basic sentences $A_1,\cdots,A_n$ over $\bar{\alpha}$ and any distinct Basic sentences $B_1,\cdots,B_m$ over $\bar{\alpha}$,

$$\models A_1,\cdots,A_n\longrightarrow B_1,\cdots,B_m \text{ implies } \vdash_L A_1,\cdots,A_n\longrightarrow B_1,\cdots,B_m$$

in general.

Let $\Gamma_0=\{A_1, \cdots, A_n\}$, $\Delta_0=\{B_1, \cdots, B_m\}$ and assume that $\Gamma_0\to\Delta_0$ is not provable in L. Let Φ be the set

$$\left\{\begin{matrix}(\exists^i v)C_\varepsilon(v) \\ (Qv)C_\varepsilon(v)\end{matrix} \;\middle|\; \begin{matrix} i=0,1,\cdots,2^N(K+1)-1\,; \\ dom(\varepsilon)=\bar{\alpha} \end{matrix}\right\}.$$

Then using a cut-rule we can easily construct the sets Γ, Δ which satisfy the following properties (1), (2) and (3):

(1) $\Gamma_0\subseteq\Gamma$ and $\Delta_0\subseteq\Delta$;

(2) $\Gamma\cup\Delta=\Phi$;

(3) $\Gamma\to\Delta$ is not provable in L.

We put $\Gamma-\Gamma_0=\{A_{n+1}, \cdots, A_s\}$, $\Delta-\Delta_0=\{B_{m+1}, \cdots, B_t\}$ $(s+t=\bar{\bar{\Phi}}<\omega)$ and we consider the following condition [*].

[*] $\begin{cases} & (4)\ \ A_i=B_j \text{ for some } i,j \text{ such that } 1\leqq i\leqq n, 1\leqq j\leqq m\,; \\ \text{or} & (5)\ \ A_i=(\exists^p v)C_\varepsilon(v) \text{ and } A_j=(\exists^q v)C_\varepsilon(v) \text{ for some } p\neq q, \varepsilon, i, j\,; \\ \text{or} & (6)\ \ A_i=(Qv)C_\varepsilon(v) \text{ and } A_j=(\exists^k v)C_\varepsilon(v) \text{ for some } k, \varepsilon, i, j\,; \\ \text{or} & (7)\ \ \{(Qv)C_\varepsilon(v)\}_{dom(\varepsilon)=\bar{\alpha}}\subseteq\Delta. \end{cases}$

If [*] holds, then $\Gamma\to\Delta$ is provable in L. In fact, we assume that [*] holds and so at least one of (4)-(7) holds. If (4) or (5) holds then $\vdash_L\Gamma\to\Delta$ is clear, and if (6) holds then it is also obvious by using (viii) of lemma. In the case of (7), we can prove that $\vdash_L\to\{(Qv)C_\varepsilon(v)\}_{dom(\varepsilon)=\bar{\alpha}}$ from $\vdash_L\bigvee_{dom(\varepsilon)=\bar{\alpha}}C_\varepsilon(x)$ and (iii) of lemma, hence $\vdash_L\Gamma\to\Delta$. Therefore, $\nvdash_L\Gamma\to\Delta$ implies that the condition [*] does not hold, and so it is adequate for our purposes to show $\not\models\Gamma\to\Delta$ on the assumption that the condition [*] does not hold.

Suppose that the condition [*] does not hold (i.e., all of (4)-(7) are not true). Since (4) is not true, all of elements of $\Gamma\cup\Delta$ are mutually distinct Basic sentences over $\bar{\alpha}$ by the construction of Γ, Δ. So each of them has an uniquely corresponding type over $\bar{\alpha}$. We can assume, therefore, that ε_i corresponds to A_i $(1\leqq i\leqq s)$, and τ_j to B_j $(1\leqq j\leqq t)$. By the hypothesis, the following are easily checked:

(8) $\varepsilon_i\neq\varepsilon_j$ if $i\neq j$,

(9) $\varepsilon_i=\tau_j$ holds in the following three cases only:

9.1. $A_i=(\exists^{r_i} v)C_{\varepsilon_i}(v), B_j=(\exists^{s_j} v)C_{\tau_j}(v), r_i\neq s_j$ and $\varepsilon_i=\tau_j$;

9.2. $A_i=(\exists^{r_i} v)C_{\varepsilon_i}(v), B_j=(Qv)C_{\tau_j}(v)$ and $\varepsilon_i=\tau_j$;

9.3. $A_i=(Qv)C_{\varepsilon_i}(v), B_j=(\exists^{s_j} v)C_{\tau_j}(v)$ and $\varepsilon_i=\tau_j$.

Owing to these (8), (9); for each ε_i, τ_j we can define its degree $d(\varepsilon_i), d(\tau_j)$, respectively, as follows:

$$d(\varepsilon_i)=\begin{cases} r_i & \text{if} \quad A_i=(\exists^{r_i} v)C_{\varepsilon_i}(v)\,, \\ \kappa & \text{if} \quad A_i=(\mathrm{Q}v)C_{\varepsilon_i}(v)\,, \end{cases}$$

where κ is an arbitrary infinite cardinal number. And

$$d(\tau_j)=2^N(K+1) \text{ if } \tau_j\neq\varepsilon_i \quad \text{for any } 1\leqq i\leqq s\,.$$

Clearly $\{\varepsilon_i, \tau_j\colon 1\leqq i\leqq s, 1\leqq j\leqq t\}=\{\varepsilon\colon dom(\varepsilon)=\bar{\alpha}\}$, so we could define the degree $d(\varepsilon)$ for each ε such that $dom(\varepsilon)=\bar{\alpha}$. Then, for each ε with $dom(\varepsilon)=\bar{\alpha}$ we shall define a set X_ε of free individual variables by

$$X_\varepsilon=\begin{cases} \{a_\nu^\varepsilon\colon \nu<d(\varepsilon)\} & \text{if} \quad d(\varepsilon)>0\,, \\ \text{the empty set} & \text{if} \quad d(\varepsilon)=0\,; \end{cases}$$

and let

$$|\mathfrak{A}|=\bigcup_{dom(\varepsilon)=\bar{\alpha}} X_\varepsilon\,;$$

$$\mathfrak{A}(\alpha)=\begin{cases} \bigcup_{dom(\varepsilon)=\bar{\alpha},\, \varepsilon(\alpha)=+} X_\varepsilon \text{ for any } \alpha\in\bar{\alpha}\,, \\ \text{the empty set for any free predicate variable } \alpha\notin\bar{\alpha}\,. \end{cases}$$

Then $\overline{\overline{\mathfrak{A}}}=\kappa$ because $(\mathrm{Q}v)C_\varepsilon(v)\in\Gamma$ for some ε with $dom(\varepsilon)=\bar{\alpha}$ on the assumption that (7) does not hold, so $\mathfrak{A}$ is an L-structure. And $\Gamma\to\Delta$ is not satisfied in $\mathfrak{A}$. In order to show this, it suffices to prove that $\mathfrak{A}\models A_i$ for any i and $\mathfrak{A}\not\models B_j$ for any j. We first show that the following [+] holds:

[+] $\quad \{x\in|\mathfrak{A}|\colon \mathfrak{A}\models C_\varepsilon(x)\}=X_\varepsilon \qquad$ for any ε with $dom(\varepsilon)=\bar{\alpha}$.

[Proof] ($\supseteqq$) Let $x\in X_\varepsilon$ and let α be in $\bar{\alpha}$. If $\varepsilon(\alpha)=+$, then $x\in\mathfrak{A}(\alpha)$ by the definition of $\mathfrak{A}(\alpha)$ and so $\mathfrak{A}\models\alpha(x)$. Hence $\mathfrak{A}\models\alpha^{\varepsilon(\alpha)}(x)$. If $\varepsilon(\alpha)=-$, then $x\notin\mathfrak{A}(\alpha)$ by the definition of $\mathfrak{A}(\alpha)$ and so $\mathfrak{A}\not\models\alpha(x)$. Hence $\mathfrak{A}\models\alpha^{\varepsilon(\alpha)}(x)$. We can therefore conclude that $\mathfrak{A}\models\alpha^{\varepsilon(\alpha)}(x)$. This shows that

$$\mathfrak{A}\models \bigwedge_{\alpha\in\bar{\alpha}}\alpha^{\varepsilon(\alpha)}(x) \text{ i.e., } \quad \mathfrak{A}\models C_\varepsilon(x)$$

since α is an arbitrary member in $\bar{\alpha}$.

($\subseteqq$) Let $x\notin X_\varepsilon$ and $x\in|\mathfrak{A}|$. Then $x\in X_{\varepsilon'}$ for some $\varepsilon'\neq\varepsilon$ with $dom(\varepsilon')=\bar{\alpha}$. By $\varepsilon'\neq\varepsilon$, $\varepsilon'(\alpha)\neq\varepsilon(\alpha)$ for some α in $\bar{\alpha}$. Therefore if $\varepsilon(\alpha)=+$, then $\varepsilon'(\alpha)=-$ and so $x\notin\mathfrak{A}(\alpha)$ by the definition of $\mathfrak{A}(\alpha)$. Hence $\mathfrak{A}\not\models\alpha(x)$, that is, $\mathfrak{A}\not\models\alpha^{\varepsilon(\alpha)}(x)$. If $\varepsilon(\alpha)=-$, then $\varepsilon'(\alpha)=+$ and so $x\in\mathfrak{A}(\alpha)$ by the definition of $\mathfrak{A}(\alpha)$. Hence $\mathfrak{A}\models\alpha(x)$, that is, $\mathfrak{A}\not\models\alpha^{\varepsilon(\alpha)}(x)$. We can therefore conclude that $\mathfrak{A}\not\models\alpha^{\varepsilon(\alpha)}(x)$. This shows that $\mathfrak{A}\not\models C_\varepsilon(x)$, so our proof of [+] is completed.

Now suppose $1\leqq i\leqq s$. We can divide our proof into two cases according to the form of A_i, since A_i is a Basic sentence.

Case 1. $A_i=(\exists^{r_i}v)C_{\varepsilon_i}(v)$. $\overline{\overline{X}}_{\varepsilon_i}=r_i$ because $d(\varepsilon_i)=r_i$. Hence by [+],

$$\overline{\overline{\{x\in|\mathfrak{A}|\colon \mathfrak{A}\models C_{\varepsilon_i}(x)\}}}=r_i, \text{ that is, } \mathfrak{A}\models A_i\,.$$

Case 2. $A_i=(Qv)C_{\varepsilon_i}(v)$. $\overline{\overline{X}}_{\varepsilon_i}=\kappa$ because $d(\varepsilon_i)=\kappa$. Hence by [+],

$$\overline{\overline{\{x\in|\mathfrak{A}|:\ \mathfrak{A}\models C_{\varepsilon_i}(x)\}}}=\kappa\ .$$

If Q is interpreted as the infinite quantifier, then clearly $\mathfrak{A}\models A_i$ as $\kappa\geqq\omega$. If Q is interpreted as the Chang quantifier, then also $\mathfrak{A}\models A_i$ by $\overline{\overline{\mathfrak{A}}}=\kappa$. In any case, it follows therefore that $\mathfrak{A}\models A_i$ for any $1\leqq i\leqq s$.

Also, suppose $1\leqq j\leqq t$. If $\tau_j\neq\varepsilon_i$ for any $1\leqq i\leqq s$, then $d(\tau_j)=2^N(K+1)$ without reference to the form of B_j. So, $d(\tau_j)$ is not an integer less than $2^N(K+1)$ and furthermore it is not κ. Hence $\mathfrak{A}\not\models B_j$. If $\tau_j=\varepsilon_i$ for some $1\leqq i\leqq s$, then the possibility is limited to the previous case (9). But $\mathfrak{A}\models A_i$ holds in any subcase 9.1.-9.3. of case (9), and so $\mathfrak{A}\not\models B_j$. In any case, it follows therefore that $\mathfrak{A}\not\models B_j$ for any $1\leqq j\leqq t$.

Thus we have shown $\mathfrak{A}\not\models\Gamma\to\Delta$ and $\mathfrak{A}\not\models\Gamma_0\to\Delta_0$. Hence $\Gamma_0\to\Delta_0$ is not valid. This completes our proof of Completeness theorem.

3. Compactness of the Chang quantifier

For any infinite set Γ of sentences in L of power $\kappa\geqq\omega$, the following theorem holds.

THEOREM (Compactness of the Chang quantifier): *If Q is interpreted as the Chang quantifier, then Γ has a model if and only if any finite subset of Γ has a model.*

PROOF. As "only if" part is obviously shown, we need only prove "if" part.

Suppose that any finite subset of Γ has a model. Let Ω be the set of all predicate variables that occur free in Γ. Clearly $\overline{\overline{\Omega}}\leqq\kappa$ but if $\overline{\overline{\Omega}}<\kappa$ then we can make $\overline{\overline{\Omega}}=\kappa$ by adding to Ω new free predicate variables which do not occur in Γ. Let Φ be the set of Basic sentences over each finite subset of Ω, so it is clear that $\overline{\overline{\Phi}}=\kappa$. Then let $\{C_\nu:\ \nu<\kappa\}$ be a repetition-free enumeration of all elements of Φ, and we shall construct an extension $\tilde{\Gamma}$ of Γ as follows:

$$\Gamma_0=\Gamma\ ;$$

$$\Gamma_{\nu+1}=\begin{cases}\Gamma_\nu\cup\{C_\nu\} & \text{if any finite subset of } \Gamma_\nu\cup\{C_\nu\} \text{ has a model,}\\ \Gamma_\nu\cup\{\neg C_\nu\} & \text{otherwise}\ ;\end{cases}$$

$$\Gamma_\nu=\cup_{\mu<\nu}\Gamma_\mu \qquad \text{if } \nu \text{ is a limit ordinal}\ ;$$

and

$$\tilde{\Gamma}=\cup_{\nu<\kappa}\Gamma_\nu\ .$$

By the construction of $\tilde{\Gamma}$, it is easily seen that $\tilde{\Gamma}$ has the following properties:

(1) $\Gamma \subseteq \tilde{\Gamma}$,

(2) any finite subset of $\tilde{\Gamma}$ has a model,

(3) $\tilde{\Gamma}$ is complete about Φ; that is, $A \in \tilde{\Gamma}$ or $\neg A \in \tilde{\Gamma}$ for any $A \in \Phi$.

Now, for each type ε over a finite subset of Ω, we define the degree $d(\varepsilon)$ by

$$d(\varepsilon)=\begin{cases} i & \text{if } (\exists^{i} v)C_{\varepsilon}(v) \in \tilde{\Gamma}\,, \\ \kappa^{+} & \text{if } (Qv)C_{\varepsilon}(v) \in \tilde{\Gamma}\,, \\ \kappa & \text{otherwise}\,. \end{cases}$$

Note that this definition is well-defined from the properties of $\tilde{\Gamma}$. For any two types ε, τ, by writing $\varepsilon \prec \tau$ we mean that τ is an extension of ε as a mapping. Then $\varepsilon \prec \tau$ implies $d(\varepsilon) \geqq d(\tau)$. Because if $\varepsilon \prec \tau$ and $d(\varepsilon) < d(\tau)$, then there exists a finite subset of $\tilde{\Gamma}$ which has no models by trivial classification.

A *full-type* is an element of $\{+, -\}^{\Omega}$. For any full-type f, we have a type $\varepsilon \prec f$ with the following property [*]:

[*] $\varepsilon \prec \tau \prec f$ implies $d(\tau) = d(\varepsilon)$, for any type τ.

We take such a type ε at will and fix it. We call this ε *a fixed type of* f, write ε_f, and define the degree of the full-type f by

$$d(f) = d(\varepsilon_f)\,.$$

Then the following hold:

(I) For any type ε over a finite subset of Ω such that $d(\varepsilon) < \omega$,

$$d(\varepsilon) = \Sigma_{f \succ \varepsilon} d(f)\,.$$

(II) Let ε be either a fixed type of some full-type or a type with $d(\varepsilon) = \kappa^{+}$ or κ. Then there exists a full-type $f_{\varepsilon} \succ \varepsilon$ such that $d(\varepsilon) = d(f_{\varepsilon})$.

[Proof of (I)] We first note that if f is a full-type and $f \succ \varepsilon$, then $d(f) \leqq d(\varepsilon)$. Let ε be a type over a finite subset of Ω with $d(\varepsilon) < \omega$, and let $\mathcal{F}_{\varepsilon}$ be a set $\{f\colon f \succ \varepsilon$ and $d(f) > 0\}$. Then $\mathcal{F}_{\varepsilon}$ is a finite set. Otherwise, there exists k-elements $f_1, \cdots, f_k$ ($k > d(\varepsilon)$) of $\mathcal{F}_{\varepsilon}$ such that

$$\vdash_L (\forall v)(C_{\tau_i}(v) \supset C_{\varepsilon}(v))$$

and

$$\vdash_L \neg(\exists v)(C_{\tau_i}(v) \wedge C_{\tau_j}(v)) \text{ if } i \neq j\,,$$

where $\tau_i = \varepsilon_{f_i} \cup \varepsilon$; ε_{f_i} is a fixed type of f_i ($i = 1, \cdots, k$). Then consider a set

$$\{(\exists^{d(\tau_i)} v)C_{\tau_i}(v)\colon 1 \leqq i \leqq k\} \cup \{\exists^{d(\varepsilon)} v)C_{\varepsilon}(v)\}\,.$$

This set is a finite subset of $\tilde{\Gamma}$ but has no models. This is contrary to the property of $\tilde{\Gamma}$. Hence $\mathcal{F}_{\varepsilon}$ is a finite set.

Moreover if $d(\varepsilon) \neq \Sigma_{f \succ \varepsilon}\, d(f)$ then there is a finite subset of $\tilde{\Gamma}$ which has no models, contradicting to the property of $\tilde{\Gamma}$.

[Proof of (II)] If ε is a fixed type of some full-type, we have nothing to prove. Assume first that $d(\varepsilon)=\kappa^{+}$. From the definition of $d(\varepsilon)$, $(Qv)C_{\varepsilon}(v)\in\tilde{\Gamma}$.

Now, let $X_{\kappa}=\{\alpha_{\nu}:\nu<\kappa\}=\Omega-\mathrm{dom}(\varepsilon)$, and we shall define a set $Y_{\kappa}=\{\beta_{\nu}:\nu<\kappa\}$ which satisfies the following conditions $(1)_{\kappa}$ and $(2)_{\kappa}$:

$(1)_{\kappa}$: for each $\nu<\kappa, \beta_{\nu}(*)=\alpha_{\nu}(*)$ or $\neg\alpha_{\nu}(*)$.

$(2)_{\kappa}$: for each finite subset ν_0 of κ,

$$\tilde{\Gamma}\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge_{\nu\in\nu_0}\beta_{\nu}(v)).$$

Suppose $\zeta<\kappa$ and we have defined $Y_{\zeta}=\{\beta_{\nu}:\nu<\zeta\}$ so as to satisfy the conditions $(1)_{\zeta}$ and $(2)_{\zeta}$, but that $(2)_{\zeta+1}$ is not satisfied if we put $\beta_{\zeta}(*)=\alpha_{\zeta}(*)$. We show that in this case it is satisfied if we put $\beta_{\zeta}(*)=\neg\alpha_{\zeta}(*)$.

On the assumption that $(2)_{\zeta+1}$ is not satisfied, there is some finite subset ζ_0 of ζ such that

$$\tilde{\Gamma}\not\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge_{\nu\in\zeta_0}\beta_{\nu}(v)\wedge\alpha_{\zeta}(v)). \qquad [1]$$

Let η_0 be any finite subset of ζ and let $\sigma_0=\zeta_0\cup\eta_0$, then as σ_0 is a finite subset of ζ, it follows that

$$\tilde{\Gamma}\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge_{\nu\in\sigma_0}\beta_{\nu}(v)),$$

by the hypothesis. So, by the properties of $\tilde{\Gamma}$, either

$$\tilde{\Gamma}\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge_{\nu\in\sigma_0}\beta_{\nu}(v)\wedge\alpha_{\zeta}(v)) \qquad [2]$$

or

$$\tilde{\Gamma}\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge_{\nu\in\sigma_0}\beta_{\nu}(v)\wedge\neg\alpha_{\zeta}(v)). \qquad [3]$$

If [2] holds, then since $\zeta_0\subseteq\sigma_0$,

$$\tilde{\Gamma}\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge_{\nu\in\zeta_0}\beta_{\nu}(v)\wedge\alpha_{\zeta}(v)),$$

contradicting to [1]. Therefore [3] holds. But $\eta_0\subseteq\sigma_0$, so

$$\tilde{\Gamma}\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge_{\nu\in\eta_0}\beta_{\nu}(v)\wedge\neg\alpha_{\zeta}(v)).$$

Thus we can define $Y_{\zeta+1}$ so as to satisfy the conditions $(1)_{\zeta+1}$ and $(2)_{\zeta+1}$. If ζ is a limit ordinal, then we define $Y_{\zeta}=\bigcup_{\nu<\zeta}Y_{\nu}$, and $Y_{\kappa}=\bigcup_{\zeta<\kappa}Y_{\zeta}$. This Y_{κ} satisfies $(1)_{\kappa}$ and $(2)_{\kappa}$. The full-type is now defined as follows:

$$f_{\varepsilon}\upharpoonright_{dom(\varepsilon)}=\varepsilon$$

$$f_{\varepsilon}(\alpha_{\nu})=\begin{cases}+ & \text{if } \beta_{\nu}(*)=\alpha_{\nu}(*)\\ - & \text{if } \beta_{\nu}(*)=\neg\alpha_{\nu}(*)\end{cases} \quad \text{for any } \alpha_{\nu}\in X_{\kappa};$$

Then it is obvious $\varepsilon\prec f_{\varepsilon}$ and $d(f_{\varepsilon})=\kappa^{+}=d(\varepsilon)$ by our construction. Therefore (II) holds if $d(\varepsilon)=\kappa^{+}$.

Suppose next that $d(\varepsilon)=\kappa$. By the definition of $d(\varepsilon)$,

$$\tilde{\Gamma}\ni\neg(\exists^{i}v)C_{\varepsilon}(v) \text{ for any } i<\omega \text{ and } \tilde{\Gamma}\ni\neg(Qv)C_{\varepsilon}(v)\,.$$

Now let $X_{\kappa}=\{\alpha_{\nu}:\nu<\kappa\}=\Omega\text{-dom}(\varepsilon)$, and we shall define a set $Y_{\kappa}=\{\beta_{\nu}:\nu<\kappa\}$ which satisfies the following conditions $(1)_{\kappa}$ and $(2)_{\kappa}$:

$(1)_{\kappa}$: for each $\nu<\kappa$, $\beta_{\nu}(*)=\alpha_{\nu}(*)$ or $\neg\alpha_{\nu}(*)$.

$(2)_{\kappa}$: for each finite subset ν_0 of κ,

$$\tilde{\Gamma}\ni\neg(\exists^{i}v)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\nu_0}\beta_{\nu}(v)) \text{ for any } i<\omega$$

and

$$\tilde{\Gamma}\ni\neg(Qv)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\nu_0}\beta_{\nu}(v))\,.$$

Suppose $\zeta<\kappa$ and we have defined $Y_{\zeta}=\{\beta_{\nu}:\nu<\zeta\}$ so as to satisfy the conditions $(1)_{\zeta}$ and $(2)_{\zeta}$, but that $(2)_{\zeta+1}$ is not satisfied if we put $\beta_{\zeta}(*)=\alpha_{\zeta}(*)$. We show that in this case it is satisfied if we put $\beta_{\zeta}(*)=\neg\alpha_{\zeta}(*)$. Let η_0 be any finite subset of ζ. It is sufficient to show that

$$\tilde{\Gamma}\ni\neg(\exists^{i}v)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\eta_0}\beta_{\nu}(v)\wedge\neg\alpha_{\zeta}(v)) \text{ for any } i<\omega \qquad [1]$$

and

$$\tilde{\Gamma}\ni\neg(Qv)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\eta_0}\beta_{\nu}(v)\wedge\neg\alpha_{\zeta}(v))\,. \qquad [2]$$

If [2] does not hold, then by the properties of $\tilde{\Gamma}$,

$$\tilde{\Gamma}\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\eta_0}\beta_{\nu}(v)\wedge\neg\alpha_{\zeta}(v))\,.$$

While, since η_0 is a finite subset of ζ and Y_{ζ} satisfies $(2)_{\zeta}$,

$$\tilde{\Gamma}\ni\neg(Qv)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\eta_0}\beta_{\nu}(v))\,.$$

Hence we have a finite subset

$$\{(Qv)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\eta_0}\beta_{\nu}(v)\wedge\neg\alpha_{\zeta}(v)),\ \neg(Qv)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\eta_0}\beta_{\nu}(v))\}$$

of $\tilde{\Gamma}$ which has no models in contradiction to the property of $\tilde{\Gamma}$. Therefore [2] holds and so we need only to prove [1].

By assumption, there exists a finite subset ζ_0 of ζ such that, either

$$\tilde{\Gamma}\ni(\exists^{i}v)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\zeta_0}\beta_{\nu}(v)\wedge\alpha_{\zeta}(v)) \text{ for some } i<\omega \qquad [3]$$

or

$$\tilde{\Gamma}\ni(Qv)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\zeta_0}\beta_{\nu}(v)\wedge\alpha_{\zeta}(v))\,. \qquad [4]$$

It is easily seen that [4] does not hold in the same manner in which we have shown that [2] holds. Therefore [3] only holds. Now let $\sigma_0=\zeta_0\cup\eta_0$. Since σ_0 is a finite subset of ζ,

$$\tilde{\Gamma}\ni\neg(\exists^{i}v)(C_{\varepsilon}(v)\wedge\bigwedge\nolimits_{\nu\in\sigma_0}\beta_{\nu}(v)) \text{ for any } i<\omega \qquad [5]$$

by the hypothesis of $(2)_\zeta$. Hence, the following [6] or [7] holds:

$$\tilde{\Gamma} \ni \neg(\exists^i v)(C_\varepsilon(v) \wedge \bigwedge_{\nu\in\sigma_0}\beta_\nu(v) \wedge \alpha_\zeta(v)) \text{ for any } i<\omega \quad [6]$$

$$\tilde{\Gamma} \ni \neg(\exists^i v)(C_\varepsilon(v) \vee \bigwedge_{\nu\in\sigma_0}\beta_\nu(v) \wedge \neg\alpha_\zeta(v)) \text{ for any } i<\omega \quad [7]$$

Because; suppose that neither [6] nor [7] hold there are some $i, j<\omega$ such that,

$$\tilde{\Gamma} \ni (\exists^i v)(C_\varepsilon(v) \wedge \bigwedge_{\nu\in\sigma_0}\beta_\nu(v) \wedge \alpha_\zeta(v)),$$

and

$$\tilde{\Gamma} \ni (\exists^j v)(C_\varepsilon(v) \wedge \bigwedge_{\nu\in\sigma_0}\beta_\nu(v) \wedge \neg\alpha_\zeta(v)).$$

Hence,

$$\tilde{\Gamma} \ni (\exists^{i+j} v)(C_\varepsilon(v) \wedge \bigwedge_{\nu\in\sigma_0}\beta_\nu(v)).$$

On the other hand, by [5],

$$\tilde{\Gamma} \ni \neg(\exists^{i+j} v)(C_\varepsilon(v) \wedge \bigwedge_{\nu\in\sigma_0}\beta_\nu(v))$$

also holds. So we have a finite subset of $\tilde{\Gamma}$ which has no models, contradicting to the property of $\tilde{\Gamma}$.

Suppose [6] holds. Since $\zeta_0 \subseteq \sigma_0$ and [3] holds, if we take $j<\omega$ which satisfies [3], we have a finite subset

$$\{(\exists^j v)(C_\varepsilon(v) \wedge \bigwedge_{\nu\in\zeta_0}\beta_\nu(v) \wedge \alpha_\zeta(v)), \neg(\exists^i v)(C_\varepsilon(v) \wedge \bigwedge_{\nu\in\sigma_0}\beta_\nu(v) \wedge \alpha_\zeta(v)) : i \leqq j\}$$

which has no models. This also contradicts to the property of $\tilde{\Gamma}$. Hence [7] holds. Therefore [1] holds as $\eta_0 \subseteq \sigma_0$.

We have now defined $Y_{\zeta+1}$ so as to satisfy the conditions $(1)_{\zeta+1}$ and $(2)_{\zeta+1}$. The definitions of Y_ε and f_ε are similar to the case $d(\varepsilon)=\kappa^+$, and they lead to $f_\varepsilon \succ \varepsilon$ and $d(f_\varepsilon)=\kappa=d(\varepsilon)$; so our proof of (II) is completed.

Now we return to the proof of the theorem. Let $\mathscr{E}$ be a set $\{\varepsilon : \varepsilon$ is a fixed type or a type of $d(\varepsilon)=\kappa^+, \kappa\}$; and let $\mathscr{F}=\{f_\varepsilon : \varepsilon\in\mathscr{E}\}$, then clearly $\overline{\overline{\mathscr{F}}}=\kappa$. For each $f\in\mathscr{F}$, we define a set X_f of free individual variables by

$$X_f=\begin{cases}\{a_\nu^f : \nu<d(f)\} & \text{if } d(f)>0,\\ \text{the empty set} & \text{if } d(f)=0;\end{cases}$$

and let

$$|\mathfrak{A}| = \cup_{f\in\mathscr{F}} X_f;$$

$$\mathfrak{A}(\alpha)=\begin{cases}\cup_{f\in\mathscr{F}, f(\alpha)=+} X_f & \text{for any } \alpha\in\Omega,\\ \text{the empty set for any free predicate variable } \alpha\notin\Omega.\end{cases}$$

Clearly $(Qv)C_\varepsilon(v)\in\tilde{\Gamma}$ for some ε and so $\overline{\overline{\mathfrak{A}}}=\kappa^+$, therefore $\mathfrak{A}$ is an L-structure. Moreover $\mathfrak{A}$ is a model of $\tilde{\Gamma}$. We shall show this in the following way. We first prove that the following (III) and (IV) hold.

(III) $\{x\in|\mathfrak{A}|: \mathfrak{A}\models C_\varepsilon(x)\}=\bigcup_{\varepsilon<f\in\mathscr{F}}X_f$ for any type ε.

(IV) If $D\in\Phi$, then $D\in\tilde{\Gamma}$ if and only if $\mathfrak{A}\models D$.

[Proof of (III)] Fix an arbitrary ε. ($\supseteq$) Suppose $x\in\bigcup_{\varepsilon<f\in\mathscr{F}}X_f$, then $x\in X_f$ for some $\varepsilon\prec f\in\mathscr{F}$. Let $\alpha\in\mathrm{dom}(\varepsilon)$. Then either $f(\alpha)=+$ or $f(\alpha)=-$. If $f(\alpha)=+$, then $x\in\mathfrak{A}(\alpha)$ and so $\mathfrak{A}\models\alpha(x)$ i.e., $\mathfrak{A}\models\alpha^{f(\alpha)}(x)$. Hence $\mathfrak{A}\models\alpha^{\varepsilon(\alpha)}(x)$. If $f(\alpha)=-$, then $x\notin\mathfrak{A}(\alpha)$ and so $\mathfrak{A}\not\models\alpha(x)$ i.e., $\mathfrak{A}\models\alpha^{f(\alpha)}(x)$. Hence $\mathfrak{A}\models\alpha^{\varepsilon(\alpha)}(x)$. In any case it follows that $\mathfrak{A}\models\alpha^{\varepsilon(\alpha)}(x)$. Since α is an arbitrary element of dom(ε),

$$\mathfrak{A}\models\bigwedge_{\alpha\in dom(\varepsilon)}\alpha^{\varepsilon(\alpha)},\ \text{that is, } \mathfrak{A}\models C_\varepsilon(x)\,.$$

($\subseteq$) Suppose $x\notin\bigcup_{\varepsilon<f\in\mathscr{F}}X_f$, then $x\notin X_f$ for any $\varepsilon\prec f\in\mathscr{F}$. So $x\in X_{f'}$ for some $f'\in\mathscr{F}$ and clearly this $f'\not\succ\varepsilon$. Hence $f'(\alpha)\neq\varepsilon(\alpha)$ for some $\alpha\in\mathrm{dom}(\varepsilon)$. If $f'(\alpha)=+$ (so, $\varepsilon(\alpha)=-$), then $x\in\mathfrak{A}(\alpha)$ i.e., $\mathfrak{A}\models\alpha(x)$. Hence $\mathfrak{A}\not\models\alpha^{\varepsilon(\alpha)}(x)$. If $f'(\alpha)=-$ (so, $\varepsilon(\alpha)=+$), then $x\notin\mathfrak{A}(\alpha)$ i.e., $\mathfrak{A}\not\models\alpha(x)$. Hence $\mathfrak{A}\not\models\alpha^{\varepsilon(\alpha)}(x)$. In any case, it follows also that $\mathfrak{A}\not\models\alpha^{\varepsilon(\alpha)}(x)$ for some α in dom(ε). That is,

$$x\notin\bigcap_{\alpha\in dom(\varepsilon)}\{x\in|\mathfrak{A}|: \mathfrak{A}\models\alpha^{\varepsilon(\alpha)}(x)\}=\{x\in|\mathfrak{A}|: \mathfrak{A}\models C_\varepsilon(x)\}\,.$$

[Proof of (IV)] From now on, for each type ε let X_ε be a set $\{x\in|\mathfrak{A}|: \mathfrak{A}\models C_\varepsilon(x)\}$. Suppose $D\in\Phi$. We can divide our proof into the following two cases according to the form of D.

Case 1. $D=(\exists^i v)C_\varepsilon(v)$.

$D\in\tilde{\Gamma}$ iff $d(\varepsilon)=i$ by the definition of $d(\varepsilon)$,
iff $\Sigma_{f\succ\varepsilon}d(f)=i$ by (I),
iff $\overline{\overline{\bigcup_{f\succ\varepsilon}X_f}}=i$,
iff $\overline{\overline{X}}_\varepsilon=i$ by (III) and the definition of X_ε,
iff $\mathfrak{A}\models D$.

Case 2. $D=(Qv)C_\varepsilon(v)$. $d(\varepsilon)=\kappa^+$ by $D\in\tilde{\Gamma}$. From (II), $d(\varepsilon)=d(f_\varepsilon)$ for some $f_\varepsilon\succ\varepsilon$. So,

$$\kappa^+=d(\varepsilon)=d(f_\varepsilon)=\overline{\overline{X}}_{f_\varepsilon}\leqq\overline{\overline{\bigcup_{f\succ\varepsilon}X_f}}=\Sigma_{f\succ\varepsilon}\overline{\overline{X}}_f\leqq\kappa\kappa^+=\kappa^+\,.$$

Hence,

$$\overline{\overline{X}}_\varepsilon=\overline{\overline{\bigcup_{f\succ\varepsilon}X_f}}=\kappa^+$$

by (III). Therefore $\overline{\overline{\mathfrak{A}}}=\overline{\overline{X}}_\varepsilon$, that is, $\mathfrak{A}\models D$.

Conversely, suppose $D\notin\tilde{\Gamma}$. Then $\neg D\in\tilde{\Gamma}$ by the completeness of $\tilde{\Gamma}$ about Φ. If $(\exists^i v)C_\varepsilon(v)\in\tilde{\Gamma}$, then $\mathfrak{A}\models(\exists^i v)C_\varepsilon(v)$ by case 1. For this reason $\mathfrak{A}\not\models(Qv)C_\varepsilon(v)$ i.e., $\mathfrak{A}\not\models D$. If $(\exists^i v)C_\varepsilon(v)\notin\tilde{\Gamma}$, then $\neg(\exists^i v)C_\varepsilon(v)\in\tilde{\Gamma}$ and $\neg D$ i.e. $\neg(Qv)C_\varepsilon(v)\in\tilde{\Gamma}$, so $d(\varepsilon)=\kappa$ by the definition of $d(\varepsilon)$. Hence $\overline{\overline{X}}_\varepsilon\leqq\kappa$. Because

$$\overline{\overline{X}}_\varepsilon=\overline{\overline{\bigcup_{f\succ\varepsilon}X_f}}=\Sigma_{f\succ\varepsilon}d(f)\leqq\kappa$$

by $d(f)\leqq d(\varepsilon)=\kappa$ if $f\succ\varepsilon$. Thus $\overline{\overline{X}}_\varepsilon$ is not equal to $\overline{\overline{\mathfrak{A}}}$ as $\overline{\overline{\mathfrak{A}}}=\kappa^+$. This means that $\mathfrak{A}\not\models(Qv)C_\varepsilon(v)$, that is, $\mathfrak{A}\not\models D$. Now (IV) has been proved.

Finally we must prove that $\mathfrak{A}$ is a model of $\tilde{\Gamma}$, but it is easily shown by using our Main theorem.

4. Compactness of the infinite quantifier

Let Γ be an infinite set of sentences in L of power $\kappa \geqq \omega$, and Γ satisfies the following conditions (1) and (2):

(1) The predicate variables which occur free in Γ are finite.

(2) The number of second order quantifiers and the number of first order quantifiers, which occur in the sentences in Γ, are both bounded.

Then the following theorem holds.

THEOREM (Compactness of the infinite quantifier): *If Q is interpreted as the infinite quantifier, then Γ has a model if and only if any finite subset of Γ has a model.*

PROOF. It is obvious by our Main theorem.

Counterexamples: (i) The following example of Γ shows that the condition (1) is necessary to prove the above-mentioned theorem.

$$\Gamma = \left\{ \begin{array}{l|l} \neg(Qv)\alpha(v), (\forall v)(\alpha_n(v) \supset \alpha(v)) & n, m < \omega \\ (\exists v)\alpha_n(v), \neg(\exists v)(\alpha_n(v) \wedge \alpha_m(v)) & n \neq m \end{array} \right\},$$

where $\{\alpha, \alpha_n\}_{n<\omega}$ are countable free predicate variables.

(ii) The following example of Γ shows that the condition (2) is necessary to the above-metioned theorem.

$$\Gamma = \{\neg(Qv)\alpha(v), \neg(\exists^i v)\alpha(v) : i < \omega\}.$$

5. Interpolation theorem.

In this system L, we cannot hope that a cut-elimination theorem holds, but we can prove the following interpolation theorem.

INTERPOLATION THEOREM: *Suppose that $A(\alpha, \beta)$ and $B(\beta, \gamma)$ are first order sentences in L, and the only predicate variables which occur in A and in B are α, β and β, γ, respectively. If $A(\alpha, \beta) \rightarrow B(\beta, \gamma)$ is provable in L, then there is a first order sentence $C(\beta)$ in L such that the only predicate variable which occurs in C is β, and moreover $A(\alpha, \beta) \rightarrow C(\beta)$ and $C(\beta) \rightarrow B(\beta, \gamma)$ are both provable in L.*

PROOF. Let $(\exists \xi)A(\xi, \beta)$ be the sentence obtained from $A(\alpha, \beta)$ by binding all α which occur in $A(\alpha, \beta)$. Suppose $nfq(A) \leqq K$, then $nfq((\exists \xi)A(\xi, \beta)) \leqq K$ and

$nsq((\exists\xi)A(\xi,\beta))=1$. By our Main theorem, $(\exists\xi)A(\xi,\beta)$ is equivalent in L to a Boolean combination $C(\beta)$ of elements in the following set:

$$\{(\exists^i v)C_\varepsilon(v), (Qv)C_\varepsilon(v):\ i=0,1,\cdots,2K+1;\ \mathrm{dom}(\varepsilon)=\{\beta\}\}.$$

§4. A proof of Main theorem.

Let L^1, L^2 be two object logics obtained from L by attaching 1, 2 to every symbol, except individual variables or logical constants, in every formula in L, respectively. If A is a formula in L, by A^i we shall denote the formula in L^i obtained from A by applying the operation stated above $(i=1,2)$. If Γ is a set of formulas in L, then by Γ^i we shall denote the set $\{A^i:\ A\in\Gamma\}$ of formulas in L^i.

From these two object logics L^1, L^2 we construct the morphism logic $\mathcal{L}=\mathcal{L}(L^1,L^2)$ by the method which has explicitly defined in [4]. That is, we define $\mathcal{L}$ as follows:

1. Formation rules for formulas in $\mathcal{L}$.

1.1) If A is a formula in L and $i=1,2$, then A^i is a formula in $\mathcal{L}$.

1.2) If A and B are formulas in $\mathcal{L}$, then $\neg A$, $A\wedge B$ and $A\vee B$ are formulas in $\mathcal{L}$.

1.3) If $A(x)$ is a formula in $\mathcal{L}$ and v is a bound variable which does not occur in $A(x)$, then $(\forall v)A(v)$ and $(\exists v)A(v)$ are formulas in $\mathcal{L}$.

1.4) All the formulas in $\mathcal{L}$ are obtained from 1.1)–1.3).

2. Axioms and inference rules for $\mathcal{L}$.

2.1) The axiom sequents in $\mathcal{L}$ are the sequents of the form $\Gamma^i\longrightarrow\Delta^i$, where $\Gamma\longrightarrow\Delta$ is a provable sequent in L and $i=1,2$.

2.2) The rules of inference of $\mathcal{L}$ are defined as those for LK which are formulated in Gentzen's style: only the following should be added to those.

$$(\forall\rightarrow)\ \frac{A(\lambda vB(v))^i,\Gamma\longrightarrow\Delta}{(\forall\xi)A(\xi)^i,\Gamma\longrightarrow\Delta}; \qquad (\rightarrow\forall)\ \frac{\Gamma\longrightarrow\Delta, A(\alpha)^i}{\Gamma\longrightarrow\Delta,(\forall\xi)A(\xi)^i},$$

$$(\exists\rightarrow)\ \frac{A(\alpha)^i,\Gamma\longrightarrow\Delta}{(\exists\xi)A(\xi)^i,\Gamma\longrightarrow\Delta}; \qquad (\rightarrow\exists)\ \frac{\Gamma\longrightarrow\Delta, A(\lambda vB(v))^i}{\Gamma\longrightarrow\Delta,(\exists\xi)A(\xi)^i},$$

where $i=1,2$ and α^i does not occur in the lower sequent.

Now, our Main theorem is easily proved by the following Theorem 1 and the Interpolation theorem in MOTOHASHI [3].

MOTOHASHI'S INTERPOLATION THEOREM: *Let S be the set of sentences in L and let $\Psi=\{E^1\supset E^2:\ E\in S\}$. Suppose that $\Delta(\Psi)$ be the set of sentences which is constructed from $S\cup\{\top,\bot\}$ by applying $\wedge$, $\vee$ a finite number of times. For any sen-*

tence A, B in L, if the proof-figure of

$$\Psi^*, A^1 \longrightarrow B^2 \ (\Psi^* \text{ is a finite subset of } \Psi)$$

is given concretely, then we can obtain a sentence C in $\Delta(\Psi)$ concretely (primitive recursively) such that

$$\vdash_L A \longrightarrow C \quad \text{and} \quad \vdash_L C \longrightarrow B.$$

Let k be a non-negative integer and let $\bar{\alpha}$ be a finite sequence of distinct free predicate variables. Then we put

$$\Gamma(k;\bar{\alpha}) = \left\{ \begin{matrix} (\exists^i v)C_\varepsilon(v)^1 \equiv (\exists^i v)C_\varepsilon(v)^2 \\ (Qv)C_\varepsilon(v)^1 \equiv (Qv)C_\varepsilon(v)^2 \end{matrix} \ \middle| \ \begin{matrix} i=0,\cdots,k; \\ \mathrm{dom}(\varepsilon)=\bar{\alpha} \end{matrix} \right\}$$

Notice that $\bar{\alpha}$ in theorems below is the same sequence which occurs in Main theorem.

THEOREM 1: *Suppose that A is a sentence in L whose free predicate variables are all among $\bar{\alpha}$. Let $nsq(A) \leqq N$ and $nfq(A) \leqq K$. Then*

$$\Gamma(2^N(K+1)-1;\bar{\alpha}) \vdash_{\mathcal{L}} A^1 \equiv A^2.$$

Note: Theorem 1 means that $A^1 \equiv A^2$ is provable from $\Gamma(2^N(K+1)-1;\bar{\alpha})$ in $\mathcal{L}$ primitive recursively according to the form of A. (In the following theorems and lemmas, the similar notes work.)

In order to prove Theorem 1, it is sufficient to show that the following Theorem 2 holds. Let $\bar{x} = \langle x_1, \cdots, x_p\rangle, \bar{y} = \langle y_1, \cdots, y_p\rangle$ be finite sequences of distinct free individual variables which have no common variables. Then we put

$$\Gamma(\bar{\alpha};\bar{x},\bar{y}) = \{x_i = {}^1x_i \equiv y_i = {}^2y_j : 1 \leqq i,j \leqq p\}$$
$$\cup \{C_\varepsilon(x_i,\bar{\alpha})^1 \equiv C_\varepsilon(y_i,\bar{\alpha})^2 : i=1,\cdots,p;\ \mathrm{dom}(\varepsilon)=\bar{\alpha}\}$$

THEOREM 2: *Suppose that $A(\bar{x})$ is a formula in L whose free predicate variables and free individual variables are all among $\bar{\alpha}, \bar{x}$, respectively. Let $nsq(A(\bar{x})) \leqq N$ and $nfq(A(\bar{x}))+p \leqq K$. Then*

$$\Gamma(2^N(K+1)-1;\bar{\alpha}), \Gamma(\bar{\alpha};\bar{x},\bar{y}) \vdash_{\mathcal{L}} A(\bar{x})^1 \equiv A(\bar{y})^2.$$

Theorem 2 can be easily proved by induction on the complexity of $A(\bar{x})$, using the following two lemmas. In lemmas below, we take $\bar{\alpha}, \bar{x}$ and $\bar{y}$ as in Theorem 2.

LEMMA 3: *Let x, y be two distinct free individual variables which do not occur in $\bar{x}, \bar{y}$, respectively. Suppose $A(\bar{x}, x)$ is a formula in L whose free variables are all among $\bar{x}, x$ and $\bar{\alpha}$. Moreover let $p+1 \leqq k$. If*

$$\Gamma(k;\bar{\alpha}), \Gamma(\bar{\alpha};\bar{x}^\wedge x, \bar{y}^\wedge y) \vdash_{\mathcal{L}} A(\bar{x},x)^1 \equiv A(\bar{y},y)^2,$$

then the following (1) *and* (2) *hold*:

(1) $\Gamma(k;\ \bar{\alpha}), \Gamma(\bar{\alpha};\ \bar{x}, \bar{y}) \vdash_{\mathcal{L}} (Qv)A(\bar{x}, v)^1 \equiv (Qv)A(\bar{y}, v)^2$,

(2) $\Gamma(k;\ \bar{\alpha}), \Gamma(\bar{\alpha};\ \bar{x}, \bar{y}) \vdash_{\mathcal{L}} (\exists v)A(\bar{x}, v)^1 \equiv (\exists v)A(\bar{y}, v)^2$.

LEMMA 4: *Let* α *be a new free predicate variable which does not occur in* $\bar{\alpha}$. *Suppose* $A(\bar{x}, \bar{\alpha}, \alpha)$ *is a formula in L whose free variables are all among* $\bar{x}, \bar{\alpha}$ *and* α. *Moreover let* $p \leqq k$. *If*

$$\Gamma(k;\ \bar{\alpha}^\wedge\alpha), \Gamma(\bar{\alpha}^\wedge\alpha;\ \bar{x}, \bar{y}) \vdash_{\mathcal{L}} A(\bar{x}, \bar{\alpha}, \alpha)^1 \equiv A(\bar{y}, \bar{\alpha}, \alpha)^2\,,$$

then

$$\Gamma(2k+1;\ \bar{\alpha}), \Gamma(\bar{\alpha};\ \bar{x}, \bar{y}) \vdash_{\mathcal{L}} (\exists \xi)A(\bar{x}, \bar{\alpha}, \xi)^1 \equiv (\exists \xi)A(\bar{y}, \bar{\alpha}, \xi)^2\,.$$

PROOF OF LEMMA 3. We put $\Gamma = \Gamma(k;\ \bar{\alpha}) \cup \Gamma(\bar{\alpha};\ \bar{x}, \bar{y})$ and assume that all of the hypothesis of Lemma 3.

[Proof of (1)] In order to prove (1), by the symmetry it suffices to show that

$$\Gamma \vdash_{\mathcal{L}} (Qv)A(\bar{x}, v)^1 \longrightarrow (Qv)A(\bar{y}, v)^2 \qquad [1]$$

By induction on the length of $\bar{\alpha}$ it is easily seen that

$$\bigvee_{dom(\varepsilon)=\bar{\alpha}} C_\varepsilon(x)$$

is provable in L, so we have

$$\vdash_L A(\bar{x}, x) \equiv \bigvee_{dom(\varepsilon)=\bar{\alpha}} (A(\bar{x}, x) \wedge C_\varepsilon(x))\,.$$

On the other hand, we know by (iii) of Lemma in §1 that for any formulas B, C in L

$$\vdash_L (Qv)(B(v) \vee C(v)) \equiv (Qv)B(v) \vee (Qv)C(v)$$

holds. Therefore we can get

$$\vdash_L (Qv)A(\bar{x}, v) \equiv \bigvee_{dom(\varepsilon)=\bar{\alpha}} (Qv)(A(\bar{x}, v) \wedge C_\varepsilon(v))\,.$$

Hence, all we need to prove is that

$$\Gamma \vdash_{\mathcal{L}} \bigvee_{dom(\varepsilon)=\bar{\alpha}} (Qv)(A(\bar{x}, v) \wedge C_\varepsilon(v))^1 \longrightarrow (Qv)A(\bar{y}, v)^2 \qquad [2]$$

To show this [2], we must prove that for any arbitrary fixed type ε such that $\mathrm{dom}(\varepsilon) = \bar{\alpha}$,

$$\Gamma \vdash_{\mathcal{L}} (Qv)(A(\bar{x}, v) \wedge C_\varepsilon(v))^1 \longrightarrow (Qv)A(\bar{y}, v)^2\,. \qquad [3]$$

From (Q)-rule and (viii) of Lemma in §1, it follows obviously that

$$\vdash_L (Qv)(A(\bar{x}, v) \wedge C_\varepsilon(v)) \longrightarrow (\exists v)(v \neq \bar{x} \wedge A(\bar{x}, v) \wedge C_\varepsilon(v)) \wedge (Qv)C_\varepsilon(v)\,,$$

and hence it is enough to prove the following [4] in order to show [3], where $v\neq\bar{x}$ is an abbreviation for $v\neq x_1\wedge\cdots\wedge v\neq x_p$. (We shall use similar abbreviations below.)

$$\Gamma\vdash_{\mathcal{L}} x\neq^1\bar{x},\ A(\bar{x},x)^1,\ C_\varepsilon(x)^1,\ (\mathrm{Q}v)C_\varepsilon(v)^1\longrightarrow(\mathrm{Q}v)A(\bar{y},v)^2 \qquad [4]$$

It is easily proved that for any B in $\Gamma(\bar{\alpha}\,;\ \bar{x}^\wedge x,\bar{y}^\wedge y)$,

$$\Gamma\vdash_{\mathcal{L}} x\neq^1\bar{x},\ y\neq^2\bar{y},\ C_\varepsilon(x)^1,\ C_\varepsilon(y)^2\longrightarrow B$$

holds, and since the hypothesis states that:

$$\Gamma(\boldsymbol{k}\,;\ \bar{\alpha}),\Gamma(\bar{\alpha}\,;\ \bar{x}^\wedge x,\bar{y}^\wedge y)\vdash_{\mathcal{L}} A(\bar{x},x)^1\longrightarrow A(\bar{y},y)^2\,,$$

it follows that

$$\Gamma\vdash_{\mathcal{L}} x\neq^1\bar{x},\ y\neq^2\bar{y},\ C_\varepsilon(x)^1,\ C_\varepsilon(y)^2,\ A(\bar{x},x)^1\longrightarrow A(\bar{y},y)^2 \qquad [5]$$

Whence we get the following [6]:

$$\Gamma\vdash_{\mathcal{L}} x\neq^1\bar{x},\ C_\varepsilon(x)^1,\ A(\bar{x},x)^1\longrightarrow(\forall v)(v\neq\bar{y}\wedge C_\varepsilon(v)\supset A(\bar{y},v))^2 \qquad [6]$$

Moreover, by (iv) and (vii) of Lemma in §1

$$(\mathrm{Q}v)(v\neq\bar{y}\wedge C_\varepsilon(v)),\ (\forall v)(v\neq\bar{y}\wedge C_\varepsilon(v)\supset A(\bar{y},v))\longrightarrow(\mathrm{Q}v)A(\bar{y},v)$$

and

$$(\mathrm{Q}v)C_\varepsilon(v)\longrightarrow(\mathrm{Q}v)(v\neq\bar{y}\wedge C_\varepsilon(v))$$

are both provable in L, then the following [7] and [8] hold.

$$\vdash_{\mathcal{L}}(\mathrm{Q}v)(v\neq\bar{y}\wedge C_\varepsilon(v))^2,\ (\forall v)(v\neq\bar{y}\wedge C_\varepsilon(v)\supset A(\bar{y},v))^2\longrightarrow(\mathrm{Q}v)A(\bar{y},v)^2 \qquad [7]$$

$$\vdash_{\mathcal{L}}(\mathrm{Q}v)C_\varepsilon(v)^2\longrightarrow(\mathrm{Q}v)(v\neq\bar{y}\wedge C_\varepsilon(v))^2 \qquad [8]$$

Applying a cut-rule between [6] and [7], and next between the obtained sequent and [8], we get

$$\Gamma\vdash_{\mathcal{L}} x\neq^1\bar{x},\ C_\varepsilon(x)^1,\ A(\bar{x},x)^1,\ (\mathrm{Q}v)C_\varepsilon(v)^2\longrightarrow(\mathrm{Q}v)A(\bar{y},v)^2 \qquad [9]$$

Since,

$$\Gamma\vdash_{\mathcal{L}}(\mathrm{Q}v)C_\varepsilon(v)^1\longrightarrow(\mathrm{Q}v)C_\varepsilon(v)^2 \qquad [10]$$

obviously holds, we can get [4] by applying a cut-rule between [9] and [10].

[Proof of (2)] To show (2), by the symmetry it is sufficient to prove that

$$\Gamma\vdash_{\mathcal{L}}(\exists v)A(\bar{x},v)^1\longrightarrow(\exists v)A(\bar{y},v)^2\,,$$

and so,

$$\Gamma\vdash_{\mathcal{L}} A(\bar{x},x)^1\longrightarrow(\exists v)A(\bar{y},v)^2\,. \qquad [1]$$

Since $x=\bar{x}\vee x\neq\bar{x}$ is obviously provable in L, it suffices to prove the following [2] and [3] in order to show [1], where $x=\bar{x}$ is an abbreviation for $x=x_1\vee\cdots\vee x=x_p$.

$$\Gamma\vdash_{\mathcal{L}} x=^1\bar{x},\ A(\bar{x},x)^1\longrightarrow(\exists v)A(\bar{y},v)^2 \qquad [2]$$

$$\Gamma \vdash_{\mathcal{L}} x \neq {}^1\bar{x}, A(\bar{x}, x)^1 \longrightarrow (\exists v)A(\bar{y}, v)^2 \qquad [3]$$

While, it follows from the assumption that for any $i \in \{1, \cdots, p\}$,

$$\Gamma \vdash_{\mathcal{L}} A(\bar{x}, x_i)^1 \longrightarrow A(\bar{y}, y_i)^2$$

and so,

$$\Gamma \vdash_{\mathcal{L}} x = {}^1 x_i, A(\bar{x}, x)^1 \longrightarrow A(\bar{y}, y_i)^2 .$$

Applying $(\rightarrow \exists)$, for each $i \in \{1, \cdots, p\}$

$$\Gamma \vdash_{\mathcal{L}} x = {}^1 x_i, A(\bar{x}, x)^1 \longrightarrow (\exists v)A(\bar{y}, v)^2$$

holds, and we get

$$\Gamma \vdash_{\mathcal{L}} x = {}^1 \bar{x}, A(\bar{x}, x)^1 \longrightarrow (\exists v)A(\bar{y}, v)^2$$

by $(\bigvee \rightarrow)$.

Therefore, all we have to do is to prove [3] only. Since $\bigvee_{dom(\varepsilon)=\bar{\alpha}} C_\varepsilon(x)$ is provable in L, in order to show [3], it suffices to prove that for each type ε such that $\mathrm{dom}(\varepsilon) = \bar{\alpha}$,

$$\Gamma \vdash_{\mathcal{L}} x \neq {}^1\bar{x}, C_\varepsilon(x)^1, A(\bar{x}, x)^1 \longrightarrow (\exists v)A(\bar{y}, v)^2 \qquad [4]_\varepsilon$$

holds. Let ε be a type such that $\mathrm{dom}(\varepsilon) = \bar{\alpha}$. Above $[4]_\varepsilon$ is proved if we show the following [5]:

$$\Gamma \vdash_{\mathcal{L}} x \neq {}^1\bar{x}, C_\varepsilon(x)^1 \longrightarrow (\exists v)(v \neq \bar{y} \wedge C_\varepsilon(v))^2 \qquad [5]$$

The reason is as follows: Since it is obviously seen that

$$\Gamma \vdash_{\mathcal{L}} y \neq {}^2\bar{y}, C_\varepsilon(y)^2, x \neq {}^1\bar{x}, C_\varepsilon(x)^1, A(\bar{x}, x)^1 \longrightarrow A(\bar{y}, y)^2 ,$$

if we first apply $(\rightarrow \exists)$ to this sequent, next do $(\bigwedge \rightarrow)$ and last do $(\exists \rightarrow)$, then we get

$$\Gamma \vdash_{\mathcal{L}} (\exists v)(v \neq \bar{y} \wedge C_\varepsilon(v))^2, x \neq {}^1\bar{x}, C_\varepsilon(x)^1, A(\bar{x}, x)^1 \longrightarrow (\exists v)A(\bar{y}, v)^2 . \qquad [6]$$

Applying a cut-rule between [5] and [6], we can conclude that $[4]_\varepsilon$ holds. In order to show [5], since

$$\bigvee_{i=0}^{k} (\exists^i v) C_\varepsilon(v) \vee (\bigwedge_{i=0}^{k} \neg (\exists^i v) C_\varepsilon(v))$$

is provable in L, it is adequate to prove that for each i $(0 \leqq i \leqq k)$,

$$\Gamma \vdash_{\mathcal{L}} (\exists^i v) C_\varepsilon(v)^1, x \neq {}^1\bar{x}, C_\varepsilon(x)^1 \longrightarrow (\exists v)(v \neq \bar{y} \wedge C_\varepsilon(v))^2 \qquad [7]_i$$

holds and

$$\Gamma \vdash_{\mathcal{L}} \bigwedge_{i=0}^{k} \neg (\exists^i v) C_\varepsilon(v)^1, x \neq {}^1\bar{x}, C_\varepsilon(x)^1 \longrightarrow (\exists v)(v \neq \bar{y} \wedge C_\varepsilon(v))^2 \qquad [7]$$

[7] obviously holds because the length of $\bar{x}$ (=length of $\bar{y}$) is equal to p, which we know to be less than or equal to k, and $[7]_0$ also holds. The proof is therefore completed if we prove $[7]_i$ for each i such that $1 \leqq i \leqq k$.

Fix an arbitrary i such that $1 \leqq i \leqq k$. To prove $[7]_i$, we need only show that for any new free individual variables $z_1, \cdots, z_i$,

$$\Gamma \vdash_{\mathcal{L}} Iq(z_1, \cdots, z_i)^2, C_\varepsilon(z_1)^2, \cdots, C_\varepsilon(z_i)^2, (\exists^i v) C_\varepsilon(v)^1, x \neq^1 \bar{x}, C_\varepsilon(x)^1 \longrightarrow z_1 \neq^2 \bar{y}, \cdots, z_i \neq^2 \bar{y} .$$

Let $z_1, \cdots, z_i$ be new free individual variables. To show this, it suffices to prove that

$$\Gamma \vdash_{\mathcal{L}} Iq(z_1, \cdots, z_i)^2, C_\varepsilon(z_1)^2, \cdots, C_\varepsilon(z_i)^2, (\exists^i v) C_\varepsilon(v)^1, x \neq^1 \bar{x}, C_\varepsilon(x)^1, z_1 =^2 y_1 \vee \cdots \vee z_1 =^2 y_p, \cdots, z_i =^2 y_1 \vee \cdots \vee z_i =^2 y_p \longrightarrow ,$$

which will follow if we show that: if $r_1, \cdots, r_i$ are any elements in $\{1, \cdots, p\}$,

$$\Gamma \vdash_{\mathcal{L}} Iq(z_1, \cdots, z_i)^2, C_\varepsilon(z_1)^2, \cdots, C_\varepsilon(z_i)^2, (\exists^i v) C_\varepsilon(v)^1, x \neq^1 \bar{x}, C_\varepsilon(x)^1, z_1 =^2 y_{r_1}, \cdots, z_i =^2 y_{r_i} \longrightarrow . \qquad [8]_i$$

Let $r_1, \cdots, r_i$ be elements in $\{1, \cdots, p\}$. If $r_s = r_t$ for some s, t such that $s \neq t$, then $[8]_i$ is obviously true. Hence we may assume that $r_s \neq r_t$ if $s \neq t$, that is, $r_1 = 1, \cdots, r_i = i$, after some exchanging if necessary. Then $[8]_i$ is

$$\Gamma \vdash_{\mathcal{L}} Iq(z_1, \cdots, z_i)^2, C_\varepsilon(z_1)^2, \cdots, C_\varepsilon(z_i)^2, (\exists^i v) C_\varepsilon(v)^1, x \neq^1 \bar{x}, C_\varepsilon(x)^1, z_1 =^2 y_1, \cdots, z_i =^2 y_i \longrightarrow . \qquad [8]_i'$$

By the way,

$$\vdash_{\mathcal{L}} z_1 =^2 y_1, \cdots, z_i =^2 y_i, Iq(z_1, \cdots, z_i)^2, C_\varepsilon(z_1)^2, \cdots, C_\varepsilon(z_i)^2 \longrightarrow (Iq(y_1, \cdots, y_i) \wedge C_\varepsilon(y_1) \wedge \cdots \wedge C_\varepsilon(y_i))^2$$

clearly holds, and hence, in order to show $[8]_i'$, it is enough to prove that

$$\Gamma \vdash_{\mathcal{L}} Iq(y_1, \cdots, y_i)^2, C_\varepsilon(y_1)^2, \cdots, C_\varepsilon(y_i)^2, (\exists^i v) C_\varepsilon(v)^1, x \neq^1 \bar{x}, C_\varepsilon(x)^1 \longrightarrow . \qquad [9]_i$$

Moreover, it is easily seen that

$$\Gamma \vdash_{\mathcal{L}} Iq(y_1, \cdots, y_i)^2 \longrightarrow x_m \neq^1 x_n \text{ for any } 1 \leqq m < n \leqq i$$

and

$$\Gamma \vdash_{\mathcal{L}} C_\varepsilon(y_r)^2 \longrightarrow C_\varepsilon(x_r)^1 \text{ for any } 1 \leqq r \leqq i$$

hold. Hence we get

$$\Gamma \vdash_{\mathcal{L}} Iq(y_1, \cdots, y_i)^2, C_\varepsilon(y_1)^2, \cdots, C_\varepsilon(y_i)^2 \longrightarrow (Iq(x_1, \cdots, x_i) \wedge C_\varepsilon(x_1) \wedge \cdots \wedge C_\varepsilon(x_i))^1 .$$

Then, to show $[9]_i$, it suffices to prove that

$$\Gamma \vdash_{\mathcal{L}} Iq(x_1, \cdots, x_i)^1, C_\varepsilon(x_1)^1, \cdots, C_\varepsilon(x_i)^1, (\exists^i v) C_\varepsilon(v)^1 \; x \neq^1 \bar{x}, C_\varepsilon(x)^1 \longrightarrow ,$$

which will follow immediately if we show that

$$\vdash_L Iq(x_1, \cdots, x_i), C_\varepsilon(x_1), \cdots, C_\varepsilon(x_i), (\exists^i v)C_\varepsilon(v), x \neq \bar{x}, C_\varepsilon(x) \longrightarrow ,$$

but this holds evidently. Therefore Lemma 3 is here proved.

Next we shall first give a sublemma and a definition which we need to prove Lemma 4.

SUBLEMMA: *Let k be a non-negative integer. Suppose that $Y=\{y_1, \cdots, y_p\}(p \leqq k)$ is a set of distinct free individual variables and let $Y=Y_+ \cup Y_-$ (disjoint union). If $A(x)$ is a formula in L and $D(\alpha)$ is a formula*

$$(\forall v)(\alpha(v) \supset A(v)) \wedge \bigwedge_{y \in Y_+} \alpha(y) \wedge \bigwedge_{y \in Y_-} \neg \alpha(y),$$

then the following sequents (1)-(10) are provable in L.

(1) $(\exists^i v)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)\xi(v))$,
where $\overline{Y}_+ \leqq j \leqq i - \overline{Y}_-$.

(2) $(\exists^i v)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)(A(v) \wedge \neg \xi(v)))$,
where $\overline{Y}_- \leqq j \leqq i - \overline{Y}_+$.

(3) $\{\neg(\exists^i v)A(v)\}_{i=0}^{2k+1}, \neg(Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y}$
$\longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)\xi(v))$,
where $\overline{Y}_+ \leqq j \leqq k$.

(4) $\{\neg(\exists^i v)A(v)\}_{i=0}^{2k+1}, \neg(Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y}$
$\longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)(A(v) \wedge \neg \xi(v)))$,
where $\overline{Y}_+ \leqq j \leqq k$.

(5) $\{\neg(\exists^i v)A(v)\}_{i=0}^{2k+1}, \neg(Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y}$
$\longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^{>k} v)\xi(v) \wedge (\exists^{>k} v)(A(v) \wedge \neg \xi(v)))$.

(6) $(Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)\xi(v))$,
where $\overline{Y}_+ \leqq j$.

(7) $(Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)(A(v) \wedge \neg \xi(v)))$,
where $\overline{Y}_- \leqq j$.

(8) $(Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^{>k} v)\xi(v) \wedge (Qv)(A(v) \wedge \neg \xi(v)))$.

(9) $(Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (Qv)\xi(v) \wedge (\exists^{>k} v)(A(v) \wedge \neg \xi(v)))$.

(10) $(Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y \in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (Qv)\xi(v) \wedge (Qv)(A(v) \wedge \neg \xi(v)))$.

where $(\exists^{>k} v)A(v)$ is an abbreviation for

$$(\exists v_1) \cdots (\exists v_{k+1})(Iq(v_1, \cdots, v_{k+1}) \wedge A(v_1) \wedge \cdots \wedge A(v_{k+1})).$$

PROOF OF SUBLEMMA. (1) We may assume $p \leqq i$. Let $m=i-p$ and $n=j-\overline{Y}_{+}$, so $n \leqq m$. We take m distinct free individual variables $a_1, \cdots a_n, \cdots, a_m$ which are not in Y and do not occur in $A(x)$. Let $F(a)$ be a formula

$$(\bigvee_{y\in Y+} a=y \vee \bigvee_{r=1}^{n} a=a_r) \wedge \bigwedge_{y\in Y-} a\neq y \wedge \bigwedge_{r=n+1}^{m} a\neq a_r .$$

Then, it is easily shown that

$$\vdash_L (\exists^i v)A(v), Iq(y_1, \cdots, y_p, a_1, \cdots, a_m), \{A(y)\}_{y\in Y}, \{A(a_r)\}_{r=1}^{m} \longrightarrow D(\lambda v F(v)) \wedge (\exists^j v)F(v) .$$

Applying $(\rightarrow \exists)$, we get

$$\vdash_L (\exists^i v)A(v), Iq(y_1, \cdots, y_p, a_1, \cdots, a_m), \{A(y)\}_{y\in Y}, \{A(a_r)\}_{r=1}^{m} \longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)\xi(v)) .$$

We divide $Iq(y_1, \cdots, y_p, a_1, \cdots, a_m)$ into $Iq(y_1, \cdots, y_p)$, $Iq(a_1, \cdots, a_m) \wedge \bigwedge_{\substack{1\leqq i\leqq m \\ 1\leqq j\leqq p}} a_i \neq y_j$ in the antecedent in this sequent. Next, connecting $Iq(a_1, \cdots, a_m) \wedge \bigwedge_{i,j} a_i \neq y_j$ and all of the elements in $\{A(a_r)\}_{r=1}^{m}$ by $\wedge$ in the antecedent and applying $(\exists \rightarrow)$ in turn, since $a_1, \cdots, a_m$ do not occur in any of $(\exists^i v)A(v), Iq(y_1, \cdots, y_p)$, any $A(y), (y\in Y)$, we get

$$\vdash_L (\exists^i v)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y\in Y}, (\exists v_1)\cdots(\exists v_m)(Iq(v_1, \cdots, v_m) \wedge \bigwedge_{\substack{1\leqq i\leqq m \\ 1\leqq j\leqq p}} v_i \neq y_j \wedge \bigwedge_{1\leqq i\leqq m} A(v_i)) \longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)\xi(v)) .$$

On the other hand, we have

$$\vdash_L (\exists^i v)A(v) \longrightarrow (\exists v_1)\cdots(\exists v_m)(Iq(v_1, \cdots, v_m) \wedge \bigwedge_{\substack{1\leqq i\leqq m \\ 1\leqq j\leqq p}} v_i \neq y_j \wedge \bigwedge_{1\leqq i\leqq m} A(v_i)) .$$

by $m+p=i$. Hence, by using a cut-rule, it follows that

$$\vdash_L (\exists^i v)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y\in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (\exists^j v)\xi(v)) .$$

We can prove (2)-(7) in the same manner in which we could (1).

(10) We take a free unary predicate variable α which does not occur in $A(x)$ and consider the following formula $F(a)$:

$$A(a) \wedge (\alpha(a) \vee \bigvee_{y\in Y+} a=y) \wedge \bigwedge_{y\in Y-} a\neq y .$$

Then, we get

$$\vdash_L (Qv)(A(v) \wedge \alpha(v)) \wedge (Qv)(A(v) \wedge \neg\alpha(v)), (Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y\in Y} \longrightarrow D(\lambda v F(v)) \wedge (Qv)F(v) \wedge (Qv)(A(v) \wedge \neg F(v)) .$$

Applying $(\rightarrow \exists)$,

$$\vdash_L (Qv)(A(v) \wedge \alpha(v)) \wedge (Qv)(A(v) \wedge \neg\alpha(v)), (Qv)A(v), Iq(y_1, \cdots, y_p), \{A(y)\}_{y\in Y} \longrightarrow (\exists \xi)(D(\xi) \wedge (Qv)\xi(v) \wedge (Qv)(A(v) \wedge \neg\xi(v))) .$$

Since $\alpha(*)$ does not occur in $(Qv)A(v)$ or any $A(y)$ $(y\in Y)$, using $(\exists\rightarrow)$, we get

$$\vdash_L (\exists\xi)((Qv)(A(v)\wedge\xi(v))\wedge(Qv)(A(v)\wedge\neg\xi(v))), (Qv)A(v), Iq(y_1,\cdots,y_p), \{A(y)\}_{y\in Y} \longrightarrow (\exists\xi)(D(\xi)\wedge(Qv)\xi(v)\wedge(Qv)(A(v)\wedge\neg\xi(v))).$$

Therefore, by applying a cut-rule between this sequent and our axiom sequent for Q and the second order quantifier $\exists$:

$$(Qv)A(v)\longrightarrow(\exists\xi)((Qv)(A(v)\wedge\xi(v))\wedge(Qv)(A(v)\wedge\neg\xi(v)))$$

we get

$$\vdash_L (Qv)A(v), Iq(y_1,\cdots,y_p), \{A(y)\}_{y\in Y} \longrightarrow (\exists\xi)(D(\xi)\wedge(Qv)\xi(v)\wedge(Qv)(A(v)\wedge\neg\xi(v))).$$

(8), (9) are easily shown from (10) by using (viii) of Lemma in §1.

DEFINITION. Let X be a finite set of free individual variables. A set Δ of formulas in L is *a complete equality set* (we call an *X-set* for short) if Δ satisfies the following conditions (1), (2):

(1) $x=y\in\Delta$ or $x\neq y\in\Delta$ for any $x, y\in X$.

(2) If we define the relation $\tilde{\Delta}$ by

$$x\tilde{\Delta}y \text{ if and only if } x=y\in\Delta, \text{ for any } x, y\in X,$$

then $\tilde{\Delta}$ is an equivalence relation on X.

If Δ is an X-set, then by $n(\Delta)$ we mean the number of equivalence classes of X by $\tilde{\Delta}$.

PROOF OF REMMA 4. In the following, we assume that all of the hypothesis of Lemma 4 and let $\Gamma=\Gamma(2k+1;\bar{\alpha})\cup\Gamma(\bar{\alpha};\bar{x},\bar{y})$ for simplicity. By the symmetry, it suffices to show that

$$\Gamma\vdash_{\mathcal{L}}(\exists\xi)A(\bar{x},\bar{\alpha},\xi)^1\longrightarrow(\exists\xi)A(\bar{y},\bar{\alpha},\xi)^2, \quad [1]$$

If we prove the following [2], [1] is shown from [2] by applying an inference rule $(\exists\rightarrow)$ for the second order quantifier in our morphism logic $\mathcal{L}$:

$$\Gamma\vdash_{\mathcal{L}}A(\bar{x},\bar{\alpha},\alpha)^1\longrightarrow(\exists\xi)A(\bar{y},\bar{\alpha},\xi)^2 \quad [2]$$

So, we shall show [2]. Let $\mathcal{T}$ be $\{\tau: \mathrm{dom}(\tau)=\bar{\alpha}^\wedge\alpha\}$. In order to prove [2], since $x=y\vee x\neq y$ and $\bigvee_{\tau\in\mathcal{T}}C_\tau(x)$ are both provable in L, it is enough to demonstrate that for any partition $\{X_\tau\}_{\tau\in\mathcal{T}}$ of $X=\{x_1,\cdots,x_p\}$ and any X-set Δ,

$$\Gamma\vdash_{\mathcal{L}}\Delta^1, \{C_\tau(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau,\tau\in\mathcal{T}}, A(\bar{x},\bar{\alpha},\alpha)^1\longrightarrow(\exists\xi)A(\bar{y},\bar{\alpha},\xi)^2 \quad [3]$$

In the antecedent of [3], if $x\in X_\tau$, $x'\in X_{\tau'}$, $\tau\neq\tau'$ and $x={}^1x'\in\Delta^1$, then [3] clearly holds;

and if $x \in X_\tau, x' \in X_{\tau'}, \tau \neq \tau'$ and $x \neq^1 x' \in \Delta^1$, then $x \neq^1 x'$ need not occur in Δ^1. Hence we may take Δ as a set of X_τ-sets ($\tau \in \mathcal{T}$), that is, all we need to prove is that for any set $\{\Delta_\tau\}_{\tau \in \mathcal{T}}$ such that Δ_τ is a X_τ-set for each $\tau \in \mathcal{T}$,

$$\Gamma \vdash_{\mathcal{L}} \{\Delta_\tau{}^1\}_{\tau \in \mathcal{T}}, \{C_\tau(x, \bar{\alpha}^\wedge\alpha)^1\}_{x \in X_\tau, \tau \in \mathcal{T}}, A(\bar{x}, \bar{\alpha}, \alpha)^1 \longrightarrow (\exists \xi) A(\bar{y}, \bar{\alpha}, \xi)^2 \qquad [4]$$

holds. Then we shall define $\Gamma_f(\bar{\beta}, n)$ for each map f from a finite set $\{\tau\colon \mathrm{dom}(\varepsilon) = \bar{\beta}\}$ of types to $\{0, 1, \cdots, n, n+1, \omega\}$ as follows:

$$\begin{aligned}\Gamma_f(\bar{\beta}, n) = &\{(\exists^{f(\tau)} v) C_\tau(v, \bar{\beta})\colon \mathrm{dom}(\tau) = \bar{\beta}, f(\tau) \leqq n\}\\ &\cup \{(\mathrm{Q}v) C_\tau(v, \bar{\beta})\colon \mathrm{dom}(\tau) = \bar{\beta}, f(\tau) = \omega\}\\ &\cup \{\neg(\exists^0 v) C_\tau(v, \bar{\beta}), \cdots, \neg(\exists^n v) C_\tau(v, \bar{\beta}), \neg(\mathrm{Q}v) C_\tau(v, \bar{\beta})\colon \mathrm{dom}(\tau) = \bar{\beta}, f(\tau) = n+1\}\,.\end{aligned}$$

Since for any $\tau \in \mathcal{T}$,

$$(\textstyle\bigvee_{i=0}^k (\exists^i v) C_\tau(v) \vee (\mathrm{Q}v) C_\tau(v)) \vee (\textstyle\bigwedge_{i=0}^k \neg(\exists^i v) C_\tau(v) \wedge \neg(\mathrm{Q}v) C_\tau(v))$$

is provable in L, in order to prove [4] it is sufficient to show that for any map f from $\{\tau\colon \mathrm{dom}(\tau) = \bar{\alpha}^\wedge\alpha\}$ to $\{0, 1, \cdots, k, k+1, \omega\}$,

$$\Gamma \vdash_{\mathcal{L}} \{\Delta_\tau{}^1\}_{\tau \in \mathcal{T}}, \{C_\tau(x, \bar{\alpha}^\wedge\alpha)^1\}_{x \in X_\tau, \tau \in \mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^1, A(\bar{x}, \bar{\alpha}, \alpha)^1 \longrightarrow (\exists \xi) A(\bar{y}, \bar{\alpha}, \xi)^2\,. \qquad [5]_f$$

For each $\tau \in \mathcal{T}$, let Y_τ be the set of y's which corresponds to x's in X_τ. Then $\{Y_\tau\}_{\tau \in \mathcal{T}}$ is obviously a partition of $Y = \{y_1, \cdots, y_p\}$. Let $\Delta_\tau{}^2$ be the set of formulas obtained from $\Delta_\tau{}^1$ by replacing x's which occurs in formulas in $\Delta_\tau{}^1$ by y's and 1 by 2. (This obtained set Δ_τ is a Y_τ-set.) We fix a map f from $\{\tau\colon \mathrm{dom}(\tau) = \bar{\alpha}^\wedge\alpha\}$ to $\{0, 1, \cdots, k, k+1, \omega\}$. Clearly, for any B in $\Gamma(k;\ \bar{\alpha}^\wedge\alpha) \cup \Gamma(\bar{\alpha}^\wedge\alpha;\ \bar{x}, \bar{y})$

$$\begin{aligned}\vdash_{\mathcal{L}} &\{\Delta_\tau{}^1\}_{\tau \in \mathcal{T}}, \{C_\tau(x, \bar{\alpha}^\wedge\alpha)^1\}_{x \in X_\tau, \tau \in \mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^1,\\ &\{\Delta_\tau{}^2\}_{\tau \in \mathcal{T}}, \{C_\tau(y, \bar{\alpha}^\wedge\alpha)^2\}_{y \in Y_\tau, \tau \in \mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^2 \longrightarrow B\end{aligned}$$

holds. Therefore, by using the hypothesis which states that

$$\Gamma(k;\ \bar{\alpha}^\wedge\alpha), \Gamma(\bar{\alpha}^\wedge\alpha;\ \bar{x}, \bar{y}) \vdash_{\mathcal{L}} A(\bar{x}, \bar{\alpha}, \alpha)^1 \longrightarrow A(\bar{y}, \bar{\alpha}, \alpha)^2\,,$$

we can get

$$\begin{aligned}\vdash_{\mathcal{L}} &\{\Delta_\tau{}^1\}_{\tau \in \mathcal{T}}, \{C_\tau(x, \bar{\alpha}^\wedge\alpha)^1\}_{x \in X_\tau, \tau \in \mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^1,\\ &\{\Delta_\tau{}^2\}_{\tau \in \mathcal{T}}, \{C_\tau(y, \bar{\alpha}^\wedge\alpha)^2\}_{y \in Y_\tau, \tau \in \mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^2,\\ &A(\bar{x}, \bar{\alpha}, \alpha)^1 \longrightarrow A(\bar{y}, \bar{\alpha}, \alpha)^2\,.\end{aligned} \qquad [6]$$

Applying $(\rightarrow \exists)$ to [6], it follows that

$$\begin{aligned}\vdash_{\mathcal{L}} &\{\Delta_\tau{}^1\}_{\tau \in \mathcal{T}}, \{C_\tau(x, \bar{\alpha}^\wedge\alpha)^1\}_{x \in X_\tau, \tau \in \mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^1,\\ &\{\Delta_\tau{}^2\}_{\tau \in \mathcal{T}}, \{C_\tau(y, \bar{\alpha}^\wedge\alpha)^2\}_{y \in Y_\tau, \tau \in \mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^2,\\ &A(\bar{x}, \bar{\alpha}, \alpha)^1 \longrightarrow (\exists \xi) A(\bar{y}, \bar{\alpha}, \xi)^2\,.\end{aligned} \qquad [7]$$

In the antecedent of [7], we may replace

$$\{C_\tau(\bar{y}, \bar{\alpha}^\wedge\alpha)^2\}_{y\in Y_\tau, \tau\in\mathcal{T}}$$

by

$$\{C_\varepsilon(y, \bar{\alpha})^2\}_{y\in Y_\varepsilon, \varepsilon\in\mathcal{E}}, \{\alpha^2(y)\}_{y\in Y_{\varepsilon^+}, \varepsilon\in\mathcal{E}} \cup \{\neg\alpha^2(y)\}_{y\in Y_{\varepsilon^-}, \varepsilon\in\mathcal{E}},$$

where $\mathcal{E}=\{\varepsilon: \mathrm{dom}(\varepsilon)=\bar{\alpha}\}$, $\varepsilon^+=\varepsilon\cup\{\langle\alpha, +\rangle\}$, $\varepsilon^-=\varepsilon\cup\{\langle\alpha, -\rangle\}$ and so $Y_\varepsilon=Y_{\varepsilon+}\cup Y_{\varepsilon-}$. Then we write the formula, which is obtained by connecting all of the elements in $\{\alpha^2(y)\}_{y\in Y_{\varepsilon^+}, \varepsilon\in\mathcal{E}} \cup \{\neg\alpha^2(y)\}_{y\in Y_{\varepsilon^-}, \varepsilon\in\mathcal{E}}$ and $\Gamma_f(\bar{\alpha}^\wedge\alpha, k)^2$ by $\wedge$, as

$$\wedge(\{\alpha(y)\}_{y\in Y_{\varepsilon^+}, \varepsilon\in\mathcal{E}} \cup \{\neg\alpha(y)\}_{y\in Y_{\varepsilon^-}, \varepsilon\in\mathcal{E}} \cup \Gamma_f(\bar{\alpha}^\wedge\alpha, k))^2 .$$

Since α^2 occurs in this formula only in the antecedent of [7], by applying $(\exists\rightarrow)$ to [7] we get

$$\begin{aligned}&\vdash_{\mathcal{L}}\{\Delta_\tau^1\}_{\tau\in\mathcal{T}}, \{C_\tau(x, \bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau, \tau\in\mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^1, \\ &\quad\{\Delta_\tau^2\}_{\tau\in\mathcal{T}}, \{C_\varepsilon(y, \bar{\alpha})^2\}_{y\in Y_\varepsilon, \varepsilon\in\mathcal{E}}, \\ &\quad(\exists\xi)\wedge(\{\xi(y)\}_{y\in Y_{\varepsilon^+}, \varepsilon\in\mathcal{E}} \cup \{\neg\xi(y)\}_{y\in Y_{\varepsilon^-}, \varepsilon\in\mathcal{E}} \cup \Gamma_f(\bar{\alpha}^\wedge\xi, k))^2, \\ &\qquad A(\bar{x}, \bar{\alpha}, \alpha)^1 \longrightarrow (\exists\xi)A(\bar{y}, \alpha, \xi)^2 .\end{aligned} \qquad [8]$$

Therefore, if we show that

$$\begin{aligned}&\Gamma\vdash_{\mathcal{L}}\{\Delta_\tau^1\}_{\tau\in\mathcal{T}}, \{C_\tau(x, \bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau, \tau\in\mathcal{T}}, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^1 \\ &\quad\longrightarrow(\exists\xi)\wedge(\{\xi(y)\}_{y\in Y_{\varepsilon^+}, \varepsilon\in\mathcal{E}} \cup \{\neg\xi(y)\}_{y\in Y_{\varepsilon^-}, \varepsilon\in\mathcal{E}} \cup \Gamma_f(\bar{\alpha}^\wedge\xi, k))^2 ,\end{aligned} \qquad [9]$$

then $[5]_f$ is proved in the following manner. It is easily seen that

$$\Gamma(\bar{\alpha};\ \bar{x}, \bar{y})\vdash_{\mathcal{L}}\{\Delta_\tau^1\}_{\tau\in\mathcal{T}}\longrightarrow E_\tau^2 \text{ for any } E_\tau^2\in\Delta_\tau^2, \text{ any } \tau\in\mathcal{T}$$

and

$$\Gamma(\bar{\alpha};\ \bar{x}, \bar{y})\vdash_{\mathcal{L}}\{C_\varepsilon(x, \bar{\alpha})^1\}_{x\in X_\varepsilon, \varepsilon\in\mathcal{E}}\longrightarrow C_\varepsilon(y, \bar{\alpha})^2 \text{ for any } y\in Y_\varepsilon, \text{ any } \varepsilon\in\mathcal{E} .$$

Whence we get

$$\Gamma(\bar{\alpha};\ \bar{x}, \bar{y})\vdash_{\mathcal{L}}\{\Delta_\tau^1\}_{\tau\in\mathcal{T}}\longrightarrow(\wedge\{\Delta_\tau\}_{\tau\in\mathcal{T}})^2$$

and

$$\Gamma(\bar{\alpha};\ \bar{x}, \bar{y})\vdash_{\mathcal{L}}\{C_\varepsilon(x, \bar{\alpha})^1\}_{x\in X_\varepsilon, \varepsilon\in\mathcal{E}}\longrightarrow(\wedge\{C_\varepsilon(y, \bar{\alpha})\}_{y\in Y_\varepsilon, \varepsilon\in\mathcal{E}})^2 ,$$

where $\wedge\{\Delta_\tau\}_{\tau\in\mathcal{T}}$, $\wedge\{C_\varepsilon(y, \bar{\alpha})\}_{y\in Y_\varepsilon, \varepsilon\in\mathcal{E}}$ are formulas obtained by connecting all of the elements $\cup_{\tau\in\mathcal{T}}\Delta_\tau$, $\cup_{\varepsilon\in\mathcal{E}}\{C_\varepsilon(y, \bar{\alpha})\}_{y\in Y_\varepsilon}$, respectively, by $\wedge$. Since we can divide

$$\{C_\tau(x, \bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau, \tau\in\mathcal{T}}$$

into

$$\{C_\varepsilon(x, \bar{\alpha})^1\}_{x\in X_\varepsilon, \varepsilon\in\mathcal{E}}, \{\alpha^1(x)\}_{x\in X_{\varepsilon^+}, \varepsilon\in\mathcal{E}} \cup \{\neg\alpha^1(x)\}_{x\in X_{\varepsilon^-}, \varepsilon\in\mathcal{E}}$$

in the antecedent in a sequent similarly to the case of

$$\{C_\tau(y, \bar{\alpha}^\wedge\alpha)^2\}_{y\in Y_\tau, \tau\in\mathcal{T}} ,$$

it follows that

$$\Gamma(\bar{\alpha}\,;\ \bar{x},\bar{y})\vdash_{\mathcal{L}}\{\Delta_\tau{}^1\}_{\tau\in\mathcal{T}},\{C_\tau(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau,\tau\in\mathcal{T}} \longrightarrow(\bigwedge\{\Delta_\tau\}_{\tau\in\mathcal{T}}\wedge\bigwedge\{C_\varepsilon(y,\bar{\alpha})\}_{y\in Y_\varepsilon,\varepsilon\in\mathcal{E}})^2 \qquad [10]$$

By using a cut-rule between [10] and the sequent obtained from [8] by applying $(\wedge\rightarrow)$ a finite number of times, we get

$$\Gamma(\bar{\alpha}\,;\ \bar{x},\bar{y})\vdash_{\mathcal{L}}\{\Delta_\tau{}^1\}_{\tau\in\mathcal{T}},\{C_\tau(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau,\tau\in\mathcal{T}},\Gamma_f(\bar{\alpha}^\wedge\alpha,k)^1,$$
$$(\exists\xi)\wedge(\{\xi(y)\}_{y\in Y_{\varepsilon^+},\varepsilon\in\mathcal{E}}\cup\{\neg\xi(y)\}_{y\in Y_{\varepsilon^-},\varepsilon\in\mathcal{E}}\cup\Gamma_f(\bar{\alpha}^\wedge\xi,k))^2,$$
$$A(\bar{x},\bar{\alpha},\alpha)^1\longrightarrow(\exists\xi)A(\bar{y},\bar{\alpha},\xi)^2 \qquad [11]$$

Applying a cut-rule between [11] and [9], $[5]_f$ is shown if we notice that $\Gamma(\bar{\alpha}\,;\ \bar{x},\bar{y})$ is a subset of Γ.

We shall prove [9]. It is adequate for our purposes to prove that for any map g from $\{\varepsilon\colon \mathrm{dom}(\varepsilon)=\bar{\alpha}\}$ to $\{0,1,\cdots,2k+1,2(k+1),\omega\}$,

$$\Gamma\vdash_{\mathcal{L}}\{\Delta_\tau{}^1\}_{\tau\in\mathcal{T}},\{C_\tau(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau,\tau\in\mathcal{T}},\Gamma_f(\bar{\alpha}^\wedge\alpha,k)^1,\Gamma_g(\bar{\alpha},2k+1)^1 \longrightarrow(\exists\xi)\wedge(\{\xi(y)\}_{y\in Y_{\varepsilon^+},\varepsilon\in\mathcal{E}}\cup\{\neg\xi(y)\}_{y\in Y_{\varepsilon^+},\varepsilon\in\mathcal{E}}\cup\Gamma_f(\bar{\alpha}^\wedge\xi,k))^2 \qquad [12]_g$$

We fix an arbitrary map g from $\{\varepsilon\colon \mathrm{dom}(\varepsilon)=\bar{\alpha}\}$ to $\{0,1,\cdots,2(k+1),\omega\}$.

For each $\varepsilon\in\mathcal{E}$, we take a free unary predicate variable β_ε and define a formula $F_\varepsilon(\beta_\varepsilon)$ in the following way. First let $D_\varepsilon(\beta_\varepsilon)$ be a formula

$$(\forall v)(\beta_\varepsilon(v)\supset C_\varepsilon(v))\wedge\bigwedge_{y\in Y_{\varepsilon^+}}\beta_\varepsilon(y)\wedge\bigwedge_{y\in Y_{\varepsilon^-}}\neg\beta_\varepsilon(y)\,.$$

for each $\varepsilon\in\mathcal{E}$.

Case 1. $f(\varepsilon^+)\leqq k$.

$$F_\varepsilon(\beta_\varepsilon)\colon\ D_\varepsilon(\beta_\varepsilon)\wedge(\exists^{f(\varepsilon^+)}v)\beta_\varepsilon(v)\,.$$

Case 2. $f(\varepsilon^-)\leqq k$.

$$F_\varepsilon(\beta_\varepsilon)\colon\ D_\varepsilon(\beta_\varepsilon)\wedge(\exists^{f(\varepsilon^-)}v)(C_\varepsilon(v)\wedge\neg\beta_\varepsilon(v))\,.$$

Case 3. $f(\varepsilon^+)=f(\varepsilon^-)=k+1$.

$$F_\varepsilon(\beta_\varepsilon)\colon\ D_\varepsilon(\beta_\varepsilon)\wedge(\exists^{>k}v)\beta_\varepsilon(v)\wedge(\exists^{>k}v)(C_\varepsilon(v)\wedge\neg\beta_\varepsilon(v))\,.$$

Case 4. $f(\varepsilon^+)=k+1, f(\varepsilon^-)=\omega$.

$$F_\varepsilon(\beta_\varepsilon)\colon\ D_\varepsilon(\beta_\varepsilon)\wedge(\exists^{>k}v)\beta_\varepsilon(v)\wedge(\mathrm{Q}v)(C_\varepsilon(v)\wedge\neg\beta_\varepsilon(v)).$$

Case 5. $f(\varepsilon^+)=\omega, f(\varepsilon^-)=k+1$.

$$F_\varepsilon(\beta_\varepsilon)\colon\ D_\varepsilon(\beta_\varepsilon)\wedge(\mathrm{Q}v)\beta_\varepsilon(v)\wedge(\exists^{>k}v)(C_\varepsilon(v)\wedge\neg\beta_\varepsilon(v))\,.$$

Case 6. $f(\varepsilon^+)=f(\varepsilon^-)=\omega$.

$$F_\varepsilon(\beta_\varepsilon)\colon\ D_\varepsilon(\beta_\varepsilon)\wedge(\mathrm{Q}v)\beta_\varepsilon(v)\wedge(\mathrm{Q}v)(C_\varepsilon(v)\wedge\neg\beta_\varepsilon(v))\,.$$

Let $F(a)=\bigvee_{\varepsilon\in\mathcal{E}}\beta_\varepsilon(a)$. Then we get

$$\begin{aligned}&\Gamma\vdash_{\mathcal{L}}\{\Delta_\tau{}^1\}_{\tau\in\mathcal{T}},\{C_\tau(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau,\tau\in\mathcal{T}},\Gamma_f(\bar{\alpha}^\wedge\alpha,k)^1,\\&\quad\Gamma_g(\bar{\alpha},2k+1)^1,\{F_\varepsilon(\beta_\varepsilon)^2\}_{\varepsilon\in\mathcal{E}}\\&\quad\longrightarrow\bigwedge(\{F(y)\}_{y\in Y_{\varepsilon+},\varepsilon\in\mathcal{E}}\cup\{\neg F(y)\}_{y\in Y_{\varepsilon-},\varepsilon\in\mathcal{E}}\cup\Gamma_f(\bar{\alpha}^\wedge\lambda vF(v),k))^2.\end{aligned}\qquad[13]$$

By applying ($\to\exists$), it follows that

$$\begin{aligned}&\Gamma\vdash_{\mathcal{L}}\{\Delta_\tau{}^1\}_{\tau\in\mathcal{T}},\{C_\tau(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau,\tau\in\mathcal{T}},\Gamma_f(\bar{\alpha}^\wedge\alpha,k)^1,\\&\quad\Gamma_g(\bar{\alpha},2k+1)^1,\{F_\varepsilon(\beta_\varepsilon)^2\}_{\varepsilon\in\mathcal{E}}\\&\quad\longrightarrow(\exists\xi)\bigwedge(\{\xi(y)\}_{y\in Y_{\varepsilon+},\varepsilon\in\mathcal{E}}\cup\{\neg\xi(y)\}_{y\in Y_{\varepsilon-},\varepsilon\in\mathcal{E}}\cup\Gamma_f(\bar{\alpha}^\wedge\xi,k))^2.\end{aligned}\qquad[14]$$

Since all of the elements of $\{\beta_\varepsilon{}^2\}_{\varepsilon\in\mathcal{E}}$ are mutually distinct in the antecedent in [14], by applying ($\exists\to$) to every $\beta_\varepsilon{}^2$ one by one, we can get

$$\begin{aligned}&\Gamma\vdash_{\mathcal{L}}\{\Delta_\tau{}^1\}_{\tau\in\mathcal{T}},\{C_\tau(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_\tau,\tau\in\mathcal{T}},\Gamma_f(\bar{\alpha}^\wedge\alpha,k)^1,\\&\quad\Gamma_g(\bar{\alpha},2k+1)^1,\{(\exists\xi)F_\varepsilon(\xi)^2\}_{\varepsilon\in\mathcal{E}}\\&\quad\longrightarrow(\exists\xi)\bigwedge(\{\xi(y)\}_{y\in Y_{\varepsilon+},\varepsilon\in\mathcal{E}}\cup\{\neg\xi(y)\}_{y\in Y_{\varepsilon-},\varepsilon\in\mathcal{E}}\cup\Gamma_f(\bar{\alpha}^\wedge\xi,k))^2.\end{aligned}\qquad[15]$$

In order to show $[12]_g$, therefore, it is sufficient to prove that for each $\varepsilon\in\mathcal{E}$,

$$\begin{aligned}&\Gamma\vdash_{\mathcal{L}}\Delta_\varepsilon{}^1{}_+,\Delta_\varepsilon{}^1{}_-,\{C_{\varepsilon+}(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_{\varepsilon+}},\{C_{\varepsilon-}(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_{\varepsilon-}},\\&\quad\Gamma_f(\bar{\alpha}^\wedge\alpha,k)^1,\Gamma_g(\bar{\alpha},2k+1)^1\longrightarrow(\exists\xi)F_\varepsilon(\xi)^2.\end{aligned}\qquad[16]_\varepsilon$$

Now we fix an arbitrary $\varepsilon\in\mathcal{E}$ and we shall prove $[16]_\varepsilon$. We first note the following fact. It is enough to check that $[16]_\varepsilon$ holds for each case of Case 1—Case 6 in accordance with the definition of $F_\varepsilon(*)$, and further we can divide the proofs of Case 1 and Case 2 into three subcases, respectively, according to the value of $g(\varepsilon)$. Then, in subcase 1.1 of Case 1 (i.e. $f(\varepsilon^+)\leqq k$ and $g(\varepsilon)\leqq 2k+1$), $[16]_\varepsilon$ clearly holds if $n(\Delta_{\varepsilon+})>f(\varepsilon^+)$ or $n(\Delta_{\varepsilon-})>g(\varepsilon)-f(\varepsilon^+)$, and hence we may assume that $n(\Delta_{\varepsilon+})\leqq f(\varepsilon^+)$ and $n(\Delta_{\varepsilon-})\leqq g(\varepsilon)-f(\varepsilon^+)$; similarly in subcase 2.1 of Case 2 (i.e. $f(\varepsilon^-)\leqq k$ and $g(\varepsilon)\leqq 2k+1$), we may assume that $n(\Delta_{\varepsilon-})\leqq f(\varepsilon^-)$ and $n(\Delta_{\varepsilon+})\leqq g(\varepsilon)-f(\varepsilon^-)$.

Now, in the antecedent in $[16]_\varepsilon$, we may rewrite

$$\{C_{\varepsilon+}(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_{\varepsilon+}},\{C_{\varepsilon-}(x,\bar{\alpha}^\wedge\alpha)^1\}_{x\in X_{\varepsilon-}}$$

by

$$\{C_\varepsilon(x,\bar{\alpha})^1\}_{x\in X_\varepsilon},\{\alpha^1(x)\}_{x\in X_{\varepsilon+}},\{\neg\alpha^1(x)\}_{x\in X_{\varepsilon-}},$$

then we can get the following sequent in the same way in which we get [10]:

$$\begin{aligned}&\Gamma(\bar{\alpha};\ \bar{x},\bar{y})\vdash_{\mathcal{L}}\Delta_\varepsilon{}^1{}_+,\Delta_\varepsilon{}^1{}_-,\{C_\varepsilon(x,\bar{\alpha})^1\}_{x\in X_\varepsilon}\\&\qquad\longrightarrow(\bigwedge\Delta_{\varepsilon+}\cup\Delta_{\varepsilon-}\wedge\bigwedge_{y\in Y_\varepsilon}C_\varepsilon(y,\bar{\alpha}))^2,\end{aligned}$$

where in the succedent $\Delta_{\varepsilon+},\Delta_{\varepsilon-}$ are $Y_{\varepsilon+}$-set, $Y_{\varepsilon-}$-set, respectively, and $\bigwedge\Delta_{\varepsilon+}\cup\Delta_{\varepsilon-}$ means the conjunction of all the elements in $\Delta_{\varepsilon+}\cup\Delta_{\varepsilon-}$. Moreover, if we express the formula obtained by connecting all of the elements of $\Gamma_g(\bar{\alpha}, 2k+1)$ with $\wedge$ by $\wedge\Gamma_g(\bar{\alpha}, 2k+1)$, then

$$\Gamma(2k+1;\ \bar{\alpha}) \vdash_{\mathcal{L}} \Gamma_g(\bar{\alpha}, 2k+1)^1 \longrightarrow (\wedge\Gamma_g(\bar{\alpha}, 2k+1))^2$$

holds. Hence we get

$$\Gamma \vdash_{\mathcal{L}} \Delta_{\varepsilon+}^1, \Delta_{\varepsilon-}^1, \{C_\varepsilon(x, \bar{\alpha})^1\}_{x \in X_\varepsilon}, \Gamma_g(\bar{\alpha}, 2k+1)^1$$
$$\longrightarrow (\bigwedge \Delta_{\varepsilon+} \cup \Delta_{\varepsilon-} \wedge \bigwedge_{y \in Y_\varepsilon} C_\varepsilon(y, \bar{\alpha}) \wedge \bigwedge \Gamma_g(\bar{\alpha}, 2k+1))^2 .$$

In order to show $[16]_\varepsilon$, therefore, it suffices to prove that

$$\Gamma \vdash_{\mathcal{L}} (\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-})^2, \{C_\varepsilon(y, \bar{\alpha})^2\}_{y \in Y_\varepsilon}, \Gamma_g(\bar{\alpha}, 2k+1)^2, \Gamma_f(\bar{\alpha}^\wedge\alpha, k)^1 ,$$
$$\{\alpha^1(x)\}_{x \in X_{\varepsilon+}}, \{\neg\alpha^1(x)\}_{x \in X_{\varepsilon-}} \longrightarrow (\exists \xi)F_\varepsilon(\xi)^2 , \qquad [17]$$

and so

$$\vdash_{\mathcal{L}} (\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-})^2, \{C_\varepsilon(y, \bar{\alpha})^2\}_{y \in Y_\varepsilon}, \Gamma_g(\bar{\alpha}, 2k+1)^2 \longrightarrow (\exists \xi)F_\varepsilon(\xi)^2 ,$$

Hence all we have to do is to prove that

$$\vdash_L \Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}, \{C_\varepsilon(y, \bar{\alpha})\}_{y \in Y_\varepsilon}, \Gamma_g(\bar{\alpha}, 2k+1) \longrightarrow (\exists \xi)F_\varepsilon(\xi) . \qquad [18]$$

Let

$$\tilde{Y}_{\varepsilon+} = \{y_1^{\varepsilon+}, \cdots, y_{n(\Delta_\varepsilon+)}^{\varepsilon+}\}, \quad \tilde{Y}_{\varepsilon-} = \{y_1^{\varepsilon-}, \cdots, y_{n(\Delta_\varepsilon-)}^{\varepsilon-}\}$$

be the sets consisting of all representative elements of equivalence classes of $Y_{\varepsilon+}$, $Y_{\varepsilon-}$, respectively and let $\tilde{Y}_\varepsilon = \tilde{Y}_{\varepsilon+} \cup \tilde{Y}_{\varepsilon-}$. Then we may substitute Y_ε by $\tilde{Y}_\varepsilon$ in the antecedent in [18]. The reason is in the following. Since, for any $y \in Y_\varepsilon$, there is $\tilde{y} \in \tilde{Y}_\varepsilon$ such that $y = \tilde{y} \in \Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}$, it follows that

$$\vdash_L \Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}, \{C_\varepsilon(\tilde{y}, \bar{\alpha})\}_{\tilde{y} \in \tilde{Y}_\varepsilon} \longrightarrow C_\varepsilon(y, \bar{\alpha})$$

for any $y \in Y_\varepsilon$, and so, we get

$$\vdash_L \Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}, \{C_\varepsilon(\tilde{y}, \bar{\alpha})\}_{\tilde{y} \in \tilde{Y}_\varepsilon} \longrightarrow \wedge_{y \in Y_\varepsilon} C_\varepsilon(y, \bar{\alpha}) .$$

Hence, applying a cut-rule between this sequent and the sequent obtained from [18] by connecting all of the elements of

$$\{C_\varepsilon(y, \bar{\alpha})\}_{y \in Y_\varepsilon}$$

by $\wedge$ we get the sequent obtained from [18] by substituting $\tilde{Y}_\varepsilon$ for Y_ε. Further, we may replace $\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}$ by $\widetilde{\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}}$, where $\widetilde{\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}}$ is the set obtained from $\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}$ by eliminating all of the formulas of form $x = y$. In other words, to show [18], it is sufficient to prove that

$$\vdash_L \widetilde{\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}}, \{C_\varepsilon(y, \bar{\alpha})\}_{y \in \tilde{Y}_\varepsilon}, \Gamma_g(\bar{\alpha}, 2k+1) \longrightarrow (\exists \xi)F_\varepsilon(\xi) . \qquad [19]$$

Clearly, for any i, j such that $1 \leqq i \leqq n(\Delta_{\varepsilon+}), 1 \leqq j \leqq n(\Delta_{\varepsilon-})$,

$$\vdash_L \widetilde{\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}}, \{C_\varepsilon(y, \bar{\alpha})\}_{y \in \tilde{Y}_\varepsilon} \longrightarrow y_i^{\varepsilon+} \neq y_j^{\varepsilon-}$$

holds, hence in order to prove [19] it is adequate to show that

$$\vdash_L \bigwedge_{\substack{1 \leq i \leq n(\Delta_\varepsilon+) \\ 1 \leq j \leq n(\Delta_\varepsilon-)}} y_i^{\varepsilon+} \neq y_j^{\varepsilon-}, \widetilde{\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}}, \{C_\varepsilon(y, \bar{\alpha})\}_{y \in \tilde{Y}_\varepsilon},$$
$$\Gamma_g(\bar{\alpha}, 2k+1) \longrightarrow (\exists \xi) F_\varepsilon(\xi)\,.$$

The formula which is obtained by connecting $\bigwedge_{i,j} y_i^{\varepsilon+} \neq y_j^{\varepsilon-}$ and all of the elements of $\widetilde{\Delta_{\varepsilon+} \cup \Delta_{\varepsilon-}}$ by $\wedge$ is equivalent to

$$Iq(y_1^{\varepsilon+}, \cdots, y_{n(\Delta_\varepsilon+)}^{\varepsilon+}, y_1^{\varepsilon-}, \cdots, y_{n(\Delta_\varepsilon-)}^{\varepsilon-})$$

in L, and hence all we need to prove is that the following [20] holds:

$$\vdash_L Iq(y_1^{\varepsilon+}, \cdots, y_{n(\Delta_\varepsilon+)}^{\varepsilon+}, y_1^{\varepsilon-}, \cdots, y_{n(\Delta_\varepsilon-)}^{\varepsilon-}), \{C_\varepsilon(y, \bar{\alpha})\}_{y \in \tilde{Y}_\varepsilon},$$
$$\Gamma_g(\bar{\alpha}, 2k+1) \longrightarrow (\exists \xi) F_\varepsilon(\xi)\,. \qquad [20]$$

Case 1. $f(\varepsilon^+) \leqq k$. In this case, $(\exists \xi) F_\varepsilon(\xi)$ is

$$(\exists \xi)(D_\varepsilon(\xi) \wedge (\exists^{f(\varepsilon^+)} v) \xi(v))\,.$$

[Subcase 1.1] $g(\varepsilon) \leqq 2k+1$. Then [20] is

$$\vdash_L Iq(y_1^{\varepsilon+}, \cdots, y_{n(\Delta_\varepsilon-)}^{\varepsilon-}), \{C_\varepsilon(y, \bar{\alpha})\}_{y \in \tilde{Y}_\varepsilon}, (\exists^{g(\varepsilon)} v) C_\varepsilon(v, \bar{\alpha})$$
$$\longrightarrow (\exists \xi)(D_\varepsilon(\xi) \wedge (\exists^{f(\varepsilon^+)} v) \xi(v))\,.$$

From the previous notes, we know that $n(\Delta_{\varepsilon+}) \leqq f(\varepsilon^+)$ and $n(\Delta_{\varepsilon-}) \leqq g(\varepsilon) - f(\varepsilon^+)$ and so, $\bar{\bar{\tilde{Y}}}_{\varepsilon+} = n(\Delta_{\varepsilon+}) \leqq f(\varepsilon^+)$ and $\bar{\bar{\tilde{Y}}}_{\varepsilon-} = n(\Delta_{\varepsilon-}) \leqq g(\varepsilon) - f(\varepsilon^+)$. Hence [20] holds clearly by (1) of Sublemma.

[Subcase 1.2] $g(\varepsilon) = 2(k+1)$. Then [20] is

$$\vdash_L Iq(y_1^{\varepsilon+}, \cdots, y_{n(\Delta_\varepsilon-)}^{\varepsilon-}), \{C_\varepsilon(y, \bar{\alpha})\}_{y \in \tilde{Y}_\varepsilon}, \{\neg(\exists^i v) C_\varepsilon(v)\}_{i=0}^{2k+1},$$
$$\neg(Qv) C_\varepsilon(v, \bar{\alpha}) \longrightarrow (\exists \xi)(D_\varepsilon(\xi) \wedge (\exists^{f(\varepsilon^+)} v) \xi(v))\,.$$

This is clear by (3) of Sublemma.

[Subcase 1.3] $g(\varepsilon) = \omega$. Then [20] is

$$\vdash_L Iq(y_1^{\varepsilon+}, \cdots, y_{n(\Delta_\varepsilon-)}^{\varepsilon-}), \{C_\varepsilon(y, \bar{\alpha})\}_{y \in \tilde{Y}_\varepsilon}, (Qv) C_\varepsilon(v, \bar{\alpha})$$
$$\longrightarrow (\exists \xi)(D_\varepsilon(\xi) \wedge (\exists^{f(\varepsilon^+)} v) \xi(v))\,.$$

This is also trivial by (6) of Sublemma.

Case 2. $f(\varepsilon^-) \leqq k$. In this case, $(\exists \xi) F_\varepsilon(\xi)$ is

$$(\exists \xi)(D_\varepsilon(\xi) \wedge (\exists^{f(\varepsilon^-)} v)(C_\varepsilon(v, \bar{\alpha}) \wedge \neg \xi(v)))\,.$$

It suffices to check three subcases according to the value of $g(\varepsilon)$ in the same manner to Case 1.

Case 3. $f(\varepsilon^+) = f(\varepsilon^-) = k+1$. In this case, $(\exists \xi) F_\varepsilon(\xi)$ is

$$(\exists\xi)(D_\varepsilon(\xi)\wedge(\exists^{>k}v)\xi(v)\wedge(\exists^{>k}v)(C_\varepsilon(v,\bar{\alpha})\wedge\neg\xi(v))),$$

and $g(\varepsilon)=2(k+1)$. Hence [20] is

$$\vdash_L Iq(y_1^{\varepsilon+},\cdots,y_{n(\Delta_\varepsilon-)}^{\varepsilon-}),\{C_\varepsilon(y,\bar{\alpha})\}_{y\in\bar{Y}_\varepsilon},\{\neg(\exists^i v)C_\varepsilon(v,\bar{\alpha})\}_{i=0}^{2k+1},$$
$$\neg(Qv)C_\varepsilon(v,\bar{\alpha})\longrightarrow(\exists\xi)(D_\varepsilon(\xi)\wedge(\exists^{>k}v)\xi(v)\wedge(\exists^{>k}v)(C_\varepsilon(v,\bar{\alpha})\wedge\neg\xi(v))).$$

This is also clear by (5) of Sublemma.

Case 4. $f(\varepsilon^+)=k+1, f(\varepsilon^-)=\omega$. In this case, $(\exists\xi)F_\varepsilon(\xi)$ is

$$(\exists\xi)(D_\varepsilon(\xi)\wedge(\exists^{>k})\xi(v)\wedge(Qv)(C_\varepsilon(v,\bar{\alpha})\wedge\neg\xi(v))).$$

and $g(\varepsilon)=\omega$. Hence [20] is

$$\vdash_L Iq(y_1^{\varepsilon+},\cdots,y_{n(\Delta_\varepsilon-)}^{\varepsilon-}),\{C_\varepsilon(y,\bar{\alpha})\}_{y\in\bar{Y}_\varepsilon},(Qv)C_\varepsilon(v,\bar{\alpha})$$
$$\longrightarrow(\exists\xi)(D_\varepsilon(\xi)\wedge(\exists^{>k}v)\xi(v)\wedge(Qv)(C_\varepsilon(v,\bar{\alpha})\wedge\neg\xi(v))).$$

This is obvious by (8) of Sublemma.

Case 5. $f(\varepsilon^+)=\omega, f(\varepsilon^-)=k+1$. This case is a dual case of Case 4 and so it is easily shown by (9) of Sublemma.

Case 6. $f(\varepsilon^+)=f(\varepsilon^-)=\omega$. In this case, $(\exists\xi)F_\varepsilon(\xi)$ is

$$(\exists\xi)(D_\varepsilon(\xi)\wedge(Qv)\xi(v)\wedge(Qv)(C_\varepsilon(v,\bar{\alpha})\wedge\neg\xi(v))),$$

and $g(\varepsilon)=\omega$. Hence [20] is

$$\vdash_L Iq(y_1^{\varepsilon+},\cdots,y_{n(\Delta_\varepsilon-)}^{\varepsilon-}),\{C_\varepsilon(y,\bar{\alpha})\}_{y\in\bar{Y}_\varepsilon},(Qv)C_\varepsilon(v,\bar{\alpha})$$
$$\longrightarrow(\exists\xi)(D_\varepsilon(\xi)\wedge(Qv)\xi(v)\wedge(Qv)(C_\varepsilon(v,\bar{\alpha})\wedge\neg\xi(v))).$$

This is also clear by (10) of Sublemma.

Our proof of Lemma 4 is here completed and so is our proof of Main theorem.

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