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A SMALL REMARK ON THE FILTERED φ-MODULE OF FERMAT VARIETIES AND STICKELBERGER'S THEOREM

By

Go Yamashita†

Abstract. We show that the weakly admissibility of the filtered φ-module with coefficients of Fermat varieties in the sense of Fontaine essentially expresses Stickelberger's theorem in Iwasawa theory. In particular, it gives us a simple re-proof of the weakly admissibility of it.

This short paper is essentially a letter to Noriyuki Otsubo in July/2013.

1. Introduction

Let $V = V_d^n$ be the relatively n-dimensional Fermat variety of degree d over $\mathbf{Z}[\mu_d]$, i.e., the variety defined by the equation

$$X_0^d + X_1^d + \cdots + X_{n+1}^d = 0$$

in $\mathbf{P}^{n+1}_{\mathbf{Z}[\mu_d]}$. For a prime number p which does not divide d, we fix an embedding of μ_d into an algebraic closure $\overline{\mathbf{Q}_p}$ of $\mathbf{Q}_p$. Let $\mathbf{F}_q$ denote the residue field of $K := \mathbf{Q}_p(\mu_d)$ and $q = p^f$. Note that d divides $q - 1$. The fractional field of the ring of Witt vectors $W := W(\mathbf{F}_q)$ with coefficients in $\mathbf{F}_q$ is canonically isomorphic to K since p does not divide d. Let σ denote the Frobenius of K.

We consider the following two cohomology groups: Firstly, the de Rham cohomology group $H_{\mathrm{dR}} := H^n_{\mathrm{dR}}(V_K/K)$ of $V_K := V \otimes_{\mathbf{Z}[\mu_d]} K$, which is a finite dimensional K-vector space equipped with the Hodge filtration $\{\mathrm{Fil}^i H_{\mathrm{dR}}\}_i$. Secondly, the crystalline cohomology group $H_{\mathrm{crys}} := H^n_{\mathrm{crys}}(V_0/W) \otimes_W K$ of

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$V_0 := V \otimes_{\mathbf{Z}[\mu_d]} \mathbf{F}_q$, which is a finite dimensional K-vector space equipped with the σ-semi-linear crystalline Frobenius action φ. Via Berthelot-Ogus isomorphism [BO], the pair $(H_{\mathrm{dR}}, H_{\mathrm{crys}})$ is a filtered φ-module in the sense of Fontaine ([F]). Note that the cohomology groups of other degrees $H_\bullet^q$ ($q \neq n$, $\bullet = \mathrm{dR}, \mathrm{crys}$) are just zero ($q$: odd) or Tate objects ($q$: even) by the weak Lefschetz theorem and the Poincaré duality, hence the structure is well-known.

Both of H_{dR}, H_{crys} have natural actions of

$$S := \left(\bigoplus_{i=0}^{n+1} \mu_d \right) \Big/ \Delta(\mu_d),$$

which are induced by the action on V given by $[X_0 : \cdots : X_{n+1}] \mapsto [\zeta_0 X_0 : \cdots : \zeta_{n+1} X_{n+1}]$ for $(\zeta_0, \dots, \zeta_{n+1}) \bmod \Delta(\mu_d) \in S$, where Δ is the diagonal homomorphism. This action of S induces an action of S on V_0 as well. However, the crystalline Frobenius (i.e., induced by the p-power map) is not compatible with this action of S. Hence, we consider cohomology groups $H_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} K$ and $H_{\mathrm{crys}} \otimes_{\mathbf{Q}_p} K$ over $K \otimes_{\mathbf{Q}_p} K$, where we consider the Hodge filtration and the crystalline Frobenius via the left K of $K \otimes_{\mathbf{Q}_p} K$, and the action of S via the right K of $K \otimes_{\mathbf{Q}_p} K$. To lighten the notations, we put $\otimes := \otimes_{\mathbf{Q}_p}$ in the following. Then, these cohomology groups $H_\bullet \otimes K$ ($\bullet = \mathrm{dR}, \mathrm{crys}$) are decomposed as filtered φ-modules with coefficient K by this action:

$$H_\bullet \otimes K = \bigoplus_{\underline{a} \in X(S)} (H_\bullet \otimes K)_{\underline{a}},$$

where $X(S)$ denotes the group of the characters of S, and $(H_\bullet \otimes K)_{\underline{a}}$ is the $\underline{a}$-part of $H_\bullet \otimes K$, i.e., the sub-$K \otimes K$-module of $H_\bullet \otimes K$ on which $\underline{\zeta} \in S$ acts by $\underline{a}(\underline{\zeta})$. Note that $X(S) = \{\underline{a} = (a_0, \dots, a_{n+1}) \in \bigoplus_{i=0}^{n+1} \mathbf{Z}/d\mathbf{Z} \mid \sum_{i=0}^{n+1} a_i = 0\}$ and $\underline{a}(\underline{\zeta}) = \underline{\zeta}^{\underline{a}} := \prod_{i=0}^{n+1} \zeta_i^{a_i}$. It is well-known that $(H_\bullet \otimes K)_{\underline{a}}$ is free $K \otimes K$-module of rank

$$\mathrm{rank}_{K \otimes K} (H_\bullet \otimes K)_{\underline{a}} = \begin{cases} 1 & a_0, \dots, a_{n+1} \neq 0, \text{ or } (a_0 = \cdots = a_{n+1} = 0 \text{ and } n : \text{even}), \\ 0 & \text{otherwise.} \end{cases}$$

(See [D, Proposition 7.4] and its erratum). (Note that the total chern class $c(V_{\mathbf{Q}}) = c_0(V_{\mathbf{Q}}) + \cdots + c_n(V_{\mathbf{Q}})$ of the tangent bundle of $V_{\mathbf{Q}} := V \otimes_{\mathbf{Z}} \mathbf{Q}$ is given by $c(V_{\mathbf{Q}}) = (1+x)^{n+2}(1+dx)^{-1} = \left(1 + (n+2)x + \cdots + \binom{n+2}{n} x^n\right)(1 - da + \cdots + (-1)^n d^n x^n)$ with $x := c_1(\mathcal{O}(1))$, hence the Euler characteristic $c_n(V_{\mathbf{Q}})$ is equal to $c_n(V_{\mathbf{Q}}) = \binom{n+2}{n} x^n - \binom{n+2}{n-1} dx^n + \cdots + (-1)^n d^n x^n = x^n \frac{(1-d)^{n+2} - 1 + (n+2)d}{d^2} = \frac{1}{d}((1-d)^{n+2} - 1) + n + 2$. Thus, the dimension of the primitive part of the cohomology group is $h^n_{\mathrm{prim}} = (-1)^n (c_n(V_{\mathbf{Q}}) - (n+1)) = \frac{1}{d}((d-1)^{n+2} - (-1)^n) + (-1)^n$. On the other hand, the number of $\underline{a} \in X(S)$ with $a_0, \dots, a_{n+1} \neq 0$

is $(d-1)^{n+1}-(d-1)^n+\cdots+(-1)^n(d-1)=\frac{1}{d}((d-1)^{n+2}-(-1)^n)+(-1)^n$, which coincides with h^n_{prim}, as expected.) In the case where $a_0=\cdots=a_{n+1}=0$ and n is even, $(H_\bullet\otimes K)_{\underline{a}}$ is also a Tate object (See the proof of [D, Proposition 7.4]), hence the structure is well-known. Thus, the remaining part is the case where $a_0,\dots,a_{n+1}\neq 0$.

In the rest of the paper, we assume that $a_0,\dots,a_{n+1}\neq 0$. The pair

$$((H_{\mathrm{dR}}\otimes K)_{\underline{a}},(H_{\mathrm{crys}}\otimes K)_{\underline{a}})$$

is a filtered φ-module with coefficient K with $\mathrm{rank}_{K\otimes K}=1$. By using the main theorem of the p-adic Hodge theory (See [T]), we have:

Theorem 1.1 (a consequence of [T]). *The filtered φ-module $((H_{\mathrm{dR}}\otimes K)_{\underline{a}}$, $(H_{\mathrm{crys}}\otimes K)_{\underline{a}})$ with coefficient in K is weakly admissible in the sense of Fontaine* ([F]).

(Strictly speaking, now we are considering the with-coefficient-version of the weakly admissibility.) In this short paper, we show the weakly admissibility of it without using the difficult theorem of [T]. Our (re-)proof shows that the weakly admissibility of it expresses essentially *Stickelberger's theorem* in Iwasawa theory (Note that Stickelberger's theorem is based on elementary calculations, and not difficult), and that the concrete content of a special case (i.e., Fermat variety case) of such a general difficult **geometric** theorem (i.e., the main theorem of the p-adic Hodge theory [T]) is of highly **arithmetic** nature.

2. de Rham Side (Hodge Polygon)

First, we introduce some notations. For $\underline{a}=(a_0,\dots,a_{n+1})\in X(S)$, let $\langle a_i\rangle\in\mathbf{N}$ denote the representative of a_i with $1\le\langle a_i\rangle\le d$, and $\langle\underline{a}\rangle:=\frac{1}{d}\sum_{i=0}^{n+1}\langle a_i\rangle\in\mathbf{N}$. For $\alpha\in\mathbf{Q}$, let $0\le\{\alpha\}<1$ denote the fractional part of α. For $n=n_0+n_1p+\cdots+n_{f-1}p^{f-1}$ with $0\le n_0,\dots,n_{f-1}\le p-1$, we set $s(n):=n_0+\cdots+n_{f-1}$.

Note that $(H_{\mathrm{dR}}\otimes K)_{\underline{a}}$ has rank one over $K\otimes K\cong K\times K^\sigma\times\cdots\times K^{\sigma^{f-1}}$, and $H_{\mathrm{dR}}\otimes K\cong H^n_{\mathrm{dR}}(V_K/K)\oplus H^n_{\mathrm{dR}}(V_{K^\sigma}/K^\sigma)\oplus\cdots\oplus H^n_{\mathrm{dR}}(V_{K^{\sigma^{f-1}}}/K^{\sigma^{f-1}})$. Thus, $\bigwedge_K(H_{\mathrm{dR}}\otimes K)_{\underline{a}}$ is isomorphic to

$$\bigwedge_K((H_{\mathrm{dR}})_{\underline{a}}\oplus(H_{\mathrm{dR}})_{p\underline{a}}\oplus\cdots\oplus(H_{\mathrm{dR}})_{p^{f-1}\underline{a}}),$$

where $p^k\underline{a}:=(p^ka_0,\dots,p^ka_{n+1})$ and $(H_{\mathrm{dR}})_{p^k\underline{a}}$ is the $p^k\underline{a}$-part of H_{dR} (not of $H_{\mathrm{dR}}\otimes K$ as before), i.e., the sub-K-vector space of H_{dR} on which $\underline{\zeta}\in S$ acts by

$(p^k\underline{a})(\underline{\zeta})$. Note that $\dim_K(H_{\mathrm{dR}})_{p^k\underline{a}} = 1$. By [D, Proposition 7.6], we have

$$\mathrm{Fil}^i(H_{\mathrm{dR}})_{\underline{a}} = \begin{cases} (H_{\mathrm{dR}})_{\underline{a}} & i \le \langle \underline{a} \rangle - 1, \\ 0 & i \ge \langle \underline{a} \rangle. \end{cases}$$

Therefore, the only jump of the Hodge filtration on $\bigwedge_K (H_{\mathrm{dR}} \otimes K)_{\underline{a}}$ happens at the degree

$$\begin{aligned}&(\langle \underline{a} \rangle - 1) + \cdots + (\langle p^{f-1}\underline{a} \rangle - 1) \\ &\quad = \sum_{i=0}^{n+1} \left\{ \frac{\langle a_i \rangle}{d} \right\} + \sum_{i=0}^{n+1} \left\{ \frac{p\langle a_i \rangle}{d} \right\} + \cdots + \sum_{i=0}^{n+1} \left\{ \frac{p^{f-1}\langle a_i \rangle}{d} \right\} - f.\end{aligned}$$

On the other hand, elementary calculation [W, Lemma 6.14] says that $\left\{\frac{n}{q-1}\right\} + \left\{\frac{pn}{q-1}\right\} + \cdots + \left\{\frac{p^{f-1}n}{q-1}\right\} = \frac{1+p+\cdots+p^{f-1}}{q-1} s(n) = \frac{1}{p-1} s(n)$. Then, by noting $\frac{p^k\langle a_i \rangle}{d} = \frac{((q-1)/d)p^k\langle a_i \rangle}{q-1}$ and d divides $q-1$, the above quantity is equal to

$$(*)_{\mathrm{Hodge}} \qquad \frac{1}{p-1} \sum_{i=0}^{n+1} s\left(\frac{(q-1)\langle a_i \rangle}{d} \right) - f.$$

3. Crystalline Side (Newton Polygon)

First, we introduce some notations (See also [D, pp. 84–85]). Let $\mathfrak{p}$ denote the prime ideal in $\mathbf{Q}(\mu_d)$ over p corresponding to the embedding $\mathbf{Q}(\mu_d) \hookrightarrow \mathbf{Q}_p(\mu_d)$. The reduction modulo $\mathfrak{p}$ defines an isomorphism $(\mathbf{Q}(\mu_d) \supset) \mu_d(\mathbf{Q}(\mu_d)) \xrightarrow{\sim} \mu_d(\mathbf{F}_q) (\subset \mathbf{F}_q)$. We set t to be its inverse. We fix $\underline{a} \in X(S)$ with $a_0, \ldots, a_{n+1} \neq 0$. We define a character $\varepsilon_i : \mathbf{F}_q^\times \to \mu_d$ to be

$$\varepsilon_i(x) := t(x^{(1-q)/d})^{a_i}$$

(Note that $x^{(1-q)/d}$ lives in $\mu_d(\mathbf{F}_q)$). Then $\prod_{i=0}^{n+1} \varepsilon_i(x_i)$ is well-defined for $(x_0 : \cdots : x_{n+1}) \in \mathbf{P}^{n+1}(\mathbf{F}_q)$, since $\prod_{i=0}^{n+1} \varepsilon_i = 1$. We define a *Jacobi sum*

$$J(\varepsilon_0, \ldots, \varepsilon_{n+1}) := (-1)^n \sum_{(x_0:\cdots:x_{n+1}) \in \mathbf{P}^{n+1}(\mathbf{F}_q),\, x_0+\cdots+x_{n+1}=0} \; \prod_{i=0}^{n+1} \varepsilon_i(x_i) \in \mathbf{Q}(\mu_d),$$

where we put $\varepsilon_i(0) := 0$. Let ψ be a non-trivial additive character $\psi : \mathbf{F}_q \to \mathbf{Q}(\mu_q)$, and we define *Gauss sums*

$$g(\mathfrak{p}, a_i, \psi) := -\sum_{x \in \mathbf{F}_q} \varepsilon_i(x)\psi(x) \in \mathbf{Q}(\mu_d, \mu_q),$$

and

$$g(\mathfrak{p}, \underline{a}) := q^{-\langle \underline{a} \rangle} \prod_{i=0}^{n+1} g(\mathfrak{p}, a_i, \psi).$$

Then [D, Lemma 7.9] says that $J(\varepsilon_0, \ldots, \varepsilon_{n+1}) = q^{\langle \underline{a} \rangle - 1} g(\mathfrak{p}, \underline{a})$, hence in particular $g(\mathfrak{p}, \underline{a})$ is independent of ψ, and lands in $\mathbf{Q}(\mu_d)$. By [D, Proposition 7.10] and the comparison of the Lefschetz trace formula for crystalline cohomology and étale cohomology, the K-linear Frobenius action φ^f on $(H_{\mathrm{crys}} \otimes K)_{\underline{a}}$ is given by the multiplication by

$$q^{\langle \underline{a} \rangle - 1} g(\mathfrak{p}, \underline{a}) = q^{-1} \prod_{i=0}^{n+1} g(\mathfrak{p}, a_i, \psi).$$

Then, (an essential part of) Stickelberger's theorem [W, Proposition 6.13] says that

$$(*)_{\mathrm{Newton}} \qquad v_p\left(q^{-1} \prod_{i=0}^{n+1} g(\mathfrak{p}, a_i, \psi)\right) = \frac{1}{p-1} \sum_{i=0}^{n+1} s\left(\frac{(q-1)\langle a_i \rangle}{d}\right) - f,$$

where the valuation v_p is normalised as $v_p(p) = 1$. Then, the quantity $(*)_{\mathrm{Hodge}}$ in Section 2 coincides with the quantity $(*)_{\mathrm{Newton}}$ in Section 3. This means the weakly admissibility of the filtered φ-module $((H_{\mathrm{dR}} \otimes K)_{\underline{a}}, (H_{\mathrm{crys}} \otimes K)_{\underline{a}})$.

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Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-8502, Japan
E-mail: gokun@kurims.kyoto-u.ac.jp