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ESSENTIAL m-SECTORIALITY AND ESSENTIAL SPECTRUM OF THE SCHRÖDINGER OPERATORS WITH RAPIDLY OSCILLATING COMPLEX-VALUED POTENTIALS

By

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Abstract. Schrödinger operators $T_0 = -\triangle + q(x)$ with rapidly oscillating complex-valued potentials $q(x)$ are considered. Each of such operators is sectorial and hence has Friedrichs extension. We prove that T_0 is essentially m-sectorial in the sense that the closure of T_0 coincides with its Friedrichs extension T. In particular, T_0 is essentially self-adjoint if the rapidly oscillating potential $q(x)$ is real-valued. Further, we prove $\sigma_{ess}(T) = [0, \infty)$ under somewhat stricter condition on the potentials $q(x)$.

1 Introduction

It is well known (see Theorem X.38 and its corollary in Reed-Simon [4]) that the Schrödinger operator $-\triangle + q(x)$ $(x \in \mathbf{R}^N)$ is essentially self-adjoint if the real potential $q(x)$ satisfies $q(x) \geq -c|x|^2$ for some positive constant c. However, there are still many potentials for which the essential self-adjointness of the Schrödinger operators have not been fully studied. Rapidly oscillating potentials are among such ones and typical examples are

$$\varphi\left(\frac{x}{|x|}\right)|x|^3 \sin(|x|^5), (1 + |x|^2)^{-1} e^{|x|} \cos(e^{|x|}).$$

Here $\varphi(\omega)$ is a bounded function on the unit sphere $S^{N-1} = \{\omega \in \mathbf{R}^N : |\omega| = 1\}$. Skriganov [6] (see also Mateev and Skriganov [3]) studies such potentials and also

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provides sufficient conditions for the essential self-adjointness of the operators. However, he assumes that the potentials are continuous and satisfy some additional properties. Removing the continuity conditions on the potentials, Sasaki [5] proves that the essential spectrum of their Friedrichs extension is $[0, \infty)$ though he does not consider their essential self-adjointness.

It should be noted that the above authors use argument applicable only to the real potentials. In this paper, we study complex-valued rapidly oscillating potentials. To mention our results, we define the essential m-sectoriality of operators.

Let S_0 be a densely defined sectorial operator in a Hilbert space. Then S_0 has an m-sectorial extension S which is called its Friedrichs extension. (See Kato [2, p325].) If this S coincides with the closure of S_0, then S_0 is called *essentially m-sectorial*. In the special case where S_0 is a symmetric operator bounded from below, the essential m-sectoriality becomes the essential self-adjointness.

In Section 2, we prove the essential self-adjointness or rather the essential m-sectoriality of the operators with complex-valued rapidly oscillating potentials, avoiding continuity conditions. It is guaranteed that, for example, $T_0 = -\triangle + q(x)$, $\mathrm{Dom}(T_0) = C_0^\infty$ with $q(x) = |x|^3 e^{i|x|^4}$ or $e^{|x|} \exp(ie^{|x|})$ is essentially m-sectorial and its closure coincides with its Friedrichs extension.

In Section 3, we prove that the essential spectrum of such operators equals $[0, \infty)$ under somewhat stricter condition on the potentials. It is guaranteed that, for example, the Friedrichs extension T of $T_0 = -\triangle + q(x)$ with $q(x) = |x|^3 e^{i|x|^5}$ or $(1 + |x|^2)^{-1} e^{|x|} \exp(ie^{|x|})$ satisfies $\sigma_{ess}(T) = [0, \infty)$.

Our main tools are sectorial sesquilinear forms and associated m-sectorial operators. See Kato [2] for their definitions and basic properties.

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2 Essential m-Sectoriality

In this section, we consider the essential m-sectoriality of the operator

$$T_0 u = -\triangle u + q(x)u, \quad (x \in \mathbf{R}^N)$$

with domain $\mathrm{Dom}(T_0) = C_0^\infty(\mathbf{R}^N)$.

Throughout this section, we always assume

$$q(x) = q_1(x) + q_2(x)$$

with $q_1(x) \in L^\infty_{loc}(\mathbf{R}^N)$, $q_2(x) \in L^\infty(\mathbf{R}^N)$ and

$$\sup_{r>0,\,\omega\in S^{N-1}} \left| \int_0^r q_1(\rho\omega)\,d\rho \right| < \infty.$$

Therefore, by setting

$$Q_1(r\omega) = \int_0^r q_1(\rho\omega)\,d\rho,$$

$q_1(x) \in L^\infty_{loc}(\mathbf{R}^N)$ implies

$$|Q_1(r\omega)| \le M \min\{1, r\},$$

$$\sup_{r>0,\,\omega\in S^{N-1}} |q_2(r\omega)| \le M$$

for some constant $M > 0$ independent of $\omega \in S^{N-1}$.

Note that $q(x) = |x|^3 e^{i|x|^4}$ are $e^{|x|}\exp(ie^{|x|})$ typical examples for the above $q_1(x)$.

Lemma 1. *For* $u \in H^1(\mathbf{R}^N)$, $v \in C_0^\infty(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} q_1(x)u(x)\overline{v(x)}\,dx = -(N-1)\int_{\mathbf{R}^N} (Q_1(x)/|x|)u(x)\overline{v(x)}\,dx$$
$$- \int_{\mathbf{R}^N} Q_1(x)\sum_{j=1}^N (x_j/|x|)\{u(\partial\bar{v}/\partial x_j) + \{(\partial u/\partial x_j)\bar{v}\}\,dx.$$

Proof. We may assume $u \in C_0^\infty(\mathbf{R}^N)$. Using $(\partial/\partial r)Q_1(r\omega) = q_1(r\omega)$ and integration by parts, we have

$$\int_{\mathbf{R}^N} q_1(x)u(x)\overline{v(x)}\,dx = \int_{S^{N-1}}\int_0^\infty r^{N-1}q_1(r\omega)u(r\omega)\overline{v(r\omega)}\,drd\omega$$
$$= -(N-1)\int_{\mathbf{R}^N} (Q_1(x)/|x|)u(x)\overline{v(x)}\,dx$$
$$- \int_{\mathbf{R}^N} Q_1(x)\sum_{j=1}^N (x_j/|x|)\{u(\partial\bar{v}/\partial x_j) + (\partial u/\partial x_j)\bar{v}\}\,dx. \qquad \blacksquare$$

Lemma 2. *Define the sesqulinear form* $s[u,v]$ *by*

$$s[u,v] = -(N-1)\int_{\mathbf{R}^N} \frac{Q_1(x)}{|x|} u(x)\overline{v(x)}\,dx$$
$$-\int_{\mathbf{R}^N} Q_1(x) \sum_{j=1}^{N} \frac{x_j}{|x|} \left\{ \frac{\partial u(x)}{\partial x_j}\overline{v(x)} + u(x)\overline{\frac{\partial v(x)}{\partial x_j}} \right\} dx$$
$$+\int_{\mathbf{R}^N} q_2(x) u(x)\overline{v(x)}\,dx$$

for $u, v \in H^1(\mathbf{R}^N)$. *Then we have*

$$|s[u,v]| \le M\|\nabla u\|_{L^2}\|v\|_{L^2} + M\|u\|_{L^2}\|\nabla v\|_{L^2} + MN\|u\|_{L^2}\|v\|_{L^2}$$

for all $u, v \in H^1(\mathbf{R}^N)$.

Proof. Recalling $|Q_1(x)| \le M$ and using the Cauchy-Schwartz inequality, we have

$$\left| \int_{\mathbf{R}^N} Q_1(x) \sum_{j=1}^{N} \frac{x_j}{|x|} \left\{ \frac{\partial u(x)}{\partial x_j}\overline{v(x)} + u(x)\overline{\frac{\partial v(x)}{\partial x_j}} \right\} dx \right| \le M\|\nabla u\|\,\|v\| + M\|u\|\,\|\nabla v\|.$$

We apply $|Q_1(x)| \le M|x|$ and $|q_2(x)| \le M$ to the first and third terms of $s[u,v]$ to obtain

$$|s[u,v]| \le M\|\nabla u\|\,\|v\| + M\|u\|\,\|\nabla v\| + MN\|u\|\,\|v\|. \qquad \blacksquare$$

Remark. Combining Lemma 1 and Lemma 2 we know that the multiplication operator $u \to (q_1(x) + q_2(x))u$ can be extended to a bounded map from H^1 to H^{-1}.

Theorem 3.

$$t[u,v] = (\nabla u, \nabla v) + s[u,v]$$

is a closed sectorial sesquilinear form with domain H^1. *The associated m-sectorial operator is*

$$Tu = -\triangle u + q(x)u$$

with domain

$$\mathrm{Dom}(T) = \{u \in H^1 \cap H^2_{loc} : -\triangle u + q(x)u \in L^2\}.$$

PROOF. From the previous Lemma 2, we have

$$|s[u,u]| \le (1/2)\|\nabla u\|^2 + (8M^2 + MN)\|u\|^2.$$

Hence

$$\operatorname{Re} t[u,u] = \|\nabla u\|^2 + \operatorname{Re} s[u,u] \ge (1/2)\|\nabla u\|^2 - (8M^2 + MN)\|u\|^2.$$

We also have

$$|t[u,v]| \le \|\nabla u\|\,\|\nabla v\| + M\|\nabla u\|\,\|v\| + M\|u\|\,\|\nabla v\| + MN\|u\|\,\|v\|.$$

Therefore $t[u,v]$ is a closed sectorial sesquilinear form with domain H^1. Thus the representation theorem (Theorem 2.1 of Kato, [2; p322]) ensures that there exists a unique associated m-sectorial operator T such that for an arbitrary $u \in \mathrm{Dom}(T)$, $t[u,v] = (Tu,v)$ holds for all $v \in \mathrm{Dom}(t)$.

Finally, considering the special case where $v \in C_0^\infty$, it is easy to prove $\mathrm{Dom}(T) = \{u \in H^1 \cap H^2_{loc} : -\triangle u + q(x)u \in L^2\}$. ■

We have now proved the minimal operator $T_0 u = -\triangle u + q(x)u$ with domain $\mathrm{Dom}(T_0) = C_0^\infty$ has an m-sectorial extension (i.e., Friedrichs extension). We shall show this extension is unique by proving the closure of T_0 is exaclty the Friedrichs extension T we have just obtained. Let us begin with a lemma.

LEMMA 4. *For any $u \in H^1_{loc}$ and any constant $R \ge 1$, the following holds.*

$$\left|\int_{|x|\le R} q_1(x)|u(x)|^2\,dx\right| \le (1/2)\int_{|x|\le R} |\nabla u|^2\,dx + (8M^2 + 2MN)\int_{|x|\le R} |u|^2\,dx.$$

PROOF. We may assume $u \in H^1_{loc} \cap C^\infty$. Note that

$$\begin{aligned}\int_{|x|\le R} q_1(x)|u(x)|^2\,dx = &\int_{S^{N-1}} Q_1(R\omega)R^{N-1}|u(R\omega)|^2\,d\omega \\ &- \int_{S^{N-1}}\int_0^R (N-1)r^{N-2}Q_1(r\omega)|u(r\omega)|^2\,dr d\omega \\ &- \int_{S^{N-1}}\int_0^R r^{N-1}Q_1(r\omega)\frac{\partial}{\partial r}|u(r\omega)|^2\,dr d\omega.\end{aligned}$$

Recalling $|Q_1(r\omega)| \le M \min\{r, 1\}$, we further have

$$\left|\int_{|x|\le R} q_1(x)|u(x)|^2\,dx\right| \le M\int_{S^{N-1}} R^{N-1}|u(R\omega)|^2\,d\omega + M(N-1)\int_{|x|\le R}|u(x)|^2\,dx$$

$$+ 2M\int_{|x|\le R}|u(x)|\,|\nabla u(x)|\,dx$$

$$\le M\int_{S^{N-1}} R^{N-1}|u(R\omega)|^2\,d\omega + (1/4)\int_{|x|\le R}|\nabla u(x)|^2\,dx$$

$$+ (4M^2 + MN)\int_{|x|\le R}|u(x)|^2\,dx.$$

Now we have only to estimate $\int_{S^{N-1}} R^{N-1}|u(R\omega)|^2\,d\omega$. Indeed,

$$\int_{S^{N-1}} R^{N-1}|u(R\omega)|^2\,d\omega$$

$$= \int_{S^{N-1}}\int_1^R \frac{\partial}{\partial r} r^{N-1}|u(r\omega)|^2\,drd\omega + \int_{S^{N-1}}\int_0^1 \frac{\partial}{\partial r} r^N|u(r\omega)|^2\,drd\omega$$

$$= \int_{S^{N-1}}\int_1^R (N-1)r^{N-2}|u(r\omega)|^2\,drd\omega$$

$$+ \int_{S^{N-1}}\int_1^R 2r^{N-1}\,\mathrm{Re}\,u(r\omega)(\omega\cdot\overline{\nabla u(r\omega)})\,drd\omega$$

$$+ \int_{S^{N-1}}\int_0^1 Nr^{N-1}|u(r\omega)|^2\,drd\omega$$

$$+ \int_{S^{N-1}}\int_0^1 2r^N\,\mathrm{Re}\,u(r\omega)(\omega\cdot\overline{\nabla u(r\omega)})\,drd\omega$$

$$\le \int_{1\le|x|\le R}(N-1)|x|^{-1}|u(x)|^2\,dx + \int_{1\le|x\le R} 2|u(x)|\,|\nabla u|\,dx$$

$$+ \int_{|x|\le 1} N|u(x)|^2\,dx + \int_{|x|\le 1} 2|u(x)|\,|\nabla u|\,dx$$

$$\le N\int_{|x|\le R}|u(x)|^2\,dx + 2\int_{|x|\le R}|u(x)|\,|\nabla u|\,dx$$

$$\le \frac{1}{4M}\int_{|x|\le R}|\nabla u|^2\,dx + (N+4M)\int_{|x|\le R}|u(x)|^2\,dx. \qquad \blacksquare$$

Proposition 5. *Suppose* $u \in L^2$ *satisfies*

$$-\triangle u + (q(x) - \lambda)u = w \in L^2.$$

in the distributional sense, for some complex constant λ. *Then* $u \in H^1 \cap H^2_{loc}$.

Proof. First notice that

$$-\triangle u = -(q(x) - \lambda)u + w \in L^2_{loc}$$

since $q(x) = q_1(x) + q_2(x)$ is locally bounded. Hence $u \in H^2_{loc}$.

Observe now that

$$\int_{|x| \le R} \bar{u} \triangle u = -\int_{|x| \le R} |\nabla u|^2 \, dx + \int_{S^{N-1}} R^{N-1} \bar{u}(R\omega) \frac{\partial u}{\partial R}(R\omega) \, d\omega$$

since $u \in H^2_{loc}$. Note that the integral on S^{N-1} converges. From this equation and $\triangle u = (q_1(x) + q_2(x) - \lambda)u - w$,

$$\begin{aligned}
-\int_{|x| \le R} &|\nabla u|^2 \, dx + \int_{S^{N-1}} R^{N-1} |u(R\omega)| \, |\nabla u(R\omega)| \, d\omega \\
&\ge \mathrm{Re} \int_{|x| \le R} \bar{u} \triangle u \, dx \\
&= \mathrm{Re} \int_{|x| \le R} \{(q_1(x) + q_2(x) - \lambda)|u(x)|^2 - \overline{u(x)} w(x)\} \, dx \\
&\ge -\left| \int_{|x| \le R} q_1(x) |u(x)|^2 \, dx \right| - \int_{|x| \le R} \{|q_2(x) - \lambda| \, |u(x)|^2 + |u(x)| \, |w(x)|\} \, dx.
\end{aligned}$$

By Lemma 4 and $|q_2(x) - \lambda| \le M + |\lambda|$, this is estimated from below by

$$-(1/2) \int_{|x| \le R} |\nabla u|^2 \, dx - (8M^2 + 2MN + M + |\lambda|) \|u\|^2 - \|u\| \, \|w\|.$$

Therefore, we have

$$\begin{aligned}
(1/2) \int_{|x| \le R} &|\nabla u|^2 \, dx - (8M^2 + 2MN + M + |\lambda|) \|u\|^2 - \|u\| \, \|w\| \\
&\le \int_{S^{N-1}} R^{N-1} |u(R\omega)| \, |\nabla u(R\omega)| \, d\omega.
\end{aligned}$$

Let us now prove $u \in H^1$ by contradiction, assuming to the contrary that

$$\lim_{R\to\infty} \int_{|x|\le R} |\nabla u|^2 \, dx = \infty.$$

Thus

$$(1/4)\int_{|x|\le R} |\nabla u|^2 \, dx \le \int_{S^{N-1}} R^{N-1}|u(R\omega)|\,|\nabla u(R\omega)|\, d\omega$$

for $R \ge R_0$ with sufficiently large R_0. Putting

$$G(R) = \int_{|x|\le R} |\nabla u|^2 \, dx$$

and integrating the last inequality over $[R_0, R]$, we have

$$\begin{aligned}(1/4)\int_{R_0}^{R} G(r)\, dr &\le \int_{R_0}^{R}\int_{S^{N-1}} r^{N-1}|u(r\omega)|\,|\nabla u(r\omega)|\, d\omega dr \\ &\le \int_{R_0\le|x|\le R} |u(x)|\,|\nabla u(x)|\, dx \\ &\le \|u\|(G(R))^{1/2}.\end{aligned}$$

Hence we have

$$\frac{\|u\|^{-2}}{16} \le \frac{G(R)}{\left(\int_{R_0}^{R} G(r)\, dr\right)^2} \quad (R \ge R_0).$$

Integrating this inequality over $[2R_0, R]$, we have

$$\frac{(R-2R_0)\|u\|^{-2}}{16} \le \left(\int_{R_0}^{2R_0} G(r)\, dr\right)^{-1} - \left(\int_{R_0}^{R} G(r)\, dr\right)^{-1} < \infty.$$

If $R \to \infty$, the left side goes to ∞ while the right side remains bounded. A contradiction. ∎

Now we are ready to prove the main theorem of this section.

THEOREM 6. *Let*

$$T_0 u = -\triangle u + q(x)u, \quad \mathrm{Dom}(T_0) = C_0^\infty(\mathbf{R}^N).$$

Assume

$$q(x) = q_1(x) + q_2(x), \quad q_1(x) \in L^\infty_{loc}(\mathbf{R}^N), \quad q_2(x) \in L^\infty(\mathbf{R}^N)$$

and

$$\sup_{r>0,\, \omega \in S^{N-1}} \left| \int_0^r q_1(\rho\omega)\, d\rho \right| < \infty.$$

Then the closure $\overline{T_0}$ *of* T_0 *coincides with its Friedrichs extension* T.

Proof. Let λ be outside the sectorial regions (which are larger than the numerical ranges) of the sesquilinear forms $t[u, v]$ and $t^*[u, v] = \overline{t[v, u]}$. Note that $\lambda \in \rho(T) \cap \rho(T^*)$. Suppose there exists $v \in L^2 \backslash \{0\}$ such that

$$(v, (T_0 - \lambda)u) = 0$$

for all $u \in C_0^\infty$. This means $v \in L^2$ is the distributional solution of

$$-\triangle v + \overline{(q(x) - \lambda)}v = 0.$$

Therefore, the previous Proposition 5 implies that $v \in H^1 = \mathrm{Dom}(t^*) = \mathrm{Dom}(t)$. Hence

$$(t^* - \bar{\lambda})[v, v] = \overline{(t - \lambda)[v, v]} = (-\triangle v + \overline{(q(x) - \lambda)}v, v) = 0.$$

Recalling that λ is outside the sectorial region of the sesquilinear form $t[u, v]$, we obtain $v = 0$. Thus we have proved that

$$\overline{\mathrm{Ran}(\overline{T_0} - \lambda)} = \overline{\mathrm{Ran}(T_0 - \lambda)} = L^2.$$

Since $(\overline{T_0} - \lambda) \subseteq (T - \lambda)$ and $(T - \lambda)^{-1}$ is bounded, so is $(\overline{T_0} - \lambda)^{-1}$ on its domain $\mathrm{Ran}(\overline{T_0} - \lambda)$. Recalling that $\overline{T_0} - \lambda$ is a closed operator, we obtain

$$\mathrm{Ran}(\overline{T_0} - \lambda) = \overline{\mathrm{Ran}(\overline{T_0} - \lambda)} = L^2 = \mathrm{Ran}(T - \lambda).$$

This implies $(\overline{T_0} - \lambda) = (T - \lambda)$ and $\overline{T_0} = T$. ■

Remark. If, in addition, $q_1(x)$ and $q_2(x)$ in $q(x) = q_1(x) + q_2(x)$ are both real-valued, then Theorem 6 ensures that $T_0 u = -\triangle u + q(x)u$ is essentially self-adjoint.

3 Essential Spectrum

In this section, imposing somewhat stricter conditions on the potential $q(x) = q_1(x) + q_2(x)$, we study the essential spectrum of the Friedrichs extension T of $-\triangle + q(x)$. More specifically, we assume that $q_1(x) \in L^\infty_{loc}(\mathbf{R}^N)$, $q_2(x) \in L^\infty(\mathbf{R}^N)$ satisfy

$$\lim_{r_1, r_2 \to \infty} \sup_{\omega \in S^{N-1}} \left| \int_{r_1}^{r_2} q_1(\rho\omega)\, d\rho \right| = 0$$

and

$$\lim_{r \to \infty} \sup_{\omega \in S^{N-1}} |q_2(r\omega)| = 0$$

throughout this section. In other words, for any $\omega \in S^{N-1}$ and $r_2 > r_1 \geq R$, we assume

$$|Q_1(r_2\omega) - Q_1(r_1\omega)| + |q_2(r_1\omega)| < \varepsilon(R)$$

with some $\varepsilon(R)$ such that $\lim_{R\to\infty} \varepsilon(R) = 0$.

Note that $q(x) = |x|^3 e^{i|x|^5}$ and $(1 + |x|^2)^{-1} e^{|x|} \exp(ie^{|x|})$ are typical examples for the aove $q_1(x)$.

From the result of the previous section, we already know that the multiplication operator $u \mapsto (q_1(x) + q_2(x))u$ is bounded from H^1 to H^{-1}. We consider its further property under the stricter assumption of this section.

LEMMA 7. *For any $u \in H^1$, $v \in C_0^\infty$ and any constant $R \geq 1$, the following holds.*

$$\left| \int_{|x| \geq R} q_1(x) u(x) \overline{v(x)}\, dx \right| \leq \varepsilon(R)(\|u\|\, \|\nabla v| + \|\nabla u\|\, \|v\|) + (N-1)\varepsilon(R)\|u\|\, \|v\|.$$

PROOF. We may assume $u \in H^1 \cap C^\infty$. Notice that $v \in C_0^\infty$.

$$\begin{aligned}\int_{|x| \geq R} q_1(x) u(x) \overline{v(x)}\, dx &= \int_{S^{N-1}} \int_R^\infty \left\{ \frac{\partial}{\partial r}(Q_1(r\omega) - Q_1(R\omega)) \right\} r^{N-1} u(r\omega) \overline{v(r\omega)}\, dr d\omega \\ &= -\int_{S^{N-1}} \int_R^\infty \{Q_1(r\omega) - Q_1(R\omega)\} \frac{\partial}{\partial r} r^{N-1} u(r\omega) \overline{v(r\omega)}\, dr d\omega\end{aligned}$$

Since $|Q_1(r\omega) - Q_1(R\omega)| < \varepsilon(R)$ for $r \geq R \geq 1$,

$$\left| \int_{|x| \geq R} q_1(x) u(x) \overline{v(x)} \, dx \right|$$

$$\leq \varepsilon(R) \int_{S^{N-1}} \int_R^\infty r^{N-1} \{(N-1) r^{-1} |u(r\omega)| \, |v(r\omega)|$$

$$+ |(\partial/\partial r) u(r\omega)| \, |v(r\omega)| + |u(r\omega)| \, |(\partial/\partial r) v(r\omega)|\} \, dr d\omega$$

$$\leq (N-1) R^{-1} \varepsilon(R) \int_{|x| \geq R} |u(x)| \, |v(x)| \, dx$$

$$+ \varepsilon(R) \int_{|x| \geq R} (|\nabla u(x)| \, |v(x)| + |u(x)| \, |\nabla v(x)|) \, dx$$

$$\leq (N-1) \varepsilon(R) \|u\| \, \|v\| + \varepsilon(R) (\|u\| \, \|\nabla v| + \|\nabla u\| \, \|v\|). \qquad \blacksquare$$

Lemma 8. *The multiplication operator*

$$u \mapsto (1 - \chi_R(x)) q(x) u$$

defines a bounded map from H^1 *to* H^{-1} *with norm not larger than* $2N\varepsilon(R)$. *Here* $\chi_R(x)$ *is the characteristic function of the open ball* $B_R = \{x \in \mathbf{R}^N : |x| < R\}$.

Proof. Since $|q_2(r\omega)| < \varepsilon(R)$ for $r \geq R$, we have

$$\left| \int_{|x| \geq R} q_2(x) u(x) \overline{v(x)} \, dx \right| \leq \varepsilon(R) \|u\| \, \|v\|$$

for $u, v \in H^1$. Using this inequality and the previous lemma,

$$\left| \int_{\mathbf{R}^N} (1 - \chi_R(x)) q(x) u(x) \overline{v(x)} \, dx \right| = \left| \int_{|x| \geq R} (q_1(x) + q_2(x)) u(x) \overline{v(x)} \, dx \right|$$

$$\leq N\varepsilon(R) \|u\| \, \|v\| + \varepsilon(R) (\|u\| \, \|\nabla v| + \|\nabla u\| \, \|v\|)$$

$$\leq (N+1) \varepsilon(R) (\|u\|^2 + \|\nabla u|^2)^{1/2} (\|v\|^2 + \|\nabla v\|)^{1/2}.$$

This implies the claim. $\blacksquare$

Proposition 9. *Let* $\{u_n\} \subset H^1$ *be an arbitrary bounded sequence. Then* $\{q(x) u_n(x)\} \subset H^{-1}$ *has a converging subsequence.* $\square$

Remark. In other words, the multiplication operator $u \mapsto q(x) u$ from H^1 to H^{-1} is compact. However, it is generally unbounded as a map from H^2 to L^2.

(e.g., $q(x) = |x|^3 \sin(|x|^5)$, $u(x) = (1 + |x|^2)^{-(N+1)/4}$). That is, it may be relatively unbounded with respect to $(-\triangle)$ in the usual framework.

Proof. The Rellich theorem and $q(x) \in L^\infty_{loc}$ imply $u \mapsto \chi_j(x)q(x)u$ is a compact operator from H^1 to $L^2 \subseteq H^{-1}$ for each $j = 1, 2, \dots$.

Let us choose $R = 1, 2, \dots, j, \dots$ in Lemma 8. Then we have

$$\|(1 - \chi_j(x))q(x)u_n\|_{H^{-1}} \le 2N\varepsilon(j)\|u_n\|, \quad \lim_{j\to\infty} 2N\varepsilon(j) = 0.$$

Let us assume from now on that $\|u_n\|_{H^1} \le 1$ for simplicity.

By the compactness of $u \mapsto \chi_1(x)q(x)u$, we can choose a subsequence $\{u_j^{(1)}\}_j$ of $\{u_n\}_n$ such that $\chi_1(x)q(x)u_j^{(1)}$ converges in $L^2 \subset H^{-1}$ and

$$\begin{aligned}\limsup_{j,k\to\infty} \|q(x)u_j^{(1)} - q(x)u_k^{(1)}\|_{H^{-1}} &\le \limsup_{j,k\to\infty} \|(1 - \chi_1(x))(q(x)u_j^{(1)} - q(x)u_k^{(1)})\|_{H^{-1}} \\ &\le 4N\varepsilon(1).\end{aligned}$$

In the same way, we choose a subsequence $\{u_j^{(2)}\}_j$ of $\{u_j^{(1)}\}_j$ such that $\chi_2(x)q(x)u_j^{(2)}$ converges in $L^2 \subset H^{-1}$ and

$$\limsup_{j,k\to\infty} \|q(x)u_j^{(2)} - q(x)u_k^{(2)}\|_{H^{-1}} \le 4N\varepsilon(2).$$

Repeating the same procedure, we finally have $\{u_j^{(\ell)}\}$ $(\ell, j = 1, 2, \dots)$ such that

$$\limsup_{j,k\to\infty} \|q(x)u_j^{(\ell)} - q(x)u_k^{(\ell)}\|_{H^{-1}} \le 4N\varepsilon(\ell).$$

Using the diagonal process, we have

$$\limsup_{j,k\to\infty} \|q(x)u_j^{(j)} - q(x)u_k^{(k)}\|_{H^{-1}} = 0$$

since $\lim_{j\to\infty} 4N\varepsilon(j) = 0$. In other words, the subsequence $\{q(x)u_j^{(j)}\}$ converges in H^{-1}. ■

Theorem 10. *Let*

$$T_0 u = -\triangle u + q(x)u, \quad \mathrm{Dom}(T_0) = C_0^\infty(\mathbf{R}^N).$$

Assume

$$q(x) = q_1(x) + q_2(x), \quad q_1(x) \in L^\infty_{loc}(\mathbf{R}^N), \quad q_2(x) \in L^\infty(\mathbf{R}^N),$$

$$\lim_{r_1, r_2\to\infty} \sup_{\omega\in S^{N-1}} \left| \int_{r_1}^{r_2} q_1(\rho\omega)\, d\rho \right| = 0$$

and

$$\lim_{r\to\infty} \sup_{\omega\in S^{N-1}} |q_2(r\omega)| = 0.$$

Then the Friedrichs extension T of T_0 satisfies

$$\sigma_{ess}(T) = [0, \infty).$$

PROOF. Let $\mu > 0$ be sufficiently large. Then $-\mu \in \rho(T) \cap \rho(-\triangle)$ holds and $T + \mu$, $-\triangle + \mu$ are isomorphic maps from H^1 to H^{-1}. (Strictly speaking, consider both of closed sectorial forms with domain H^1 which are associated with $T + \mu$ and $-\triangle + \mu$.)

Let us prove $(T+\mu)^{-1} - (-\triangle+\mu)^{-1}$ is a compact operator in L^2. Suppose that $\{u_n\} \subset L^2$ is an arbitrary bounded sequence. Note that

$$\begin{aligned}
&(T+\mu)^{-1} - (-\triangle+\mu)^{-1} \\
&\quad = (T+\mu)^{-1}(-\triangle+\mu)(-\triangle+\mu)^{-1} - (T+\mu)^{-1}(T+\mu)(-\triangle+\mu)^{-1} \\
&\quad = (T+\mu)^{-1}\{-q(x)\}(-\triangle+\mu)^{-1}.
\end{aligned}$$

Note also that $\{(-\triangle+\mu)^{-1}u_n\}$ is a bounded sequence in $H^2 \subset H^1$. By Lemma 9, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $q(x)(-\triangle+\mu)^{-1}u_{n_j}$ converges in H^{-1}, hence

$$\{(T+\mu)^{-1} - (-\triangle+\mu)^{-1}\}u_{n_j} = -(T+\mu)^{-1}q(x)(-\triangle+\mu)^{-1}u_{n_j}$$

converges in $H^1 \subset L^2$. Since $\{u_n\} \subset L^2$ is an arbitrary bounded sequence in L^2, this implies $(T+\mu)^{-1} - (-\triangle+\mu)^{-1}$ is a compact operator from L^2 into itself. Therefore

$$\sigma_{ess}(T) = \sigma_{ess}(-\triangle) = [0, \infty). \qquad \blacksquare$$

REMARK. The present theorem can be extended as follows by the result of F. Gesztesy et al. [1] and a rather lengthy argument, although the minimal operator has not yet been proved essentially m-sectorial.

Let $q_1(x)$, $q_2(x)$ in $q(x) = q_1(x) + q_2(x)$ belong to L^2_{loc}, $q_2(x)$ be $(-\triangle)$-compact and $\chi_R(x)q_1(x)$ be $(-\triangle)$-compact for all $R > 0$. Let the other assumption be the same. Then the result remain the same.

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