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ON THE CAUCHY PROBLEM FOR A CLASS OF HYPERBOLIC OPERATORS WHOSE COEFFICIENTS DEPEND ONLY ON THE TIME VARIABLE

By

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Abstract. In this paper we investigate the Cauchy problem for hyperbolic operators with double characteristics and hyperbolic operators of third order whose coefficients depend only on the time variable. And we give sufficient conditions for C^∞ well-posedness.

1. Introduction

We say that a (partial differential) operator is an operator with time-dependent coefficients if the coefficients of the operator depend only on the time variable. In [16] we studied the Cauchy problem for hyperbolic operators of second order with time-dependent coefficients. And we gave sufficient conditions for the Cauchy problem to be C^∞ well-posed, assuming that the coefficients of the principal parts are real analytic functions of the time variable. These conditions are also necessary conditions if the space dimension is less than 3, or if the coefficients of the principal parts of the operators are semi-algebraic functions (*e.g.*, polynomials) of the time variable (see, also, [17]).

In this paper we shall deal with hyperbolic operators with time-dependent coefficients and double characteristics and give sufficient conditions for the Cauchy problem to be C^∞ well-posed, imposing some conditions on the sub-principal symbols. Our conditions are generalizations of the conditions given in [16]. If one considers the Cauchy problem for hyperbolic operators of m-th

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order, then one must impose some conditions not only on the subprincipal symbols but on the lower order symbols of order k $(1 \le k \le m-2)$, in general. So one needs to define the symbols of order k $(1 \le k \le m-2)$ corresponding to the subprincipal symbols in order to describe the conditions for C^∞ well-posedness. To clarify the situation we consider hyperbolic operators of third order with time-dependent coefficients in this paper. In doing so, we shall define symbols of first order for operators of third order with time-dependent coefficients, which are called the sub-sub-principal symbols. We should note that Jackson [8] showed that the sub-sub-principal symbol can not be defined invariantly under canonical transformations. We shall prove C^∞ well-posedness of the Cauchy problem for hyperbolic operators of third order with time-dependent coefficients, imposing some conditions on the subprincipal symbols and the sub-sub-principal symbols.

Let $m \in \mathbf{N}$ and $P(t,\tau,\xi) \equiv \tau^m + \sum_{j=1}^m \sum_{|\alpha| \le j} a_{j,\alpha}(t)\tau^{m-j}\xi^\alpha$ be a polynomial of τ and $\xi = (\xi_1, \dots, \xi_n)$ of degree m whose coefficients $a_{j,\alpha}(t)$ are C^∞ functions of $t \in [0,\infty)$. Here $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{Z}_+)^n$ is a multi-index, $|\alpha| = \sum_{j=1}^n \alpha_j$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, where $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$ $(= \{0,1,2,3,\dots\})$. We consider the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} P(t,D_t,D_x)u(t,x) = f(t,x) & \text{in } [0,\infty) \times \mathbf{R}^n, \\ D_t^j u(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (0 \le j \le m-1) \end{cases}$$

in the framework of the space of C^∞ functions, where $D_t = -i\partial/\partial t$ $(= -i\partial_t)$, $D_x = (D_1, \dots, D_n) = -i(\partial/\partial x_1, \dots, \partial/\partial x_n)$, $f(t,x) \in C^\infty([0,\infty) \times \mathbf{R}^n)$ and $u_j(x) \in C^\infty(\mathbf{R}^n)$ $(0 \le j \le m-1)$.

Definition 1.1. The Cauchy problem (CP) is said to be C^∞ well-posed if the following conditions (E) and (U) are satisfied:

(E) For any $f \in C^\infty([0,\infty) \times \mathbf{R}^n)$ and $u_j \in C^\infty(\mathbf{R}^n)$ $(0 \le j \le m-1)$ there is $u \in C^\infty([0,\infty) \times \mathbf{R}^n)$ satisfying (CP).

(U) If $s > 0$, $u \in C^\infty([0,\infty) \times \mathbf{R}^n)$, $D_t^j u(t,x)|_{t=0} = 0$ $(0 \le j \le m-1)$ and $P(t,D_t,D_x)u(t,x)$ vanishes for $t < s$, then $u(t,x)$ also vanishes for $t < s$.

We assume throughout the paper that

(A-1) $a_{j,\alpha}(t)$ $(1 \le j \le m,\ |\alpha| = j)$ are real analytic on $[0,\infty)$, *i.e.*, the coefficients of the principal part of $P(t,D_t,D_x)$ are real analytic on $[0,\infty)$.

From (A-1) there are a complex neighborhood Ω of $[0, \infty)$ (in $\mathbf{C}$) and $\delta > 0$ such that $[-\delta, \infty) \subset \Omega$ and $a_{j,\alpha}(t)$ $(1 \le j \le m,\ |\alpha| = j)$ are regarded as analytic functions defined in Ω. Put

$$p(t, \tau, \xi) = \tau^m + \sum_{j=1}^{m} a_j^0(t, \xi)\tau^{m-j} \quad (\equiv P_m(t, \tau, \xi)),$$

$$a_j^0(t, \xi) = \sum_{|\alpha|=j} a_{j,\alpha}(t)\xi^{\alpha},$$

$$P_k(t, \tau, \xi) = \sum_{j=m-k}^{m} \sum_{|\alpha|=k+j-m} a_{j,\alpha}(t)\tau^{m-j}\xi^{\alpha} \quad (0 \le k \le m-1).$$

We also assume that

(A-2) $p(t, \tau, \xi)$ is hyperbolic with respect to $\vartheta \equiv (1, 0, \dots, 0) \in \mathbf{R}^{n+1}$ for $t \in [-\delta, \infty)$, *i.e.*,

$$p(t, \tau - i, \xi) \ne 0 \quad \text{for any } (t, \tau, \xi) \in [-\delta, \infty) \times \mathbf{R} \times \mathbf{R}^n.$$

Let $\Gamma(p(t, \cdot, \cdot), \vartheta)$ be the connected component of the set $\{(\tau, \xi) \in \mathbf{R}^{n+1} \setminus \{0\};\ p(t, \tau, \xi) \ne 0\}$ which contains ϑ, and define the generalized flows $K^{\pm}_{(t_0, x^0)}$ for $p(t, \tau, \xi)$ by

$$\begin{aligned} K^{\pm}_{(t_0, x^0)} = \{&(t(s), x(s)) \in [0, \infty) \times \mathbf{R}^n;\ \pm s \ge 0 \text{ and } \{(t(s), x(s))\} \text{ is} \\ &\text{a Lipschitz continuous curve in } [0, \infty) \times \mathbf{R}^n \text{ satisfying} \\ &(d/ds)(t(s), x(s)) \in \Gamma(p(t, \cdot, \cdot), \vartheta)^* \ (a.e.\ s) \text{ and } (t(0), x(0)) = (t_0, x^0)\}, \end{aligned}$$

where $(t_0, x^0) \in [0, \infty) \times \mathbf{R}^n$ and $\Gamma^* = \{(t, x) \in \mathbf{R}^{n+1};\ t\tau + x \cdot \xi \ge 0$ for any $(\tau, \xi) \in \Gamma\}$. To describe conditions on the lower order terms we define the polynomials $h_j(t, \tau, \xi)$ $(\equiv h_j(t, \tau, \xi; p))$ of (τ, ξ) by

$$|p(t, \tau - i\gamma, \xi)|^2 = \sum_{j=0}^{m} \gamma^{2j} h_{m-j}(t, \tau, \xi) \quad \text{for } (t, \tau, \xi) \in [0, \infty) \times \mathbf{R} \times \mathbf{R}^n \text{ and } \gamma \in \mathbf{R}.$$

Since $|p(t, \tau - i\gamma, \xi)|^2 = \prod_{j=1}^{m}((\tau - \lambda_j(t, \xi))^2 + \gamma^2)$, we have

$$h_k(t, \xi) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le m} \prod_{l=1}^{k} (\tau - \lambda_{j_l}(t, \xi))^2 \quad (1 \le k \le m),$$

where $p(t,\tau,\xi) = \prod_{j=1}^{m}(\tau - \lambda_j(t,\xi))$. Let $\mathscr{R}(\xi)$ be a set-valued function, whose values are discrete subsets of $[0,\infty)$, defined for $\xi \in S^{n-1}$ satisfying the following:

(i) $\mathscr{R}(\xi) \subset [0,\infty)$ for $\xi \in S^{n-1} \equiv \{\xi \in \mathbf{R}^n; |\xi| = 1\}$.
(ii) For any $T > 0$ there is $N_T \in \mathbf{Z}_+$ such that

$$\#\{\mathscr{R}(\xi) \cap [0,T]\} \le N_T \quad \text{for } \xi \in S^{n-1}.$$

Here $\#A$ denotes the number of the elements of a set A. First we consider the case where the characteristic roots are at most double, *i.e.*,

(D) If $(t,\tau,\xi) \in [0,\infty) \times \mathbf{R} \times S^{n-1}$ and $p(t,\tau,\xi) = \partial_\tau p(t,\tau,\xi) = 0$, then $\partial_\tau^2 p(t,\tau,\xi) \ne 0$.

We assume that the following condition (D-L) is satisfied, which is corresponding to a so-called Levi condition:

(D-L) For any $T > 0$ there is $C > 0$ satisfying

$$\min\left\{\min_{s \in \mathscr{R}(\xi)} |t-s|, 1\right\} |sub\,\sigma(P)(t,\tau,\xi)|$$
$$\le C h_{m-1}(t,\tau,\xi)^{1/2} \quad \text{for } (t,\tau,\xi) \in [0,T] \times \mathbf{R} \times S^{n-1},$$

where $\min_{s \in \mathscr{R}(\xi)} |t-s| = 1$ if $\mathscr{R}(\xi) = \emptyset$.

Here $sub\,\sigma(P)(t,\tau,\xi)$ denotes the subprincipal symbol of $P(t,D_t,D_x)$, *i.e.*,

$$sub\,\sigma(P)(t,\tau,\xi) = P_{m-1}(t,\tau,\xi) + (i/2)\partial_t\partial_\tau p(t,\tau,\xi).$$

Then we have the following

THEOREM 1.2. *We assume that the conditions* (A-1), (A-2), (D) *and* (D-L) *are satisfied. Then the Cauchy problem* (CP) *is* C^∞ *well-posed. Moreover, if* $(t_0,x^0) \in (0,\infty) \times \mathbf{R}^n$ *and* $u \in C^\infty([0,\infty) \times \mathbf{R}^n)$ *satisfies* (CP), $u_j(x) = 0$ *near* $\{x \in \mathbf{R}^n; (0,x) \in K^-_{(t_0,x^0)}\}$ $(0 \le j \le m-1)$ *and* $f = 0$ *near* $K^-_{(t_0,x^0)}$ (*in* $[0,\infty) \times \mathbf{R}^n$), *then* $(t_0,x^0) \notin \operatorname{supp} u$.

REMARK. The condition (D-L) is necessary for C^∞ well-posedness if $p(t,\tau,\xi) \equiv p(\tau,\xi)$ (see [12]). Moreover, (D-L) is the same condition as given in [16] if $m = 2$.

Next we consider the third order case, *i.e.*,

(T) $m = 3$.

We define the sub-sub-principal symbol $sub^2\,\sigma(P)(t,\tau,\xi)$ of $P(t,D_t,D_x)$ by

$$sub^2\,\sigma(P)(t,\tau,\xi) = P_1(t,\tau,\xi) + (1/6)\partial_t^2\partial_\tau^2 p(t,\tau,\xi) + (i/12)\partial_\tau^2 P_2(t,\tau,\xi)\cdot\partial_t\partial_\tau^2 p(t,\tau,\xi), \tag{1.1}$$

and assume that the following condition (T-L) is satisfied:

(T-L) For any $T > 0$ there is $C > 0$ satisfying

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|, 1\right\}|sub\,\sigma(P)(t,\tau,\xi)| \le Ch_2(t,\tau,\xi)^{1/2},$$

$$\min\left\{\min_{s\in\mathscr{R}(\xi)}|t-s|^2, 1\right\}|sub^2\,\sigma(P)(t,\tau,\xi)| \le Ch_1(t,\tau,\xi)^{1/2} \quad \text{for } (t,\tau,\xi)\in[0,T]\times\mathbf{R}\times S^{n-1}. \tag{1.2}$$

Now we can state our main result.

Theorem 1.3. *We assume that the conditions* (A-1), (A-2), (T) *and* (T-L) *are satisfied. Then the conclusion of* Theorem 1.2 *also holds.*

Remark. If $p(t,\tau,\xi) \equiv p(\tau,\xi)$, then the condition (T-L) is necessary for C^∞ well-posedness (see [12]).

We should note that Colombini-Orrú [1], D'Ancona-Kinoshita [3], Colombini-Taglialatela [2] and Ishida [7] investigated the Cauchy problem for higher-order hyperbolic operators with time-dependent coefficients and gave sufficient conditions for C^∞ well-posedness. In their sufficient conditions they also imposed restrictions on

$$(|\partial_t\lambda_j(t,\xi)| + |\partial_t\lambda_k(t,\xi)|)/|\lambda_j(t,\xi) - \lambda_k(t,\xi)|.$$

This means that the principal parts of the operators must satisfy some conditions in general. On the other hand, one believes that the Cauchy problem for hyperbolic operators with time-dependent coefficients is C^∞ well-posed with suitable choices of the lower order terms if, for example, the coefficients of the principal parts are real analytic.

The remainder of this paper is organized as follows. In §2 we shall give preliminary lemmas. Theorem 1.2 will be proved in §3. Theorem 1.3 will be proved in §4. In §5 some remarks and examples will be given.

2. Preliminaries

We begin with a simple lemma concerning polynomials with real analytic coefficients.

LEMMA 2.1. *Let $f(t,\zeta)$ be a polynomial of $\zeta = (\zeta_1, \dots, \zeta_d)$ whose coefficients are real analytic functions of t in $[0,\infty)$. Then, for any $T > 0$ there is $N_T \in \mathbf{Z}_+$ such that*

$$\sum_{k=0}^{N_T} |\partial_t^k f(t,\zeta)| \neq 0 \quad \textit{for} \ \ t \in [0,T],$$

$$\#\{t \in [0,T]; f(t,\zeta) = 0\} \le N_T$$

if $\zeta \in \mathbf{R}^d$ and $f(t,\zeta) \not\equiv 0$ in t.

PROOF. Write

$$f(t,\zeta) = \sum_{|\alpha| \le m} f_\alpha(t)\zeta^\alpha,$$

where $m \in \mathbf{N}$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{Z}_+^d$. We put $L = \#\{\alpha \in \mathbf{Z}_+^d; |\alpha| \le m\}$, *i.e.*, $L = \binom{d+m}{m}$, and

$$F(t,Z) = \sum_{|\alpha| \le m} f_\alpha(t) Z_\alpha,$$

where $Z = (Z_\alpha)_{|\alpha| \le m} \in \mathbf{R}^L$. Define

$$V = \{Z \in \mathbf{R}^L; F(t,Z) \equiv 0 \ \text{in} \ t\}.$$

Then V is a subspace of $\mathbf{R}^L$. So there are $r \in \mathbf{Z}_+$ and an $L \times L$ non-singular matrix Q such that

$$V = \{\tilde{Z}Q; \tilde{Z}' = 0 \in \mathbf{R}^r\},$$

where $\tilde{Z}' = (\tilde{Z}_1, \dots, \tilde{Z}_r) \in \mathbf{R}^r$ for $\tilde{Z} = (\tilde{Z}_1, \dots, \tilde{Z}_L) \in \mathbf{R}^L$, and $\tilde{Z}' = 0$ and $V = \mathbf{R}^L$ if $r = 0$. Then we can write

$$F(t,Z) = \tilde{F}(t,\tilde{Z}') = \sum_{j=1}^{r} \tilde{f}_j(t)\tilde{Z}_j,$$

where $Z = \tilde{Z}Q$. Note that

$$\tilde{F}(t, \lambda\tilde{Z}') = \lambda\tilde{F}(t, \tilde{Z}') \quad \text{for } \lambda \in \mathbf{R},$$

$$\tilde{Z}' \neq 0 \Leftrightarrow Z \ (\equiv \tilde{Z}Q) \notin V \Leftrightarrow F(t, Z) \not\equiv 0 \text{ in } t.$$

Let $T > 0$ and $(t_0, \tilde{Z}^{0\prime}) \in [0, T] \times S^{r-1}$. From the Weierstrass preparation theorem it follows that there are $\delta_0 > 0$, a neighborhood U_0 of $\tilde{Z}^{0\prime}$ in $\mathbf{R}^r$, $\mu_0 \in \mathbf{Z}_+$, a real analytic function $c_0(t, \tilde{Z}')$ defined in $[t_0 - \delta_0, t_0 + \delta_0] \times \overline{U}_0$ and real analytic functions $a_{0,k}(\tilde{Z}')$ defined in $\overline{U}_0$ $(1 \le k \le \mu_0)$ such that $c_0(t, \tilde{Z}') \neq 0$,

$$\tilde{F}(t, \tilde{Z}') = c_0(t, \tilde{Z}')\{(t - t_0)^{\mu_0} + a_{0,1}(\tilde{Z}')(t - t_0)^{\mu_0 - 1} + \cdots + a_{0,\mu_0}(\tilde{Z}')\}$$

for $(t, \tilde{Z}') \in ([t_0 - \delta_0, t_0 + \delta_0] \cap [0, T]) \times \overline{U}_0$, and

$$a_{0,k}(\tilde{Z}^{0\prime}) = 0 \quad (1 \le k \le \mu_0),$$

where $\tilde{F}(t, \tilde{Z}') = c_0(t, \tilde{Z}')$ if $\mu_0 = 0$. Then we have

$$\partial_t^{\mu_0}\tilde{F}(t, \tilde{Z}^{0\prime})|_{t=t_0} = \mu_0! c_0(t_0, \tilde{Z}^{0\prime}) \neq 0.$$

So we may assume that

$$\partial_t^{\mu_0}\tilde{F}(t, \tilde{Z}') \neq 0 \quad \text{for } (t, \tilde{Z}') \in ([t_0 - \delta_0, t_0 + \delta_0] \cap [0, T]) \times \overline{U}_0,$$

modifying δ_0 and U_0 if necessary. Since $[0, T] \times S^{r-1}$ is compact and $\tilde{F}(t, \tilde{Z}')$ is homogeneous of degree 1 in $\tilde{Z}'$, there is $N_T \in \mathbf{Z}_+$ satisfying

$$\sum_{k=0}^{N_T} |\partial_t^k \tilde{F}(t, \tilde{Z}')| \neq 0 \quad \text{for } (t, \tilde{Z}') \in [0, T] \times (\mathbf{R}^r \backslash \{0\}), \tag{2.1}$$

$$\#\{t \in [0, T]; \tilde{F}(t, \tilde{Z}') = 0\} \le N_T \quad \text{for } \tilde{Z}' \in \mathbf{R}^r \backslash \{0\}. \tag{2.2}$$

Put

$$Z(\zeta) = (\zeta^\alpha)_{|\alpha| \le m}, \quad \tilde{Z}(\zeta) \equiv (\tilde{Z}_1(\zeta), \dots, \tilde{Z}_L(\zeta)) = Z(\zeta)Q^{-1},$$

$$\tilde{Z}'(\zeta) = (\tilde{Z}_1(\zeta), \dots, \tilde{Z}_r(\zeta)).$$

Since $f(t, \zeta) = \tilde{F}(t, \tilde{Z}'(\zeta))$, and $\tilde{Z}'(\zeta) \neq 0$ if $f(t, \zeta) \not\equiv 0$ in t, (2.1) and (2.2) prove the lemma. □

Write

$$p(\tau) = \prod_{j=1}^{m} (\tau - \lambda_j),$$

$$p_\varepsilon(\tau) = (1 + \varepsilon(d/d\tau))p(\tau),$$

$$|p(\tau - i\gamma)|^2 = \sum_{j=0}^{m} \gamma^{2j} h_{m-j}(\tau; p) \quad \text{for } \tau \in \mathbf{R} \text{ and } \gamma \in \mathbf{R},$$

where $\lambda_j \in \mathbf{R}$ $(1 \le j \le m)$ and $\varepsilon \in \mathbf{R}$. Then we have the following

LEMMA 2.2. *For* $\tau \in \mathbf{R}$, $\varepsilon \in \mathbf{R}$ *and* $1 \le j \le m$ *we have*

$$h_{m-j}(\tau; p_\varepsilon) \ge \left\{ 3 \binom{m}{j-1} \right\}^{-1} h_{m-j}(\tau; p).$$

PROOF. Lemma 2.1 of Svensson [11] gives

$$\begin{aligned} (2.3) \qquad &(m-r)!(k-r)!/(m!k!) \\ &\le h_{m-k}(\tau; p)/h_{m-k}(\tau; p^{(r)}) \\ &\le (k-1-r)!(k-r)!/((k-1)!k!) \quad (0 \le r < k \le m), \end{aligned}$$

where $p^{(r)}(\tau) = \partial_\tau^r p(\tau)$. In particular, we have

$$\begin{aligned} (2.4) \qquad (m-r)!/(m!(r+1)!) &\le h_{m-r-1}(\tau; p)/h_{m-r-1}(\tau; p^{(r)}) \\ &\le 1/(r!(r+1)!) \quad (0 \le r < m). \end{aligned}$$

Therefore, it suffices to show that

$$(2.5) \qquad h_{m-1}(\tau; p_\varepsilon) \ge h_{m-1}(\tau; p)/3 \quad \text{for } \tau \in \mathbf{R} \text{ and } \varepsilon \in \mathbf{R},$$

in order to prove the lemma. Indeed, (2.4) and (2.5) with p replaced by $p^{(r)}$ yield

$$\begin{aligned} h_{m-r-1}(\tau; p_\varepsilon) &\ge (m-r)! h_{m-r-1}(\tau; p_\varepsilon^{(r)})/(m!(r+1)!) \\ &\ge (m-r)! h_{m-r-1}(\tau; p^{(r)})/(3m!(r+1)!) \\ &\ge (m-r)! r! h_{m-r-1}(\tau; p)/(3m!) \end{aligned}$$

$(0 \le r < m)$. Put $\sigma_j = \tau - \lambda_j$. Since

$$-\lim_{\gamma\downarrow 0} \operatorname{Im}\{p(\tau - i\gamma)\cdot p^{(1)}(\tau + i\gamma)\}/\gamma = \sum_{j=1}^{m}\prod_{k\neq j}\sigma_k^2 = h_{m-1}(\tau; p),$$

a simple calculation yields

$$(2.6) \qquad h_{m-1}(\tau; p_\varepsilon) = -\lim_{\gamma\downarrow 0} \operatorname{Im}\{(p(\tau - i\gamma) + \varepsilon p^{(1)}(\tau - i\gamma)) \times (p^{(1)}(\tau + i\gamma) + \varepsilon p^{(2)}(\tau + i\gamma))\}/\gamma$$

$$= \sum_{1\le j<k<l\le m}\left(\prod_{\substack{1\le\mu\le m,\\ \mu\neq j,k,l}}\sigma_\mu^2\right) I^{\varepsilon}_{j,k,l},$$

where

$$I^{\varepsilon}_{j,k,l} = 2\varepsilon^2\{2\sigma_j^2 + 2\sigma_k^2 + 2\sigma_l^2 + \sigma_j\sigma_k + \sigma_j\sigma_l + \sigma_k\sigma_l\}$$
$$+ 2\varepsilon\{\sigma_j^2\sigma_k + \sigma_j^2\sigma_l + \sigma_j\sigma_k^2 + \sigma_k^2\sigma_l + \sigma_j\sigma_l^2 + \sigma_k\sigma_l^2\}$$
$$+ \sigma_j^2\sigma_k^2 + \sigma_j^2\sigma_l^2 + \sigma_k^2\sigma_l^2.$$

Indeed, for example, we have

$$-\lim_{\gamma\downarrow 0} \operatorname{Im}\{p^{(1)}(\tau - i\gamma)p^{(2)}(\tau + i\gamma)\}/\gamma$$

$$= -\lim_{\gamma\downarrow 0} 2\operatorname{Im}\left\{\sum_{j=1}^{m}\prod_{v\neq j}(\sigma_v - i\gamma)\cdot\sum_{1\le k<l\le m}\prod_{\mu\neq k,l}(\sigma_\mu + i\gamma)\right\}\Big/\gamma$$

$$= 2\left\{\sum_{1\le j<k<l\le m} + \sum_{1\le k<j<l\le m} + \sum_{1\le k<l<j\le m}\right\}(-\sigma_k\sigma_l + \sigma_j\sigma_k + \sigma_j\sigma_l)\prod_{\mu\neq j,k,l}\sigma_\mu^2$$
$$+ 4\sum_{1\le k<l\le m}\prod_{\mu\neq k,l}\sigma_\mu^2$$

$$= 2\sum_{1\le j<k<l\le m}((\sigma_j\sigma_k + \sigma_j\sigma_l + \sigma_k\sigma_l) + 2(\sigma_j^2 + \sigma_k^2 + \sigma_l^2))\prod_{\mu\neq j,k,l}\sigma_\mu^2$$

Let $1 \le j < k < l \le m$, and put

$$\Xi_{j,k,l} = 3(\sigma_j^2 + \sigma_k^2 + \sigma_l^2) + (\sigma_j + \sigma_k + \sigma_l)^2$$
$$(= 2\{2\sigma_j^2 + 2\sigma_k^2 + 2\sigma_l^2 + \sigma_j\sigma_k + \sigma_j\sigma_l + \sigma_k\sigma_l\}).$$

If $\Xi_{j,k,l} = 0$, then $\sigma_j = \sigma_k = \sigma_l = 0$. Therefore, we have

$$(0 =)\, 3I^{\varepsilon}_{j,k,l} \geq \sigma_j^2\sigma_k^2 + \sigma_j^2\sigma_l^2 + \sigma_k^2\sigma_l^2\, (= 0) \quad \text{if } \Xi_{j,k,l} = 0.$$

Now assume that $\Xi_{j,k,l} \neq 0$. Then we have

$$\text{(2.7)} \qquad I^{\varepsilon}_{j,k,l} = \Xi_{j,k,l}\{\varepsilon + (\sigma_j^2(\sigma_k + \sigma_l) + \sigma_k^2(\sigma_j + \sigma_l) + \sigma_l^2(\sigma_j + \sigma_k))/\Xi_{j,k,l}\}^2$$
$$+ J_{j,k,l}/\Xi_{j,k,l} \geq J_{j,k,l}/\Xi_{j,k,l},$$

where

$$J_{j,k,l} = -\{\sigma_j^2(\sigma_k + \sigma_l) + \sigma_k^2(\sigma_j + \sigma_l) + \sigma_l^2(\sigma_j + \sigma_k)\}^2$$
$$+ \Xi_{j,k,l}(\sigma_j^2\sigma_k^2 + \sigma_j^2\sigma_l^2 + \sigma_k^2\sigma_l^2).$$

A simple calculation yields

$$J_{j,k,l} = \sigma_j^4(\sigma_k - \sigma_l)^2 + \sigma_k^4(\sigma_j - \sigma_l)^2 + \sigma_l^4(\sigma j - \sigma_l)^2$$
$$+ 2\sigma_j^4(\sigma_k^2 + \sigma_l^2) + 2\sigma_k^4(\sigma_j^2 + \sigma_l^2) + 2\sigma_l^4(\sigma_j^2 + \sigma_k^2) + 6\sigma_j^2\sigma_k^2\sigma_l^2$$
$$\geq 2\sigma_j^4(\sigma_k^2 + \sigma_l^2) + 2\sigma_k^4(\sigma_j^2 + \sigma_l^2) + 2\sigma_l^4(\sigma_j^2 + \sigma_k^2) + 6\sigma_j^2\sigma_k^2\sigma_l^2.$$

This yields

$$\text{(2.8)} \qquad \Xi_{j,k,l}(\sigma_j^2\sigma_k^2 + \sigma_j^2\sigma_l^2 + \sigma_k^2\sigma_l^2)$$
$$= 4\sigma_j^4(\sigma_k^2 + \sigma_l^2) + 4\sigma_k^4(\sigma_j^2 + \sigma_l^2) + 4\sigma_l^4(\sigma_j^2 + \sigma_k^2) + 12\sigma_j^2\sigma_k^2\sigma_l^2$$
$$+ 2\sigma_j\sigma_k^3\sigma_l^2 + 2\sigma_j\sigma_k^2\sigma_l^3 + 2\sigma_j^3\sigma_k\sigma_l^2 + 2\sigma_j^2\sigma_k\sigma_l^3 + 2\sigma_j^3\sigma_k^2\sigma_l$$
$$+ 2\sigma_j^2\sigma_k^3\sigma_l + 2\sigma_j^3\sigma_k^3 + 2\sigma_j^3\sigma_l^3 + 2\sigma_k^3\sigma_l^3$$
$$\leq 4\sigma_j^4(\sigma_k^2 + \sigma_l^2) + 4\sigma_k^4(\sigma_j^2 + \sigma_l^2) + 4\sigma_l^4(\sigma_j^2 + \sigma_k^2) + 12\sigma_j^2\sigma_k^2\sigma_l^2$$
$$+ \sigma_k^2\sigma_l^2(\sigma_j^2 + \sigma_k^2) + \sigma_k^2\sigma_l^2(\sigma_j^2 + \sigma_l^2) + \sigma_j^2\sigma_l^2(\sigma_j^2 + \sigma_k^2)$$
$$+ \sigma_j^2\sigma_l^2(\sigma_k^2 + \sigma_l^2) + \sigma_j^2\sigma_k^2(\sigma_j^2 + \sigma_l^2) + \sigma_j^2\sigma_k^2(\sigma_k^2 + \sigma_l^2)$$
$$+ \sigma_k^2\sigma_l^2(\sigma_k^2 + \sigma_l^2) + \sigma_j^2\sigma_l^2(\sigma_j^2 + \sigma_l^2) + \sigma_j^2\sigma_k^2(\sigma_j^2 + \sigma_k^2)$$
$$= 3(2\sigma_j^4(\sigma_k^2 + \sigma_l^2) + 2\sigma_k^4(\sigma_j^2 + \sigma_l^2) + 2\sigma_l^4(\sigma_j^2 + \sigma_k^2) + 6\sigma_j^2\sigma_k^2\sigma_l^2)$$
$$\leq 3J_{j,k,l},$$

since $2\sigma_j\sigma_k \le \sigma_j^2 + \sigma_k^2, \cdots$. Therefore, from (2.6) and (2.7) we have

$$\begin{aligned} h_{m-1}(\tau; p_\varepsilon) &\ge \sum_{1\le j<k<l\le m} \left(\prod_{\substack{1\le\mu\le m,\\ \mu\ne j,k,l}} \sigma_\mu^2 \right) (\sigma_j^2\sigma_k^2 + \sigma_j^2\sigma_l^2 + \sigma_k^2\sigma_l^2)/3 \\ &= h_{m-1}(\tau; p)/3, \end{aligned}$$

which proves (2.5) and the lemma. □

Now we assume that (A-1) and (A-2) are satisfied, and define

$$p_\varepsilon(t,\tau,\xi) = (1 - \varepsilon^2|\xi|^2\partial_\tau^2)p(t,\tau,\xi)$$

for $\varepsilon \in \mathbf{R}$, $(t,\tau,\xi) \in [0,\infty) \times \mathbf{R} \times \mathbf{R}^n$, changing the notation. We note that $p_\varepsilon(t,\tau,\xi)$ is strictly hyperbolic with respect to ϑ for $\varepsilon \in \mathbf{R}\backslash\{0\}$ if $p(t,\tau,\xi)$ has at most triple characteristics (see [10]). Lemma 2.2 gives

$$h_{m-j}(t,\tau,\xi; p_\varepsilon) \ge \left\{ 3\binom{m}{j-1} \right\}^{-2} h_{m-j}(t,\tau,\xi; p)$$

for $1 \le j \le m$, $\varepsilon \in \mathbf{R}$, $(t,\tau,\xi) \in [0,\infty) \times \mathbf{R} \times \mathbf{R}^n$. We note that one can directly prove that

$$h_j(t,\tau,\xi; p_\varepsilon) \ge h_j(t,\tau,\xi; p) \quad (0 \le j \le 2)$$

if $m = 3$. Write

$$p_\varepsilon(t,\tau,\xi) = \prod_{j=1}^m (\tau - \lambda_j(t,\xi;\varepsilon)).$$

Lemma 2.3. *For each fixed $\xi \in S^{n-1}$ and $\varepsilon \in \mathbf{R}$ we can enumerate $\{\lambda_j(t,\xi;\varepsilon)\}$ so that the $\lambda_j(t,\xi;\varepsilon)$ are real analytic in $t \in [0,\infty)$. Moreover, for any $\nu \in \mathbf{Z}_+$ there are $\mathcal{N}_\nu^0$ $(\equiv \mathcal{N}_\nu^0(p)) \subset \mathbf{R}$ and $\mathcal{N}_\nu(\varepsilon)$ $(\equiv \mathcal{N}_\nu(\varepsilon; p)) \subset \mathbf{R}^n\backslash\{0\}$ for $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_\nu^0$ satisfying the following*:

(i) *$\lambda\xi \in \mathcal{N}_\nu(\varepsilon)$ if $\lambda > 0$ and $\xi \in \mathcal{N}_\nu(\varepsilon)$.*
(ii) *$\mu_1(\mathcal{N}_\nu^0) = 0$.*
(iii) *$\mu_n(\mathcal{N}_\nu(\varepsilon)) = 0$ for $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_\nu^0$.*
(iv) *For any $T > 0$ there is $N_{T,\nu} \in \mathbf{Z}_+$ such that*

$$\#\{t \in [0,T]; \partial_t^\nu(\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon)) = 0\} \le N_{T,\nu}$$

$$\textit{if} \;\; 1 \le j < k \le m \;\; \textit{and} \;\; \partial_t^\nu(\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon)) \not\equiv 0 \;\; \textit{in} \;\; t,$$

$$\#\{t \in [0,T]; \partial_t^\nu \lambda_j(t,\xi;\varepsilon) = 0\} \le N_{T,\nu}$$

$$\textit{if} \;\; 1 \le j \le m \;\; \textit{and} \;\; \partial_t^\nu \lambda_j(t,\xi;\varepsilon) \not\equiv 0 \;\; \textit{in} \;\; t$$

for $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_\nu^0$ *and* $\xi \in S^{n-1}\backslash\mathcal{N}_\nu(\varepsilon)$.

Here μ_n *denotes the Lebesgue measure in* $\mathbf{R}^n$.

REMARK. (i) The $\lambda_j(t,\xi;\varepsilon)$ in the lemma are not necessarily continuous in (ξ,ε). (ii) If the conditions (D) or (T) are satisfied, then $p_\varepsilon(t,\tau,\xi)$ is strictly hyperbolic for $\varepsilon \neq 0$, and the assertion of the first part of the lemma is obvious for $\varepsilon \neq 0$.

PROOF. First fix $(\xi,\varepsilon) \in S^{n-1} \times \mathbf{R}$. To simplify the notations we write $p(t,\tau) = p_\varepsilon(t,\tau,\xi)$. For $t_0 \in [0,\infty)$ $\mathcal{A}_{t_0}$ denotes the convergent power series ring of $(t-t_0)$. Since $\mathcal{A}_{t_0}$ is a unique factorization domain, $\mathcal{A}_{t_0}[\tau]$ is also a unique factorization domain. Therefore, we can write

$$p(t,\tau) = p_1(t,\tau)^{r_1} p_2(t,\tau)^{r_2} \cdots p_\sigma(t,\tau)^{r_\sigma}, \tag{2.9}$$

where $\sigma, r_j \in \mathbf{N}$, the $p_j(t,\tau)$ $(\in \mathcal{A}_{t_0}[\tau])$ are irreducible in $\mathcal{A}_{t_0}[\tau]$, and $p_j(t,\tau)$ and $p_k(t,\tau)$ are mutually prime if $j \neq k$. Since the leading coefficient of $p(t,\tau)$ is equal to 1, we may assume that the leading coefficients of the $p_j(t,\tau)$ are also equal to 1. Put

$$q(t,\tau) = \prod_{j=1}^{\sigma} p_j(t,\tau).$$

We denote by $D(t)$ the discriminant of $q(t,\tau) = 0$ in τ. Then we have $D(t) \not\equiv 0$. Indeed, suppose that $D(t) \equiv 0$. Then $q(t,\tau)$ and $\partial_\tau q(t,\tau)$ are not mutually prime as polynomials in $\mathcal{A}_{t_0}[\tau]$ (see, *e.g.*, Chap. 5 of [5] and § A.1 of [6]). This leads a contradiction. When $D(t_0) \neq 0$, $q(t,\tau)$ is strictly hyperbolic in τ near $t = t_0$ and, therefore, we may assume that the $\lambda_j(t,\xi;\varepsilon)$ are analytic in a complex neighborhood of t_0. Next assume that $D(t_0) = 0$. Since the zeros of $D(t)$ are discrete, the $\lambda_j(t,\xi;\varepsilon)$ are analytic in a complex neighborhood of t_0 except for t_0. Fix j_0 so that $1 \le j_0 \le m$. Analytic continuations of $\lambda_{j_0}(t,\xi;\varepsilon)$ around t_0 and Riemann's theorem on removable singularities show that there is $r \in \mathbf{N}$ such that $\lambda_{j_0}(t_0 + z^r, \xi;\varepsilon)$ is analytic in a complex neighborhood of $z = 0$. Hyperbolicity implies that $\lambda_{j_0}(t_0 + z^r, \xi; \varepsilon)$ is real if z^r is real, and that one can take $r = 1$, *i.e.*, $\lambda_{j_0}(t, \xi; \varepsilon)$ is analytic in t near t_0. Starting from $t = 0$ and continuing analytically along $[0, \infty)$, we can enumerate $\{\lambda_j(t, \xi; \varepsilon)\}$ so that the $\lambda_j(t, \xi; \varepsilon)$ are real analytic. This proves the first part of the assertions of the lemma. Next let us prove the second part. Let $\mathscr{A}$ be the ring of the real analytic functions of t defined in $[0, \infty)$. Note that $\mathscr{A}$ is an integral domain and that $\mathscr{A}$ is not a unique factorization domain. We denote by Σ the quotient field of $\mathscr{A}[\xi, \varepsilon]$. Then $\Sigma[\tau]$ is a unique factorization domain and $p_\varepsilon(t, \tau, \xi) \in \Sigma[\tau]$. Write

$$\tau p_\varepsilon(t, \tau, \xi) = p_\varepsilon^1(t, \tau, \xi)^{r_1} \cdots p_\varepsilon^\sigma(t, \tau, \xi)^{r_\sigma},$$

where $\sigma, r_j \in \mathbf{N}$, the $p_\varepsilon^j(t, \tau, \xi)$ $(\in \Sigma[\tau])$ are irreducible in $\Sigma[\tau]$ and $p_\varepsilon^j(t, \tau, \xi)$ and $p_\varepsilon^k(t, \tau, \xi)$ are mutually prime if $j \neq k$. Here σ and the r_j are different from those as appeared in (2.9), in general. Define $q(t, \tau, \xi; \varepsilon) = \prod_{j=1}^\sigma p_\varepsilon^j(t, \tau, \xi)$, and let $D(t, \xi; \varepsilon)$ be the discriminant of $q(t, \tau, \xi; \varepsilon) = 0$ in τ. We note that

$$\{\tau \in \mathbf{C}; q(t, \tau, \xi; \varepsilon) = 0\} = \{\tau \in \mathbf{C}; p_\varepsilon(t, \tau, \xi) = 0\} \cup \{0\}.$$

Write

$$D(t, \xi; \varepsilon) = d_0(t, \xi; \varepsilon)/d_1(t, \xi; \varepsilon),$$

where $d_k(t, \xi; \varepsilon) \in \mathscr{A}[\xi, \varepsilon]$ and $d_k(t, \xi; \varepsilon) \neq 0$ in $\mathscr{A}[\xi, \varepsilon]$, *i.e.*, $d_k(t, \xi; \varepsilon) \not\equiv 0$ in (t, ξ, ε) $(k = 0, 1)$. We may assume that the $d_k(t, \xi; \varepsilon)$ are homogeneous in ξ. Indeed, assume that $a_k(\xi)$ $(k = 0, 1)$ are polynomials of ξ and $a_0(\xi)/a_1(\xi)$ is homogeneous in ξ. Write $a_k(\xi) = a_k^0(\xi) + (a_k(\xi) - a_k^0(\xi))$ $(k = 0, 1)$, where $a_k^0(\xi)$ is the principal part of $a_k(\xi)$. Then we have, with some $\kappa \in \mathbf{Z}$,

$$a_0(\xi)/a_1(\xi) = \lambda^{-\kappa} a_0(\lambda\xi)/a_1(\lambda\xi) \to a_0^0(\xi)/a_1^0(\xi) \quad (\lambda \to \infty)$$

for ξ with $a_1^0(\xi) \neq 0$ and, therefore, $a_0(\xi)/a_1(\xi) = a_0^0(\xi)/a_1^0(\xi)$. So we may assume that the $a_k(\xi)$ are homogeneous in ξ. Put

$$\tilde{\mathscr{N}_0} = \{(\xi, \varepsilon) \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}; d_0(t, \xi; \varepsilon) d_1(t, \xi; \varepsilon) \equiv 0 \text{ in } t \ (\in [0, \infty))\}.$$

Then we have $\mu_{n+1}(\tilde{\mathscr{N}_0}) = 0$, since

$$\mu_{n+2}(\{(t, \xi, \varepsilon) \in [0, \infty) \times (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}; d_0(t, \xi; \varepsilon) d_1(t, \xi; \varepsilon) = 0\}) = 0.$$

Define

$$\mathscr{N}_0^0 = \{\varepsilon \in \mathbf{R}; \mu_n(\{\xi \in \mathbf{R}^n \setminus \{0\}; (\xi, \varepsilon) \in \tilde{\mathscr{N}_0}\}) > 0\}.$$

Then it is obvious that $\mu_1(\mathscr{N}_0^0) = 0$. For $\varepsilon \in \mathbf{R} \setminus \mathscr{N}_0^0$ we define

$$\mathscr{N}_0(\varepsilon) = \{\xi \in \mathbf{R}^n \setminus \{0\}; (\xi, \varepsilon) \in \tilde{\mathscr{N}_0}\}.$$

By definition we see that $\mu_n(\mathcal{N}_0(\varepsilon)) = 0$ for $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_0^0$ and that $d_0(t,\xi;\varepsilon) \times d_1(t,\xi;\varepsilon) \not\equiv 0$ in t, *i.e.*, $D(t,\xi;\varepsilon) \not\equiv 0$ in t, for $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_0^0$ and $\xi \in S^{n-1}\backslash\mathcal{N}_0(\varepsilon)$. Since the $d_k(t,\xi;\varepsilon)$ are homogeneous in ξ, $\mathcal{N}_0(\varepsilon)$ is a cone, *i.e.*, $\lambda\xi \in \mathcal{N}_0(\varepsilon)$ if $\lambda > 0$ and $\xi \in \mathcal{N}_0(\varepsilon)$. For fixed $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_0^0$ and $\xi \in S^{n-1}\backslash\mathcal{N}_0(\varepsilon)$ the roots of $q(t,\tau,\xi;\varepsilon) = 0$ in τ are simple if $t \in D_{(\xi,\varepsilon)}$, where $D_{(\xi,\varepsilon)} = \{t \in [0,\infty); d_0(t,\xi;\varepsilon)d_1(t,\xi;\varepsilon) \neq 0\}$. So, enumerating $\{\lambda_j(t,\xi;\varepsilon)\}$, we can write

$$q(t,\tau,\xi;\varepsilon) = \prod_{j=1}^{\hat{m}}(\tau - \lambda_j(t,\xi;\varepsilon)) \quad \text{for } t \in [0,\infty),\ \varepsilon \in \mathbf{R}\backslash\mathcal{N}_0^0 \text{ and } \xi \in S^{n-1}\backslash\mathcal{N}_0(\varepsilon),$$

where $\hat{m} = \deg_\tau q(t,\tau,\xi;\varepsilon)$ and the $\lambda_j(t,\xi;\varepsilon)$ are real analytic in $t \in [0,\infty)$ for $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_0^0$ and $\xi \in S^{n-1}\backslash\mathcal{N}_0(\varepsilon)$. We may assume that $\lambda_{\hat{m}}(t,\xi;\varepsilon) \equiv 0$. Write

$$q(t,\tau,\xi;\varepsilon) = \tau^{\hat{m}} + a_1(t,\xi;\varepsilon)\tau^{\hat{m}-1} + \cdots + a_{\hat{m}-1}(t,\xi;\varepsilon)\tau,$$

where $a_j(t,\xi;\varepsilon) \in \Sigma$. Note that the $a_j(t,\xi;\varepsilon)$ are real analytic in $t \in [0,\infty)$ for $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_0^0$ and $\xi \in S^{n-1}\backslash\mathcal{N}_0(\varepsilon)$. Let $\varepsilon \in \mathbf{R}\backslash\mathcal{N}_0^0$ and $\xi \in S^{n-1}\backslash\mathcal{N}_0(\varepsilon)$. Then we have

$$\partial_\tau q(t,\tau,\xi;\varepsilon)|_{\tau=\lambda_j(t,\xi;\varepsilon)} \cdot \partial_t\lambda_j(t,\xi;\varepsilon) + \partial_t q(t,\tau,\xi;\varepsilon)|_{\tau=\lambda_j(t,\xi;\varepsilon)} = 0$$

for $1 \le j \le \hat{m}$. So, for $t \in D_{(\xi,\varepsilon)}$ we have $\partial_\tau q(t,\tau,\xi;\varepsilon)|_{\tau=\lambda_j(t,\xi;\varepsilon)} \neq 0$ and

$$\partial_t\lambda_j(t,\xi;\varepsilon) = -\partial_t q(t,\tau,\xi;\varepsilon)|_{\tau=\lambda_j(t,\xi;\varepsilon)}/\partial_\tau q(t,\tau,\xi;\varepsilon)|_{\tau=\lambda_j(t,\xi;\varepsilon)}$$

for $1 \le j \le \hat{m}$. Since $\lambda_{\hat{m}}(t,\xi;\varepsilon) \equiv 0$, we have

$$\prod_{j=1}^{\hat{m}} \partial_t\lambda_j(t,\xi;\varepsilon) \equiv 0.$$

Noting that $\prod_{1\le j,k\le\hat{m},j\neq k}(\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon)) = (-1)^{\hat{m}(\hat{m}-1)/2}D(t,\xi;\varepsilon)$, we can write the other fundamental symmetric expressions as follows;

$$\begin{aligned}
&\sum_{j=1}^{\hat{m}}\prod_{k\neq j}\partial_t\lambda_k(t,\xi;\varepsilon)\\
&\quad = (-1)^{\hat{m}-1+\hat{m}(\hat{m}-1)/2}\\
&\qquad \times \sum_{j=1}^{\hat{m}}\prod_{k\neq j}\{(\lambda_k(t,\xi;\varepsilon) - \lambda_j(t,\xi;\varepsilon))\partial_t q(t,\tau,\xi;\varepsilon)|_{\tau=\lambda_k(t,\xi;\varepsilon)}\}/D(t,\xi;\varepsilon)\\
&\quad = E_{\hat{m}-1}(t,\xi;\varepsilon)/D(t,\xi;\varepsilon),\\
&\qquad \cdots\\
&\sum_{j=1}^{\hat{m}}\partial_t\lambda_j(t,\xi;\varepsilon) = E_1(t,\xi;\varepsilon)/D(t,\xi;\varepsilon),
\end{aligned}$$

where the $E_k(t,\xi;\varepsilon)$ are polynomials of $\{a_j(t,\xi;\varepsilon)\}_{1\le j\le\hat{m}}$ and $\{\partial_t a_j(t,\xi;\varepsilon)\}_{1\le j\le\hat{m}}$. Put

$$\tilde{p}(t,\tau,\xi;\varepsilon) = \tau^{\hat{m}} - E_1(t,\xi;\varepsilon)D(t,\xi;\varepsilon)^{-1}\tau^{\hat{m}-1} + E_2(t,\xi;\varepsilon)D(t,\xi;\varepsilon)^{-1}\tau^{\hat{m}-2} + \cdots + (-1)^{\hat{m}-1}E_{\hat{m}-1}(t,\xi;\varepsilon)D(t,\xi;\varepsilon)^{-1}\tau$$

$$\left(= \prod_{j=1}^{\hat{m}}(\tau - \partial_t\lambda_j(t,\xi;\varepsilon))\right).$$

Let us repeat the above argument with τp_ε replaced by $\tilde{p}$. We write

$$\tilde{p}(t,\tau,\xi;\varepsilon) = \tilde{p}^1(t,\tau,\xi;\varepsilon)^{r'_1}\cdots\tilde{p}^{\sigma'}(t,\tau,\xi;\varepsilon)^{r'_{\sigma'}},$$

where $\sigma', r'_j \in \mathbf{N}$, the $\tilde{p}^j(t,\tau,\xi;\varepsilon)$ $(\in\Sigma[\tau])$ are irreducible in $\Sigma[\tau]$ and $\tilde{p}^j(t,\tau,\xi;\varepsilon)$ and $\tilde{p}^k(t,\tau,\xi;\varepsilon)$ are mutually prime if $j\neq k$. Put

$$\tilde{q}(t,\tau,\xi;\varepsilon) = \prod_{j=1}^{\sigma'}\tilde{p}^j(t,\tau,\xi;\varepsilon),$$

and let $\tilde{D}(t,\xi;\varepsilon)$ be the discriminant of $\tilde{q}(t,\tau,\xi;\varepsilon)=0$ in τ. Then we can write

$$\tilde{D}(t,\xi;\varepsilon) = \tilde{d}_0(t,\xi;\varepsilon)/\tilde{d}_1(t,\xi;\varepsilon),$$

where $\tilde{d}_k(t,\xi;\varepsilon)\in\mathscr{A}[\xi,\varepsilon]$ and $\tilde{d}_k(t,\xi;\varepsilon)\neq 0$ in $\mathscr{A}[\xi,\varepsilon]$ $(k=0,1)$. Put

$$\tilde{\mathscr{N}_1} = \tilde{\mathscr{N}_0}\cup\{(\xi,\varepsilon)\in(\mathbf{R}^n\backslash\{0\})\times\mathbf{R};\tilde{d}_0(t,\xi;\varepsilon)\tilde{d}_1(t,\xi;\varepsilon)\equiv 0 \text{ in } t\}.$$

Then we have, similarly, $\mu_{n+1}(\tilde{\mathscr{N}_1})=0$. Define

$$\mathscr{N}_1^0 = \{\varepsilon\in\mathbf{R};\mu_n(\{\xi\in\mathbf{R}^n\backslash\{0\};(\xi,\varepsilon)\in\tilde{\mathscr{N}_1}\})>0\}.$$

Then we have $\mu_1(\mathscr{N}_1^0)=0$. For $\varepsilon\in\mathbf{R}\backslash\mathscr{N}_1^0$ we define

$$\mathscr{N}_1(\varepsilon) = \{\xi\in\mathbf{R}^n\backslash\{0\};(\xi,\varepsilon)\in\tilde{\mathscr{N}_1}\}.$$

By definition we have

$$\mu_n(\mathscr{N}_1(\varepsilon)) = 0 \quad \text{for } \varepsilon\in\mathbf{R}\backslash\mathscr{N}_1^0,$$

$$\tilde{d}_0(t,\xi;\varepsilon)\tilde{d}_1(t,\xi;\varepsilon)\not\equiv 0 \quad \text{in } t \text{ for } \varepsilon\in\mathbf{R}\backslash\mathscr{N}_1^0 \text{ and } \xi\in S^{n-1}\backslash\mathscr{N}_1(\varepsilon).$$

We may assume that the $\tilde{d}_k(t,\xi;\varepsilon)$ are homogeneous in ξ. So $\mathscr{N}_1(\varepsilon)$ is a cone. For $\varepsilon\in\mathbf{R}\backslash\mathscr{N}_1^0$ and $\xi\in S^{n-1}\backslash\mathscr{N}_1(\varepsilon)$ the roots of $\tilde{q}(t,\tau,\xi;\varepsilon)=0$ in τ are simple if $t \in [0, \infty)$ and $\tilde{d}_0(t, \xi; \varepsilon)\tilde{d}_1(t, \xi; \varepsilon) \neq 0$. Therefore, the multiplicities of the roots $\partial_t \lambda_j(t, \xi; \varepsilon)$ of $\tilde{p}(t, \tau, \xi; \varepsilon) = 0$ in τ $(1 \leq j \leq \hat{m})$ are constant for $\varepsilon \in \mathbf{R} \backslash \mathcal{N}_1^0$, $\xi \in S^{n-1} \backslash \mathcal{N}_1(\varepsilon)$ and $t \in [0, \infty)$ with $\tilde{d}_0(t, \xi; \varepsilon)\tilde{d}_1(t, \xi; \varepsilon) \neq 0$. Since $\tilde{d}_0(t, \xi; \varepsilon)\tilde{d}_1(t, \xi; \varepsilon) \in \mathscr{A}[\xi, \varepsilon]$, it follows from Lemma 2.1 that for any $T > 0$ there is $N_{T,1} \in \mathbf{Z}_+$ such that

$$\#\{t \in [0, T]; \partial_t(\lambda_j(t, \xi; \varepsilon) - \lambda_k(t, \xi; \varepsilon)) = 0\}$$

$$(\leq \#\{t \in [0, T]; \tilde{d}_0(t, \xi; \varepsilon)\tilde{d}_1(t, \xi; \varepsilon) = 0\}) \leq N_{T,1}$$

if $\varepsilon \in \mathbf{R} \backslash \mathcal{N}_1^0$, $\xi \in S^{n-1} \backslash \mathcal{N}_1(\varepsilon)$, $1 \leq j < k \leq m$ and $\partial_t(\lambda_j(t, \xi; \varepsilon) - \lambda_k(t, \xi; \varepsilon)) \not\equiv 0$ in t. This proves the second part of the assertions of the lemma in the case where $\nu = 1$. Repeating the above arguments we can prove the lemma for $\nu = 2, 3, \ldots,$ inductively. □

LEMMA 2.4. *Let $T > 0$, Γ be a cone (with its vertex at 0) in $\mathbf{R}^n \backslash \{0\}$, and let $a(t, \xi)$ be a function defined for $(t, \xi) \in [0, T] \times \Gamma$ satisfying the following:*

(i) *$a(t, \xi)$ is continuously differentiable in $t \in [0, T]$ and positively homogeneous of degree 1 in ξ.*
(ii) *$\#\{t \in [0, T]; \partial_t a(t, \xi) = 0\} \leq N$ if $\xi \in \Gamma$ and $\partial_t a(t, \xi) \not\equiv 0$ in t.*
(iii) *$|a(t, \xi)| \leq C_0 |\xi|$ for $t \in [0, T]$ and $\xi \in \Gamma$.*

Here $N \in \mathbf{Z}_+$ and $C_0 \geq 0$. Then there is a positive constant $C(N, C_0)$, which depends only on N and C_0, such that

$$\int_0^T |\partial_t a(t, \xi)|/|\xi| \, dt \leq C(N, C_0),$$

$$\int_0^T |\partial_t a(t, \xi)|/(|a(t, \xi)| + 1) \, dt \leq C(N, C_0)(\log\langle \xi \rangle + 1)$$

for $\xi \in \Gamma$.

PROOF. Fix $\xi \in \Gamma$. We may assume that $\partial_t a(t, \xi) \not\equiv 0$ in t. Noting that $\#\{t \in [0, T]; a(t, \xi) = 0\} \leq N + 1$, we write

$$\#\{t \in [0, T]; a(t, \xi)\partial_t a(t, \xi) = 0\} = \{t_1, t_2, \ldots, t_{N(\xi)}\},$$

where $N(\xi) \in \mathbf{Z}_+$ and $0 \leq t_1 < t_2 < \cdots < t_{N(\xi)} \leq T$. It is obvious that $N(\xi) \leq 2N + 1$. In each sub-interval $[t_{j-1}, t_j]$ $(1 \leq j \leq N(\xi) + 1)$ we have "$a(t, \xi) \geq 0$ or

$a(t,\xi) \le 0$" and "$\partial_t a(t,\xi) \ge 0$ or $\partial_t a(t,\xi) \le 0$", where $t_0 = 0$ and $t_{N(\xi)+1} = T$. Then we have

$$\int_{t_{j-1}}^{t_j} |\partial_t a(t,\xi)|/|\xi|\, dt = |a(t_j, \xi/|\xi|) - a(t_{j-1}, \xi/|\xi|)| \le 2C_0.$$

Moreover, we have

$$\int_{t_{j-1}}^{t_j} |\partial_t a(t,\xi)|/(|a(t,\xi)|+1)\, dt \le \int_{t_{j-1}}^{t_j} |\partial_t a(t,\xi/|\xi|)|\, dt \le 2C_0 \quad \text{if } |\xi| \le 1.$$

If $|\xi| \ge 1$, then

$$\begin{aligned}
&\int_{t_{j-1}}^{t_j} |\partial_t a(t,\xi)|/(|a(t,\xi)|+1)\, dt \\
&\qquad = |\log(|a(t_j, \xi/|\xi|)| + |\xi|^{-1}) - \log(|a(t_{j-1}, \xi/|\xi|)| + |\xi|^{-1})| \\
&\qquad \le \log(C_0+1) + \log|\xi|.
\end{aligned}$$

Therefore, we have

$$\int_0^T |\partial_t a(t,\xi)|/|\xi|\, dt \le 2C_0(2N+1),$$

$$\int_0^T |\partial_t a(t,\xi)|/(|a(t,\xi)|+1)\, dt \le \begin{cases} 2C_0(2N+1) & \text{if } |\xi| \le 1, \\ (2N+1)(\log(C_0+1) + \log\langle\xi\rangle) & \text{if } |\xi| \ge 1, \end{cases}$$

which proves the lemma. □

Put

$$p_\varepsilon^{(j)}(t,\tau,\xi) = \partial_\tau^j p_\varepsilon(t,\tau,\xi) \;\; (= (1 - \varepsilon|\xi|^2\partial_\tau^2)\partial_\tau^j p(t,\tau,\xi))$$

$(1 \le j \le m)$ and write

$$p_\varepsilon^{(j)}(t,\tau,\xi) = (m!/(m-j)!)\prod_{k=1}^{m-j}(\tau - \lambda_k^{(j)}(t,\xi;\varepsilon)) \quad (1 \le j \le m-1).$$

Here we enumerate so that the $\lambda_k^{(j)}(t,\xi;\varepsilon)$ are real analytic in $t \in [0,\infty)$. Recall that

$$|p_\varepsilon^{(j)}(t,\tau - i\gamma,\xi)|^2 = \sum_{k=0}^{m-j} \gamma^{2k} h_{m-j-k}(t,\tau,\xi;p_\varepsilon^{(j)})$$

for $t \in [0,\infty)$, $(\tau,\xi) \in \mathbf{R}^{n+1}$, $\varepsilon \in \mathbf{R}$ and $\gamma \in \mathbf{R}$. Then it follows from Lemma 2.1 of [11] that

$$(m-r)!(k-r)!/(m!k!) \le h_{m-k}(t,\tau,\xi;p_\varepsilon)/h_{m-k}(t,\tau,\xi;p_\varepsilon^{(r)}) \le (k-1-r)!(k-r)!/((k-1)!k!) \tag{2.10}$$

for $0 \le r < k \le m$ (see, also, (2.3)). We put

$$p_I(t,\tau,\xi;\varepsilon) = \prod_{j\in\{1,2,\dots,m\}\setminus I} (\tau-\lambda_j(t,\xi;\varepsilon)) \quad \text{for } I \subset \{1,2,\dots,m\},$$

$$p_j(t,\tau,\xi;\varepsilon) = p_{\{j\}}(t,\tau,\xi;\varepsilon),$$

$$p_{j,k}(t,\tau,\xi;\varepsilon) = p_{\{j,k\}}(t,\tau,\xi;\varepsilon) \quad (j\ne k), \quad \cdots,$$

$$p_j^{(k)}(t,\tau,\xi;\varepsilon) = (m!/(m-k)!)\prod_{l\ne j}(\tau-\lambda_l^{(k)}(t,\xi;\varepsilon)).$$

Note that

$$h_{m-j}(t,\tau,\xi;p_\varepsilon) = \sum_{\substack{I\subset\{1,2,\dots,m\},\\ \#I=j}} p_I(t,\tau,\xi;\varepsilon)^2. \tag{2.11}$$

Lemma 2.5. *Let $r\in\mathbf{N}$, $\tau_j\in\mathbf{R}$ $(1\le j\le r)$ and $a_k\in\mathbf{C}$ $(0\le k\le r-1)$, and put $p(\tau)=a_0\tau^{r-1}+a_1\tau^{r-2}+\cdots+a_{r-1}$. If*

$$|p(\tau)| \le A\sum_{j=1}^r \prod_{1\le k\le r,k\ne j}|\tau-\tau_k| \quad \textit{for any } \tau\in\mathbf{R},$$

then there are b_j $(1\le j\le r)$ such that $|b_j|\le A$ $(1\le j\le r)$ and

$$p(\tau) = \sum_{j=1}^r b_j \prod_{1\le k\le r,k\ne j}(\tau-\tau_k). \tag{2.12}$$

Proof. Write

$$\prod_{k=1}^r(\tau-\tau_k) = \prod_{j=1}^{r_0}(\tau-\lambda_j)^{m_j},$$

where $r_0\in\mathbf{N}$, $m_j\in\mathbf{N}$, $\lambda_j\ne\lambda_{j'}$ $(j\ne j')$ and $\{\tau_1,\dots,\tau_r\}=\{\lambda_1,\dots,\lambda_{r_0}\}$. Then we have $m_1+\cdots+m_{r_0}=r$ and

$$|p(\tau)| \le A\prod_{j=1}^{r_0}|\tau-\lambda_j|^{m_j-1}\sum_{j=1}^{r_0} m_j\prod_{k\ne j}|\tau-\lambda_k|.$$

Therefore, there are a polynomial $\tilde{p}(\tau)$ of τ such that $\deg \tilde{p} = r_0 - 1$ and

$$p(\tau) = \tilde{p}(\tau) \prod_{j=1}^{r_0} (\tau - \lambda_j)^{m_j - 1}.$$

Since

$$|\tilde{p}(\tau)| \le A \sum_{j=1}^{r_0} m_j \prod_{k \ne j} |\tau - \lambda_k|, \tag{2.13}$$

Lagrange's interpolation formula gives

$$\tilde{p}(\tau) = \sum_{j=1}^{r_0} \left\{ \tilde{p}(\lambda_j) \Big/ \prod_{k \ne j} (\lambda_j - \lambda_k) \right\} \prod_{k \ne j} (\tau - \lambda_k).$$

Thus, putting

$$b_j = \tilde{p}(\lambda_l) \Big/ \left(m_l \prod_{k \ne l} (\lambda_l - \lambda_k) \right) \quad \text{if } \tau_j = \lambda_l, \tag{2.14}$$

we have (2.12). (2.13) with $\tau = \lambda_l$ and (2.14) give $|b_j| \le A$ $(1 \le j \le r)$. □

Lemma 2.6. (i) *The condition* (D-L) *is equivalent to the following condition* (D-L)$'$:

(D-L)$'$ *There are* $b_{1,j}(t, \xi; \varepsilon)$ $(1 \le j \le m)$ *defined for* $\xi \in \mathbf{R}^n \backslash \{0\}$, $t \in [0, \infty) \backslash \mathscr{R}(\xi/|\xi|)$ *and* $\varepsilon \in \mathbf{R}$ *such that the* $b_{1,j}(t, \xi; \varepsilon)$ *are positively homogeneous in* ξ *of degree* 0,

$$\text{sub } \sigma(P)(t, \tau, \xi) = \sum_{j=1}^{m} b_{1,j}(t, \xi; \varepsilon) p_j(t, \tau, \xi; \varepsilon)$$

$$\text{for } \xi \in S^{n-1},\ t \in [0, \infty) \backslash \mathscr{R}(\xi) \text{ and } \varepsilon \in \mathbf{R},$$

and for any $T > 0$ *there is* $C > 0$ *satisfying*

$$\min \left\{ \min_{s \in \mathscr{R}(\xi)} |t - s|, 1 \right\} |b_{1,j}(t, \xi; \varepsilon)| \le C$$

$$\text{for } 1 \le j \le m,\ \xi \in S^{n-1},\ t \in [0, \infty) \backslash \mathscr{R}(\xi) \text{ and } \varepsilon \in \mathbf{R}.$$

(ii) *Assume that* $m = 3$. *Then the condition* (T-L) *is equivalent to the following condition* (T-L)$'$:

(T-L)$'$ *The condition* (D-L)$'$ *is satisfied, and there are* $b_{2,j}(t,\xi;\varepsilon)$ $(j=1,2)$ *defined for* $\xi\in\mathbf{R}^n\backslash\{0\}$, $t\in[0,\infty)\backslash\mathscr{R}(\xi/|\xi|)$ *and* $\varepsilon\in\mathbf{R}$ *such that the* $b_{2,j}(t,\xi;\varepsilon)$ *are positively homogeneous in* ξ *of degree* 0,

$$sub^2\,\sigma(P)(t,\tau,\xi)=\sum_{j=1}^{2}b_{2,j}(t,\xi;\varepsilon)p_j^{(1)}(t,\tau,\xi;\varepsilon)$$

$$\text{for }\ \xi\in S^{n-1},\ t\in[0,\infty)\backslash\mathscr{R}(\xi)\ \text{ and }\ \varepsilon\in\mathbf{R},$$

and for any $T>0$ *there is* $C>0$ *satisfying*

$$\min\Big\{\min_{s\in\mathscr{R}(\xi)}|t-s|^2,1\Big\}|b_{2,j}(t,\xi;\varepsilon)|\le C \tag{2.15}$$

$$\text{for }\ j=1,2,\ \xi\in S^{n-1},\ t\in[0,\infty)\backslash\mathscr{R}(\xi)\ \text{ and }\ \varepsilon\in\mathbf{R}.$$

Proof. It is obvious that (D-L) with C replaced by $\sqrt{m}C$ is valid if (D-L)$'$ with $\varepsilon=0$ is valid, since we have, by (2.11),

$$\sum_{j=1}^{m}|p_j(t,\tau,\xi;0)|\le\sqrt{m}h_{m-1}(t,\tau,\xi;p)^{1/2}.$$

Similarly, from (2.10) and (2.11) it follows that (1.2) with C replaced by $2\sqrt{3}C$ in (T-L) is valid if (T-L)$'$ with $\varepsilon=0$ is valid. The converses in (i) and (ii) easily follow from Lemma 2.5. □

Corollary 2.7. *Let* $\delta>0$ *and* $\tau_j\in\mathbf{R}$ $(1\le j\le r)$ *satisfy* $|\tau_j-\tau_k|\ge\delta$ $(1\le j<k\le r)$. *Then there are* $b_{l,j}\in\mathbf{C}$ $(0\le l\le r-1,\ 1\le j\le r)$ *such that*

$$|b_{l,j}|\le\delta^{-r+1}|\tau_j|^l \tag{2.16}$$

$$\tau^l=\sum_{j=1}^{r}b_{l,j}\prod_{k\ne j}(\tau-\tau_k). \tag{2.17}$$

Proof. Take $b_{l,j}=\tau_j^l/\prod_{k\ne j}(\tau_j-\tau_k)$. Then (2.16) and (2.17) are satisfied. □

Define

$$\mathscr{P}_j(t,\tau,\xi;\varepsilon)=p_j(t,\tau,\xi;\varepsilon)-(i/2)\partial_t\partial_\tau p_j(t,\tau,\xi;\varepsilon)\quad(1\le j\le m). \tag{2.18}$$

Then a simple calculation yields

$$(2.19)\qquad \begin{aligned}&(\tau - \lambda_j(t,\xi;\varepsilon)) \circ \mathscr{P}_j(t,\tau,\xi;\varepsilon)\\ &= p_\varepsilon(t,\tau,\xi) - (i/2)\partial_t\partial_\tau p_\varepsilon(t,\tau,\xi)\\ &\quad - (i/2)\sum_{k\neq j} \partial_t(\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon)) \cdot p_{j,k}(t,\tau,\xi;\varepsilon)\\ &\quad - \partial_t^2\partial_\tau p_j(t,\tau,\xi;\varepsilon)/2 \quad \text{for } 1 \le j \le m,\end{aligned}$$

where $a(t,\tau,\xi) \circ b(t,\tau,\xi)$ denotes the symbol of $a(t,D_t,\xi)b(t,D_t,\xi)$. Indeed, we have

$$\begin{aligned}&(\tau - \lambda_j(t,\xi;\varepsilon)) \circ p_j(t,\tau,\xi;\varepsilon) = p_\varepsilon(t,\tau,\xi) - i\partial_t p_j(t,\tau,\xi;\varepsilon),\\ &(\tau - \lambda_j(t,\xi;\varepsilon)) \circ \partial_t\partial_\tau p_j(t,\tau,\xi;\varepsilon) = \partial_t\partial_\tau p_\varepsilon(t,\tau,\xi) - \partial_t p_j(t,\tau,\xi;\varepsilon)\\ &\qquad\qquad + \partial_t\lambda_j(t,\xi;\varepsilon) \cdot \partial_\tau p_j(t,\tau,\xi;\varepsilon) - i\partial_t^2\partial_\tau p_j(t,\tau,\xi;\varepsilon).\end{aligned}$$

From (2.19) we have

$$(2.20)\qquad \begin{aligned}&(\tau - \lambda_j(t,\xi;\varepsilon)) \circ \mathscr{P}_j(t,\tau,\xi;\varepsilon)\\ &= p_\varepsilon(t,\tau,\xi) - (i/2)\partial_t\partial_\tau p_\varepsilon(t,\tau,\xi) - (2m)^{-1}\partial_t^2\partial_\tau^2 p_\varepsilon(t,\tau,\xi)\\ &\quad - (i/2)\sum_{k\neq j} \partial_t(\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon)) \cdot p_{j,k}(t,\tau,\xi;\varepsilon)\\ &\quad - (2m)^{-1}\partial_t^2\left\{\sum_{k\neq j}\sum_{l\neq j,k} (\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon)) p_{j,k,l}(t,\tau,\xi;\varepsilon)\right\}\end{aligned}$$

for $1 \le j \le m$. Note that $p_{j,k,l}(t,\tau,\xi;\varepsilon) = 1$ if $m = 3$. We have also

$$(2.21)\qquad (\tau - \lambda_j^{(r)}(t,\xi;\varepsilon)) \circ p_j^{(r)}(t,\tau,\xi;\varepsilon) = p_\varepsilon^{(r)}(t,\tau,\xi) - i\partial_t p_j^{(r)}(t,\tau,\xi;\varepsilon)$$

for $1 \le r \le m-1$ and $1 \le j \le m-r$. In particular, we have

$$(2.22)\qquad \begin{aligned}&(\tau - \lambda_j^{(1)}(t,\xi;\varepsilon)) \circ p_j^{(1)}(t,\tau,\xi;\varepsilon)\\ &= \sum_{l=1}^{3} \mathscr{P}_l(t,\tau,\xi;\varepsilon) + (i/2)\partial_t(\lambda_j^{(1)}(t,\xi;\varepsilon) - \lambda_k^{(1)}(t,\xi;\varepsilon))\end{aligned}$$

if $m = 3$ and $\{j,k\} = \{1,2\}$. Indeed, we have

$$\partial_t \partial_\tau^2 p_\varepsilon(t,\tau,\xi) = \partial_t \partial_\tau p_\varepsilon^{(1)}(t,\tau,\xi) = \sum_{k=1}^{2} \partial_t p_k^{(1)}(t,\tau,\xi;\varepsilon),$$

$$p_\varepsilon^{(1)}(t,\tau,\xi;\varepsilon) - (i/2)\partial_t \partial_\tau^2 p_\varepsilon(t,\tau,\xi) = \sum_{l=1}^{3} \mathscr{P}_l(t,\tau,\xi;\varepsilon).$$

3. Proof of Theorem 1.2

In this section we assume that the hypotheses of Theorem 1.2 are fulfilled and we shall prove Theorem 1.2. Let $\{\varepsilon_j\}_{j=1,2,\dots}$ be a sequence satisfying $\varepsilon_j \in (0,1] \backslash \mathscr{N}_2^0$ and $\varepsilon_j \downarrow 0$ as $j \to \infty$, where $\mathscr{N}_2^0$ is as in Lemma 2.3. Put

$$E_0 = \{\varepsilon_j; j = 1,2,\dots\} \cup \{0\},$$

$$\mathscr{N} = \bigcup_{j=1}^{\infty} \mathscr{N}_2(\varepsilon_j) \cup \mathscr{N}_2(0) \cup \{0\} \quad (\subset \mathbf{R}^n),$$

where $\mathscr{N}_2(\varepsilon)$ is as in Lemma 2.3. Note that $\mu_n(\mathscr{N}) = 0$. We define

$$P_\varepsilon(t,\tau,\xi) = P(t,\tau,\xi) + p_\varepsilon(t,\tau,\xi) - p(t,\tau,\xi)$$
$$\qquad - (i/2)\partial_t \partial_\tau (p_\varepsilon(t,\tau,\xi) - p(t,\tau,\xi)).$$

Note that

$$sub\,\sigma(P_\varepsilon)(t,\tau,\xi) = sub\,\sigma(P)(t,\tau,\xi),$$

$$P_\varepsilon(t,\tau,\xi) = P(t,\tau,\xi) + p_\varepsilon(t,\tau,\xi) - p(t,\tau,\xi) \quad \text{if } m = 3.$$

Consider the Cauchy problem

$$(\mathrm{CP})_\varepsilon \qquad \begin{cases} P_\varepsilon(t,D_t,D_x)u_\varepsilon(t,x) = f(t,x) & \text{in } [0,\infty) \times \mathbf{R}^n, \\ D_t^j u_\varepsilon(t,x)|_{t=0} = u_j(x) & \text{in } \mathbf{R}^n \ (0 \le j \le m-1) \end{cases}$$

for $\varepsilon \in E_0$, where $f \in C^\infty([0,\infty); H^\infty(\mathbf{R}_x^n))$ and $u_j \in H^\infty(\mathbf{R}^n)$ $(0 \le j \le m-1)$. Here $H^s(\mathbf{R}^n)$ denotes the Sobolev space over $\mathbf{R}^n$ of order s and $H^\infty(\mathbf{R}^n) = \bigcap_{s \in \mathbf{R}} H^s(\mathbf{R}^n)$. By partial Fourier transformation in x, the Cauchy problem $(\mathrm{CP})_\varepsilon$ is reduced to the Cauchy problem for an ordinary differential operator with parameters ξ:

$$(3.1)\qquad \begin{cases} P_\varepsilon(t, D_t, \xi)v_\varepsilon(t,\xi) = \hat{f}(t,\xi) & \text{for } (t,\xi) \in [0,\infty) \times \mathbf{R}^n, \\ D_t^j v_\varepsilon(t,\xi)|_{t=0} = \hat{u}_j(\xi) & \text{for } \xi \in \mathbf{R}^n \ (0 \le j \le m-1) \end{cases}$$

for $\varepsilon \in E_0$, where $\hat{f}(t,\xi)$ and $\hat{u}_j(\xi)$ $(0 \le j \le m-1)$ denotes the partial Fourier transforms of $f(t,x)$ and $u_j(x)$ with respect to x, respectively, for example, $\hat{f}(t,\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(t,x)\,dx$. We note that the Cauchy problem (3.1) has a unique solution $v_\varepsilon(t,\xi) \in C^\infty([0,\infty); C^\infty(\mathbf{R}^n_\xi))$. If it can be shown that $v_\varepsilon(t,\xi) \in C^m([0,\infty); \mathscr{S}'(\mathbf{R}^n_\xi))$, then $u_\varepsilon(t,x) = \mathscr{F}_\xi^{-1}[v_\varepsilon(t,\xi)](x)$ $(\in C^m([0,\infty); \mathscr{S}'(\mathbf{R}^n_x))$ is a unique solution to the Cauchy problem $(\mathrm{CP})_\varepsilon$, where $\mathscr{F}_\xi^{-1}[v(t,\xi)](x)$ denotes the inverse partial Fourier transform of $v(t,\xi)$ in ξ. We fix $T > 0$. Define

$$W_0(t,\xi) = \sum_{s \in \mathscr{R}(\xi/|\xi|)\cap[0,T+1]} \langle\xi\rangle^{1/2} / \sqrt{(t-s)^2\langle\xi\rangle + 1} + 1,$$

$$W_1(t,\xi;\varepsilon) = \sum_{1 \le j < k \le m} |\partial_t(\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon))| / (|\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon)| + 1)$$

$$+ \sum_{j=1}^m |\partial_t^2 \lambda_j(t,\xi;\varepsilon)| / |\xi|,$$

$$\Lambda(t,\xi;\varepsilon) = \int_0^t (W_0(s,\xi) + W_1(s,\xi;\varepsilon))\,ds$$

for $(t,\xi,\varepsilon) \in [0,T] \times (\mathbf{R}^n \backslash \mathscr{N}) \times E_0$, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. Note that

$$(3.2)\qquad \partial_t \log(\sqrt{(t-s)^2\langle\xi\rangle + 1} + (t-s)\langle\xi\rangle^{1/2}) = \langle\xi\rangle^{1/2} / \sqrt{(t-s)^2\langle\xi\rangle + 1},$$

$$(3.3)\qquad |\partial_t W_0(t,\xi)| \le \sum_{s \in \mathscr{R}(\xi/|\xi|)\cap[0,T+1]} \langle\xi\rangle / ((t-s)^2\langle\xi\rangle + 1) \le W_0(t,\xi)^2.$$

From (3.2) and Lemmas 2.3 and 2.4 it follows that there is $C_T > 0$ satisfying

$$(3.4)\qquad 0 \le \Lambda(t,\xi;\varepsilon) \le C_T(\log\langle\xi\rangle + 1) \quad \text{for } t \in [0,T],\ \xi \in \mathbf{R}^n \backslash \mathscr{N} \text{ and } \varepsilon \in E_0.$$

Here we have used the fact that, with some $C_T' > 0$,

$$(3.5)\qquad |\partial_t \lambda_j(t,\xi;\varepsilon)| \le C_T' |\xi| \quad \text{for } (t,\xi,\varepsilon) \in [0,T] \times \mathbf{R}^n \times [-1,1]$$

(see, *e.g.*, Theorem 1 of [14]). For $(t,\xi,\varepsilon) \in [0,T] \times (\mathbf{R}^n \backslash \mathscr{N}) \times E_0$ and $A \ge 1$ we define

$$\mathscr{E}(t,\xi;\varepsilon;A) = \sum_{j=1}^{m} e^{-A\Lambda(t,\xi;\varepsilon)} |\mathscr{P}_j(t,D_t,\xi;\varepsilon)v_\varepsilon(t,\xi)|^2 + \sum_{k=1}^{m-1}\sum_{j=1}^{m-k} W_0(t,\xi)^2 e^{-A\Lambda(t,\xi;\varepsilon)} |p_j^{(k)}(t,D_t,\xi;\varepsilon)v_\varepsilon(t,\xi)|^2.$$

Let $(t,\xi)\in[0,T]\times(\mathbf{R}^n\backslash\mathscr{N})$ and $\varepsilon\in E_0$. It is obvious that

$$\begin{aligned}(3.6)\quad D_t\mathscr{E}(t,\xi;\varepsilon;A) &= i\sum_{j=1}^{m}[A\Lambda_t e^{-A\Lambda}|\mathscr{P}_j v_\varepsilon|^2 + 2\,\mathrm{Im}\{e^{-A\Lambda}(D_t\mathscr{P}_j v_\varepsilon)\cdot(\overline{\mathscr{P}_j v_\varepsilon})\}] \\ &\quad + i\sum_{k=1}^{m-1}\sum_{j=1}^{m-k}[AW_0^2\Lambda_t e^{-A\Lambda}|p_j^{(k)}v_\varepsilon|^2 - 2W_0W_{0t}e^{-A\Lambda}|p_j^{(k)}v_\varepsilon|^2 \\ &\quad + 2\,\mathrm{Im}\{W_0^2 e^{-A\Lambda}(D_t p_j^{(k)}v_\varepsilon)\cdot(\overline{p_j^{(k)}v_\varepsilon})\}].\end{aligned}$$

Here we write $\Lambda=\Lambda(t,\xi;\varepsilon)$, $\Lambda_t=\partial_t\Lambda(t,\xi;\varepsilon)$, $\mathscr{P}_j=\mathscr{P}_j(t,\tau,\xi;\varepsilon)$, $v_\varepsilon=v_\varepsilon(t,\xi)$, $W_0=W_0(t,\xi)$, $W_{0t}=\partial_t W_0(t,\xi)$ and so forth. Since the $\lambda_j(t,\xi;\varepsilon)$ are real-valued, from (2.19) we have

$$\begin{aligned}(3.7)\quad &\mathrm{Im}\{e^{-A\Lambda}(D_t\mathscr{P}_j v_\varepsilon)\cdot(\overline{\mathscr{P}_j v_\varepsilon})\} \\ &= \mathrm{Im}\{e^{-A\Lambda}((D_t-\lambda_j(t,\xi;\varepsilon))\mathscr{P}_j v_\varepsilon)\cdot(\overline{\mathscr{P}_j v_\varepsilon})\} \\ &= \mathrm{Im}\{e^{-A\Lambda}((p_\varepsilon-(i/2)(\partial_t\partial_\tau p_\varepsilon)(t,D_t,\xi))v_\varepsilon)\cdot(\overline{\mathscr{P}_j v_\varepsilon})\} \\ &\quad - \mathrm{Re}\left\{e^{-A\Lambda}\sum_{k\neq j}(\lambda_{jt}-\lambda_{kt})(p_{j,k}v_\varepsilon)\cdot(\overline{\mathscr{P}_j v_\varepsilon})\right\}\Big/2 \\ &\quad - \mathrm{Im}\{e^{-A\Lambda}((\partial_t^2\partial_\tau p_j)(t,D_t,\xi;\varepsilon)v_\varepsilon)\cdot(\overline{\mathscr{P}_j v_\varepsilon})\}/2 \quad (1\le j\le m).\end{aligned}$$

Similarly, (2.21) gives

$$\begin{aligned}(3.8)\quad &\mathrm{Im}\{W_0^2 e^{-A\Lambda}(D_t p_j^{(k)}v_\varepsilon)\cdot(\overline{p_j^{(k)}v_\varepsilon})\} \\ &= \mathrm{Im}\{W_0^2 e^{-A\Lambda}(p_\varepsilon^{(k)}v_\varepsilon)\cdot(\overline{p_j^{(k)}v_\varepsilon})\} \\ &\quad - \mathrm{Re}\{W_0^2 e^{-A\Lambda}(\partial_t p_j^{(k)})(t,D_t,\xi;\varepsilon)v_\varepsilon)\cdot(\overline{p_j^{(k)}v_\varepsilon})\}\end{aligned}$$

$(1 \le k \le m-1,\ 1 \le j \le m-k)$. Therefore, (3.1), (3.3) and (3.6)–(3.8) yield

$$\begin{aligned}(3.9)\quad \partial_t \mathscr{E}(t,\xi;\varepsilon;A) \le{}& m\Lambda_t^{-1} e^{-A\Lambda}|\hat{f}(t,\xi)|^2 - \sum_{j=1}^{m}\Bigg[(A-2m-1)\Lambda_t e^{-A\Lambda}|\mathscr{P}_j v_\varepsilon|^2 \\ &+ \Lambda_t^{-1}e^{-A\Lambda}\Bigg\{|sub\ \sigma(P)(t,D_t,\xi)v_\varepsilon|^2 + \sum_{k\ne j}|(\lambda_{jt}-\lambda_{kt})p_{j,k}v_\varepsilon|^2/4 \\ &+ \sum_{k=2}^{m}|P_{m-k}(t,D_t,\xi)v_\varepsilon|^2 + |(\partial_t^2\partial_\tau p_j)(t,D_t,\xi;\varepsilon)v_\varepsilon|^2/4\Bigg\}\Bigg] \\ &- \sum_{k=1}^{m-1}\sum_{j=1}^{m-k}[\{(A-2)W_0^2\Lambda_t - 2W_0^3\}e^{-A\Lambda}|p_j^{(k)}v_\varepsilon|^2 \\ &- W_0^2\Lambda_t^{-1}e^{-A\Lambda}\{|p_\varepsilon^{(k)}v_\varepsilon|^2 + |(\partial_t p_j^{(k)})(t,D_t,\xi;\varepsilon)v_\varepsilon|^2\}],\end{aligned}$$

since

$$\begin{aligned}&(p_\varepsilon - (i/2)(\partial_t\partial_\tau p_\varepsilon)(t,D_t,\xi))v_\varepsilon \\ &\quad = P_\varepsilon v_\varepsilon - sub\ \sigma(P)(t,D_t,\xi)v_\varepsilon - \sum_{k=2}^{m} P_{m-k}(t,D_t,\xi)v_\varepsilon.\end{aligned}$$

From (D-L), Lemma 2.6 and Corollary 2.7 we have, with some $C, C' > 0$,

$$\begin{aligned}(3.10)\qquad &\min\left\{\min_{s\in\mathscr{R}(\xi/|\xi|)}|t-s|^2, 1\right\}|sub\ \sigma(P)(t,D_t,\xi)v_\varepsilon|^2 \\ &\quad \le mC^2\sum_{j=1}^{m}|p_j v_\varepsilon|^2 \le C'\left(\sum_{j=1}^{m}|\mathscr{P}_j v_\varepsilon|^2 + \sum_{k=1}^{m-1}|p_k^{(1)}v_\varepsilon|^2\right)\end{aligned}$$

for $t \in [0,T]$, $\xi \in \mathbf{R}^n \backslash \mathscr{N}$ and $\varepsilon \in E_0$. Here and after the constants do not depend on the parameter A unless stated. Indeed, $p^{(1)}(t,\tau,\xi;\varepsilon)$ is stictly hyperbolic with respect to ϑ for $t \in [0,\infty)$ and $\varepsilon \in \mathbf{R}$. By Corollary 2.7 with $r = m-1$ and (3.5) we have, with some $C'' > 0$,

$$|(\partial_t\partial_\tau p_j)(t,D_t,\xi;\varepsilon)v_\varepsilon| \le C''\sum_{k=1}^{m-1}|p_k^{(1)}v_\varepsilon|.$$

When $\min\{\min_{s\in\mathscr{R}(\xi/|\xi|)}|t-s|, 1\} \le \langle\xi\rangle^{-1/2}$, we can not use (3.10). It follows from Corollary 2.7 that

$$sub\ \sigma(P)(t,\tau,\xi) = c_0(t)\tau^{m-1} + \sum_{k=1}^{m-1} c_k(t,\xi)\tau^{m-k-1}$$

$$= c_0(t)\mathscr{P}_j(t,\tau,\xi;\varepsilon) + \sum_{k=1}^{m-1} \tilde{c}_k(t,\xi;\varepsilon) p_k^{(1)}(t,\tau,\xi;\varepsilon),$$

where $1 \le j \le m$, $\deg_\xi c_k(t,\xi) = k$ and the $\tilde{c}_k(t,\xi;\varepsilon)$ are functions, determined by Corollary 2.7, satisfying, with some $C > 0$,

$$|\tilde{c}_k(t,\xi;\varepsilon)| \le C\langle\xi\rangle \quad \text{for } t \in [0,T],\ \xi \in \mathbf{R}^n \backslash \mathscr{N} \text{ and } \varepsilon \in E_0.$$

This gives

$$|sub\ \sigma(P)(t,D_t,\xi)v_\varepsilon|^2 \le C\left(|\mathscr{P}_j v_\varepsilon|^2 + \langle\xi\rangle^2 \sum_{k=1}^{m-1} |p_k^{(1)} v_\varepsilon|^2\right) \tag{3.11}$$

for $t \in [0,T]$, $\xi \in \mathbf{R}^n \backslash \mathscr{N}$ and $\varepsilon \in E_0$. We have also, with some $C > 0$,

$$\begin{aligned} (3.12)\quad & |(\lambda_j - \lambda_k) p_{j,k} v_\varepsilon|^2 \\ & \le |\{\mathscr{P}_j - \mathscr{P}_k + (i/2)((\partial_t\partial_\tau p_j)(t,D_t,\xi;\varepsilon) - (\partial_t\partial_\tau p_k)(t,D_t,\xi;\varepsilon))\}v_\varepsilon|^2 \\ & \le 3\left\{|\mathscr{P}_j v_\varepsilon|^2 + |\mathscr{P}_k v_\varepsilon|^2 + C\sum_{l=1}^{m-1} |p_l^{(1)} v_\varepsilon|^2\right\}, \end{aligned}$$

$$\sum_{k=2}^{m} |P_{m-k}(t,D_t,\xi)v_\varepsilon|^2 \le C\sum_{k=1}^{m-1}\sum_{j=1}^{m-k} |p_j^{(k)} v_\varepsilon|^2, \tag{3.13}$$

$$\begin{aligned} (3.14)\quad & |(\partial_t^2\partial_\tau p_j)(t,D_t,\xi;\varepsilon)v_\varepsilon|^2 \\ & \le C\sum_{k=1}^{m-1} |p_k^{(1)} v_\varepsilon|^2 + C\sum_{k=1}^{m} |\lambda_{ktt}(t,\xi;\varepsilon)|^2 |\xi|^{-2} \sum_{l=1}^{m-1} |p_l^{(1)} v_\varepsilon|^2, \end{aligned}$$

$$\begin{aligned} (3.15)\quad & W_0^2 \Lambda_t^{-1} e^{-A\Lambda} |p_\varepsilon^{(1)} v_\varepsilon|^2 \\ & = W_0^2 \Lambda_t^{-1} e^{-A\Lambda} \left|\sum_{j=1}^{m} (\mathscr{P}_j + (i/2)(\partial_t\partial_\tau p_j)(t,D_t,\xi;\varepsilon))v_\varepsilon\right|^2 \\ & \le C W_0^2 \Lambda_t^{-1} e^{-A\Lambda} \left\{\sum_{j=1}^{m} |\mathscr{P}_j v_\varepsilon|^2 + \sum_{k=1}^{m-1} |p_k^{(1)} v_\varepsilon|^2\right\}, \end{aligned}$$

$$(3.16) \qquad W_0^2 \Lambda_t^{-1} e^{-A\Lambda} |(\partial_t p_\varepsilon^{(1)})(t, D_t, \xi; \varepsilon) v_\varepsilon|^2 \le C W_0^2 \Lambda_t^{-1} e^{-A\Lambda} \sum_{k=1}^{m-1} |p_k^{(1)} v_\varepsilon|^2,$$

$$(3.17) \qquad W_0^2 \Lambda_t^{-1} e^{-A\Lambda} \{ |p_\varepsilon^{(k)} v_\varepsilon|^2 + (\partial_t p_j^{(k)})(t, D_t, \xi; \varepsilon) v_\varepsilon|^2 \}$$
$$\le C W_0^2 \Lambda_t^{-1} e^{-A\Lambda} \left\{ \sum_{l=1}^{m-k+1} |p_l^{(k-1)} v_\varepsilon|^2 + \sum_{l=1}^{m-k} |p_l^{(k)} v_\varepsilon|^2 \right\}$$
$$\text{for } 2 \le k \le m-1 \text{ and } 1 \le j \le m-k$$

for $t \in [0, T]$, $\xi \in \mathbf{R}^n \setminus \mathcal{N}$ and $\varepsilon \in E_0$. Indeed, (3.12), (3.13) and (3.15) follow from Corollary 2.7, applying the argument as in (3.10), since $p^{(k)}(t, \tau, \xi; \varepsilon)$ $(1 \le k \le m-1)$ are strictly hyperbolic with respect to ϑ. We have

$$\partial_t^2 \partial_\tau p_j(t, \tau, \xi; \varepsilon) = -\sum_{k \ne j} \sum_{l \ne j,k} \lambda_{ltt}(t, \xi; \varepsilon) p_{j,k,l}(t, \tau, \xi; \varepsilon)$$
$$+ \sum_{k \ne j} \sum_{l \ne j,k} \sum_{i \ne j,k,l} \lambda_{lt}(t, \xi; \varepsilon) \lambda_{it}(t, \xi; \varepsilon) p_{j,k,l,i}(t, \tau, \xi; \varepsilon)$$

By (3.5) and the same argument as in (3.10) we obtain (3.14), (3.16) and (3.17). Let us estimate $\Lambda_t^{-1} e^{-A\Lambda} |sub\, \sigma(P)(t, D_t, \xi) v_\varepsilon|^2$. First assume that $\min\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|, 1\} \le \langle \xi \rangle^{-1/2}$. Then we have

$$W_0(t, \xi) \ge \langle \xi \rangle^{1/2} / \sqrt{2}.$$

Therefore, from (3.11) we have

$$(3.18) \qquad \Lambda_t^{-1} e^{-A\Lambda} |sub\, \sigma(P)(t, D_t, \xi) v_\varepsilon|^2$$
$$\le C \Lambda_t^{-1} e^{-A\Lambda} \left(|\mathscr{P}_j v_\varepsilon|^2 + 4 W_0^4 \sum_{k=1}^{m-1} |p_k^{(1)} v_\varepsilon|^2 \right)$$
$$\le C \Lambda_t^{-1} e^{-A\Lambda} |\mathscr{P}_j v_\varepsilon|^2 + 4C \sum_{k=1}^{m-1} W_0^2 \Lambda_t e^{-A\Lambda} |p_k^{(1)} v_\varepsilon|^2$$

for $t \in [0, T]$, $\xi \in \mathbf{R}^n \setminus \mathcal{N}$ and $\varepsilon \in E_0$. Next assume that $\min\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|, 1\}$ $\ge \langle \xi \rangle^{-1/2}$. Then we have

$$W_0(t, \xi) \ge \left(\sqrt{2} \min\left\{ \min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|, 1 \right\} \right)^{-1}.$$

This, together with (3.10), yields

$$\begin{aligned}(3.19)\qquad &\Lambda_t^{-1}e^{-A\Lambda}|sub\ \sigma(P)(t,D_t,\xi)v_\varepsilon|^2\\ &\le 2C'W_0^2\Lambda_t^{-1}e^{-A\Lambda}\left(\sum_{j=1}^m|\mathscr{P}_jv_\varepsilon|^2+\sum_{k=1}^{m-1}|p_k^{(1)}v_\varepsilon|^2\right)\\ &\le 2C'\Lambda_te^{-A\Lambda}\sum_{j=1}^m|\mathscr{P}_jv_\varepsilon|^2+2C'W_0^2\Lambda_te^{-A\Lambda}\sum_{k=1}^{m-1}|p_k^{(1)}v_\varepsilon|^2\end{aligned}$$

for $t\in[0,T]$, $\xi\in\mathbf{R}^n\backslash\mathscr{N}$ and $\varepsilon\in E_0$. Thus, by (3.18) and (3.19) we have

$$\begin{aligned}(3.20)\qquad &\Lambda_t^{-1}e^{-A\Lambda}|sub\ \sigma(P)(t,D_t,\xi)v_\varepsilon|^2\\ &\le C\Lambda_te^{-A\Lambda}\sum_{j=1}^m|\mathscr{P}_jv_\varepsilon|^2+CW_0^2\Lambda_te^{-A\Lambda}\sum_{k=1}^{m-1}|p_k^{(1)}v_\varepsilon|^2\end{aligned}$$

for $t\in[0,T]$, $\xi\in\mathbf{R}^n\backslash\mathscr{N}$ and $\varepsilon\in E_0$. It follows from (3.12) and the definition of $W_1(t,\xi;\varepsilon)$ that

$$\begin{aligned}(3.21)\qquad &\Lambda_t^{-1}e^{-A\Lambda}|(\lambda_{jt}-\lambda_{kt})p_{j,k}v_\varepsilon|^2\\ &\le 2W_1e^{-A\Lambda}(|\lambda_j-\lambda_k|^2+1)|p_{j,k}v_\varepsilon|^2\\ &\le 6\Lambda_te^{-A\Lambda}(|\mathscr{P}_jv_\varepsilon|^2+|\mathscr{P}_kv_\varepsilon|^2)+C'W_0^2\Lambda_te^{-A\Lambda}\sum_{k=1}^m|p_k^{(1)}v_\varepsilon|^2\end{aligned}$$

for $t\in[0,T]$, $\xi\in\mathbf{R}^n\backslash\mathscr{N}$ and $\varepsilon\in E_0$. From (3.13)–(3.17) we have

$$(3.22)\qquad \Lambda_t^{-1}e^{-A\Lambda}\sum_{k=2}^m|P_{m-k}(t,D_t,\xi)v_\varepsilon|^2\le CW_0^2\Lambda_te^{-A\Lambda}\sum_{k=1}^{m-1}\sum_{j=1}^{m-k}|p_j^{(k)}v_\varepsilon|^2,$$

$$(3.23)\qquad \Lambda_t^{-1}e^{-A\Lambda}|(\partial_t^2\partial_\tau p_j)(t,D_t,\xi;\varepsilon)v_\varepsilon|^2\le 2CW_0^2\Lambda_te^{-A\Lambda}\sum_{k=1}^{m-1}|p_k^{(1)}v_\varepsilon|^2,$$

$$(3.24)\qquad W_0^2\Lambda_t^{-1}e^{-A\Lambda}|p_\varepsilon^{(1)}v_\varepsilon|^2\le C\Lambda_te^{-A\Lambda}\sum_{j=1}^m|\mathscr{P}_jv_\varepsilon|^2+CW_0^2\Lambda_te^{-A\Lambda}\sum_{k=1}^{m-1}|p_k^{(1)}v_\varepsilon|^2,$$

$$(3.25)\qquad W_0^2\Lambda_t^{-1}e^{-A\Lambda}|(\partial_tp_j^{(1)})(t,D_t,\xi;\varepsilon)v_\varepsilon|^2\le CW_0^2\Lambda_te^{-A\Lambda}\sum_{k=1}^{m-1}|p_k^{(1)}v_\varepsilon|^2,$$

$$W_0^2\Lambda_t^{-1}e^{-A\Lambda}\{|p_\varepsilon^{(k)}v_\varepsilon| + |(\partial_t p_j^{(k)})(t, D_t, \xi; \varepsilon)v_\varepsilon|^2\} \tag{3.26}$$

$$\leq CW_0^2\Lambda_t e^{-A\Lambda}\left\{\sum_{l=1}^{m-k+1}|p_l^{(k-1)}v_\varepsilon|^2 + \sum_{l=1}^{m-k}|p_l^{(k)}v_\varepsilon|^2\right\}$$

$$\text{for } 2 \leq k \leq m-1 \text{ and } 1 \leq j \leq m-k$$

for $t \in [0, T]$, $\xi \in \mathbf{R}^n \backslash \mathcal{N}$ and $\varepsilon \in E_0$, since $|\lambda_{ktt}|^2|\xi|^{-2} \leq W_1^2$. So it follows from (3.9) and (3.20)–(3.26) that there is $A_0 \geq 1$ satisfying

$$\partial_t\mathscr{E}(t, \xi; \varepsilon; A) \leq m\Lambda_t^{-1}e^{-A\Lambda}|\hat{f}(t, \xi)|^2$$

for $A \geq A_0$, $t \in [0, T]$, $\xi \in \mathbf{R}^n \backslash \mathcal{N}$ and $\varepsilon \in E_0$. This gives

$$\mathscr{E}(t, \xi; \varepsilon; A) \leq \mathscr{E}(0, \xi; \varepsilon; A) + m\int_0^t |\hat{f}(s, \xi)|^2\,ds \tag{3.27}$$

for $A \geq A_0$, $t \in [0, T]$, $\xi \in \mathbf{R}^n \backslash \mathcal{N}$ and $\varepsilon \in E_0$. From (3.4) we have

$$1 \geq e^{-A\Lambda(t, \xi; \varepsilon)} \geq e^{-AC_T}\langle\xi\rangle^{-AC_T}$$

for $t \in [0, T]$, $\xi \in \mathbf{R}^n \backslash \mathcal{N}$ and $\varepsilon \in E_0$.

Lemma 3.1. *For a fixed $T > 0$ there are $c > 0$ and $C_A > 0$, which depends on A, such that*

$$c\mathscr{E}(t, \xi; \varepsilon; A) \leq \sum_{k=0}^{m-1}\langle\xi\rangle^{2k}|D_t^{m-1-k}v_\varepsilon|^2 \leq C_A\langle\xi\rangle^{2+AC_T}\mathscr{E}(t, \xi; \varepsilon; A) \tag{3.28}$$

for $A \geq 1$, $t \in [0, T]$, $\xi \in \mathbf{R}^n$ and $\varepsilon \in [0, 1]$. In particular, we have $v_\varepsilon(t, \xi) \in C^{m-1}([0, \infty); \mathscr{S}'(\mathbf{R}_\xi^n))$ and $u_\varepsilon(t, x)$ $(\equiv \mathscr{F}_\xi^{-1}[v_\varepsilon(t, \xi)](x)) \in C^{m-1}([0, \infty); H^\infty(\mathbf{R}^n))$ for $\varepsilon \in E_0$.

Proof. We can write

$$D_t^{m-1}v_\varepsilon(t, \xi) = \mathscr{P}_1(t, D_t, \xi; \varepsilon)v_\varepsilon + \sum_{j=1}^{m-1}c_j(t, \xi; \varepsilon)D_t^{m-1-j}v_\varepsilon + (i/2)\partial_t\partial_\tau p_1(t, D_t, \xi; \varepsilon)v_\varepsilon,$$

where $c_j(t, \xi; \varepsilon)$ $(1 \leq j \leq m-1)$ satisfy $|c_j(t, \xi; \varepsilon)| \leq C|\xi|^j$ for $\varepsilon \in [0, 1]$. So Corollary 2.7 gives

$$|D_t^{m-1}v_\varepsilon(t, \xi)| \leq |\mathscr{P}_1(t, D_t, \xi; \varepsilon)v_\varepsilon| + C\langle\xi\rangle\sum_{j=1}^{m-1}|p_j^{(1)}(t, D_t, \xi; \varepsilon)v_\varepsilon|.$$

Similarly, we have

$$\langle\xi\rangle^k |D_t^{m-1-k} v_\varepsilon| \le C\left(|\xi| \sum_{j=1}^{m-1} |p_j^{(1)}(t, D_t, \xi; \varepsilon) v_\varepsilon| + \sum_{j=1}^{m-k} |p_j^{(k)} v_\varepsilon|\right)$$

$(1 \le k \le m-1)$. Therefore, we have

$$\sum_{k=0}^{m-1} \langle\xi\rangle^{2k} |D_t^{m-1-k} v_\varepsilon|^2 \le C_A \langle\xi\rangle^{2+AC_T} e^{-A\Lambda}\left(|\mathscr{P}_1 v_\varepsilon|^2 + \sum_{k=1}^{m-1}\sum_{j=1}^{m-k} W_0^2 |p_j^{(k)} v_\varepsilon|^2\right) \tag{3.29}$$

$$\le C_A \langle\xi\rangle^{2+AC_T} \mathscr{E}(t, \xi; \varepsilon; A)$$

for $A \ge 1$, $t \in [0, T]$, $\xi \in \mathbf{R}^n$ and $\varepsilon \in [0, 1]$, where $C_A (> 0)$ depends on A. It is obvious that, with some $C, C' > 0$,

$$\mathscr{E}(t, \xi; \varepsilon; A) \le \sum_{j=1}^{m} |\mathscr{P}_j v_\varepsilon|^2 + \sum_{k=1}^{m-1}\sum_{j=1}^{m-k} C\langle\xi\rangle |p_j^{(k)} v_\varepsilon|^2$$

$$\le C' \sum_{k=0}^{m-1} \langle\xi\rangle^{2k} |D_t^{m-1-k} v_\varepsilon|^2$$

for $t \in [0, T]$, $\xi \in \mathbf{R}^n$ and $\varepsilon \in [0, 1]$, since $W_0(t, \xi)^2 \le C\langle\xi\rangle$. This, together with (3.29), proves the lemma. $\square$

Fix $T > 0$ and put

$$E_{k,l}[u](t) = \sum_{\mu=0}^{k} \|\langle D_x\rangle^{l+k-\mu} D_t^\mu u(t, x)\|_{L^2}^2,$$

where $k \ge m-1$, $l \in \mathbf{R}$, $u(t, x) \in C^k([0, T]; H^\infty(\mathbf{R}_x^n))$ and $\|u(t, x)\|_{L^2} = (\int_{\mathbf{R}^n} |u(t, x)|^2 \, dx)^{1/2}$. It follows from (3.27), (3.28) and Plancherel's theorem that

$$E_{m-1,l}[u_\varepsilon](t) \le C_{A_0}\left\{\sum_{\nu=0}^{m-1} \|\langle D_x\rangle^{l+m+A_0C_T/2-\nu} u_\nu(x)\|_{L^2}^2 + \int_0^t \|\langle D_x\rangle^{l+1+A_0C_T/2} f(s, x)\|_{L^2}^2 \, ds\right\} \tag{3.30}$$

for $l \in \mathbf{R}$, $t \in [0, T]$ and $\varepsilon \in E_0$.

Lemma 3.2. *For* $\varepsilon \in E_0$ *we have* $u_\varepsilon(t,x) \in C^\infty([0,\infty); H^\infty(\mathbf{R}^n))$. *Fix* $T > 0$. *Then, for any* $k \in \mathbf{Z}_+$ *with* $k \geq m-1$ *there is* $C_k > 0$ *such that*

$$E_{k,l}[u_\varepsilon](t) \leq C_k \Biggl\{ \sum_{\nu=0}^{m-1} \|\langle D_x \rangle^{l+k+1+A_0 C_T/2-\nu} u_\nu(x)\|_{L^2}^2 + \int_0^t \|\langle D_x \rangle^{l+k+2-m+A_0 C_T/2} f(s,x)\|_{L^2}^2 \, ds + \sum_{\mu=0}^{k-m} \|\langle D_x \rangle^{l+k-m-\mu} D_t^\mu f(t,x)\|_{L^2}^2 \Biggr\} \tag{3.31}$$

for $l \in \mathbf{R}$, $t \in [0,T]$ *and* $\varepsilon \in E_0$, *where* $\sum_{\mu=0}^{k-m} \cdots = 0$ *if* $k = m-1$.

Proof. By (3.30) (3.31) with $k = m-1$ is valid. Let $K \geq m$, and assume that $u_\varepsilon(t,x) \in C^{K-1}([0,T]; H^\infty(\mathbf{R}^n))$ and that (3.31) is valid if $(m-1 \leq) k \leq K-1$. Write

$$P_\varepsilon(t,\tau,\xi) = \tau^m + \sum_{j=1}^m a_{j,\varepsilon}(t,\xi)\tau^{m-j}.$$

Then we have

$$D_t^K v_\varepsilon(t,\xi) = -\sum_{j=1}^m \sum_{\mu=0}^{K-m} \binom{K-m}{\mu} (D_t^{K-m-\mu} a_{j,\varepsilon}(t,\xi)) D_t^{m-j+\mu} v_\varepsilon(t,\xi) + D_t^{K-m} \hat{f}(t,\xi). \tag{3.32}$$

Since the right-hand side of (3.32) belongs to $C([0,T]; \mathscr{F}^{-1}(H^\infty(\mathbf{R}^n)))$, we have $u_\varepsilon \in C^K([0,T]; H^\infty(\mathbf{R}^n))$. Moreover, we have

$$\|\langle D_x \rangle^l D_t^K u_\varepsilon(t,x)\|_{L^2(\mathbf{R}_x^n)}^2 \leq C_K \Biggl\{ \sum_{\mu=0}^{K-1} \|\langle D_x \rangle^{l+K-\mu} D_t^\mu u_\varepsilon(t,x)\|_{L^2(\mathbf{R}_x^n)}^2 + \|\langle D_x \rangle^l D_t^{K-m} f(t,x)\|_{L^2(\mathbf{R}_x^n)}^2 \Biggr\}$$

for $l \in \mathbf{R}$, $t \in [0,T]$ and $\varepsilon \in E_0$. Therefore, (3.31) is valid for $k = K$. □

Put $u(t,x) = u_0(t,x)$ and $u_j(t,x) = u_{\varepsilon_j}(t,x)$. Applying the same argument as in §3 of [16], we can prove that

$$D_t^k D_x^\alpha u_j(t,x) \to D_t^k D_x^\alpha u(t,x) \quad \text{uniformly in } [0,T] \times \mathbf{R}^n \text{ as } j \to \infty.$$

Denote by $K^{\pm}_{j,(t_0,x^0)}$ the generalized flows for $p_{\varepsilon_j}(t,\tau,\xi)$. Then it follows from § 3 of [13] (or [15]) that for any $(t_0,x^0) \in (0,\infty)\times\mathbf{R}^n$ and any neighborhood V of $K^-_{(t_0,x^0)} \cap \{t\geq 0\}$ there is $J\in\mathbf{N}$ such that

$$K^-_{j,(t_0,x^0)} \cap \{t\geq 0\} \subset V \quad \text{if} \ \ j\geq J.$$

Since $P_{\varepsilon_j}(t,D_t,D_x)$ is strictly hyperbolic with respect to ϑ, we can show that $(t_0,x^0)\notin \operatorname{supp} w$ if $j\in\mathbf{N}$, $(t_0,x^0)\in(0,\infty)\times\mathbf{R}^n$, $w(t,x)\in C^\infty([0,\infty)\times\mathbf{R}^n)$, $\operatorname{supp} P_{\varepsilon_j}(t,D_t,D_x)w(t,x)\cap K^-_{j,(t_0,x^0)}\cap\{t\geq 0\}=\varnothing$ and $\{0\}\times(\bigcup_{k=0}^{m-1}\operatorname{supp}(D_t^k w)(0,x))\cap K^-_{j,(t_0,x^0)}=\varnothing$ (see, *e.g.*, [9]). So we can repeat the same arguments as in the end of § 3 of [16] and prove Theorem 1.2.

4. Proof of Theorem 1.3

In this section we assume that the hypotheses of Theorem 1.3 are fulfilled and we shall prove Theorem 1.3. We shall change the definitions of E_0, $\mathcal{N}$, W_0, W_1, Λ and $\mathscr{E}(t,\xi;\varepsilon;A)$. Let $\{\varepsilon_j\}_{j=1,2,\ldots}$ be a sequence satisfying $\varepsilon_j\in(0,1]\backslash(\mathcal{N}_2^0(p)\cup\mathcal{N}_1^0(p^{(1)}))$ and $\varepsilon_j\downarrow 0$ as $j\to\infty$, where $\mathcal{N}_2^0(p)$ and $\mathcal{N}_1^0(p^{(1)}$ are as in Lemma 2.3. Put

$$E_0=\{\varepsilon_j;j=1,2,\ldots\}\cup\{0\},$$

$$\mathcal{N}=\bigcup_{j=1}^{\infty}(\mathcal{N}_2(\varepsilon_j;p)\cup\mathcal{N}_1(\varepsilon_j;p^{(1)}))\cup\mathcal{N}_2(0;p)\cup\mathcal{N}_1(0;p^{(1)})\cup\{0\},$$

where $\mathcal{N}_2(\varepsilon;p)$ and $\mathcal{N}_1(\varepsilon;p^{(1)})$ are as in Lemma 2.3. We note that $\mu_n(\mathcal{N})=0$. Consider the Cauchy problem $(\mathrm{CP})_\varepsilon$ and (3.1) with $m=3$. Fix $T>0$. Define

$$\begin{aligned}
W_0(t,\xi;\varepsilon) &= \sum_{s\in\mathscr{R}(\xi/|\xi|)\cap[0,T+1]}\langle\xi\rangle^{2/3}/\sqrt{(t-s)^2\langle\xi\rangle^{4/3}+1}\\
&\quad+\sum_{1\leq j<k\leq 3}\sqrt{(\partial_t(\lambda_j(t,\xi;\varepsilon)-\lambda_k(t,\xi;\varepsilon)))^2+1}\\
&\quad\times 1/\sqrt{(\lambda_j(t,\xi;\varepsilon)-\lambda_k(t,\xi;\varepsilon))^2+1}+1,\\
W_1(t,\xi;\varepsilon) &= \sum_{1\leq j<k\leq 3}|\partial_t^2(\lambda_j(t,\xi;\varepsilon)-\lambda_k(t,\xi;\varepsilon))|/(|\partial_t(\lambda_j(t,\xi;\varepsilon)-\lambda_k(t,\xi;\varepsilon))|+1)\\
&\quad+|\partial_t(\lambda_2^{(1)}(t,\xi;\varepsilon)-\lambda_1^{(1)}(t,\xi;\varepsilon))|/(|\lambda_2^{(1)}(t,\xi;\varepsilon)-\lambda_1^{(1)}(t,\xi;\varepsilon)|+1),\\
\Lambda(t,\xi;\varepsilon) &= \int_0^t(W_0(s,\xi;\varepsilon)+W_1(s,\xi;\varepsilon))\,ds
\end{aligned}$$

for $(t, \xi, \varepsilon) \in [0, T] \times (\mathbf{R}^n \backslash \mathcal{N}) \times E_0$. It is easy to see that

$$
\begin{aligned}
(4.1) \qquad |\partial_t W_0(t, \xi; \varepsilon)| &\leq W_0(t, \xi; \varepsilon)(W_0(t, \xi; \varepsilon) + \sqrt{2} W_1(t, \xi; \varepsilon)) \\
&\leq 3 W_0(t, \xi; \varepsilon) \partial_t \Lambda(t, \xi; \varepsilon)
\end{aligned}
$$

for $(t, \xi, \varepsilon) \in [0, T] \times (\mathbf{R}^n \backslash \mathcal{N}) \times E_0$. By Lemmas 2.3 and 2.4 there is $C_T > 0$ satisfying

$$
0 \leq \Lambda(t, \xi; \varepsilon) \leq C_T(\log\langle\xi\rangle + 1)
$$

for $(t, \xi, \varepsilon) \in [0, T] \times (\mathbf{R}^n \backslash \mathcal{N}) \times E_0$, since

$$
\partial_t \log\left(\sqrt{(t-s)^2 \langle\xi\rangle^{4/3} + 1} + (t-s)\langle\xi\rangle^{2/3}\right) = \langle\xi\rangle^{2/3} / \sqrt{(t-s)^2 \langle\xi\rangle^{4/3} + 1}.
$$

For $(t, \xi, \varepsilon) \in [0, T] \times (\mathbf{R}^n \backslash \mathcal{N}) \times E_0$ and $A \geq 1$ we define

$$
\mathscr{E}(t, \xi; \varepsilon; A) = \sum_{j=1}^{3} e^{-A\Lambda} |\mathscr{P}_j v_\varepsilon|^2 + \sum_{j=1}^{2} W_0^2 e^{-A\Lambda} |p_j^{(1)} v_\varepsilon|^2 + W_0^4 e^{-A\Lambda} |v_\varepsilon|^2.
$$

Here we write $\Lambda = \Lambda(t, \xi; \varepsilon)$, $\mathscr{P}_j = \mathscr{P}_j(t, D_t, \xi; \varepsilon)$, $p_j^{(1)} = p_j^{(1)}(t, D_t, \xi; \varepsilon)$, $v_\varepsilon = v_\varepsilon(t, \xi)$ and $W_0 = W_0(t, \xi; \varepsilon)$. Let $(t, \xi) \in [0, T] \times (\mathbf{R}^n \backslash \mathcal{N})$ and $\varepsilon \in E_0$. It is obvious that

$$
\begin{aligned}
(4.2) \quad D_t \mathscr{E}(t, \xi; \varepsilon; A) = i &\sum_{j=1}^{3} [A \Lambda_t e^{-A\Lambda} |\mathscr{P}_j v_\varepsilon|^2 + 2 \operatorname{Im}\{e^{-A\Lambda}(D_t \mathscr{P}_j v_\varepsilon) \cdot (\overline{\mathscr{P}_j v_\varepsilon})\}] \\
&+ i \sum_{j=1}^{2} [(A W_0^2 \Lambda_t - 2 W_0 W_{0t}) e^{-A\Lambda} |p_j^{(1)} v_\varepsilon|^2 \\
&\qquad + 2 \operatorname{Im}\{W_0^2 e^{-A\Lambda}(D_t p_j^{(1)} v_\varepsilon) \cdot (\overline{p_j^{(1)} v_\varepsilon})\}] \\
&+ i[(A W_0^4 \Lambda_t - 4 W_0^3 W_{0t}) e^{-A\Lambda} |v_\varepsilon|^2 \\
&\qquad + 2 \operatorname{Im}\{W_0^4 e^{-A\Lambda}(D_t v_\varepsilon) \cdot \overline{v_\varepsilon}\}],
\end{aligned}
$$

where $\Lambda_t = \partial_t \Lambda(t, \xi; \varepsilon)$ and $W_{0t} = \partial_t W_0(t, \xi; \varepsilon)$. Since the $\lambda_j(t, \xi; \varepsilon)$ and the $\lambda_j^{(1)}(t, \xi; \varepsilon)$ are real-valued, $\partial_t \partial_\tau p_\varepsilon(t, \tau, \xi) = \partial_t \partial_\tau p(t, \tau, \xi)$ and $\partial_t^2 \partial_\tau^2 p_\varepsilon(t, \tau, \xi) = \partial_t^2 \partial_\tau^2 p(t, \tau, \xi)$, it follows from (2.20) and (2.22) that

$$
\begin{aligned}
(4.3) \qquad &\operatorname{Im}\{e^{-A\Lambda}(D_t \mathscr{P}_j v_\varepsilon) \cdot (\overline{\mathscr{P}_j v_\varepsilon})\} \\
&= \operatorname{Im}\{e^{-A\Lambda}((D_t - \lambda_j) \mathscr{P}_j v_\varepsilon) \cdot (\overline{\mathscr{P}_j v_\varepsilon})\} \\
&= \operatorname{Im}\{e^{-A\Lambda}((p_\varepsilon - (i/2)(\partial_t \partial_\tau p)(t, D_t, \xi)) v_\varepsilon) \cdot (\overline{\mathscr{P}_j v_\varepsilon})\}
\end{aligned}
$$

$$- \operatorname{Im}\{e^{-A\Lambda}((\partial_t^2 \partial_\tau^2 p)(t, D_t, \xi) v_\varepsilon) \cdot (\overline{\mathscr{P}_j v_\varepsilon})\}/6$$

$$- \operatorname{Re}\left\{ e^{-A\Lambda} \sum_{k \neq j} (\lambda_{jt} - \lambda_{kt})(p_{j,k} v_\varepsilon) \cdot (\overline{\mathscr{P}_j v_\varepsilon}) \right\} \Big/ 2$$

$$+ \operatorname{Im}\left\{ e^{-A\Lambda} \sum_{k \neq j} (\lambda_{jtt} - \lambda_{ktt}) v_\varepsilon \cdot (\overline{\mathscr{P}_j v_\varepsilon}) \right\} \Big/ 6,$$

$$\text{(4.4)} \qquad \operatorname{Im}\{ W_0^2 e^{-A\Lambda} (D_t p_j^{(1)} v_\varepsilon) \cdot (\overline{p_j^{(1)} v_\varepsilon}) \}$$

$$= \operatorname{Im}\{ W_0^2 e^{-A\Lambda} ((D_t - \lambda_j^{(1)}) p_j^{(1)} v_\varepsilon) \cdot (\overline{p_j^{(1)} v_\varepsilon}) \}$$

$$= \sum_{k=1}^{3} \operatorname{Im}\{ W_0^2 e^{-A\Lambda} (\mathscr{P}_k v_\varepsilon) \cdot (\overline{p_j^{(1)} v_\varepsilon}) \}$$

$$+ \operatorname{Re}\{ (-1)^j W_0^2 e^{-A\Lambda} (\lambda_{2t}^{(1)} - \lambda_{1t}^{(1)}) v_\varepsilon \cdot (\overline{p_j^{(1)} v_\varepsilon}) \}/2,$$

$$\text{(4.5)} \qquad \operatorname{Im}\{ W_0^4 e^{-A\Lambda} (D_t v_\varepsilon) \cdot \overline{v_\varepsilon} \} = \operatorname{Im}\{ W_0^4 e^{-A\Lambda} (p_1^{(1)} v_\varepsilon) \cdot \overline{v_\varepsilon} \}.$$

Here we also write $\lambda_{jt} = \partial_t \lambda_j(t, \xi; \varepsilon)$, $\lambda_{jtt} = \partial_t^2 \lambda_j(t, \xi; \varepsilon)$ and so forth. (3.1) and (4.1)–(4.5) yield

$$\text{(4.6)} \quad \partial_t \mathscr{E}(t, \xi; \varepsilon; A)$$

$$\leq 3\Lambda_t^{-1} e^{-A\Lambda} |\hat{f}(t, \xi)|^2$$

$$- \sum_{j=1}^{3} \Bigg[(A - 7) \Lambda_t e^{-A\Lambda} |\mathscr{P}_j v_\varepsilon|^2 - \Lambda_t^{-1} e^{-A\Lambda}$$

$$\times \Bigg\{ |(sub\ \sigma(P)(t, D_t, \xi) + P_1(t, D_t, \xi) + \partial_t^2 \partial_\tau^2 p(t, D_t, \xi)/6) v_\varepsilon|^2$$

$$+ |P_0(t) v_\varepsilon|^2 + \sum_{k \neq j} |(\lambda_{jt} - \lambda_{kt}) p_{j,k} v_\varepsilon|^2/4 + \sum_{k \neq j} |(\lambda_{jtt} - \lambda_{ktt}) v_\varepsilon|^2/36 \Bigg\} \Bigg]$$

$$- \sum_{j=1}^{3} \Bigg[(A - 10) W_0^2 \Lambda_t e^{-A\Lambda} |p_j^{(1)} v_\varepsilon|^2$$

$$- W_0^2 \Lambda_t^{-1} e^{-A\Lambda} \left\{ \sum_{k=1}^{3} |\mathscr{P}_k v_\varepsilon|^2 + |(\lambda_{2t}^{(1)} - \lambda_{1t}^{(1)}) v_\varepsilon|^2/4 \right\} \Bigg]$$

$$- (A - 13) W_0^4 \Lambda_t e^{-A\Lambda} |v_\varepsilon|^2 + W_0^4 \Lambda_t^{-1} |p_1^{(1)} v_\varepsilon|^2.$$

From Lemma 2.6 we can write

$$\text{(4.7)} \qquad sub\ \sigma(P)(t,\tau,\xi) = \sum_{j=1}^{3} b_{1,j}(t,\xi;\varepsilon)p_j(t,\tau,\xi;\varepsilon),$$

$$\min\left\{\min_{s\in\mathscr{R}(\xi/|\xi|)}|t-s|,1\right\}|b_{1,j}(t,\xi;\varepsilon)| \le C_T \quad (1\le j\le 3)$$

for $\xi \in \mathbf{R}^n\backslash\{0\}$, $t\in[0,T]\backslash\mathscr{R}(\xi/|\xi|)$ and $\varepsilon\in\mathbf{R}$, where $C_T>0$. Operating ∂_τ^2 in both sides of (4.7), we have

$$\text{(4.8)} \qquad \partial_\tau^2 P_2(t,\tau,\xi) = 2\sum_{j=1}^{3} b_{1,j}(t,\xi;\varepsilon).$$

Since

$$\partial_t\partial_\tau p_j(t,\tau,\xi;\varepsilon) = -\sum_{k\neq j}\lambda_{kt}(t,\xi;\varepsilon),$$

$$\partial_t\partial_\tau^2 p(t,\tau,\xi;\varepsilon) = -2\sum_{k=1}^{3}\lambda_{kt}(t,\xi;\varepsilon),$$

$$\partial_t\partial_\tau p_j(t,\tau,\xi;\varepsilon) - \partial_t\partial_\tau^2 p(t,\tau,\xi;\varepsilon)/3 = -\sum_{k\neq j}(\lambda_{kt}(t,\xi;\varepsilon)-\lambda_{jt}(t,\xi;\varepsilon))/3,$$

(1.1), (2.18), (4.8) and Lemma 2.6 give

$$\begin{aligned}
\text{(4.9)} \qquad & sub\ \sigma(P)(t,\tau,\xi) + P_1(t,\tau,\xi) + \partial_t^2\partial_\tau^2 p(t,\tau,\xi)/6 \\
&= \sum_{j=1}^{3} b_{1,j}(t,\xi;\varepsilon)\mathscr{P}_j(t,\tau,\xi;\varepsilon) + P_1(t,\tau,\xi) + \partial_t^2\partial_\tau^2 p(t,\tau,\xi)/6 \\
&\quad - (i/6)\sum_{j=1}^{3}\sum_{k\neq j} b_{1,j}(t,\xi;\varepsilon)(\lambda_{kt}(t,\xi;\varepsilon)-\lambda_{jt}(t,\xi;\varepsilon)) \\
&\quad + (i/6)\sum_{j=1}^{3} b_{1,j}(t,\xi;\varepsilon)\partial_t\partial_\tau^2 p(t,\tau,\xi) \\
&= \sum_{j=1}^{3} b_{1,j}(t,\xi;\varepsilon)\mathscr{P}_j(t,\tau,\xi;\varepsilon) + sub^2\ \sigma(P)(t,\tau,\xi) \\
&\quad - (i/6)\sum_{j=1}^{3}\sum_{k\neq j} b_{1,j}(t,\xi;\varepsilon)(\lambda_{kt}(t,\xi;\varepsilon)-\lambda_{jt}(t,\xi;\varepsilon))
\end{aligned}$$

$$= \sum_{j=1}^{3} b_{1,j}(t,\xi;\varepsilon)\mathscr{P}_j(t,\tau,\xi;\varepsilon) + \sum_{j=1}^{2} b_{2,j}(t,\xi;\varepsilon)p_j^{(1)}(t,\tau,\xi;\varepsilon)$$

$$- (i/6)\sum_{j=1}^{3}\sum_{k\neq j} b_{1,j}(t,\xi;\varepsilon)(\lambda_{kt}(t,\xi;\varepsilon) - \lambda_{jt}(t,\xi;\varepsilon)),$$

where the $b_{2,j}(t,\xi;\varepsilon)$ satisfy (2.15). If $\{j,k,l\} = \{1,2,3\}$, then we have

(4.10) $$|(\lambda_j - \lambda_k)v_\varepsilon|^2 = |(p_{j,l} - p_{k,l})v_\varepsilon|^2 \le 2|p_{j,l}v_\varepsilon|^2 + 2|p_{k,l}v_\varepsilon|^2.$$

It follows from (2.10) with $m = 3$, $k = 2$ and $r = 1$ and Lemma 2.5 that there are $b_{j,k,\mu}(t,\xi;\varepsilon)$ $(\mu = 1,2)$ satisfying

$$p_{j,k}(t,\tau,\xi;\varepsilon) = \sum_{\mu=1}^{2} b_{j,k,\mu}(t,\xi;\varepsilon)p_\mu^{(1)}(t,\tau,\xi;\varepsilon),$$

$$|b_{j,k,\mu}(t,\xi;\varepsilon)| \le 1/\sqrt{2} \quad (\mu = 1,2),$$

since

$$h_1(t,\tau,\xi;p_\varepsilon) = \sum_{1\le j<k\le 3} p_{j,k}(t,\tau,\xi;\varepsilon)^2,$$

$$h_1(t,\tau,\xi;p_\varepsilon^{(1)}) = \sum_{\mu=1}^{2} p_\mu^{(1)}(t,\tau,\xi;\varepsilon)^2.$$

Therefore, we have

(4.11) $$|p_{j,k}v_\varepsilon|^2 \le \sum_{\mu=1}^{2} |p_\mu^{(1)}v_\varepsilon|^2 \quad (1 \le j < k \le 3).$$

It is obvious that

(4.12) $$|(\lambda_2^{(1)} - \lambda_1^{(1)})v_\varepsilon|^2 \le 2\sum_{\mu=1}^{2} |p_\mu^{(1)}v_\varepsilon|^2.$$

Let us estimate $\Lambda_t^{-1}e^{-A\Lambda}|(sub\ \sigma(P)(t,D_t,\xi) + P_1(t,D_t,\xi) + (\partial_t^2\partial_\tau^2 p)(t,D_t,\xi)/6)v_\varepsilon|^2$. First assume that $\min\{\min_{s\in\mathscr{R}(\xi/|\xi|)}|t-s|, 1\} \le \langle\xi\rangle^{-2/3}$. Then we have

$$W_0(t,\xi;\varepsilon) \ge \langle\xi\rangle^{2/3}/\sqrt{2}.$$

Therefore, we have

(4.13) $$\Lambda_t^{-1}e^{-A\Lambda}|(sub\ \sigma(P)(t,D_t,\xi) + P_1(t,D_t,\xi) + (\partial_t^2\partial_\tau^2 p)(t,D_t,\xi)/6)v_\varepsilon|^2$$

$$\le C\Lambda_t e^{-A\Lambda}\{|\mathscr{P}_j v_\varepsilon|^2 + W_0^2|p_1^{(1)}v_\varepsilon|^2 + W_0^4|v_\varepsilon|^2\}$$

for $t \in [0, T]$, $\xi \in \mathbf{R}^n \backslash \mathcal{N}$ and $\varepsilon \in E_0$, since

$$sub\ \sigma(P)(t,\tau,\xi) + P_1(t,\tau,\xi) + \partial_t^2\partial_\tau^2 p(t,\tau,\xi)/6$$
$$= c_0(t)\mathscr{P}_j(t,\tau,\xi;\varepsilon) + c_{j,1}(t,\xi;\varepsilon)p_1^{(1)}(t,\tau,\xi;\varepsilon) + c_{j,2}(t,\xi;\varepsilon),$$
$$|c_0(t)| \le C, \quad |c_{j,k}(t,\xi;\varepsilon)| \le C\langle\xi\rangle^k \quad (k = 1,2)$$

for $t \in [0, T]$, $\xi \in \mathbf{R}^n \backslash \mathcal{N}$ and $\varepsilon \in E_0$. Next assume that $\min\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|, 1\} \ge \langle\xi\rangle^{-2/3}$. Then we have

$$W_0(t,\xi;\varepsilon) \ge \left(\sqrt{2}\min\left\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|, 1\right\}\right)^{-1}.$$

This, together with (4.9)–(4.11), yields

$$(4.14)\qquad \Lambda_t^{-1}e^{-A\Lambda}|(sub\ \sigma(P)(t,D_t,\xi) + P_1(t,D_t,\xi) + (\partial_t^2\partial_\tau^2 p)(t,D_t,\xi)/6)v_\varepsilon|^2$$
$$\le CW_0^2\Lambda_t^{-1}e^{-A\Lambda}\left\{\sum_{j=1}^3 |\mathscr{P}_j v_\varepsilon|^2 + W_0^2\sum_{j=1}^2 |p_j^{(1)}v_\varepsilon|^2 + \sum_{1\le j<k\le 3} |\lambda_{jt}-\lambda_{kt}|^2|v_\varepsilon|^2\right\}$$
$$\le C'\Lambda_t e^{-A\Lambda}\left\{\sum_{j=1}^3 |\mathscr{P}_j v_\varepsilon|^2 + W_0^2\sum_{j=1}^2 |p_j^{(1)}v_\varepsilon|^2 + W_0^4|v_\varepsilon|^2\right\}$$

for $t \in [0, T]$, $\xi \in \mathbf{R}^n \backslash \mathcal{N}$ and $\varepsilon \in E_0$, since

$$|\lambda_{jt}(t,\xi;\varepsilon) - \lambda_{kt}(t,\xi;\varepsilon)|^2 \le W_0(t,\xi;\varepsilon)^2((\lambda_j(t,\xi;\varepsilon) - \lambda_k(t,\xi;\varepsilon))^2 + 1).$$

By (4.13) (4.14) is also valid in the case where $\min\{\min_{s \in \mathscr{R}(\xi/|\xi|)} |t-s|, 1\} \le \langle\xi\rangle^{-2/3}$. We can easily show that

$$(4.15)\qquad \Lambda_t^{-1}e^{-A\Lambda}|P_0(t)v_\varepsilon|^2 \le CW_0^4\Lambda_t e^{-A\Lambda}|v_\varepsilon|^2,$$

$$(4.16)\qquad \Lambda_t^{-1}e^{-A\Lambda}\sum_{k\ne j}|(\lambda_{jt}-\lambda_{kt})p_{j,k}v_\varepsilon|^2$$
$$\le \Lambda_t^{-1}e^{-A\Lambda}\sum_{k\ne j}(W_0^2|(\lambda_j-\lambda_k)p_{j,k}v_\varepsilon|^2 + |p_{j,k}v_\varepsilon|^2)$$
$$\le C'\Lambda_t e^{-A\Lambda}\left\{\sum_{k=1}^3 |\mathscr{P}_k v_\varepsilon|^2 + W_0^2\sum_{k=1}^2 |p_k^{(1)}v_\varepsilon|^2\right\} \quad (1 \le j \le 3),$$

$$(4.17)\qquad \Lambda_t^{-1}e^{-A\Lambda}\sum_{k\neq j}|(\lambda_{jtt}-\lambda_{ktt})v_\varepsilon|^2$$
$$\leq \Lambda_t^{-1}e^{-A\Lambda}\sum_{k\neq j}(W_0^2W_1^2|(\lambda_j-\lambda_k)v_\varepsilon|^2+|v_\varepsilon|^2)$$
$$\leq CW_0^2\Lambda_t e^{-A\Lambda}\left\{\sum_{k=1}^2|p_k^{(1)}v_\varepsilon|^2+W_0^2|v_\varepsilon|^2\right\}\quad (1\leq j\leq 3),$$

$$(4.18)\qquad W_0^2\Lambda_t^{-1}e^{-A\Lambda}|(\lambda_{2t}^{(1)}-\lambda_{1t}^{(1)})v_\varepsilon|^2$$
$$\leq W_0^2W_1^2\Lambda_t^{-1}e^{-A\Lambda}(|(\lambda_2^{(1)}-\lambda_1^{(1)})v_\varepsilon|^2+|v_\varepsilon|^2)$$
$$\leq 2W_0^2\Lambda_t e^{-A\Lambda}\left\{\sum_{k=1}^2|p_k^{(1)}v_\varepsilon|^2+W_0^2|v_\varepsilon|^2\right\}$$

for $t\in[0,T]$, $\xi\in\mathbf{R}^n\backslash\mathcal{N}$ and $\varepsilon\in E_0$. Indeed, (4.15) is obvious. (4.16) follows from (3.12) and (4.11). In (4.17) we use (4.10) and (4.11). (4.18) follows from (4.12). So it follows from (4.6) and (4.14)–(4.18) that there is $A_0\geq 1$ satisfying

$$\partial_t\mathscr{E}(t,\tau,\xi;\varepsilon;A)\leq 3\Lambda_t^{-1}e^{-A\Lambda}|\hat{f}(t,\xi)|^2$$

for $A\geq A_0$, $t\in[0,T]$, $\xi\in\mathbf{R}^n\backslash\mathcal{N}$ and $\varepsilon\in E_0$. Therefore, repeating the same arguments as in §3, we can prove Theorem 1.3.

5. Some Remarks and Examples

Let us first consider the validity of the condition (T-L). Let $P(t,\tau,\xi)=(\tau-\lambda(t,\xi))^3+b_2(t,\tau,\xi)+b_1(t,\tau,\xi)+b_0(t)$, where

$$(5.1)\qquad \begin{cases}\lambda(t,\xi)=\sum_{j=1}^n\lambda_j(t)\xi_j,\\ b_2(t,\tau,\xi)=b_{2,0}(t)\tau^2+\sum_{j=1}^n b_{2,j}(t)\tau\xi_j+\sum_{|\alpha|=2}b_{2,\alpha}(t)\xi^\alpha,\\ b_1(t,\tau,\xi)=b_{1,0}(t)\tau+\sum_{j=1}^n b_{1,j}(t)\xi_j.\end{cases}$$

We assume that the $\lambda_j(t)$ are real-valued and that the $\lambda_j(t)$, the $b_{j,k}(t)$ and $b_0(t)$ belong to $C^\infty([0,\infty))$. It is well-known that the Cauchy problem for $P(t,D_t,D_x)$ is C^∞ well-posed if and only if $P(t,D_t,D_x)$ can be represented as follows:

$$P(t,D_t,D_x)=(D_t-\lambda(t,D_x))^3+\sum_{j=0}^2 c_j(t)(D_t-\lambda(t,D_x))^j,$$

where $c_j(t)\in C^\infty([0,\infty))$ $(0\leq j\leq 2)$ (see, *e.g.*, [4] and [18]).

The following theorem insists that the condition (T-L) is a reasonable and likely condition for C^∞ well-posedness.

Theorem 5.1. *The Cauchy problem for* $P(t, D_t, D_x)$ *is* C^∞ *well-posed if and only if the condition* (T-L) *with* $\mathscr{R}(\xi) = \emptyset$ *is satisfied.*

Proof. It is obvious that

$$(\tau - \lambda(t,\xi)) \circ (\tau - \lambda(t,\xi)) = (\tau - \lambda(t,\xi))^2 + i\partial_t\lambda(t,\xi),$$

$$(\tau - \lambda(t,\xi)) \circ (\tau - \lambda(t,\xi)) \circ (\tau - \lambda(t,\xi))$$

$$= (\tau - \lambda(t,\xi))^3 + 3i\partial_t\lambda(t,\xi) \cdot (\tau - \lambda(t,\xi)) + \partial_t^2\lambda(t,\xi).$$

Therefore, we have

$$P(t, D_t, D_x) = (D_t - \lambda(t, D_x))^3 + b_{2,0}(t)(D_t - \lambda(t, D_x))^2$$
$$+ b_{1,0}(t)(D_t - \lambda(t, D_x)) + \tilde{b}_2(t, D_t, D_x) + \tilde{b}_1(t, D_t, D_x) + b_0(t),$$

where

(5.2) $$\tilde{b}_2(t,\tau,\xi) = b_2(t,\tau,\xi) - b_{2,0}(t)(\tau - \lambda(t,\xi))^2 - 3i\partial_t\lambda(t,\xi) \cdot (\tau - \lambda(t,\xi)),$$

(5.3) $$\tilde{b}_1(t,\tau,\xi) = b_1(t,\tau,\xi) - b_{1,0}(t)(\tau - \lambda(t,\xi)) - ib_{2,0}(t)\partial_t\lambda(t,\xi) - \partial_t^2\lambda(t,\xi).$$

On the other hand, we have

$$h_2(t,\tau,\xi)^{1/2} = \sqrt{3}(\tau - \lambda(t,\xi))^2, \quad h_1(t,\tau,\xi)^{1/2} = \sqrt{3}|\tau - \lambda(t,\xi)|,$$

$$sub\, \sigma(P)(t,\tau,\xi) = b_2(t,\tau,\xi) - 3i\partial_t\lambda(t,\xi) \cdot (\tau - \lambda(t,\xi)),$$

$$sub^2\, \sigma(P)(t,\tau,\xi) = b_1(t,\tau,\xi) - \partial_t^2\lambda(t,\xi) - ib_{2,0}(t)\partial_t\lambda(t,\xi).$$

This, together with (5.2) and (5.3), shows that

$$\tilde{b}_2(t,\tau,\xi) \equiv \tilde{b}_1(t,\tau,\xi) \equiv 0$$

if and only if the condition (T-L) is satisfied, since (T-L) implies that $sub\, \sigma(P)(t,\tau,\xi)$ and $sub^2\, \sigma(P)(t,\tau,\xi)$ are divided by $(\tau - \lambda(t,\xi))^2$ and $(\tau - \lambda(t,\xi))$, respectively. □

Finally we shall give two simple examples.

Example 5.2. Let $P(t,\tau,\xi) = (\tau^2 - a(t)|\xi|^2)\tau + b_2(t,\tau,\xi) + b_1(t,\tau,\xi) + b_0(t)$, where $b_j(t,\tau,\xi)$ $(j = 1,2)$ and $b_0(t)$ are as in (5.1) and $a(t)$ is real analytic in a neighborhood of $[0,\infty)$. Assume that $a(t) \geq 0$ for $t \in [0,\infty)$. Then the conditions (A-1), (A-2) and (T) are satisfied. A simple calculation yields

$$h_2(t,\tau,\xi) = 3\tau^4 + a(t)^2|\xi|^4,$$

$$h_1(t,\tau,\xi) = 3\tau^2 + 2a(t)|\xi|^2,$$

$$sub\,\sigma(P)(t,\tau,\xi) = b_2(t,\tau,\xi) - (i/2)\partial_t a(t)\cdot|\xi|^2,$$

$$sub^2\,\sigma(P)(t,\tau,\xi) = b_1(t,\tau,\xi).$$

Therefore, we have

$$(\tau^2 + \sqrt{a(t)}|\tau|\,|\xi| + a(t)|\xi|^2)/5 \leq h_2(t,\tau,\xi)^{1/2} \leq 2(\tau^2 + \sqrt{a(t)}|\tau|\,|\xi| + a(t)|\xi|^2),$$

$$|\tau| + \sqrt{a(t)}|\xi| \leq h_1(t,\tau,\xi)^{1/2} \leq 2(|\tau| + \sqrt{a(t)}|\xi|).$$

Let $t_k \in [0,\infty)$ be a zero of $a(t)$ of order v_k $(k = 1,2,3,\ldots)$, where $0 \leq t_1 < t_2 < t_3 < \cdots$. Taking $\mathscr{R}(\xi) = \{t_1, t_2, t_3, \ldots\}$ we can see that the condition (T-L) is satisfied if and only if

$$b_{2,j}(t) = O((t - t_k)^{v_k/2-1}) \quad \text{as } t \to t_k \quad (1 \leq j \leq n),$$

$$b_{2,\alpha}(t) = O((t - t_k)^{v_k-1}) \quad \text{as } t \to t_k \quad (|\alpha| = 2),$$

$$b_{1,j}(t) = O((t - t_k)^{v_k/2-2}) \quad \text{as } t \to t_k \quad (1 \leq j \leq n)$$

for $k = 1,2,3,\ldots$.

Let S be a subset of $[0,\infty) \times S^{n-1}$, and define the condition $(\text{T-L})_S$ by replacing $[0,T] \times \mathbf{R} \times S^{n-1}$ with $\{(t,\tau,\xi) \in [0,T] \times \mathbf{R} \times S^{n-1}; (t,\xi) \in S\}$ in the condition (T-L).

Example 5.3. Let $P(t,\tau,\xi) = (\tau - \lambda(t,\xi))\tau^2 + b_2(t,\tau,\xi) + b_1(t,\tau,\xi) + b_0(t)$, where $\lambda(t,\xi) = t^{\kappa_1}\xi_1 + t^{\kappa_2}\xi_2$, $\kappa_1, \kappa_2 \in \mathbf{Z}_+$, $\kappa_1 \leq \kappa_2$ and $b_j(t,\tau,\xi)$ $(j = 1,2)$ and $b_0(t)$ are as in (5.1). A simple calculation yields

$$h_2(t,\tau,\xi) = \tau^2(\tau^2 + 2(\tau - \lambda(t,\xi))^2),$$

$$h_1(t,\tau,\xi) = 2\tau^2 + (\tau - \lambda(t,\xi))^2,$$

$$sub\ \sigma(P)(t,\tau,\xi) = b_2(t,\tau,\xi) - i\partial_t\lambda(t,\xi)\cdot\tau,$$

$$sub^2\ \sigma(P)(t,\tau,\xi) = b_1(t,\tau,\xi) - \partial_t^2\lambda(t,\xi)/3 - (i/3)b_{2,0}(t)\partial_t\lambda(t,\xi).$$

Therefore, we have

$$(\tau^2 + |\lambda(t,\xi)|\,|\tau|)/3 \le h_2(t,\tau,\xi)^{1/2} \le 3(\tau^2 + |\lambda(t,\xi)|\,|\tau|),$$

$$(|\tau| + |\lambda(t,\xi)|)/2 \le h_1(t,\tau,\xi)^{1/2} \le 2(|\tau| + |\lambda(t,\xi)|).$$

It is obvious that

$$(5.4) \quad b_{2,j}(t) \equiv b_{1,j}(t) \equiv 0 \quad (3 \le j \le n) \qquad \text{and} \qquad b_{2,\alpha}(t) \equiv 0 \quad (|\alpha| = 2)$$

if the condition (T-L) is satisfied. Let us first consider the case where $\kappa_1 = \kappa_2$. Then it is easy to see that the condition (T-L) with $\mathscr{R}(\xi) = \{0\}$ is satisfied if and only if

$$\begin{cases} b_{2,1}(t) = b_{2,2}(t) = O(t^{\kappa_1 - 1}) & \text{as } t \downarrow 0, \\ b_{1,1}(t) = b_{1,2}(t) = O(t^{\kappa_1 - 2}) & \text{as } t \downarrow 0 \end{cases}$$

and (5.4) is satisfied. Next consider the case where $\kappa_1 < \kappa_2$. Put

$$S_1 = \{(t,\xi) \in [0,\infty) \times S^{n-1}; \xi_1\xi_2 \ge 0\},$$

$$S_2 = \{(t,\xi) \in [0,\infty) \times S^{n-1}; \xi_1\xi_2 < 0 \text{ and } 0 \le t^\kappa \le |\xi_1/\xi_2|/2\},$$

$$S_3 = \{(t,\xi) \in [0,\infty) \times S^{n-1}; \xi_1\xi_2 < 0 \text{ and } t^\kappa \ge |\xi_1/\xi_2|/2\},$$

where $\kappa = \kappa_2 - \kappa_1$. If $(t,\xi) \in S_1$, then we have

$$|\lambda(t,\xi)| = t^{\kappa_1}|\xi_1| + t^{\kappa_2}|\xi_2|.$$

Therefore, the condition $(\text{T-L})_{S_1}$ with $\mathscr{R}(\xi) = \{0\}$ is satisfied if and only if

$$(5.5) \qquad b_{2,j}(t) = O(t^{\kappa_j - 1}) \quad \text{and} \quad b_{1,j}(t) = O(t^{\kappa_j - 2}) \quad \text{as } t \downarrow 0$$

for $j = 1, 2$ and (5.4) is satisfied. If $(t,\xi) \in S_2$ and $\mathscr{R}(\xi) = \{0\}$, then we have

$$(t^{\kappa_1}|\xi_1| + t^{\kappa_2}|\xi_2|)/4 \le |\lambda(t,\xi)| = t^{\kappa_1}|\xi_2|(|\xi_1/\xi_2| - t^\kappa)$$
$$\le t^{\kappa_1}|\xi_1| + t^{\kappa_2}|\xi_2|.$$

This implies that $(\text{T-L})_{S_2}$ with $\mathscr{R}(\xi) = \{0\}$ is satisfied if and only if (5.4) and (5.5) are satisfied, since

$$|\partial_t \lambda(t,\xi)| \le \kappa_2 (t^{\kappa_1 - 1}|\xi_1| + t^{\kappa_2 - 1}|\xi_2|),$$

$$|\partial_t^2 \lambda(t,\xi)| \le \kappa_2(\kappa_2 - 1)(t^{\kappa_1 - 2}|\xi_1| + t^{\kappa_2 - 2}|\xi_2|).$$

If $(t,\xi) \in S_3$ and $\mathscr{R}(\xi) = \{|\xi_1/\xi_2|^{1/\kappa}\}$, then we have

$$\min\left\{\min_{s \in \mathscr{R}(\xi)} |t - s|, 1\right\} = \min\{|t - |\xi_1/\xi_2|^{1/\kappa}|, 1\},$$

$$\min\left\{\min_{s \in \mathscr{R}(\xi)} |t - s|^2, 1\right\} = \min\{|t - |\xi_1/\xi_2|^{1/\kappa}|^2, 1\},$$

$$\begin{aligned}
&|t - |\xi_1/\xi_2|^{1/\kappa}|(t^{\kappa_1 - 1}|\xi_1| + t^{\kappa_2 - 1}|\xi_2|)/3 \\
&\quad \le |\lambda(t,\xi)| = t^{\kappa_1}|\xi_2|\,|t - |\xi_1/\xi_2|^{1/\kappa}|(t^{\kappa - 1} + |\xi_1/\xi_2|^{1/\kappa} t^{\kappa - 2} + \cdots + |\xi_1/\xi_2|^{1 - 1/\kappa}) \\
&\quad \le (2^{1/\kappa} - 1)^{-1} t^{\kappa_2 - 1}|\xi_2|\,|t - |\xi_1/\xi_2|^{1/\kappa}| \\
&\quad \le (2^{1/\kappa} - 1)^{-1}|t - |\xi_1/\xi_2|^{1/\kappa}|(t^{\kappa_1 - 1}|\xi_1| + t^{\kappa_2 - 1}|\xi_2|).
\end{aligned}$$

Therefore, the condition $(\text{T-L})_{S_3}$ with $\mathscr{R}(\xi) = \{|\xi_1/\xi_2|^{1/\kappa}\}$ is satisfied if and only if (5.4) and (5.5) are satisfied, since

$$|\partial_t \lambda(t,\xi)| \le \kappa_2 (t^{\kappa_1 - 1}|\xi_1| + t^{\kappa_2 - 1}|\xi_2|),$$

$$|t - |\xi_1/\xi_2|^{1/\kappa}|\,|\partial_t^2 \lambda(t,\xi)| \le t|\partial_t^2 \lambda(t,\xi)| \le \kappa_2(\kappa_2 - 1)(t^{\kappa_1 - 1}|\xi_1| + t^{\kappa_2 - 1}|\xi_2|).$$

Thus, taking $\mathscr{R}(\xi) = \begin{cases} \{0, |\xi_1/\xi_2|^{1/\kappa}\} & (\xi_2 \ne 0), \\ \{0\} & (\xi_2 = 0) \end{cases}$, we can show that the condition (T-L) is satisfied if and only if (5.4) and (5.5) are satisfied. Moreover, it is easy to see that (5.4) and (5.5) are satisfied if the condition (T-L) is satisfied for some $\mathscr{R}(\xi)$.

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