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A CHARACTERIZATION OF ISOPARAMETRIC HYPERSURFACES IN A SPHERE WITH $g \leq 3$

By

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Abstract. Ki and Nakagawa [Tôhoku Math. J., **39**, 27–40 (1987)] characterized the minimal isoparametric hypersurfaces in a sphere that have three distinct constant principal curvatures from the stand point of the Ricci tensor. In this paper we generalize their results and characterize the isoparametric hypersurfaces, again in a sphere and with three distinct constant principal curvatures, that are either minimal or parallel to a minimal hypersurface.

1. Introduction

Let $S^{n+1}(1)$ be a unit sphere in an $(n+2)$-dimensional Euclidean space E^{n+2}. We consider the induced Riemannian metric on $S^{n+1}(1)$. Then $S^{n+1}(1)$ comes to be a connected, complete and simply connected Riemannian manifold with constant sectional curvature 1. We call an n-dimensional Riemannian manifold M^n a hypersurface of $S^{n+1}(1)$ when M^n is isometrically immersed into $S^{n+1}(1)$. A hypersurface M^n of $S^{n+1}(1)$ is said to be an isoparametric hypersurface if M^n is locally defined by the level hypersurface of an isoparametric function on $S^{n+1}(1)$ ([3]).

E. Cartan [3] proved that a hypersurface M^n is isoparametric if and only if it is a hypersurface with constant principal curvatures.

Let M^n be an isoparametric hypersurface of $S^{n+1}(1)$ and let $\lambda_1, \dots, \lambda_g$ be all of the distinct constant principal curvatures with multiplicities $m_1, \dots, m_g$, respectively. E. Cartan [4] classified the isoparametric hypersurfaces with $g \leq 3$ and showed that all of them are homogeneous. Here we call M^n homogeneous if

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M^n is obtained as the orbit space of an analytic subgroup of the isometry group of $S^{n+1}(1)$. Later, Münzner [11] proved by a topological argument that the number of distinct principal curvatures g must be one of $g = 1, 2, 3, 4$ or 6. For $g = 6$, Abresch [1] showed that the multiplicity of each principal curvature is the same number m where m is either 1 or 2. Dorfmeister and Neher [6] proved the homogeneity of such hypersurfaces with $g = 6$ and $m = 1$. Recently, R. Miyaoka [12] finally proved that isoparametric hypersurfaces with $g = 6$ and $m = 2$ are also homogeneous. So, we know that isoparametric hypersurfaces in $S^{n+1}(1)$ are homogeneous for $g = 1, 2, 3$ or 6.

We are interested in geometric characterizations of homogeneous isoparametric hypersurfaces in a sphere. More precisely, we consider characterization of such hypersurfaces from the standpoint of the Ricci tensor. The Ricci tensor S of M^n is said to be cyclic parallel if the following equation holds for all vector fields X, Y and Z on M^n:

$$\langle(\nabla_X S)Y, Z\rangle + \langle(\nabla_Y S)Z, X\rangle + \langle(\nabla_Z S)X, Y\rangle = 0.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the Riemannian metric and ∇ the Levi-Civita connection of M^n.

A Riemannian manifold is said to be a D'Atri space if its local geodesic symmetries are volume-preserving up to sign. Such spaces were first studied by J. E. D'Atri and H. K. Nickerson ([5]). An analytic Riemannian manifold is knowen to be a D'Atri space if and only if it satisfies an infinite sequence of equations for the curvature tensor and its covariant derivatives ([2], [9]). The cyclic-parallel condition of the Ricci tensor is the first equation in the infinite sequence.

In this paper, we consider the cases where $g \leq 3$. Isoparametric hypersurfaces of $S^{n+1}(1)$ with $g = 3$ will be called Cartan hypersurfaces. Ki and Nakagawa [8] gave a characterization of minimal Cartan hypersurfaces and obtained the following theorem.

Theorem 1.1 (Ki and Nakagawa [8]). *Let M^n be a closed hypersurface in $S^{n+1}(1)$ with constant mean curvature. If the Ricci tensor S of M is cyclic-parallel but not parallel, then M^n is congruent to one of the minimal Cartan hypersurfaces.*

In this paper, we generalize this result. More precisely, we characterize all Cartan hypersurfaces in $S^{n+1}(1)$. This includes both minimal hypersurface and hypersurfaces parallel to them. We obtain the following theorem.

Theorem 1.2. *Let M^n be a closed hypersurface in $S^{n+1}(1)$ with constant mean curvature. If the covariant derivative of the Ricci tensor S of M^n satisfies*

$$(*)\quad \mathfrak{S}_{X,Y,Z}\langle(\nabla_X S)Y, Z\rangle$$
$$:= \langle(\nabla_X S)Y, Z\rangle + \langle(\nabla_Y S)Z, X\rangle + \langle(\nabla_Z S)X, Y\rangle$$
$$= 3\left(1 - \frac{2}{n}\right)h\langle(\nabla_X A)Y, Z\rangle$$

for all vector fields X, Y and Z over M^n, then M^n is congruent to one of the isoparametric hypersurfaces with $g \leq 3$. Here A and h denote the shape operator and the mean curvature of M^n, respectively. Further, if $\nabla S \neq 0$, then M^n is congruent to the Cartan hypersurface.

The corresponding local theorem also holds, which is reflected by the following theorem.

Theorem 1.3. *Let M^n be a connected hypersurface isometrically immersed in $S^{n+1}(1)$ with constant mean curvature. If the covariant derivative of the Ricci tensor S of M^n satisfies the equation $(*)$ in* Theorem 1.2, *then M^n is an open submanifold of one of the compact isoparametric hypersurfaces with $g \leq 3$. Further, if $\nabla S \neq 0$, then M^n is an open submanifold of the Cartan hypersurface.*

In this paper we assume that all manifolds are of class C^∞ and connected unless otherwise stated.

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2. Preliminaries

In this section, we present some preliminary results of hypersurfaces in a unit sphere.

Let $S^{n+1}(1)$ be an $(n+1)$-dimensional unit sphere in an $(n+2)$-dimensional Euclidean space E^{n+2} and consider the induced metric on $S^{n+1}(1)$, under which $S^{n+1}(1)$ becomes a real space form of constant sectional curvature 1. Further, let M^n be a connected submanifold of $S^{n+1}(1)$ with codimension 1. Below, we refer to this type of manifold as a hypersurface of $S^{n+1}(1)$. We will begin by establishing the structure equations of M^n.

Let $\bar{e}_1, \dots, \bar{e}_{n+1}$ be a local orthonormal frame field of $S^{n+1}(1)$ such that the restrictions of the first n vectors to M^n are tangent to M^n. Let $\bar{\omega}^1, \dots, \bar{\omega}^{n+1}$

be the corresponding coframe of $S^{n+1}(1)$. The first and the second structure equations of $S^{n+1}(1)$ are the following:

$$d\varpi^i = -\sum_j \varpi^i_j \wedge \varpi^j, \quad \varpi^i_j + \varpi^j_i = 0, \tag{2.1}$$

$$d\varpi^i_j = -\sum_k \varpi^i_k \wedge \varpi^k_j + \varpi^i \wedge \varpi^j, \quad i,j,k = 1,\dots,n+1. \tag{2.2}$$

Let ω^i and ω^i_j, respectively, be the restrictions of ϖ^i and ϖ^i_j (where $i,j = 1,\dots,n$) to M^n. Then, from (2.1) and (2.2) we have the following first and second structure equations of M^n:

$$d\omega^i = -\sum_j \omega^i_j \wedge \omega^j, \quad \omega^i_j + \omega^j_i = 0, \tag{2.3}$$

$$d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j + \Omega^i_j,$$

$$\Omega^i_j = \frac{1}{2}\sum_{k,\ell} R^i_{jk\ell}\omega^k \wedge \omega^\ell, \quad R^i_{jk\ell} + R^i_{j\ell k} = 0, \quad i,j,k,\ell = 1,\dots,n. \tag{2.4}$$

Here, Ω^i_j denotes the curvature form of M^n and $R^i_{jk\ell}$ the coefficient of the curvature tensor of M^n.

The second Bianchi identity is given by

$$R^i_{jk\ell m} + R^i_{j\ell mk} + R^i_{jmk\ell} = 0, \quad i,j,k,\ell,m = 1,\dots,n. \tag{2.5}$$

Here, $R^i_{jk\ell m}$ are the coefficients of the covariant derivative of the curvature tensor $R^i_{jk\ell}$ defined by

$$\sum_m R^i_{jk\ell m}\omega^m = dR^i_{jk\ell} + \sum_t \omega^i_t R^t_{jk\ell} - \sum_t \omega^t_j R^i_{tk\ell} - \sum_t \omega^t_k R^i_{jt\ell} - \sum_t \omega^t_\ell R^i_{jkt}.$$

However, ω^{n+1}, the restriction of ϖ^{n+1} to M^n, vanishes. This gives

$$0 = d\omega^{n+1} = -\sum_i \omega^{n+1}_i \wedge \omega^i.$$

Hence, from Cartan's lemma, ω^{n+1}_i can be expressed by

$$\omega^{n+1}_i = \sum_j h_{ij}\omega^j, \quad h_{ij} = h_{ji}, \quad i,j = 1,\dots,n. \tag{2.6}$$

The tensor field A on M^n defined by

$$A = \sum_{i,j} h_{ij}\omega^i \otimes e_j \tag{2.7}$$

is said to be the second fundamental form of M^n in $S^{n+1}(1)$.

From (2.2), (2.4) and (2.6), we deduce the following Gauss equation:

$$R^i_{jk\ell} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk} + h_{ik}h_{j\ell} - h_{i\ell}h_{jk}. \tag{2.8}$$

Covariantly differentiating both sides of (2.6), and making use of (2.2) and (2.3), we obtain

$$\sum_{j,k} h_{ijk}\omega^j \wedge \omega^k = 0, \tag{2.9}$$

where the coefficients h_{ijk} are defined by

$$\sum_k h_{ijk}\omega^k = dh_{ij} - \sum_\ell h_{\ell j}\omega^\ell_i - \sum_\ell h_{i\ell}\omega^\ell_j. \tag{2.10}$$

Comparing these coefficients to the coefficients of $\omega^j \wedge \omega^k$ in (2.9), we obtain the following Codazzi equation:

$$h_{ijk} = h_{ikj}, \quad i, j, k = 1, \dots, n. \tag{2.11}$$

Using the Gauss equation (2.8), the coefficients $R_{ij} = \sum_k R^k_{jki}$ of the Ricci tensor S of M^n are given by

$$R_{ij} = (n-1)\delta_{ij} + \left(hh_{ij} - \sum_k h_{ik}h_{kj}\right), \tag{2.12}$$

where h denotes the mean curvature of M^n, defined by $h = \sum_i h_{ii}$.

Furthermore, the scalar curvature $\rho = \sum_i R_{ii}$ of M^n satisfies the following equation:

$$\rho = n(n-1) + h^2 - \sum_{i,j} h_{ij}h_{ij}. \tag{2.13}$$

Covariantly differentiating both sides of (2.10) and using the structure equations (2.3) and (2.4), we obtain that

$$\sum_{k.\ell} h_{ijk\ell}\omega^k \wedge \omega^\ell = \frac{1}{2}\sum_{k,\ell,m} h_{mj}R^m_{ik\ell}\omega^k \wedge \omega^\ell + \frac{1}{2}\sum_{k,\ell,m} h_{im}R^m_{jk\ell}\omega^k \wedge \omega^\ell, \tag{2.14}$$

where the coefficients $h_{ijk\ell}$ are defined by

$$\sum_{\ell} h_{ijk\ell}\omega^{\ell} = dh_{ijk} - \sum_{\ell} h_{\ell jk}\omega_i^{\ell} - \sum_{\ell} h_{i\ell k}\omega_j^{\ell} - \sum_{\ell} h_{ij\ell}\omega_k^{\ell}.$$

Comparing these coefficients to the coefficients of $\omega^k \wedge \omega^{\ell}$ in (2.14), we deduce the following Ricci formula for h_{ij}:

$$h_{ijk\ell} - h_{ij\ell k} = \sum_m h_{mj} R^m_{ik\ell} + \sum_m h_{im} R^m_{jk\ell}. \tag{2.15}$$

From (2.12), our condition $(*)$ in §1 is equivalent to the following equation:

$$\frac{3}{n} h h_{ijk} = \sum_r (h_{ir} h_{rjk} + h_{jr} h_{rik} + h_{rk} h_{rij}). \tag{2.16}$$

For the sake of brevity, we will write $(h_{ij})^m$ and α_m for a tensor and function, respectively, on M^n for any integer m to mean:

$$(h_{ij})^m = \sum_{i_1,\dots,i_{m-1}} h_{ii_1} h_{i_1 i_2} \cdots h_{i_{m-1} j}, \tag{2.17}$$

$$\alpha_m = \sum_i (h_{ii})^m, \quad m = 1, 2, \dots. \tag{2.18}$$

In particular, α_1 coincides with the mean curvature h and α_2 is related to the scalar curvature ρ by

$$\rho = n(n-1) + h^2 - \alpha_2.$$

3. Cartan Hypersurfaces

E. Cartan [4] classified all isoparametric hypersurfaces in $S^{n+1}(1)$ that have at most three distinct constant principal curvatures. They are homogeneous hypersurfaces and orbits under the isotropy representations of symmetric spaces of rank 2 ([7], [16]). In this paper isoparametric hypersurfaces in $S^{n+1}(1)$ are called Cartan hypersurfaces if they have just three distinct constant principal curvatures.

REMARK 3.1. In [8], the name Cartan hypersurface refers to minimal isoparametric hypersurfaces in $S^{n+1}(1)$ with three distinct constant principal curvatures, but we use this term for both minimal one and all its parallel hypersurfaces.

R. Takagi and T. Takahashi [16] determined the principal curvatures and their multiplicities for all homogeneous hypersurfaces in $S^{n+1}(1)$. We know the following for general isoparametric hypersurfaces.

THEOREM 3.2 ([3], [4], [11]). *Let M^n be an isoparametric hypersurface in $S^{n+1}(1)$ and let $\lambda_1, \dots, \lambda_g$ with $\lambda_1 > \cdots > \lambda_g$ be the distinct constant principal curvatures with multiplicities $m_1, \dots, m_g$. Then the following properties hold.*

(1) *g is either 1, 2, 3, 4 or 6.*

(2) *If $g = 3$, then $m_1 = m_2 = m_3 = 2^r$ (for $r = 0, 1, 2, 3$).*

(3) *There is an angle $\theta \in \left(0, \frac{\pi}{g}\right)$ such that $\lambda_i = \cot\left\{(i-1)\frac{\pi}{g} + \theta\right\}$, $i = 1, \dots, g$.*

We now consider the case $g = 3$. Let M^n be the Cartan hypersurface of $S^{n+1}(1)$ and $(\mathfrak{u}, \mathfrak{k})$ be the corresponding effective orthogonal symmetric Lie algebra of compact type and rank 2. Further, let m_i $(i = 1, 2, 3)$ be the multiplicities of the principal curvatures of M^n. Then we have the following table 1 ([16]).

Table 1: Cartan hypersurfaces

$\mathfrak{u}$	$\mathfrak{k}$	n	m_i $(i = 1, 2, 3)$
$\mathfrak{su}(3) + \mathfrak{su}(3)$	$\mathfrak{su}(3)$	6	$m_i = 2$ $(i = 1, 2, 3)$
$\mathfrak{su}(3)$	$\mathfrak{so}(3)$	3	$m_i = 1$ $(i = 1, 2, 3)$
$\mathfrak{su}(6)$	$\mathfrak{sp}(3)$	12	$m_i = 4$ $(i = 1, 2, 3)$
E_6	F_4	24	$m_i = 8$ $(i = 1, 2, 3)$

We can now demonstrate the following proposition:

PROPOSITION 3.3. *Let M^n be an n-dimensional Cartan hypersurface in $S^{n+1}(1)$. Then the Ricci tensor S with components R_{ij} of M^n satisfies the following equation*:

$$(*) \qquad \mathfrak{S}_{i,j,k} R_{ijk} = 3\left(1 - \frac{2}{n}\right) h h_{ijk},$$

where R_{ijk} are the components of the covariant derivative ∇S of S and $\mathfrak{S}_{i,j,k} R_{ijk}$ is defined by

$$\mathfrak{S}_{i,j,k} R_{ijk} = R_{ijk} + R_{jki} + R_{kij}.$$

Proof. Because the second fundamental form h_{ij} of M^n is diagonalizable, a local field $\{e_i\}$ of the orthonormal frames on M^n can be chosen in such a way that $h_{ij} = \lambda_i \delta_{ij}$; namely,

$$h_{ab} = \cot \theta \delta_{ab}, \quad a, b = 1, \dots, m,$$

$$h_{rs} = \cot\left(\frac{\pi}{3} + \theta\right)\delta_{rs}, \quad r, s = m+1, \dots, 2m,$$

$$h_{xy} = \cot\left(\frac{2\pi}{3} + \theta\right)\delta_{xy}, \quad x, y = 2m+1, \dots, n,$$

$$h_{ij} = 0 \quad \text{for other } i \text{ and } j. \tag{3.1}$$

This gives us

$$0 = dh_{ab} = \sum_k h_{abk}\omega^k + (\lambda_a - \lambda_b)\omega_b^a = \sum_k h_{abk}\omega^k,$$

$$0 = dh_{rs} = \sum_k h_{rsk}\omega^k + (\lambda_r - \lambda_s)\omega_s^r = \sum_k h_{rsk}\omega^k,$$

$$0 = dh_{xy} = \sum_k h_{xyk}\omega^k + (\lambda_x - \lambda_y)\omega_y^x = \sum_k h_{xyk}\omega^k.$$

It follows that

$$h_{abk} = h_{rsk} = h_{xyk} = 0, \quad k = 1, \dots, n \tag{3.2}$$

and

$$\sum_r (h_{ir}h_{rjk} + h_{jr}h_{rik} + h_{rk}h_{rij}) = (\lambda_i + \lambda_j + \lambda_k)h_{ijk}. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we coclude that

$$\mathfrak{S}_{i,j,k} R_{ijk} = 3\left(1 - \frac{2}{n}\right) h h_{ijk}.$$

This completes the proof. ■

For hypersurfaces with at most two distinct principal curvatures Lawson [10] proved the following theorem.

Theorem 3.4 (H. B. Lawson, Jr. [10], Theorem 5). *Let M^n be a Riemannian manifold over which the Ricci tensor is covariant constant. Then, if M^n is* *isometrically immersed into* $S^{n+1}(1)$ *with constant mean curvature, it must be an open submanifold of* $S^k \times S^{n-k}$, $k = 0, \dots, \left[\frac{n}{2}\right]$ *which is canonically imbedded in* $S^{n+1}(1)$.

Remark 3.5. The submanifolds $S^k \times S^{n-k}$ $(k = 0, \dots, \left[\frac{n}{2}\right])$ of Theorem 3.4 are characterized by the hypersurfaces with parallel second fundamental form in $S^{n+1}(1)$. Lawson [10] has also classified hypersurfaces in a space form with non-positive constant sectional curvature whose Ricci tensors are parallel and whose mean curvatures are constant.

P. J. Ryan [14] generalized Lawson's results and gave the following theorem:

Theorem 3.6 (P. J. Ryan [14]). *Let* M^n *be a hypersurface in* $S^{n+1}(1)$ *with* $n > 2$. *If* M^n *is not of constant curvature* 1 *and if the Ricci tensor of* M^n *is covariant constant, then* M^n *is an open subset of* $S^k \times S^{n-k}$, $k = 0, \dots, \left[\frac{n}{2}\right]$ *and is canonically imbedded in* $S^{n+1}(1)$.

4. Proof of Theorem

In this section we shall prove our main theorem. We consider an n-dimensional hypersurface M^n in a unit sphere $S^{n+1}(1)$ with constant mean curvature satisfying the condition $(*)$ in §1. Our condition $(*)$ is equivalent to the following equations:

$$\frac{3}{n} h h_{ijk} = \sum_r (h_{ir} h_{rjk} + h_{jr} h_{rik} + h_{rk} h_{rij}), \quad i, j, k = 1, \dots, n. \tag{4.1}$$

By the rigidity of the Cartan hypersurfaces ([13], [15]), it suffices to show that M^n has at most three distinct constant principal curvatures.

First, we prove the following lemma.

Lemma 4.1. *Under the same conditions of* Theorem 1.3, *the functions* α_m $(m = 1, 2, \dots)$ *are constant.*

Proof. We prove this lemma by induction on m. The function $\alpha_1 = h =$ const is an assumption, so the lemma holds for $m = 1$.

By using the Gauss equation (2.8), we have

$$\alpha_2 = n(n-1) + \alpha_1^2 - \rho. \tag{4.2}$$

By contracting indices j and k in the equation (4.1), we deduce

$$\rho_i + 2\sum_j R_{jij} = 0, \tag{4.3}$$

where ρ_i is defined by $d\rho = \sum_i \rho_i \omega^i$. Contracting the second Bianchi identity (2.5) with respect to the indices j and ℓ, we arrive at

$$\sum_j R_{jkj} = 0.$$

Combining this with (4.2) and (4.3), we have

$$\alpha_2 = \text{const}.$$

Differentiating both sides of (2.18), we obtain

$$d\alpha_m = m\sum_{i,j} (h_{ij})^{m-1}\, dh_{ij} \quad (m \geq 2). \tag{4.4}$$

Substituting (2.10) to the right-hand side of (4.4), we have

$$d\alpha_m = m\sum_{i,j,k} h_{ijk}(h_{ij})^{m-1}\omega^k. \tag{4.5}$$

Now multiplying $(h_{ij})^{m-1}$ to both sides of (4.1) and contracting indices i and j in the resulting equation, we have

$$2\sum_{i,j,k} h_{ijk}(h_{ij})^m\omega^k + \sum_{i,j,k,r} h_{kr}h_{rij}(h_{ij})^{m-1}\omega^k = \sum_{i,j,k} h(h_{ij})^{m-1}h_{ijk}\omega^k.$$

From (4.5) and the last equation, we obtain

$$\frac{2}{m+1}\, d\alpha_{m+1} + \frac{1}{m}\sum_{k,r} h_{kr}\, d\alpha_m(e_r)\omega^k = \frac{h}{m}\, d\alpha_m \quad (m \geq 2). \tag{4.6}$$

By (4.6) and the induction hypothesis $d\alpha_m = 0$, we have $d\alpha_{m+1} = 0$. Hence we conclude that $\alpha_m = const$ $(m \geq 1)$. This gives the desired lemma. ■

Because α_m $(m \geq 1)$ are fundamental symmetric functions of the principal curvatures of M^n, we can deduce the following proposition.

Proposition 4.2. *Under the same conditions of* Theorem 1.3, M^n *is an isoparametric hypersurface in* $S^{n+1}(1)$.

Next, we demonstrate the following lemma.

Lemma 4.3. *Under the same conditions of* Theorem 1.3, *we have*

$$\sum_k (h_{ik})_k^m = 0 \quad (m \geq 1). \tag{4.7}$$

Proof. We prove this lemma by induction on m. For $m = 1$, the Codazzi equation (2.11) gives us

$$\sum_k h_{ikk} = \sum_k h_{kki} = (\alpha_1)_i = 0.$$

For $m \geq 2$, by using the induction hypotheses $\sum_k (h_{ik})_k^{m-1} = 0$, Lemma 4.1 and the equation (4.5), we can deduce

$$\sum_k (h_{ik})_k^m = \sum_{k,s} h_{isk} (h_{sk})^{m-1} = \frac{1}{m} (\alpha_m)_i = 0.$$

This completes the proof. ■

We now demonstrate the following proposition.

Proposition 4.4. *Under the same conditions of* Theorem 1.3, *the following equations are satisfied*:

$$\sum_{r,k} h_{irk} h_{jrk} = \frac{3}{n} h \sum_m \left(\sum_k h_{mk} R_{mijk} + h_{im} R_{mj} \right) - 2 \sum_{r,m} h_{ir} h_{rm} R_{mj}$$
$$- \sum_{k,m,r} h_{kr} h_{rm} R_{mijk} - \sum_{k,m,r} h_{jr} h_{mk} R_{mrik}. \tag{4.8}$$

Proof. Covariantly differentiating both sides of (4.1), we have

$$\frac{3}{n} h h_{ijk\ell} = \sum_r (h_{ir\ell} h_{rjk} + h_{jr\ell} h_{rik} + h_{rk\ell} h_{rij})$$
$$+ \sum_r (h_{ir} h_{rjk\ell} + h_{jr} h_{rik\ell} + h_{rk} h_{rij\ell}). \tag{4.9}$$

Taking the skew-symmetric part with respect to the indices i and ℓ and making use of the Ricci formula (2.15) in the last equation, we obtain

$$\frac{3}{n}h\sum_{m}(h_{mk}R_{mji\ell}+h_{jm}R_{mki\ell})$$
$$=\sum_{m,r}h_{ir}h_{rm}R_{mkj\ell}+\sum_{m,r}h_{jr}h_{rm}R_{mki\ell}+\sum_{r}h_{ir}h_{r\ell kj}$$
$$-\sum_{r}h_{\ell r}h_{rjki}+\sum_{m,r}h_{ir}h_{mk}R_{mrj\ell}+\sum_{m,r}h_{kr}h_{rm}R_{mji\ell}.$$

Summing this equation with respect to k and ℓ, and interchanging indices i and j, we get

$$\sum_{k,r}h_{kr}h_{rikj}=-\frac{3}{n}h\sum_{k,m}(h_{mk}R_{mijk}+h_{im}R_{mkjk})$$
$$+\sum_{k,m,r}h_{jr}h_{rm}R_{mkik}+\sum_{k,m,r}h_{ir}h_{rm}R_{mkjk}$$
$$+\sum_{k,m,r}h_{jr}h_{mk}R_{mrik}+\sum_{k,m,r}h_{kr}h_{rm}R_{mijk}. \tag{4.10}$$

Summing the equation (4.9) with respect to k and ℓ, we arrive at

$$\frac{3}{n}h\sum_{k}h_{ijkk}=2\sum_{k,r}h_{irk}h_{rjk}+\sum_{k,r}(h_{ir}h_{jrkk}+h_{jr}h_{rikk}+h_{rk}h_{rijk}).$$

From the Ricci formula applied to the last equation, we deduce

$$\frac{3}{n}h\sum_{k,m}(h_{mk}R_{mijk}+h_{im}R_{mkjk})=2\sum_{k,r}h_{irk}h_{jrk}$$
$$+\sum_{k,m,r}h_{ir}(h_{mk}R_{mrjk}+h_{rm}R_{mkjk})$$
$$+\sum_{k,m,r}h_{jr}(h_{mk}R_{mrik}+h_{rm}R_{mkik})$$
$$+\sum_{k,m,r}h_{rk}(h_{rikj}+h_{mi}R_{mrjk}+h_{rm}R_{mijk}). \tag{4.11}$$

By combining (4.10) with (4.11), we have

$$\sum_{k,r} h_{irk}h_{jrk} = \frac{3}{n}h\sum_{m}\left(\sum_{k} h_{mk}R_{mijk} + h_{im}R_{mj}\right) - \sum_{m,r} h_{ir}h_{rm}R_{mj} - \sum_{m,r} h_{jr}h_{rm}R_{mi} - \sum_{k,m,r} h_{rk}h_{rm}R_{mijk} - \sum_{k,m,r} h_{jr}h_{mk}R_{mrik}.$$

Additionally, the equations $\sum_k R_{ik}h_{kj} = \sum_k h_{ik}R_{kj}$ are satisfied, so we have the desired conclusion. ■

Next, we establish the following lemma.

LEMMA 4.5. *Under the same conditions of* Theorem 1.3, *the following two equations hold*:

$$(h_{ij})^2_k = \frac{3}{n}hh_{ijk} - \sum_r h_{ijr}h_{rk}, \tag{4.12}$$

$$(h_{ij})^3_k - (h_{ik})^3_j = \frac{3}{n}h\sum_r (h_{jr}h_{rki} - h_{kr}h_{rji}). \tag{4.13}$$

PROOF. From the definition of $(h_{ij})^2$, we have

$$(h_{ij})^2_k = \sum_r (h_{irk}h_{rj} + h_{ir}h_{rjk}).$$

By making use of (4.1), we get

$$\sum_r (h_{irk}h_{rj} + h_{ir}h_{rjk}) = \frac{3}{n}hh_{ijk} - \sum_r h_{kr}h_{rij}.$$

This gives (4.12) again. Further, we have

$$(h_{ij})^3_k = \left\{\sum_m (h_{im})^2 h_{mj}\right\}_k = \sum_m \left(\frac{3}{n}hh_{imk} - \sum_r h_{imr}h_{rk}\right)h_{mj} + \sum_r (h_{ir})^2 h_{rjk}. \tag{4.14}$$

Accordingly, we can deduce

$$(h_{ij})^3_k - (h_{ik})^3_j = \frac{3}{n} h \sum_r (h_{jr} h_{rki} - h_{kr} h_{rij}).$$

This proves the lemma. ■

From the Gauss equation (2.8) and (4.8), we have the following.

LEMMA 4.6. *Under the same conditions of* Theorem 1.3, *we obtain*

$$\sum_{k,r} h_{irk} h_{jrk} = \left(\alpha_2 - \frac{3}{n}\alpha_1^2\right)\delta_{ij} + \left(4\alpha_1 + \alpha_3 - \frac{3}{n}\alpha_1\alpha_2\right) h_{ij}$$
$$+ \left(\alpha_2 + \frac{3}{n}\alpha_1^2 - 2n\right)(h_{ij})^2 - 2\alpha_1 (h_{ij})^3. \tag{4.15}$$

This leads to the following proposition.

PROPOSITION 4.7. *Under the same conditions of* Theorem 1.3, *any principal curvature* λ *of* M^n *satisfies the following equation*:

$$4\alpha_1\lambda^4 + 4\left(n - \frac{3}{n}\alpha_1^2\right)\lambda^3 + \left(\frac{9}{n^2}\alpha_1^3 + \frac{6}{n}\alpha_1\alpha_2 - 3\alpha_3 - 12\alpha_1\right)\lambda^2$$
$$+ \left(\frac{15}{n}\alpha_1^2 + \frac{6}{n}\alpha_1\alpha_3 - \alpha_4 - 3\alpha_2 - \frac{9}{n^2}\alpha_1^2\alpha_2\right)\lambda$$
$$+ \frac{6}{n}\alpha_1\alpha_2 - \alpha_3 - \frac{9}{n^2}\alpha_1^3 = 0. \tag{4.16}$$

PROOF. By using (4.14), we have

$$\begin{aligned} \Delta(h_{ij})^3 &= \sum_k (h_{ij})^3_{kk} \\ &= \sum_{r,s} (h_{ir})^2_s h_{rjs} + \sum_r (h_{ir})^2 \Delta h_{rj} \\ &\quad + \frac{3}{n} h \sum_r \left\{ (\Delta h_{ir}) h_{rj} + \sum_s h_{isr} h_{jsr} \right\} \\ &\quad - \sum_{k,r,s} h_{kr}(h_{isrk} h_{sj} + h_{isr} h_{jsk}), \end{aligned} \tag{4.17}$$

where Δ denotes the Laplacian.

By means of (4.13), we obtain

$$\Delta(h_{ij})^3 - \sum_k (h_{ik})^3_{jk} = \frac{3}{n} h \sum_r \left(h_{jr} \Delta h_{ir} + \sum_s h_{rsi} h_{rsj} - \sum_s h_{ijrs} h_{rs} \right). \tag{4.18}$$

Further, making use of the Ricci formula (2.15) and Lemma 4.3, we can see that

$$\sum_k (h_{ik})^3_{jk} = \sum_{k,m} (h_{mk})^3 R_{mijk} + \sum_{k,m} (h_{im})^3 R_{mkjk}. \tag{4.19}$$

Combining (4.18) with (4.19), we are led to

$$\begin{aligned} \Delta(h_{ij})^3 = {} & \frac{3}{n} h \sum_r \left(h_{jr} \Delta h_{ir} + \sum_s h_{isr} h_{jsr} - \sum_s h_{ijrs} h_{rs} \right) \\ & + \sum_{k,m} (h_{mk})^3 R_{mijk} + \sum_{k,m} (h_{im})^3 R_{mkjk}. \end{aligned} \tag{4.20}$$

From (4.17) and (4.20), we deduce

$$\begin{aligned} & \sum_{r,s,t} h_{it} h_{trs} h_{jrs} + \sum_r (h_{ir})^2 \Delta h_{rj} - \sum_{k,r,s} h_{rk} h_{isrk} h_{sj} \\ & \quad + \frac{3}{n} h \sum_{r,s} h_{rs} h_{ijrs} - \sum_{k,m} (h_{mk})^3 R_{mijk} - \sum_m (h_{im})^3 R_{mj} = 0. \end{aligned} \tag{4.21}$$

In addition, using the Codazzi equation (2.11) and the Ricci formula (2.15), we can reduce:

$$\begin{aligned} \sum_{r,s} h_{rs} h_{ijrs} & = \sum_{r,s} h_{rs} h_{irjs} \\ & = \sum_{r,s} h_{rs} h_{rsij} + \sum_{m,s} (h_{sm})^2 R_{mijs} + \sum_{m,r,s} h_{im} h_{rs} R_{mrjs}. \end{aligned}$$

This gives

$$\begin{aligned} 0 & = (\alpha_2)_{ij} \\ & = \sum_r (h_{rr})^2_{ij} \\ & = 2 \sum_{r,s} (h_{rsij} h_{sr} + h_{rsi} h_{rsj}). \end{aligned}$$

So we have that

$$\sum_{r,s} h_{rsij} h_{sr} = -\sum_{r,s} h_{rsi} h_{rsj}.$$

We can then deduce

$$\sum_{r,s} h_{rs} h_{ijrs} = -\sum_{r,s} h_{isr} h_{jsr} + \sum_{m,s} (h_{sm})^2 R_{mijs} + \sum_{m,r,s} h_{im} h_{rs} R_{mrjs}. \tag{4.22}$$

Substituting (4.22) into (4.21), we have

$$\begin{aligned}
&\sum_{r,s,t} h_{it} h_{trs} h_{jrs} + \sum_{r} (h_{ir})^2 \Delta h_{rj} \\
&\quad - \sum_{m,r,s} \left\{ -h_{imr} h_{smr} + (h_{rm})^2 R_{misr} + \sum_{t} h_{im} h_{tr} R_{mtsr} \right\} h_{sj} \\
&\quad + \frac{3}{n} h \sum_{r,s} \left\{ -h_{isr} h_{jsr} + (h_{sr})^2 R_{rijs} + \sum_{m} h_{im} h_{rs} R_{mrjs} \right\} \\
&\quad - \sum_{r,s} (h_{rs})^3 R_{rijs} - \sum_{r} (h_{ir})^3 R_{rj} = 0.
\end{aligned} \tag{4.23}$$

By virtue of the Gauss equation (2.8), the Codazzi equation (2.11) and the Ricci formula (2.15), we obtain

$$\begin{aligned}
\Delta h_{rj} &= \sum_{k} h_{rjkk} \\
&= \sum_{k} h_{rkjk} \\
&= \sum_{m,k} h_{mk} R_{mrjk} + \sum_{m} h_{rm} R_{mj} \\
&= -\alpha_1 \delta_{rj} + (n - \alpha_2) h_{rj} + \alpha_1 (h_{rj})^2.
\end{aligned} \tag{4.24}$$

Finally, from (4.17), (4.23) and (4.24), we arrive at

$$\begin{aligned}
&\left(\alpha_3 - \frac{6}{n} \alpha_1 \alpha_2 + \frac{9}{n^2} \alpha_1^3 \right) \delta_{ij} + \left(\alpha_4 + 3\alpha_2 + \frac{9}{n^2} \alpha_1^2 \alpha_2 - \frac{6}{n} \alpha_1 \alpha_3 - \frac{15}{n} \alpha_1^2 \right) h_{ij} \\
&\quad + \left(3\alpha_3 + 12\alpha_1 - \frac{6}{n} \alpha_1 \alpha_2 - \frac{9}{n^2} \alpha_1^3 \right) (h_{ij})^2 + 4 \left(\frac{3}{n} \alpha_1^2 - n \right) (h_{ij})^3 - 4\alpha_1 (h_{ij})^4 = 0.
\end{aligned}$$

This proves the proposition. ■

As a result of Proposition 4.7, we have the following corollary.

Corollary 4.8. *Under the same conditions of* Theorem 1.3, M^n *is an isoparametric hypersurface of* $S^{n+1}(1)$ *that has at most four distinct constant principal curvatures.*

As the final part of this section, we shall give a proof of our main theorem. We assert that M^n has, in fact, at most three distinct constant principal curvatures. We use reductio ad absurdum to prove this.

Assume that M^n has four distinct constant principal curvatures λ_i $(i = 1, 2, 3, 4)$, Theorem 3.2 implies

$$\lambda_i = \cot\left\{(i-1)\frac{\pi}{4} + \theta\right\}, \quad 0 < \theta < \frac{\pi}{4}.$$

So, we have

$$\lambda_1\lambda_2\lambda_3\lambda_4 = 1,$$
$$\sum_{1 \le i < j < k \le 4} \lambda_i\lambda_j\lambda_k = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4).$$

Since λ_i $(i = 1, 2, 3, 4)$ are all distinct solutions of the algebraic equation (4.16), the relationship between the solutions and the coefficients gives

$$\alpha_3 + \alpha_1\left(4 + \frac{9}{n^2}\alpha_1^2 - \frac{6}{n}\alpha_2\right) = 0, \tag{4.25}$$

$$\alpha_4 - \frac{6}{n}\alpha_1\alpha_3 + \left(3 + \frac{9}{n^2}\alpha_1^2\right)\alpha_2 - \frac{3}{n}\alpha_1^2 - 4n = 0. \tag{4.26}$$

On the other hand, taking the trace of (4.16) yields

$$\begin{aligned}4\alpha_1\alpha_4 &+ 4\left(n - \frac{3}{n}\alpha_1^2\right)\alpha_3 + \left(\frac{9}{n^2}\alpha_1^3 + \frac{6}{n}\alpha_1\alpha_2 - 3\alpha_3 - 12\alpha_1\right)\alpha_2 \\ &+ \left(\frac{15}{n}\alpha_1^2 + \frac{6}{n}\alpha_1\alpha_3 - \alpha_4 - 3\alpha_2 - \frac{9}{n^2}\alpha_1^2\alpha_2\right)\alpha_1 \\ &+ n\left(\frac{6}{n}\alpha_1\alpha_2 - \alpha_3 - \frac{9}{n^2}\alpha_1^3\right) = 0. \end{aligned} \tag{4.27}$$

Substituting (4.26) into (4.27), we deduce

$$\left(\frac{12}{n}\alpha_1^2 - 3\alpha_2 + 3n\right)\alpha_3 + \left(\frac{6}{n}\alpha_2 - \frac{27}{n^2}\alpha_1^2 - 18\right)\alpha_1\alpha_2$$
$$+ \frac{15}{n}\alpha_1^3 + 12n\alpha_1 = 0. \tag{4.28}$$

Further, summing the equation (4.15) with respect to i and j, we get

$$\sum_{i,k,r} h_{irk}h_{irk} = -n\alpha_2 + \alpha_1^2 + \alpha_2^2 - \alpha_1\alpha_3. \tag{4.29}$$

Substituting (4.25) into (4.28), we obtain

$$\alpha_1\left(\alpha_2 - \frac{5}{n}\alpha_1^2 - \frac{9}{n^3}\alpha_1^4 + \frac{6}{n^2}\alpha_1^2\alpha_2 - \frac{1}{n}\alpha_2^2\right) = 0. \tag{4.30}$$

Finally, substituting (4.25) into the right-hand side of (4.29), we have

$$\sum_{i,k,r} h_{irk}h_{irk} = -n\left(\alpha_2 - \frac{5}{n}\alpha_1^2 - \frac{9}{n^3}\alpha_1^4 + \frac{6}{n^2}\alpha_1^2\alpha_2 - \frac{1}{n}\alpha_2^2\right). \tag{4.31}$$

Combining (4.30) and (4.31), we conclude that

$$\alpha_1\|\nabla A\| = 0.$$

Furthermore, by our assumption, the mean curvature $\alpha_1 \neq 0$. This implies $\nabla A = 0$. But, according to Theorem 3.4 (also see Remark 3.5), all parallel hypersurfaces in $S^{n+1}(1)$ have at most two distinct constant principal curvatures. This contradicts our assumption that there are four distinct constant principal curvatures. By contradiction, there must be at most three distinct constant principal curvatures. This concludes the proof of the theorem.

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