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REFINED VERSION OF HASSE'S SATZ 45 ON CLASS NUMBER PARITY

By

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Abstract. For an imaginary abelian field K, Hasse [3, Satz 45] obtained a criterion for the relative class number to be odd in terms of the narrow class number of the maximal real subfield K^+ and the prime numbers which ramify in K, by using the analytic class number formula. In [4], we gave a refined version (= "Δ-decomposed version") of Satz 45 by an algebraic method. In this paper, we give one more algebraic proof of the refined version.

1. Introduction

For a number field N, let h_N denote the class number of N. When N is an imaginary abelian field with the maximal real subfield N^+, we write $h_N^+ = h_{N^+}$ and put $h_N^- = h_N/h_N^+$. Further, let $\tilde{h}_N^+$ be the class number of N^+ in the narrow sense. Let $k/\mathbf{Q}$ be an imaginary abelian extension of 2-power degree, and $F/\mathbf{Q}$ a real abelian extension with $2 \nmid [F:\mathbf{Q}]$, and put $K = Fk$. In [3, Satz 45], Hasse proved the following theorem.

Theorem 1. *Under the above setting, assume further that the extension $K/\mathbf{Q}$ is cyclic. Then h_K^- is odd if and only if* (i) *$\tilde{h}_K^+$ is odd,* (ii) *exactly one prime number ramifies in $k/\mathbf{Q}$, say p, and* (iii) *the prime number p does not split in $F/\mathbf{Q}$.*

When $F = \mathbf{Q}$, we immediately obtain the following corollary from Satz 45 and Washington [11, Theorem 10.4(b)].

Corollary 1. *For an imaginary cyclic extension $k/\mathbf{Q}$ of 2-power degree, h_k^- is odd if and only if exactly one prime number ramifies in k.*

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In what follows, we do not assume that $K/\mathbf{Q}$ is cyclic. Using class field theory, we can easily show that the ratios h_K^-/h_k^- and $\tilde{h}_K^+/\tilde{h}_k^+$ are integers. In view of the above results, one is naturally interested in the parity of h_K^-/h_k^-, which is the subject of this note.

Hasse proved Theorem 1 heavily using the analytic class number formula. Recently, Conner and Hurrelbrink [2, Theorem 13.8] gave a purely algebraic proof of Theorem 1 using (i) their exact hexagon involving the cohomology groups $H^i(K/K^+, M)$ with $i = 0, 1$ where M is the group of units or the ideal class group of K and (ii) some fundamental properties of local norm residue symbols. In [4, Corollary 2], sharpening the method in [2], we obtained the following refined version of Theorem 1.

Let $\Delta = \mathrm{Gal}(F/\mathbf{Q}) = \mathrm{Gal}(K/k)$. For a number field N, we denote by A_N and $A_{N,\infty}$ the 2-parts of the ideal class group and the narrow class group of N, respectively. We put $A = A_K$, $A^+ = A_{K^+}$ and $A_\infty^+ = A_{K^+,\infty}$ for brevity. We define the minus class group $A^- = A_K^-$ to be the kernel of the norm map $A \to A^+$. We regard the above groups as modules over the group ring $\mathbf{Z}_2[\Delta]$. Let φ be a *nontrivial* $\overline{\mathbf{Q}}_2$-valued character of Δ, which we often regard as a primitive Dirichlet character. Here, $\mathbf{Z}_2$ denotes the ring of 2-adic integers and $\overline{\mathbf{Q}}_2$ a fixed algebraic closure of the 2-adic rationals $\mathbf{Q}_2$. For a $\mathbf{Z}_2[\Delta]$-module X, $X(\varphi)$ denotes the φ-component of X. (See §2, for the definition of the φ-component.) Let S be the set of prime numbers p such that a prime divisor of k^+ over p ramifies in k.

Theorem 2. *Under the above setting, we have* $A^-(\varphi) = \{0\}$ *if and only if* (i) $A_\infty^+(\varphi) = \{0\}$ *and* (ii) $\varphi(p) \neq 1$ *for any prime number* $p \in S$.

Corollary 2. *The ratio* h_K^-/h_k^- *is odd if and only if* (i) *the ratio* $\tilde{h}_K^+/\tilde{h}_k^+$ *is odd and* (ii) *no prime number* p *in* S *splits in* F.

The main purpose of this paper is to give one more algebraic proof of Theorem 2 using a classical reflection argument. Further, we apply Theorem 2 to show that the 2-part of the class group of the cyclotomic $\mathbf{Z}_2$-extension over a certain imaginary abelian field is trivial (Theorem 3). We show Theorem 2 in §4 after some preliminaries in §2 and 3. In §5, we show Theorem 3.

Remark. In some cases, there are two different proofs for an assertion on the 2-part of the ideal class group. For instance, a theorem of Armitage and Fröhlich [1] was generalized by Taylor [10] and Oriat [9] in two different ways. Taylor used some properties of norm residue symbols, while Oriat used a reflection argument. Recently, we gave in [5, Theorem 2] an alternative proof of [10, Assertion $(*)$] using a reflection argument. This paper gives another instance of two different proofs.

2. Kummer Duality

Let Δ be a finite abelian group whose order is odd. Let φ be a $\overline{\mathbf{Q}}_2$-valued character of Δ of order $d = d_\varphi$. Denote by e_φ the idempotent of the group ring $\mathbf{Z}_2[\Delta]$ corresponding to φ:

$$e_\varphi = \frac{1}{|\Delta|}\sum_{\delta\in\Delta} \mathrm{Tr}(\varphi(\delta^{-1}))\delta.$$

Here, Tr is the trace map from $\mathbf{Q}_2(\zeta_d)$ to $\mathbf{Q}_2$, ζ_d being a primitive dth root of unity. For a module X over $\mathbf{Z}_2[\Delta]$, we denote by $X(\varphi)$ the φ-component X^{e_φ} (or $e_\varphi X$). Let $\mathcal{O}_\varphi = \mathbf{Z}_2[\varphi]$ be the subring of $\overline{\mathbf{Q}}_2$ generated by the values of φ over $\mathbf{Z}_2$. Then the φ-component $X(\varphi)$ is naturally regarded as an $\mathcal{O}_\varphi$-module. We choose a complete set $\Gamma = \Gamma_\Delta$ of representatives of the $\mathbf{Q}_2$-conjugacy classes of the $\overline{\mathbf{Q}}_2$-valued characters of Δ. Then we have a canonical decomposition

$$X = \bigoplus_{\varphi} X(\varphi)$$

where φ runs over the characters in Γ.

Let T/N be an abelian extension over a number field N with $2 \nmid [T:N]$, and let $\Delta = \mathrm{Gal}(T/N)$. Let L/T be a pro-2 abelian extension which is Galois over N. Let $G = \mathrm{Gal}(L/T)$. Then we can naturally regard G as a module over $\mathbf{Z}_2[\Delta]$. For a character $\varphi \in \Gamma = \Gamma_\Delta$, we denote by $L(\varphi)$ the intermediate field of L/T corresponding to $\bigoplus'_{\psi} G(\psi)$ by Galois theory where ψ runs over the characters in Γ with $\psi \neq \varphi$. Then we have a natural isomorphism $\mathrm{Gal}(L(\varphi)/T) \cong G(\varphi)$ of $\mathbf{Z}_2[\Delta]$-modules.

Now, assume that the extension L/T is of exponent 2. Let V be the subgroup of $T^\times/(T^\times)^2$ such that $L = T(v^{1/2} \mid [v] \in V)$. Here, for a multiplicative abelian group X and an element $x \in X$, $[x]$ denotes the class in X/X^2 containing x. We can naturally regard V as a module over $\mathbf{Z}_2[\Delta]$. The Kummer pairing

$$V \times G \to \mu_2 = \{\pm 1\}; \quad ([v], g) \mapsto \langle v, g\rangle = (v^{1/2})^{g-1}$$

is nondegenerate and satisfies a relation $\langle v^\delta, g^\delta \rangle = \langle v, g \rangle$ for $[v] \in V$, $g \in G$ and $\delta \in \Delta$. Because of this relation, the pairing induces a nondegenerate subpairing

$$V(\varphi^{-1}) \times G(\varphi) \to \mu_2.$$

Thus, we obtain the following lemma, which we repeatedly use in this paper.

Lemma 1. *Under the above setting, the Galois group* $\mathrm{Gal}(L(\varphi)/T)$ *is canonically isomorphic to* $G(\varphi)$ *as* $\mathbf{Z}_2[\Delta]$*-modules, and*

$$L(\varphi) = T(v^{1/2} \mid [v] \in V(\varphi^{-1})).$$

3. Lemmas

We use the same notation as in Theorem 2. In particular, φ is a *nontrivial* $\overline{\mathbf{Q}}_2$-valued character of $\Delta = \mathrm{Gal}(F/\mathbf{Q}) = \mathrm{Gal}(K/k)$. For a number field N, $\mathcal{O}_N$ denotes the ring of integers of N. Let $E = E_{K^+} = \mathcal{O}_{K^+}^\times$ be the group of units of K^+. We put $P^+ = \mathrm{Gal}(k^+/\mathbf{Q})$ so that $\mathrm{Gal}(K^+/\mathbf{Q}) = P^+ \times \Delta$. We put

$$\mathfrak{X} = (K^+)^\times/((K^+)^\times)^2$$

for brevity.

Lemma 2. *Under the above setting, if* $A_\infty^+(\varphi)$ *is trivial, then both of* $A^+(\varphi)$ *and* $A^+(\varphi^{-1})$ *are trivial.*

Proof. Let $K_{>0}^+$ be the subgroup of $(K^+)^\times$ consisting of totally positive elements. Let $E_+ = E \cap K_{>0}^+$, and E_0 be the subgroup of E consisting of units ε satisfying the congruence $\varepsilon \equiv u^2 \bmod 4$ for some $u \in K^+$. We have a natural exact sequence

$$\{0\} \to (K^+)^\times/EK_{>0}^+ \to A_\infty^+ \to A^+ \to \{0\} \tag{1}$$

compatible with the action of Δ. We see that $((K^+)^\times/EK_{>0}^+)(\varphi)$ is trivial if and only if $(E_+/E^2)(\varphi)$ is trivial. This is because (i) the Galois module $(K^+)^\times/K_{>0}^+$ is isomorphic to $\mathbf{F}_2[P^+ \times \Delta]$ via the sign map, and (ii) the $\mathcal{O}_F$-module $(E/E^2)(\varphi)$ is isomorphic to $(\mathcal{O}_\varphi/2\mathcal{O}_\varphi)^{\oplus r}$ with $r = |P^+|$ by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz [8, Theorem 3.26a]). Here, $\mathbf{F}_2$ is a finite field of 2 elements. Since $A_\infty^+(\varphi)$ is trivial, it follows from (1) that $A^+(\varphi) = \{0\}$ and $((K^+)^\times/EK_{>0}^+)(\varphi) = \{0\}$. From the latter, it follows that

$$(E_+/E^2)(\varphi) = \{0\}. \tag{2}$$

Let H be the class field of K^+ corresponding to the class group A^+/A^{+2}, and V the subgroup of $\mathfrak{X}$ such that $H = K^+(v^{1/2} \mid [v] \in V)$. We see that

$$\begin{aligned}(E((K^+)^\times)^2/((K^+)^\times)^2) \cap V &= (E_+ \cap E_0)((K^+)^\times)^2/((K^+)^\times)^2 \\ &= (E_+ \cap E_0)/E^2 \qquad (3)\end{aligned}$$

by using Exercise 9.3 of Washington [11]. For each $[v]$ in V, we have $v\mathcal{O}_{K^+} = \mathfrak{A}^2$ for some ideal $\mathfrak{A}$ of K^+. By mapping $[v]$ to the ideal class $[\mathfrak{A}]$, we obtain from (3) the following Kummer sequence

$$\{0\} \to (E_+ \cap E_0)/E^2 \to V \to A^+,$$

which is compatible with the action of Δ. Assume that $A^+(\varphi^{-1})$ is nontrivial. Then it follows from Lemma 1 that $V(\varphi)$ is nontrivial. However, as $A^+(\varphi)$ is trivial, we see from the above Kummer sequence that $((E_+ \cap E_0)/E^2)(\varphi)$ is nontrivial, and hence $(E_+/E^2)(\varphi)$ is nontrivial. This contradicts (2). □

In [4, Lemma 2], we showed the following assertion by effectively using the *nontriviality* of φ.

Lemma 3. *Under the above setting, the natural map $A^+(\varphi) \to A(\varphi)$ is injective.*

We define subgroups $\bar{A}^+$ and $\bar{A}$ of A^+ and A, respectively, by

$$\bar{A}^+ = \bigoplus_{\varphi}{}' A^+(\varphi) \quad \text{and} \quad \bar{A} = \bigoplus_{\varphi}{}' A(\varphi)$$

where φ runs over the nontrivial characters in $\Gamma = \Gamma_\Delta$. By Lemma 3, we can regard $\bar{A}^+$ as a subgroup of $\bar{A}$. Then we put

$$A^* = \bar{A}/\bar{A}^+,$$

which we naturally regard as a $\mathbf{Z}_2[\Delta]$-module. Let φ_0 be the trivial character of Δ. Though the structures of the two minus class groups $A^-/A^-(\varphi_0)$ and A^* are slightly different in general, we can easily show that $|A^-(\varphi)| = |A^*(\varphi)|$. In particular, $A^-(\varphi)$ is trivial if and only if $A^*(\varphi)$ is trivial. Let M/K, M^-/K and M^+_∞/K^+ be the class fields corresponding to the class groups A, A^* and A^+_∞, respectively. Regarding the Galois groups $\mathrm{Gal}(M/K)$, $\mathrm{Gal}(M^-/K)$ and

$\mathrm{Gal}(M_\infty^+/K^+)$ as modules over $\Delta = \mathrm{Gal}(K/k) = \mathrm{Gal}(K^+/k^+)$, we define the intermediate fields $M(\varphi)$, $M^-(\varphi)$ and $M_\infty^+(\varphi)$ as in Section 2 for each nontrivial character $\varphi \in \Gamma$. In other words, $M(\varphi)/K$, $M^-(\varphi)/K$ and $M_\infty^+(\varphi)/K^+$ are the class fields corresponding to $A(\varphi)$, $A^*(\varphi)$ and $A_\infty^+(\varphi)$, respectively.

Lemma 4. *If $A^-(\varphi)$ is trivial, then $A_\infty^+(\varphi)$ is trivial.*

Proof. First, we show that $A^+(\varphi) = \{0\}$ using an argument in [7]. Because of Lemma 3 and the definition of A^-, we see that $A^-(\varphi)$ is the subgroup of $A(\varphi)$ consisting of classes c with $c^J = c^{-1}$. Here, J denotes the complex conjugation. Let B^+ be the elements c of $A^+(\varphi)$ $(\subseteq A(\varphi))$ with $c^2 = 1$. For each $c \in B^+$, we have $c^J = c = c^{-1}$. It follows that $B^+ \subseteq A^-(\varphi)$. As $A^-(\varphi) = \{0\}$, this implies that $A^+(\varphi) = \{0\}$. It follows that $A(\varphi) = \{0\}$ and hence $M(\varphi) = K$. Now assume that $A_\infty^+(\varphi)$ is nontrivial. Then there exists a quadratic subextension N_0/K^+ of $M_\infty^+(\varphi)/K^+$. We see that $N_0 \cap K = K^+$ because φ is nontrivial. Therefore, it follows that N_0K/K is an unramified quadratic extension contained in $M(\varphi)$, which is a contradiction. □

4. Proof of Theorem 2

We use the same notation as in the previous sections. Replacing F with the abelian field corresponding to $\ker \varphi$, we may as well assume that the Galois group $\Delta = \mathrm{Gal}(F/\mathbf{Q})$ is cyclic, and $\varphi : \Delta \to \overline{\mathbf{Q}}_2^\times$ is *injective*. Further, we put

$$\psi = \varphi^{-1}$$

for simplicity. Let h_0 (resp. h_1) be the 2-part (resp. odd part) of the class number h_K of K, and h_0' the least common multiple of h_0 and 2. We choose an element $\tilde{e}_\varphi \in \mathbf{Z}[\Delta]$ so that $\tilde{e}_\varphi \equiv e_\varphi \bmod h_0'$ and the coefficients of $\tilde{e}_\varphi$ are multiple of h_1. We choose $\tilde{e}_\psi \in \mathbf{Z}[\Delta]$ in a similar way. We fix an element $d \in (k^+)^\times$ such that

$$k = k^+(d^{1/2}).$$

Proof of the "only if"-part. Assume that $A^-(\varphi) = \{0\}$. By Lemma 4, we already know that $A_\infty^+(\varphi) = \{0\}$. Hence, it suffices to show that $\varphi(p) \neq 1$ for any prime number p in S. Assume that $\varphi(p) = 1$ for some $p \in S$. Let $\wp$ be a prime ideal of k^+ over p. As φ is injective, the assumption $\varphi(p) = 1$ implies that $\wp$ splits completely in K^+. We choose a prime ideal $\mathfrak{P}$ of K^+ over $\wp$. As $A_\infty^+(\varphi) = \{0\}$, it follows from Lemma 2 that $A^+(\varphi) = \{0\}$. Hence, we have $\mathfrak{P}^{\tilde{e}_\varphi} = \pi\mathcal{O}_{K^+}$ for some element π in K^+. The ideal $\mathfrak{P}^{\tilde{e}_\varphi^2}$ is not a square of an ideal of K^+ because $\tilde{e}_\varphi^2 \equiv e_\varphi \not\equiv 0 \bmod 2$ and $\wp$ splits completely in K^+. In particular, $\pi^{\tilde{e}_\varphi}$ is not a square in $(K^+)^\times$. Let

$$N_\psi = K^+((\pi^{\tilde{e}_\varphi})^{1/2}).$$

From the above, N_ψ/K^+ is a quadratic extension. As φ is nontrivial, we see that $N_\psi \cap K = K^+$.

Let $\tilde{\mathfrak{P}}$ be the prime ideal of K over $\mathfrak{P}$. As $p \in S$, we have $\mathfrak{P} = \tilde{\mathfrak{P}}^2$. We have $A(\varphi) = \{0\}$ since $A^-(\varphi)$ and $A^+(\varphi)$ are both trivial. Hence, $\tilde{\mathfrak{P}}^{\tilde{e}_\varphi} = x\mathcal{O}_K$ for some $x \in K^\times$. It follows that $\pi = \varepsilon x^2$ for some unit ε of K. Thus, we see that

$$N_\psi K = K((\pi^{\tilde{e}_\varphi})^{1/2}) = K((\varepsilon^{\tilde{e}_\varphi})^{1/2})$$

and that this is a $(2,2)$-extension over K^+. Let $J \in \mathrm{Gal}(K/K^+)$ be the complex conjugation. Then, as $N_\psi K/K^+$ is a Galois extension, we see that $(\varepsilon^{\tilde{e}_\varphi})^J = \varepsilon^{\tilde{e}_\varphi}\eta^2$ for some unit η of K. Hence, we can extend the automorphism J to that of $N_\psi K$ by

$$\tilde{J} : (\varepsilon^{\tilde{e}_\varphi})^{1/2} \mapsto (\varepsilon^{\tilde{e}_\varphi})^{1/2}\eta.$$

Since $N_\psi K/K^+$ is a $(2,2)$-extension, $\tilde{J}^2$ is the trivial automorphism and hence we obtain $\eta\eta^J = 1$. It follows that $(\varepsilon^{\tilde{e}_\varphi}\eta)^J = \varepsilon^{\tilde{e}_\varphi}\eta$, and hence $\xi = \varepsilon^{\tilde{e}_\varphi}\eta \in (K^+)^\times$. We see that η is a root of unity in K by the relation $\eta\eta^J = 1$ and a theorem ([11, Theorem 4.12]) on units of a CM field. Hence, $K((\eta^{\tilde{e}_\varphi})^{1/2})/\mathbf{Q}$ is an abelian extension. However, since φ is nontrivial, we observe using Lemma 1 that this extension would be non-abelian if the class $[\eta^{\tilde{e}_\varphi}]$ in $K^\times/(K^\times)^2$ were nontrivial. Therefore, it follows that the class $[\eta^{\tilde{e}_\varphi}]$ is trivial and hence

$$N_\psi K = K((\varepsilon^{\tilde{e}_\varphi^2})^{1/2}) = K((\xi^{\tilde{e}_\varphi})^{1/2}).$$

Then we obtain $\pi^{\tilde{e}_\varphi} = \xi^{\tilde{e}_\varphi} y^2$ or $(d\xi)^{\tilde{e}_\varphi} y^2$ for some $y \in (K^+)^\times$. As φ is nontrivial and $d \in (k^+)^\times$, we see that $d^{\tilde{e}_\varphi}$ is a square in $(k^+)^\times$. Hence, $\mathfrak{P}^{\tilde{e}_\varphi^2} = \pi^{\tilde{e}_\varphi}\mathcal{O}_{K^+}$ is a square of a principal ideal of K^+. This is a contradiction. □

To prove the "if"-part, we need to show two more lemmas.

Lemma 5. *Let Δ_0 be a nontrivial subgroup of Δ, and let $\kappa = \kappa_{\Delta/\Delta_0}$ be the restriction map $\mathbf{Z}_2[\Delta] \to \mathbf{Z}_2[\Delta/\Delta_0]$. Then we have $\kappa(e_\varphi) = 0$.*

Proof. We have $\sum_{\delta \in \Delta_0} \varphi(\delta) = 0$ since Δ_0 is nontrivial and φ is injective. The assertion follows from this. □

LEMMA 6. *Let* $\mathfrak{P}$ *be a prime ideal of* K^+, *and* $\wp = \mathfrak{P} \cap k^+$ *and* $p = \wp \cap \mathbf{Q}$. *If* $\varphi(p) \neq 1$ *and* $A^+(\varphi) = \{0\}$, *then* $\mathfrak{P}^{\tilde{e}_\varphi^2} = x^2 \mathcal{O}_{K^+}$ *for some* $x \in (K^+)^\times$.

PROOF. Let F_0 be the decomposition field of p at $F/\mathbf{Q}$, and $\Delta_0 = \mathrm{Gal}(F/F_0)$. As $\varphi(p) \neq 1$, Δ_0 is a nontrivial subgroup of Δ. We put $K_0^+ = F_0 k^+$ and $\mathfrak{P}_0 = \mathfrak{P} \cap K_0^+$. Since $\mathfrak{P}_0$ remains prime in K^+, we see that $\mathfrak{P}^{\tilde{e}_\varphi} = \mathfrak{P}_0^{\kappa(\tilde{e}_\varphi)} \mathcal{O}_{K^+}$ where $\kappa = \kappa_{\Delta/\Delta_0}$ is the restriction map in Lemma 5. By Lemma 5, we have $\kappa(\tilde{e}_\varphi) \equiv 0 \bmod 2$, and hence $\mathfrak{P}^{\tilde{e}_\varphi} = \mathfrak{A}^2$ for some ideal $\mathfrak{A}$ of K^+. It follows that $\mathfrak{P}^{\tilde{e}_\varphi^2} = (\mathfrak{A}^{\tilde{e}_\varphi})^2$. As $A^+(\varphi) = \{0\}$, $\mathfrak{A}^{\tilde{e}_\varphi}$ is principal, and hence the assertion follows. $\square$

PROOF OF THE "IF"-PART. Assume that (i) $A_\infty^+(\varphi) = \{0\}$ and that (ii) $\varphi(p) \neq 1$ for any prime number $p \in S$. Then we have

$$A^+(\psi) = \{0\}$$

by Lemma 2. To show the assertion, assume to the contrary that $A^-(\varphi)$ is nontrivial. Then the extension $M^-(\varphi)/K$ is nontrivial. Let υ be an arbitrary infinite prime divisor of K^+. By using an argument in Iwasawa [6] (or in pp. 186–187 of [11]), we see that there exists a quadratic extension $N_0 = K^+(w^{1/2})/K^+$ with $w \in K^+$ which is unramified at υ and satisfies $N_0 K \subseteq M^-(\varphi)$. In particular, $N_0 \cap K = K^+$. Let $v = w^{\tilde{e}_\psi}$ and $N = K^+(v^{1/2})$. Then we see that $NK = N_0 K \subseteq M^-(\varphi)$ from Lemma 1 and that NK/K^+ is a $(2,2)$-extension. Since the extension N/K^+ is unramified outside S and ∞, we can write

$$v\mathcal{O}_{K^+} = \prod_{\mathfrak{P}} \mathfrak{P}^{a_\mathfrak{P}} \mathfrak{A}^2$$

for some ideal $\mathfrak{A}$ of K^+. Here, $\mathfrak{P}$ runs over the prime ideals of K^+ with $\mathfrak{P} \cap \mathbf{Q} \in S$, and $a_\mathfrak{P} = 0$ or 1. As $[v] \in \mathfrak{X}(\psi)$, we may as well replace the Kummer generator v with $v^{\tilde{e}_\psi^2}$. Then, since $A^+(\psi) = \{0\}$ and $\psi(p) \neq 1$ for any $p \in S$, it follows from Lemma 6 that $v\mathcal{O}_{K^+} = x^2 \mathcal{O}_{K^+}$ for some $x \in (K^+)^\times$. Therefore, we have

$$N = K^+(\varepsilon^{1/2}) = K^+((\varepsilon^{\tilde{e}_\psi})^{1/2})$$

for some unit ε of K^+ with $[\varepsilon] \in (E/E^2)(\psi)$. It follows that N/K^+ is unramified outside ∞ and $S \cap \{2\}$. Therefore, when $2 \notin S$ or N/K^+ is unramified at 2, N/K^+ is unramified outside ∞, and hence $N \subseteq M_\infty^+(\varphi)$. Thus, we see that $A_\infty^+(\varphi)$ is nontrivial, which is a contradiction.

Assume that $2 \in S$ and that there is a prime ideal $\mathfrak{P}$ of K^+ over 2 which ramifies in N. Since $k = k^+(d^{1/2})$, the quadratic subfields of the $(2,2)$-extension NK/K^+ are K, N and

$$N' = K^+((d\varepsilon)^{1/2}) = K^+((d\varepsilon^{\tilde{e}_\psi})^{1/2}).$$

Since NK/K is unramified at $\mathfrak{P}$, and K/K^+ and N/K^+ are ramified at $\mathfrak{P}$, the third extension N'/K^+ is unramified at $\mathfrak{P}$. Therefore, $\mathrm{ord}_{\mathfrak{P}}(d)$ is even. This means that $\mathrm{ord}_{\wp}(d)$ is even as $[K^+ : k^+]$ is odd, where $\wp = \mathfrak{P} \cap k^+$. Then, replacing d with dy^2 for some $y \in (k^+)^\times$, we may as well assume that $(d, \mathfrak{P}) = 1$. Since N'/K^+ is unramified at $\mathfrak{P}$, we have $d\varepsilon^{\tilde{e}_\psi} \equiv u^2 \mod \mathfrak{P}^{2e}$ for some $u \in K^+$ by Exercise 9.3 of [11]. Here, e is the ramification index of $\mathfrak{P}$ over $\mathbf{Q}$. Let Δ_0 be the decomposition group of the prime 2 at $F/\mathbf{Q}$. Let $\psi_0 = \psi_{|\Delta_0}$ and define e_{ψ_0} and $\tilde{e}_{\psi_0}$ similarly to e_ψ and $\tilde{e}_\psi$. Since $\mathfrak{P}$ is stable under the action of Δ_0, we have

$$(d\varepsilon^{\tilde{e}_\psi})^{\tilde{e}_{\psi_0}} \equiv v^2 \mod \mathfrak{P}^{2e}$$

for some $v \in (K^+)^\times$. As $\psi(2) \neq 1$, Δ_0 is nontrivial and hence $d^{\tilde{e}_{\psi_0}}$ is a square in k^+. Further, we have $\tilde{e}_\psi \tilde{e}_{\psi_0} \equiv \tilde{e}_\psi \mod 2$ as $e_\psi e_{\psi_0} = e_\psi$. Therefore, we see that $\varepsilon^{\tilde{e}_\psi} \equiv w^2 \mod \mathfrak{P}^{2e}$ for some $w \in (K^+)^\times$. This implies that N/K^+ is unramified at $\mathfrak{P}$, a contradiction. □

5. Cyclotomic $\mathbf{Z}_2$-extension

Let F, Δ and φ be as in the previous sections. For an integer $n \geq 0$, let $k_n = \mathbf{Q}(\zeta_{2^{n+2}})$, $K_n = Fk_n$, $F_n = K_n^+$ and $\mathbf{B}_n = k_n^+$. As in the previous sections, we identify Δ with $\mathrm{Gal}(K_n/k_n)$ and $\mathrm{Gal}(F_n/\mathbf{B}_n)$. We write $\varphi \sim \varphi^{-1}$ when φ and φ^{-1} are conjugate over $\mathbf{Q}_2$. It is well known that the class groups $A_{\mathbf{B}_n,\infty}$ and $A_{k_n}^-$ are trivial for all $n \geq 0$. We generalize this fact as follows.

Theorem 3. *Under the above setting, assume that*

$$\text{(C1)}\ \varphi \sim \varphi^{-1}, \quad \text{(C2)}\ \varphi(2) \neq 1, \quad \text{(C3)}\ A_F(\varphi) = \{0\}.$$

Then the class groups $A_{F_n,\infty}(\varphi)$ and $A_{K_n}^-(\varphi)$ are trivial for all $n \geq 0$.

Proof. We write $A_{n,\infty} = A_{F_n,\infty}$ for brevity. By virtue of Theorem 2 (and the assumption (C2)), the triviality of $A_{K_n}^-(\varphi)$ follows from that of $A_{n,\infty}(\varphi)$. The assumption (C1) implies that

$$X(\varphi) = X(\varphi^{-1}) \tag{4}$$

for a $\mathbf{Z}_2[\Delta]$-module X. Further, it follows from the assumptions (C1) and (C3) that $A_{0,\infty}(\varphi) = \{0\}$ by [9, Théorème 2]. To show Theorem 3, assume to the contrary that $A_{n,\infty}(\varphi) \neq \{0\}$ for some $n \geq 1$. Let $M_{n,\infty}(\varphi)/F_n$ be the class field corresponding to $(A_{n,\infty}/A_{n,\infty}^2)(\varphi)$. The cyclic extension F_n/F is of degree 2^n and unramified outside 2. Hence, using an argument in [6], we see that there exists a quadratic extension $N' = F(w^{1/2})/F$ unramified at some prime ideal of F over 2 with $N'F_n \subseteq M_{n,\infty}(\varphi)$. In particular, we have $N' \cap F_n = F$. Put $v = w^{\tilde{e}_\varphi}$, and $N = F(v^{1/2})$. Then, we see from Lemma 1 and (4) that $NF_n = N'F_n \subseteq M_{n,\infty}(\varphi)$ and $N \cap F_n = F$. Clearly, N/F is unramified outside 2∞. Using Lemma 6 and (C2), (C3), we can show that $N = F(\varepsilon^{1/2})$ for some unit ε of F with $[\varepsilon] \in (E/E^2)(\varphi)$ by an argument similar to that in the proof of the "if part" of Theorem 2. Here, $E = \mathcal{O}_F^\times$.

We already know that the narrow class group $A_{0,\infty}(\varphi)$ is trivial. Hence, the quadratic extension N/F is ramified at some prime ideal $\mathfrak{P}$ of F over 2. We define an integer π_j of F_j inductively by $\pi_0 = 2$ and $\pi_j = 2 + \sqrt{\pi_{j-1}}$ for $j \geq 1$. Then π_j is a local parameter of each prime ideal of F_j over 2 and $F_{j+1} = F_j(\pi_j^{1/2})$. Since NF_n/F_n is unramified at 2, there exists some j with $0 \leq j \leq n-1$ such that NF_j/F_j is ramified and NF_{j+1}/F_{j+1} is unramified at the primes ideals over $\mathfrak{P}$. This implies that the intermediate extension $F_j((\varepsilon\pi_j)^{1/2})/F_j$ of the $(2,2)$-extension NF_{j+1}/F_j is unramified at the primes ideals over $\mathfrak{P}$. However, this is impossible because ε is a unit and π_j is a local parameter of the prime ideals of F_j over 2. $\square$

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