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A METHOD FOR FINDING A MINIMAL POINT OF THE LATTICE IN CUBIC NUMBER FIELDS

By

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Abstract. We give a method for finding a minimal point adjacent to 1 of the reduced lattice in cubic number fields using an isotropic vector of the quadratic form and two-dimensional lattice.

1. Introduction

Let K be a cubic algebraic number field of negative discriminant. It is known that to find all the minimal points of a reduced lattice $\mathscr{R}$ of K, it is sufficient to know how to find a minimal point adjacent to 1 in any reduced lattice of K (refer to Definition 1.1 for a rigorous definition). Williams, Cormack and Seah [6] utilized the two-dimensional lattice obtained from a reduced lattice $\mathscr{R}$ to find a minimal point adjacent to 1 in $\mathscr{R}$ (the definition of such a two-dimensional lattice is forthcoming in Section 2). Moreover, Adam [1] utilized an isotropic vector of the quadratic form obtained from a basis of reduced lattice $\mathscr{R}$ (the definition of such a quadratic form is forthcoming in Section 4). Later, Lahlou and Farhane [5] generalise the Adam's method.

In this paper, we shall prove six theorems which give candidates of a minimal point adjacent to 1 in a reduced lattice $\mathscr{R}$. In each case of the theorems, the maximum number of candidates $\varphi \in \mathscr{R}$ such that we must check whether $F(\varphi) < 1$ or not is at most four. Also, such six theorems contain all the occuring cases.

Definition 1.1. (1) Let $1, \beta, \gamma \in K$ be independent over $\mathbf{Q}$. We say that $\mathscr{R} = \langle 1, \beta, \gamma \rangle = \mathbf{Z} + \mathbf{Z}.\beta + \mathbf{Z}.\gamma$ is a *lattice* of K with basis $\{1, \beta, \gamma\}$.

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(2) For $\alpha \in \mathscr{R}$ we define $F(\alpha) = \dfrac{N_K(\alpha)}{\alpha} = \alpha'\alpha''$, where N_K denotes the norm of K over $\mathbf{Q}$, and α' and α'' the conjugates of α.

(3) Let $\mathscr{R}$ be a lattice of K, and let $\varphi(>0) \in \mathscr{R}$. We say that φ is a *minimal point* of $\mathscr{R}$ if for all α in $\mathscr{R}$ such that $0 < \alpha < \varphi$ we have $F(\alpha) > F(\varphi)$.

(4) Let $\mathscr{R}$ be a lattice of K and $\varphi, \psi \in \mathscr{R}$ be a minimal point. We say that ψ is a minimal point adjacent to φ in $\mathscr{R}$ if $\psi = \min\{\alpha \in \mathscr{R}; \varphi < \alpha, F(\varphi) > F(\alpha)\}$.

(5) If $\mathscr{R}$ is a lattice of K in which 1 is a minimal point, we call $\mathscr{R}$ a *reduced lattice*.

2. Basis of Reduced Lattice (I)

Definition 2.1. Let $\alpha \in K$. We define $Y_\alpha := Re\,\alpha'$, $Z_\alpha := Im\,\alpha'$, $X_\alpha := \alpha - Y_\alpha$. Let $\lambda \in K$, $\mu \in K \backslash \mathbf{Q}$. We define $\omega_1(\lambda, \mu) := -(Z_\lambda / Z_\mu)$, $\omega_2(\lambda, \mu) := -Y_\lambda - \omega_1(\lambda, \mu) Y_\mu$.

Remark. In [6] $Y_\alpha = Im\,\alpha'$, $Z_\alpha = Re\,\alpha'$.

Proposition 2.2. *Let* $\alpha \in K$, $c \in \mathbf{Z}$. *Then*
(1) $F(\alpha) = Y_\alpha^2 + Z_\alpha^2$.
(2) $\alpha \notin \mathbf{Q} \Rightarrow Y_\alpha,\ X_\alpha \in K - \mathbf{Q},\ Z_\alpha \notin \mathbf{Q}$.
(3) $K \ni 1, \lambda, \mu$ *are independent over* $\mathbf{Q} \Rightarrow \omega_1(\lambda, \mu) \notin \mathbf{Q}$.
(4) $K \ni 1, \lambda, \mu$ *are independent over* $\mathbf{Q} \Rightarrow 1, X_\lambda, X_\mu$ *are independent over* $\mathbf{Q}$.
(5) $K \ni 1, \lambda, \mu$ *are independent over* $\mathbf{Q} \Rightarrow \det\begin{pmatrix} X_\lambda & X_\mu \\ Z_\lambda & Z_\mu \end{pmatrix} \neq 0$.
(6) *Let* $\alpha \notin \mathbf{Q}$. *Then*
(i) $-1 < Y_{\alpha+c} < 1 \Leftrightarrow c = [-Y_\alpha]$ *or* $[-Y_\alpha] + 1$,
(ii) $Y_{[-Y_\alpha]+\alpha} < 0$, $Y_{[-Y_\alpha]+1+\alpha} > 0$,
(iii) $|Y_{[-Y_\alpha]+\alpha}| < 1/2$ *or* $|Y_{[-Y_\alpha]+1+\alpha}| < 1/2$.

Proof. (3) Let $K = \mathbf{Q}(\theta)$ and $\lambda = a_0 + a_1\theta + a_2\theta^2$ $(a_i \in \mathbf{Q})$, $\mu = b_0 + b_1\theta + b_2\theta^2$ $(b_i \in \mathbf{Q})$. Then we have

$$\begin{aligned} Z_\lambda &= \frac{1}{2i}(\lambda' - \lambda'') = \frac{1}{2i}\{a_1(\theta' - \theta'') + a_2(\theta'^2 - \theta''^2)\} \\ &= \frac{1}{2i}(\theta' - \theta'')\{a_1 + a_2(\theta' + \theta'')\} = Z_\theta\{a_1 + (T_{K/\mathbf{Q}}\theta)a_2 - a_2\theta\} \quad (i^2 = -1). \end{aligned}$$

Similarly we have $Z_\mu = Z_\theta\{b_1 + (T_{K/\mathbf{Q}}\theta)b_2 - b_2\theta\}$. Suppose that

$$\omega_1(\lambda,\mu) = -\frac{Z_\lambda}{Z_\mu} = -\frac{a_1 + pa_2 - a_2\theta}{b_1 + pb_2 - b_2\theta} = r \in \mathbf{Q} \quad (p = T_{K/\mathbf{Q}}\theta).$$

Then we have

$$r(b_1 + pb_2 - b_2\theta) = -(a_1 + pa_2 - a_2\theta), \quad rb_1 + rpb_2 + a_1 + pa_2 - (rb_2 + a_2)\theta = 0.$$

Hence $rb_2 + a_2 = 0$, $rb_1 + a_1 = 0$, so $a_0 + rb_0 - \lambda - r\mu = 0$.

Since 1, λ, μ are independent over $\mathbf{Q}$, we have reached a contradiction. Therefore we have $\omega_1(\lambda,\mu) \notin \mathbf{Q}$.

(5) Since 1, λ, μ are independent over $\mathbf{Q}$, by algebraic number theory $\det\begin{pmatrix} 1 & \lambda & \mu \\ 1 & \lambda' & \mu' \\ 1 & \lambda'' & \mu'' \end{pmatrix} \neq 0$. Moreover, $\det\begin{pmatrix} 1 & \lambda & \mu \\ 1 & \lambda' & \mu' \\ 1 & \lambda'' & \mu'' \end{pmatrix} = 2i(X_\lambda Z_\mu - X_\mu Z_\lambda)$.

Therefore we have $X_\lambda Z_\mu - X_\mu Z_\lambda \neq 0$.

Otheres are easily deduced from definitions. □

Definition 2.3. Let $\mathscr{R}$ be a reduced lattice of K. For $\mathscr{R} \ni \alpha$ we define

$$\alpha_{(1)} := [-Y_\alpha] + \alpha, \quad \alpha_{(2)} := [-Y_\alpha] + 1 + \alpha, \quad \alpha_{(3)} := \begin{cases} \alpha_{(1)} & \text{if } |Y_{\alpha_{(1)}}| < 1/2 \\ \alpha_{(2)} & \text{if } |Y_{\alpha_{(2)}}| < 1/2 \end{cases},$$

$\alpha_{(0)} := \alpha - [\alpha]$, where $[\ldots]$ is the greatest integer function.

Note that $|Z_\alpha| < \sqrt{3}/2 \Rightarrow F(\alpha_{(3)}) < 1$.

Let $\mathscr{R} = \langle 1, \beta, \gamma \rangle$ be a reduced lattice of K. Let $\tau : K \to \mathbf{R}^2$ be the $\mathbf{Q}$-linear map defined by $\alpha^\tau = (X_\alpha, Z_\alpha)$. Note that for $\alpha_1, \alpha_2 \in \mathscr{R}$, $\alpha_1^\tau = \alpha_2^\tau \Leftrightarrow$ there exists some $c \in \mathbf{Z}$ such that $\alpha_2 = c + \alpha_1$. Let $L := \mathscr{R}^\tau = \langle \beta^\tau, \gamma^\tau \rangle$. By Proposition 2.2,(5) L is a two-dimensional lattice. Moreover, by Proposition 2.2,(3)(4) L has the following property (Δ):

$$(\Delta) \quad L \cap (\{0\} \times \mathbf{R}) = L \cap (\mathbf{R} \times \{0\}) = \{(0,0)\}.$$

Now we prepare two lemmas about the two-dimensional lattice which has property (Δ) from Delone's supplement I in [2].

Definition 2.4. Let $L(\subset \mathbf{R}^2)$ be a two-dimensional lattice which has property (Δ). (1) For $\mathbf{R}^2 \ni S = (S_u, S_v) \neq (0,0)$ we define $C(S) := \{(u,v) \in \mathbf{R}^2;\ |u| < |S_u|, |v| < |S_v|\}$. Then we say that $S \in L$ is a minimal point of L if

$L \cap C(S) = \{(0,0)\}$. The system of all the minimal points of L we denote by $M(L)$. We put $M(L)_{>0} := \{P \in M(L); P_u > 0\}$.

(2) Let $S(S_u > 0)$, $Q(Q_u > 0) \in L$ be a minimal point of L. We say that Q is a minimal point adjacent to S in L if $Q_u = \min\{P_u; P \in L, S_u < P_u, |S_v| > |P_v|\}$.

LEMMA 2.5. *Let $L(\subset \mathbf{R}^2)$ be a two-dimensional lattice which has property (Δ). Let $L \ni S, Q$ $(S_u > 0, Q_u > 0)$. Then Q is a minimal point adjacent to S in L if and only if $L = \langle S, Q\rangle$, $S_u < Q_u$, $|S_v| > |Q_v|$, $S_v Q_v < 0$.*

PROOF. From Theorem XI,XII,XIII in [2, p. 467–469]. (cf. Theorem 4.1 in [9]). □

LEMMA 2.6. *Let $L(\subset \mathbf{R}^2)$ be a two-dimensional lattice which has property (Δ) and let $E, G, H \in L$. We assume that G is a minimal point adjacent to E and that H is a minimal point adjacent to G. Then we have $H = E + [-E_v/G_v]G$.*

PROOF. From supplement I, Section 3, 34 in [2, p. 470]. □

PROPOSITION 2.7. *Let $\mathscr{R}$ be a reduced lattice of K, and let $L := \mathscr{R}^\tau$. Then there exists a basis $\{1, \lambda, \mu\}$ of $\mathscr{R}$ such that λ^τ is a minimal point adjacent to μ^τ in L, $0 < X_\lambda$, $F(\lambda_{(3)}) < 1$, $F(\mu_{(3)}) > 1$.*

PROOF. Let $\mathscr{R} = \langle 1, \beta, \gamma\rangle$. For $\varepsilon > 0$, we shall consider a rectangular neighbourhood of $(0,0)$, i.e. $W(\varepsilon, \sqrt{3}/2) = \{(u,v) \in \mathbf{R}^2; |u| < \varepsilon, |v| < \sqrt{3}/2\}$. By Minkowski's convex body theorem, there exists $\varepsilon > 0$ such that $L \cap W(\varepsilon, \sqrt{3}/2) \neq \{(0,0)\}$. We take such a $\varepsilon > 0$ and fix it. We put $W = W(\varepsilon, \sqrt{3}/2)$. Then there exists $Q = (Q_u, Q_v) \in L \cap W$ such that $Q_u = \min\{P_u; P \in L \cap W, 0 < P_u\}$. Note that such a $Q \in L$ is uniquely-determined. We have $L \cap C(Q) = \{(0,0)\}$. Hence Q is a minimal point of L. There exists $S \in L$ such that Q is a minimal point adjacent to S in L. By Lemma 2.5, $\{S, Q\}$ is a basis of L. Since both $\{S, Q\}$ and $\{\beta^\tau, \gamma^\tau\}$ are a basis of L, there exists $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(\mathbf{Z})$ such that $(Q\ S) = (\beta^\tau\ \gamma^\tau)\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. We have $Q = p\beta^\tau + r\gamma^\tau = (p\beta + r\gamma)^\tau$. Similarly, we have $S = (q\beta + s\gamma)^\tau$. We define $\lambda, \mu \in K$ by $(\lambda\ \mu) = (\beta\ \gamma)\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then we have $\mathscr{R} = \langle 1, \lambda, \mu\rangle$, $Q = \lambda^\tau$, $S = \mu^\tau$. Since $Q = (Q_u, Q_v) = \lambda^\tau = (X_\lambda, Z_\lambda)$, from $|Z_\lambda| < \sqrt{3}/2$, we have $F(\lambda_{(3)}) < 1$. From this, if we put $\mathscr{R}_F := \{\alpha \in \mathscr{R}; \alpha^\tau \in M(L)_{>0}, F(\alpha_{(3)}) < 1\}$, then $\mathscr{R}_F \neq \varnothing$. Let $W(\varepsilon, 1) := \{(u,v) \in \mathbf{R}^2; |u| < \varepsilon, |v| < 1\}$. As

$W(\varepsilon, \sqrt{3}/2) \subset W(\varepsilon, 1)$, we have $1 < |\mathscr{R}_F^\tau \cap W(\varepsilon, 1)| < \infty$. Hence there exists $\lambda^\tau \in \mathscr{R}_F^\tau \cap W(\varepsilon, 1)$ such that $X_\lambda = \min\{X_\alpha; \alpha^\tau \in \mathscr{R}_F^\tau \cap W(\varepsilon, 1)\}$. Since $F(\alpha_{(3)}) < 1 \Rightarrow |Z_\alpha| < 1$, it is easily seen that $X_\lambda = \min\{X_\alpha; \alpha^\tau \in \mathscr{R}_F^\tau \cap W(\varepsilon, 1)\} = \min\{X_\alpha; \alpha^\tau \in \mathscr{R}_F^\tau\} = \min\{X_\alpha;\ \alpha \in \mathscr{R}_F\}$. For this λ, there exists $\mu \in \mathscr{R}$ such that λ^τ is a minimal point adjacent to μ^τ in L. Moreover, for such a μ we have $F(\mu_{(3)}) > 1$. □

Remark. Such a basis in Proposition 2.7 is easily found by modified version of Algorithm (A) in [6, p. 581].

Definition 2.8. Let $\mathscr{R}$ be a reduced lattice of K, and let $L := \mathscr{R}^\tau$. We say that $\lambda \in \mathscr{R}$ is a *F-point* of $M(L)_{>0}$ if $\lambda \in \mathscr{R}_F$, $X_\lambda = \min\{X_\alpha; \alpha \in \mathscr{R}_F\}$.

Lemma 2.9. *Let $\mathscr{R}$ be a reduced lattice of K. If $0 < X_\lambda$, $F(\lambda_{(3)}) < 1$, then we have $0 < \lambda_{(1)}$.*

Proof. We assume that $0 < X_\lambda$, $F(\lambda_{(3)}) < 1$. From $0 < X_\lambda = X_{\lambda_{(2)}} = \lambda_{(2)} - Y_{\lambda_{(2)}}$, we have $\lambda_{(2)} > Y_{\lambda_{(2)}} > 0$. Hence we have $\lambda_{(2)} > 0$. Suppose that $\lambda_{(1)} < 0$. We have $0 < \lambda_{(2)} = \lambda_{(1)} + 1 < 1$, so $-1 < \lambda_{(1)} < 0$. Since $\mathscr{R}$ is a reduced lattice of K, we have $F(\lambda_{(2)}) > 1$. Hence we have $\lambda_{(3)} = \lambda_{(1)}$, so $F(\lambda_{(1)}) < 1$. From this, $F(-\lambda_{(1)}) < 1$. Since $\mathscr{R}$ is a reduced lattice of K, we have reached a contradiction. Therefore, we have $\lambda_{(1)} > 0$. □

Theorem 2.10. *Let $\mathscr{R}$ be a reduced lattice of K. Then there exists a basis $\{1, \lambda, \mu\}$ of $\mathscr{R}$ such that*

(a) $0 < \lambda < 1$, $-1/2 < \mu$, $F(\mu) > 1$, $2|Y_\mu| < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$,
(b) $\omega_2(\lambda, \mu) > 0$,
(c) $F([\omega_2] + \lambda) < 1$ *or* $F([\omega_2] + 1 + \lambda) < 1$.

Proof. By Proposition 2.7, we can take a basis $\{1, \lambda, \mu\}$ of $\mathscr{R}$ such that λ^τ is a minimal point adjacent to μ^τ in L, $0 < X_\lambda$, $F(\lambda_{(3)}) < 1$, $F(\mu_{(3)}) > 1$, λ is a F-point of $M(L)_{>0}$. Clearly, $\mathscr{R} = \langle 1, \lambda_{(0)}, \mu_{(3)} \rangle$.

(a) Clearly we have $0 < \lambda_{(0)} < 1$, $F(\mu_{(3)}) > 1$, $2|Y_{\mu_{(3)}}| < 1$, $0 < X_{\mu_{(3)}} = X_\mu < X_{\lambda_{(0)}} = X_\lambda$. From $0 < X_\mu = X_{\mu_{(3)}} = \mu_{(3)} - Y_{\mu_{(3)}}$, we have $-1/2 < \mu_{(3)}$. From Remark 2.11 bellow, we have $0 < \omega_1(\lambda, \mu) < 1$. Since $\omega_1(\lambda_{(0)}, \mu_{(3)}) = -(Z_{\lambda_{(0)}}/Z_{\mu_{(3)}}) = -(Z_\lambda/Z_\mu) = \omega_1(\lambda, \mu)$, we have $0 < \omega_1(\lambda_{(0)}, \mu_{(3)}) < 1$.

(b) Proof of "$\omega_2(\lambda_{(0)}, \mu_{(3)}) > 0$".

(i) The case $\lambda_{(1)} = [-Y_\lambda] + \lambda > 1$. $\lambda_{(1)} = [-Y_\lambda] + \lambda = [-Y_{\lambda_{(0)}}] + \lambda_{(0)} > 1$. Hence $-Y_{\lambda_{(0)}} > 1$. From this and from $0 < \omega_1 < 1$, $|Y_{\mu_{(3)}}| < 1/2$ we have $\omega_2(\lambda_{(0)}, \mu_{(3)}) = -Y_{\lambda_{(0)}} - \omega_1(\lambda_{(0)}, \mu_{(3)}) Y_{\mu_{(3)}} > 0$.

(ii) The case $\lambda_{(1)} = [-Y_\lambda] + \lambda < 1$. By Lemma 2.9, we have $\lambda_{(1)} > 0$. From $0 < \lambda_{(1)} < 1$, we have $F(\lambda_{(1)}) > 1$ because $\mathscr{R}$ is a reduced lattice of K. Therefore we have $F(\lambda_{(2)}) < 1$. Since $F(\lambda_{(1)}) > 1$, we have $Y_{\lambda_{(1)}} < -1/2$. Note that $\lambda_{(1)} = \lambda_{(0)}$. Hence from $Y_{\lambda_{(0)}} = Y_{\lambda_{(1)}} < -1/2$ and from $0 < \omega_1 < 1$, $|Y_{\mu_{(3)}}| < 1/2$ we have $\omega_2(\lambda_{(0)}, \mu_{(3)}) = -Y_{\lambda_{(0)}} - \omega_1(\lambda_{(0)}, \mu_{(3)}) Y_{\mu_{(3)}} > 0$.

(c) Proof of "$F([\omega_2] + \lambda_{(0)}) < 1$ or $F([\omega_2] + 1 + \lambda_{(0)}) < 1$".

(i) The case $Y_{\mu_{(3)}} < 0$. Since $\omega_2 - (-Y_{\lambda_{(0)}}) = -\omega_1 Y_{\mu_{(3)}} > 0$, we have $-Y_{\lambda_{(0)}} < \omega_2$. From this and $|-\omega_1 Y_{\mu_{(3)}}| < 1/2$, we have $[\omega_2] = [-Y_{\lambda_{(0)}}]$ or $[-Y_{\lambda_{(0)}}] + 1$. Note that $[\omega_2] = [-Y_{\lambda_{(0)}}] + 1 \Rightarrow 0 < [-Y_{\lambda_{(0)}}] + 1 - (-Y_{\lambda_{(0)}}) < 1/2 \Rightarrow 0 < Y_{\lambda_{(2)}} = [-Y_{\lambda_{(0)}}] + 1 + Y_{\lambda_{(0)}} < 1/2$. Hence if $[\omega_2] = [-Y_{\lambda_{(0)}}] + 1$, then we have $\lambda_{(3)} = \lambda_{(2)}$. Therefore, we have "$[\omega_2] + \lambda_{(0)} = [-Y_{\lambda_{(0)}}] + \lambda_{(0)} = \lambda_{(1)}$, $[\omega_2] + 1 + \lambda_{(0)} = \lambda_{(2)}$" or "$[\omega_2] + \lambda_{(0)} = [-Y_{\lambda_{(0)}}] + 1 + \lambda_{(0)} = \lambda_{(2)}$, $F(\lambda_{(2)}) < 1$".

(ii) The case $Y_{\mu_{(3)}} > 0$. Since $\omega_2 - (-Y_{\lambda_{(0)}}) = -\omega_1 Y_{\mu_{(3)}} < 0$, we have $-Y_{\lambda_{(0)}} > \omega_2$. From this and $|-\omega_1 Y_{\mu_{(3)}}| < 1/2$, we have $[\omega_2] = [-Y_{\lambda_{(0)}}]$ or $[-Y_{\lambda_{(0)}}] - 1$. Note that $[\omega_2] = [-Y_{\lambda_{(0)}}] - 1 \Rightarrow 0 < -Y_{\lambda_{(0)}} - [-Y_{\lambda_{(0)}}] < 1/2 \Rightarrow -1/2 < Y_{\lambda_{(1)}} = [-Y_{\lambda_{(0)}}] + Y_{\lambda_{(0)}} < 0$. Hence if $[\omega_2] = [-Y_{\lambda_{(0)}}] - 1$, then we have $\lambda_{(3)} = \lambda_{(1)}$. Therefore we have "$[\omega_2] + \lambda_{(0)} = [-Y_{\lambda_{(0)}}] + \lambda_{(0)} = \lambda_{(1)}$, $[\omega_2] + 1 + \lambda_{(0)} = \lambda_{(2)}$" or "$[\omega_2] + 1 + \lambda_{(0)} = \lambda_{(1)}$, $F(\lambda_{(1)}) < 1$". □

REMARK 2.11. Let $\mathscr{R} = \langle 1, \beta, \gamma \rangle$, $0 < X_\gamma < X_\beta$. Then γ^τ is a minimal point adjacent to β^τ in $L \Leftrightarrow 0 < \omega_1(\beta, \gamma) < 1$.

3. Basis of Reduced Lattice (II)

DEFINITION 3.1. Let $\mathscr{R}$ be a lattice of K, and let $\{1, N, M\}$ be a basis of $\mathscr{R}$. We say that $\{1, N, M\}$ is *normalized* provided that

$$0 < X_M < X_N, \quad |Z_M| > 1/2, \quad |Z_N| < 1/2, \quad Z_M \cdot Z_N < 0.$$

We quote Williams [9], Theorem 8.1 as Theorem 3.2 for our convenience.

THEOREM 3.2 (Williams [9], Theorem 8.1). *Let $\mathscr{R}$ be a reduced lattice with the normalized basis $\{1, N, M\}$. If $\theta_g = x + yN + zM$ $(x, y, z \in \mathbf{Z})$ is the minimal point adjacent to* 1, *then* $(y, z) \in \{(1, 0), (0, 1), (1, 1), (1, -1), (2, 1)\}$.

In this paper, θ_g denotes the minimal point adjacent to 1 of any reduced lattice $\mathscr{R}$. We shall consider the relationship between F-point and the normalized basis.

THEOREM 3.3. *Let $\mathscr{R}$ be a reduced lattice with the normalized basis $\{1, N, M\}$. If $\mathscr{R} = \langle 1, \lambda, \mu \rangle$, λ^τ is adjacent to μ^τ, λ is a F-point of $M(L)_{>0}$ $(L = \mathscr{R}^\tau)$, then λ^τ must be one of N^τ, $(N - M)^\tau$, M^τ. Moreover,*

(1) *The case $\lambda^\tau = (N - M)^\tau$: $N^\tau = (d+1)\lambda^\tau + \mu^\tau$, $M^\tau = d\lambda^\tau + \mu^\tau$,*

(2) *The case $\lambda^\tau = M^\tau$: $N^\tau = d\lambda^\tau + \mu^\tau$,*

where $d = d(\lambda, \mu) = [1/\omega_1(\lambda, \mu)]$.

PROOF. Recall that $\mathscr{R}_F = \{\alpha \in \mathscr{R}; \alpha^\tau \in M(L)_{>0}, F(\alpha_{(3)}) < 1\}$, $X_\lambda = \min\{X_\alpha; \alpha \in \mathscr{R}_F\}$. By Lemma 2.5 and Definition 3.1, we have $N \in \mathscr{R}_F$. Hence, we have $X_\lambda \leq X_N$. Since $L = \langle N^\tau, M^\tau \rangle = \langle \lambda^\tau, \mu^\tau \rangle$, there exists $a, b \in \mathbf{Z}$ such that $\lambda^\tau = aN^\tau + bM^\tau$.

(i) The case $a < 0$. Since $X_\lambda > 0$, we have $b > 0$. Moreover, since $|Z_\lambda| = |aZ_N + bZ_M| = |a| \cdot |Z_N| + b \cdot |Z_M| < 1$ and $1/2 < |Z_M|$, we have $b \leq 1$. Therefore $b = 1$. Hence $X_\lambda = aX_N + bX_M = aX_N + X_M = X_M - |a| \cdot X_N < 0$. Therefore the case (i) is impossible.

(ii) The case $a = 0$. Since $X_\lambda = aX_N + bX_M = bX_M$, we have $b > 0$. Since $|Z_\lambda| = b|Z_M|$, we have $b = 1$. [i.e. $(a, b) = (0, 1)$]

(iii) The case $a \geq 1, b \leq 0$. Since $|Z_\lambda| = a|Z_N| + |b| \cdot |Z_M| < 1$, we have $|b| \leq 1$.

1) The case $b = -1$. Since $X_\lambda = aX_N - X_M = (a-1)X_N + (X_N - X_M)$, if $a \geq 2$, then we have $X_\lambda > X_N$, which is impossible. Therefore, we have $a = 1$. [i.e. $(a, b) = (1, -1)$]

2) The case $b = 0$. Since $X_\lambda = aX_N = (a-1)X_N + X_N$, if $a \geq 2$, then we have $X_\lambda > X_N$, which is impossible. Therefore, we have $a = 1$. [i.e. $(a, b) = (1, 0)$]

(iv) The case $a \geq 1$, $b \geq 1$. We have $X_\lambda = aX_N + bX_M > X_N$, which is impossible. Therefore, the case (iv) is impossible.

By (i) to (iv), we conclude that $\lambda^\tau = aN^\tau + bM^\tau = M^\tau$ or $(N - M)^\tau$ or N^τ.

(a) The case $|Z_\lambda| < 1/2$. Since $|Z_\mu| > \sqrt{3}/2 > 1/2$, we have $\lambda^\tau = N^\tau$, $\mu^\tau = M^\tau$.

(b) The case $|Z_\lambda| > 1/2$. Since $\lambda^\tau \neq N^\tau$, we have $0 < X_\lambda < X_N$. Hence we have $\lambda^\tau = (N - M)^\tau$ or M^τ.

(b-1) The case $\lambda^\tau = (N - M)^\tau$. We have

(1.1) $X_\lambda = X_{N-M} < X_M < X_N$.

Because if $X_M < X_\lambda = X_{N-M} < X_N$, then from $X_M < X_{N-M}$, $|Z_M| < |Z_{N-M}|$, we have $L \cap C((N-M)^\tau) = L \cap \{(u,v) \in \mathbf{R}^2; |u| < X_{N-M}, |v| < |Z_{N-M}|\} \ni M^\tau \neq (0,0)$. Since $\lambda^\tau = (N-M)^\tau \in L$ is a minimal point, we have reached a contradiction. Therefore we have $X_\lambda = X_{N-M} < X_M < X_N$. By Remark 2.11 we have $0 < \omega_1(N,M) < 1$. Since $\omega_1(M, N-M) = \dfrac{1}{\omega_1(N,M)+1}$, we have $0 < \omega_1(M, N-M) < 1$. From this, if $X_{N-M} < X_M$, then M^τ is adjacent to $(N-M)^\tau$. Note that $\mathscr{R} = \langle 1, M, N-M \rangle$. Hence we have

(1.2) $X_{N-M} < X_M \Leftrightarrow M^\tau$ is adjacent to $(N-M)^\tau$.

Since M^τ is a minimal point adjacent to λ^τ, and λ^τ is a minimal point adjacent to μ^τ, by Lemma 2.6 we have $M^\tau = \mu^\tau + [-(Z_\mu/Z_\lambda)]\lambda^\tau$. We put $d = [-(Z_\mu/Z_\lambda)] = [1/\omega_1(\lambda,\mu)]$. We have $M^\tau = \mu^\tau + d\lambda^\tau$. From $\lambda^\tau = N^\tau - M^\tau$, we have $N^\tau = \mu^\tau + (d+1)\lambda^\tau$. Therefore we obtain formulas: $M^\tau = d\lambda^\tau + \mu^\tau$, $N^\tau = (d+1)\lambda^\tau + \mu^\tau$.

(b-2) The case $\lambda^\tau = M^\tau$.

Since N^τ is a minimal point adjacent to λ^τ, and λ^τ is a minimal point adjacent to μ^τ, by Lemma 2.6 we have $N^\tau = \mu^\tau + [-(Z_\mu/Z_\lambda)]\lambda^\tau = \mu^\tau + d\lambda^\tau$. Therefore we obtain formulas: $M^\tau = \lambda^\tau$, $N^\tau = d\lambda^\tau + \mu^\tau$. □

Corollary 3.4. *Let $\mathscr{R}$ be a reduced lattice with basis $\{1,\lambda,\mu\}$ such that λ^τ is adjacent to μ^τ, λ is a F-point of $M(L)_{>0}$ $(L = \mathscr{R}^\tau)$. If $\theta_g = x + y\lambda + z\mu$ $(x,y,z \in \mathbf{Z})$, then*

the case $\lambda^\tau = N^\tau$: $(y,z) \in \{(1,0),(1,1),(1,-1),(2,1)\}$,

the case $\lambda^\tau = (N-M)^\tau$: $(y,z) \in \{(1,0),(d,1),(d+1,1),(2d+1,2),(3d+2,3)\}$,

the case $\lambda^\tau = M^\tau$: $(y,z) \in \{(1,0),(d,1),(d+1,1),(2d+1,2),(d-1,1)\}$,

where $d = [1/\omega_1(\lambda,\mu)] \geq 1$.

Proof. From Theorem 3.2. □

Remark 3.5. Since $1/(d+1) < \omega_1 < 1/d$, we have

$$[d\omega_1] = [(d-1)\omega_1] = 0, \quad [(d+1)\omega_1] = 1,$$

$$1 \leq [(2d+1)\omega_1] \leq 2, \quad 2 \leq [(3d+2)\omega_1] \leq 4.$$

Theorem 3.6. *Let $\mathscr{R}$ be a reduced lattice with basis $\{1,\lambda,\mu\}$ such that $F(\mu) > 1$, $2|Y_\mu| < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda,\mu) < 1$, $F(\lambda_{(3)}) < 1$.*

Then λ^τ must be one of N^τ, $(N-M)^\tau$, M^τ. Moreover, if $\lambda^\tau = (N-M)^\tau$ or M^τ, then λ is a F-point of $M(L)_{>0}$ $(L = \mathscr{R}^\tau)$.

PROOF. At first, we note that λ^τ is adjacent to μ^τ. Also $\lambda \in \mathscr{R}_F$. From $2|Y_\mu| < 1,\ \mu = \mu_{(3)}$.

(a) The case $|Z_\lambda| < 1/2$. Since $F(\mu_{(3)}) = F(\mu) > 1$, we have $|Z_\mu| > \sqrt{3}/2 > 1/2$. Hence we have $\lambda^\tau = N^\tau,\ \mu^\tau = M^\tau$.

(b) The case $|Z_\lambda| > 1/2$. Let λ^* be a F-point of $M(L)_{>0}$. So we have $X_{\lambda^*} \le X_\lambda$. We shall show that $\lambda^{*\tau} = \lambda^\tau$. Suppose that $\lambda^{*\tau} \ne \lambda^\tau$.

(i) The case $\lambda^\tau \ne M^\tau$. We have

(i-1) $X_{\lambda^*} < X_\mu < X_\lambda < X_M < X_N$.

Since $|Z_{\lambda^*}| > 1/2$, by Theorem 3.3, we have $\lambda^{*\tau} = M^\tau$ or $(N-M)^\tau$. Hence $\lambda^{*\tau} = (N-M)^\tau$. By (1.1) in the proof of Theorem 3.3, we have $X_{\lambda^*} = X_{N-M} < X_M$. From (i-1), we have $X_{\lambda^*} = X_{N-M} < X_\mu < X_\lambda < X_M < X_N$. Since M^τ is adjacent to $(N-M)^\tau$, we have reached a contradiction.

(ii) The case $\lambda^\tau = M^\tau$. Since $\lambda^{*\tau} \ne \lambda^\tau$, by Theorem 3.3, we have $\lambda^{*\tau} = (N-M)^\tau$. By (1.1) in the proof of Theorem 3.3, we have $X_{\lambda^*} = X_{N-M} < X_M$. Hence we have $X_{\lambda^*} = X_{N-M} < X_\mu < X_\lambda = X_M < X_N$. Since M^τ is adjacent to $(N-M)^\tau$, we have reached a contradiction.

By (i)(ii), an assumption $\lambda^{*\tau} \ne \lambda^\tau$ lead to a contradiction. Therefore we have $\lambda^{*\tau} = \lambda^\tau$.

Finally, if $\lambda^\tau = (N-M)^\tau$ or M^τ, then we must have only the case (b), so λ is a F-point of $M(L)_{>0}$. □

REMARK. $F(\lambda_{(3)}) < 1 \Leftrightarrow \exists c \in \mathbf{Z};\ F(c+\lambda) < 1$.

COROLLARY 3.7. *Let $\mathscr{R}$ be a reduced lattice with basis $\{1, \lambda, \mu\}$ such that $F(\mu) > 1$, $2|Y_\mu| < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $F(\lambda_{(3)}) < 1$. If $\theta_g = x + y\lambda + z\mu$ $(x, y, z \in \mathbf{Z})$, then $(y, z) \in \{(1,0), (1,1), (1,-1), (2,1), (d,1), (d+1,1), (2d+1,2), (d-1,1), (3d+2,3)\}$, where $d = [1/\omega_1(\lambda,\mu)] \ge 1$.*

4. Preliminaries (I)

DEFINITION 4.1. Let $\mathscr{R}$ be a lattice of K. For a basis $\{1, \lambda, \mu\}$ of $\mathscr{R}$, we define a mapping $F_{\lambda,\mu} : \mathbf{R}^3 \to \mathbf{R}$ by $F_{\lambda,\mu}(x, y, z) = x^2 + (\lambda' + \lambda'')xy + (\mu' + \mu'')xz + (\lambda'\mu'' + \lambda''\mu')yz + \lambda'\lambda''y^2 + \mu'\mu''z^2$. For any $(x, y, z) \in \mathbf{Z}^3$, we have $F_{\lambda,\mu}(x, y, z) = F(x + y\lambda + z\mu)$.

REMARK. $F_{\lambda,\mu}$ is a positive quadratic form with real coefficients of rank 2. $(\omega_2, 1, \omega_1)$ is an isotropic vector of $F_{\lambda,\mu}$.

We quote Lahlou and Farhane [5], Lemma 2.2 as Lemma 4.2 for our convenience. (cf. [1], Lemma 2.2)

LEMMA 4.2 (Lahlou and Farhane [5], Lemma 2.2). *Let* $\mathscr{R}$ *be a lattice of* K *and let* $\{1, \lambda, \mu\}$ *be a basis of* $\mathscr{R}$. *Then we can write*

(1) $F_{\lambda,\mu}(x, y, z) = a(z - \omega_1 y)^2 + 2b(z - \omega_1 y)(x - \omega_2 y) + (x - \omega_2 y)^2$

(2)

$$F_{\lambda,\mu}(x, y, z) = \frac{1}{2}(x - \omega_2 y)^2 + \frac{1}{2}(x - \omega_2 y + 2b(z - \omega_1 y))^2 + (a - 2b^2)(z - \omega_1 y)^2$$

(3)

$$F_{\lambda,\mu}(x, y, z) = \frac{a}{2}(z - \omega_1 y)^2 + \frac{a}{2}\left(z - \omega_1 y + \frac{2b}{a}(x - \omega_2 y)\right)^2 + \left(1 - \frac{2b^2}{a}\right)(x - \omega_2 y)^2$$

with $a = F(\mu)$, $b = Y_\mu$.

DEFINITION 4.3. Let $\mathscr{R}$ be a reduced lattice with basis $\{1, \lambda, \mu\}$ such that $\mu > -1/2$, $\omega_2(\lambda, \mu) > 0$, $0 < \omega_1(\lambda, \mu) < 1$. Let $y \in \mathbf{Z}$. Then we define

$$\psi_{1,y} = [\omega_2 y] - 1 + y\lambda + [\omega_1 y]\mu \qquad \psi_{7,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] - 1)\mu$$

$$\psi_{2,y} = [\omega_2 y] - 1 + y\lambda + ([\omega_1 y] + 1)\mu \qquad \psi_{8,y} = [\omega_2 y] + 1 + y\lambda + [\omega_1 y]\mu$$

$$\psi_{3,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] - 1)\mu \qquad \psi_{9,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] + 1)\mu$$

$$\psi_{4,y} = [\omega_2 y] + y\lambda + [\omega_1 y]\mu \qquad \psi_{10,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] + 2)\mu$$

$$\psi_{5,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] + 1)\mu \qquad \psi_{11,y} = [\omega_2 y] + 2 + y\lambda + [\omega_1 y]\mu$$

$$\psi_{6,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] + 2)\mu \qquad \psi_{12,y} = [\omega_2 y] + 2 + y\lambda + ([\omega_1 y] + 1)\mu$$

$$\phi_1 = \psi_{4,1} = [\omega_2] + \lambda \qquad \phi_5 = \psi_{2,1} = [\omega_2] - 1 + \lambda + \mu \qquad \phi_9 = 2\lambda + \mu$$

$$\phi_2 = \psi_{5,1} = [\omega_2] + \lambda + \mu \qquad \phi_6 = \psi_{8,1} = [\omega_2] + 1 + \lambda \qquad \phi_{10} = 3\lambda + 2\mu$$

$$\phi_3 = \psi_{3,1} = [\omega_2] + \lambda - \mu \qquad \phi_7 = \psi_{7,1} = [\omega_2] + 1 + \lambda - \mu$$

$$\phi_4 = \psi_{1,1} = [\omega_2] - 1 + \lambda \qquad \phi_8 = \psi_{9,1} = [\omega_2] + 1 + \lambda + \mu$$

REMARK 4.4. (1) If $0 < \mu < 1$, then we have

$$\psi_{1,y} < \psi_{2,y} < \psi_{4,y}; \quad \psi_{1,y} < \psi_{3,y} < \psi_{4,y}; \quad \psi_{4,y} < \psi_{5,y} < \psi_{6,y} < \psi_{9,y}$$

$$\psi_{4,y} < \psi_{5,y} < \psi_{8,y} < \psi_{9,y}; \quad \psi_{4,y} < \psi_{7,y} < \psi_{8,y} < \psi_{9,y}$$

$$\psi_{9,y} < \psi_{10,y} < \psi_{12,y}; \quad \psi_{9,y} < \psi_{11,y} < \psi_{12,y}$$

(2) If $\mu > 1$, then we have

$$\psi_{3,y} < \psi_{1,y} < \psi_{4,y}; \quad \psi_{3,y} < \psi_{7,y} < \psi_{4,y}; \quad \psi_{4,y} < \psi_{2,y} < \psi_{5,y} < \psi_{9,y}$$

$$\psi_{4,y} < \psi_{8,y} < \psi_{5,y} < \psi_{9,y}; \quad \psi_{4,y} < \psi_{8,y} < \psi_{11,y} < \psi_{9,y}$$

$$\psi_{9,y} < \psi_{6,y} < \psi_{10,y}; \quad \psi_{9,y} < \psi_{12,y} < \psi_{10,y}$$

Lemma 4.5. *Let $\mathscr{R}$ be a reduced lattice with basis $\{1, \lambda, \mu\}$ such that $\mu > -1/2$, $\omega_2(\lambda, \mu) > 0$ and $0 < \omega_1(\lambda, \mu) < 1$. Let $a > \max(1, 2b^2, 2|b|)$, where $a = F(\mu)$, $b = Y_\mu$. Then*

(1) $\theta_g \in \{\psi_{i,y}; y(\neq 0) \in \mathbf{Z}, 1 \leq i \leq 12\}$.

(2) $\lambda, \mu > 0 \Rightarrow \psi_{i,1} \leq \psi_{i,y}$ $(y \geq 1)$.

(3) (i) $b < 0 \Rightarrow F(\psi_{2,y}) > 1$, $F(\psi_{6,y}) > 1$, $F(\psi_{7,y}) > 1$, $F(\psi_{11,y}) > 1$.

(ii) $b > 0 \Rightarrow F(\psi_{1,y}) > 1$, $F(\psi_{3,y}) > 1$, $F(\psi_{10,y}) > 1$, $F(\psi_{12,y}) > 1$.

(4) $F(\psi_{3,1}) > F(\psi_{4,1})$.

(5) $(0 <)b < 1/2 \Rightarrow F(\psi_{7,1}) > F(\psi_{4,1})$.

(6) $F(\psi_{5,1}) < F(\psi_{4,1})$, $0 < b < 1 \Rightarrow F(\psi_{7,1}) > F(\psi_{4,1})$.

(7) $b > 1 \Rightarrow F(\psi_{7,1}) > 1$.

(8) $b > 0$ *or* $-1/2 < b < 0 \Rightarrow F(\psi_{1,1}) > F(\psi_{4,1})$.

(9) $F(\psi_{5,1}) > F(\psi_{8,1})$, $(0 <)b < 1 \Rightarrow F(\psi_{2,1}) > F(\psi_{4,1})$.

(10) $F(\psi_{4,1}) > F(\psi_{8,1})$, $b < 0 \Rightarrow c_2 = [\omega_2] - \omega_2 < -1/2$.

(11) $c_1 = [\omega_1] - \omega_1 < -1/2$, $b < 0 \Rightarrow F(\psi_{8,1}) > F(\psi_{9,1})$.

(12) $[2\alpha] = \begin{cases} 2[\alpha] & \textit{if } 0 \leq \alpha - [\alpha] < 1/2 \\ 2[\alpha] + 1 & \textit{if } 1/2 \leq \alpha - [\alpha] \end{cases}$.

Proof. We put $c_1 = [\omega_1] - \omega_1$, $c_2 = [\omega_2] - \omega_2$. Then $-1 < c_1, c_2 < 0$.

(1) was proved in Lahlou and Farhane [5], Theorem 2.1.

(2) obvious

(3) by Lemma 4.2,(1)

(4) By Lemma 4.2,(1), $F(\psi_{3,1}) - F(\psi_{4,1}) = -2ac_1 + a - 2bc_2 = -2ac_1 + a\left(1 - \dfrac{2b}{a}c_2\right) > 0$.

(5) By Lemma 4.2,(1), $F(\psi_{7,1}) - F(\psi_{4,1}) = -2ac_1 + a + 2bc_1 - 2bc_2 - 2b + 2c_2 + 1 = (1-2b)(1+c_2) + a + c_2 - 2(a-b)c_1 > 0$.

(6) By Lemma 4.2,(1) since $F(\psi_{5,1}) < F(\psi_{4,1})$, $F(\psi_{4,1}) - F(\psi_{5,1}) = -2ac_1 - a - 2bc_2 > 0$. So $-2bc_2 > a(1+2c_1)$. From this and $a > 2b$, we have $-2bc_2 > 2b(1+2c_1)$, $-c_2 > 1 + 2c_1$. Hence $-2c_1 > 1 + c_2$. By this,

$$\begin{aligned} F(\psi_{7,1}) - F(\psi_{4,1}) &= -2ac_1 + a + 2bc_1 - 2bc_2 - 2b + 2c_2 + 1 \\ &= (1-2b)(1+c_2) + a + c_2 - 2c_1(a-b) \\ &> (1-2b)(1+c_2) + a + c_2 + (1+c_2)(a-b) \\ &= (1-2b)(1+c_2) + a - 1 + 1 + c_2 + (1+c_2)(a-b) \\ &= (2-2b)(1+c_2) + a - 1 + (1+c_2)(a-b) > 0. \end{aligned}$$

(7) If $b > 1$, then we have $a > 2$ because $a > 2|b|$. From this and by Lemma 4.2,(3), we have $F(\psi_{7,1}) > 1$.

(8) By Lemma 4.2,(1), $F(\psi_{1,1}) - F(\psi_{4,1}) = -2bc_1 - 2c_2 + 1 > 0$.

(9) Since $F(\psi_{5,1}) > F(\psi_{8,1})$, we have $F(\psi_{5,1}) - F(\psi_{8,1}) = 2ac_1 + a + 2bc_2 - 2bc_1 - 2c_2 - 1 > 0$. From this, $F(\psi_{2,1}) - F(\psi_{4,1}) = 2ac_1 + a - 2bc_1 + 2bc_2 - 2b - 2c_2 + 1 = (2ac_1 + a + 2bc_2 - 2bc_1 - 2c_2 - 1) + 2 - 2b > 0$.

(10) Since $F(\psi_{4,1}) - F(\psi_{8,1}) > 0$, we have $bc_1 + c_2 < -1/2$. From this and $b < 0$, $c_1 < 0$, we have $c_2 < -1/2$.

(11) By Lemma 4.2,(1), $F(\psi_{9,1}) - F(\psi_{8,1}) = 2ac_1 + a + 2b(c_2+1) = a(2c_1+1) + 2b(c_2+1) < 0$.

(12) is easily deduced from the definitions. □

Some of Lemma 4.5 were proved in Lahlou and Farhane [5], Theorem 2.1.

REMARK. $a > 1,\ 2|b| < 1 \Rightarrow a > \max(1, 4b^2) \Rightarrow a > \max(1, 2b^2, 2|b|)$.

5. Preliminaries (II)

In this section, we make the following assumption;

ASSUMPTION 5.1. Let $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that
(a) $0 < \lambda < 1$, $-1/2 < \mu$, $F(\mu) > 1$, $2|Y_\mu| < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$
(b) $\omega_2(\lambda, \mu) > 0$ (c) $F(\phi_1) < 1$ or $F(\phi_6) < 1$.

By Theorem 2.10, we can take such the basis. So in next section, we shall consider six cases:

$(1A)$ $0 < \mu < 1,\ \phi_1 > 1$ $(2A)$ $\mu > 1,\ \phi_1 > 1$

$(3A)$ $\mu < 0,\ \phi_1 > 1$

$(1B)$ $0 < \mu < 1,\ \phi_1 < 1,\ F(\phi_6) < 1$ $(2B)$ $\mu > 1,\ \phi_1 < 1,\ F(\phi_6) < 1$

$(3B)$ $\mu < 0,\ \phi_1 < 1,\ F(\phi_6) < 1$

We note that
(A) $\phi_1 = [\omega_2] + \lambda > 1 \Leftrightarrow [\omega_2] \geq 1 \Leftrightarrow \omega_2 > 1$,
(B) $\phi_1 = [\omega_2] + \lambda < 1 \Leftrightarrow [\omega_2] = 0 \Leftrightarrow \omega_2 < 1$.

Lemma 5.2. *If* $\phi_1 < 1$, *then*
(1) $Y_\lambda < -1/2$ (2) $\omega_2(\lambda, \mu) > 1/2 - \omega_1 Y_\mu$.

Proof. (1) From $\phi_1 = [\omega_2] + \lambda < 1$, we have $[\omega_2] = 0$. By definition $\lambda_{(1)} = [-Y_\lambda] + \lambda$, $\lambda_{(2)} = [-Y_\lambda] + 1 + \lambda$. Since $\mathscr{R}$ is a reduced lattice, from $\phi_1 < 1$, we have $F(\phi_1) > 1$. Hence, by Assumpsion 5.1,(c), we have $F(\phi_6) < 1$. From $F(\phi_6) = F([\omega_2] + 1 + \lambda) = F(1 + \lambda) < 1$, we have $1 + \lambda = \lambda_{(1)}$ or $\lambda_{(2)}$.

(i) The case $1 + \lambda = \lambda_{(1)}$. Since $-1 < Y_\lambda + 1 = Y_{\lambda_{(1)}} < 0$, we have $-2 < Y_\lambda < -1$.

(ii) The case $1 + \lambda = \lambda_{(2)}$. We have $\lambda = \lambda_{(1)}$. Since $F(\lambda_{(2)}) < 1$, we have $0 < Y_{\lambda_{(2)}} < 1/2$. From this, $0 < Y_\lambda + 1 = Y_{\lambda_{(2)}} < 1/2$, so $-1 < Y_\lambda < -1/2$.

Finally, from (i)(ii), we have $Y_\lambda < -1/2$.

(2) From (1), we have $-Y_\lambda > 1/2$. Hence, $\omega_2(\lambda, \mu) = -Y_\lambda - \omega_1 Y_\mu > 1/2 - \omega_1 Y_\mu$. □

Corollary 5.3. $Y_\mu < 0 \Rightarrow \omega_2(\lambda, \mu) > 1/2$.

By Corollary 3.7 if $\theta_g = x + y\lambda + z\mu$ $(x, y, z \in \mathbf{Z})$, then $(y, z) \in \{(1,0), (1,1), (1,-1), (2,1), (d,1), (d+1,1), (2d+1,2), (d-1,1), (3d+2,3)\}$, where $d = [1/\omega_1(\lambda, \mu)] \geq 1$.

From Remark 3.5 and Corollary 5.3, we make the following tables in which we deside whether the possibility that $\theta_g = \psi_{i,y}$ $(1 \leq i \leq 10,\ i = 12)$ exists. Note that $y \geq 1 \Rightarrow [y\omega_2] \geq y[\omega_2]$.

Table 1

(y,z)	$\psi_{1,y} = [\omega_2 y] - 1 + y\lambda + [\omega_1 y]\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu < 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	$\mu < 0$ $\omega_2 < 1$	No.
$(1,0)$	$[\omega_2] - 1 + \lambda$			< 1	< 1	(1-1)
$(1,1)$	$[\omega_2] - 1 + \lambda$	impossible	impossible	impossible	impossible	
$(1,-1)$	$[\omega_2] - 1 + \lambda$	impossible	impossible	impossible	impossible	
$(2,1)$	$[2\omega_2] - 1 + 2\lambda + [2\omega_1]\mu$					(1-2)
$(d,1)$	$[d\omega_2] - 1 + d\lambda$	impossible	impossible	impossible	impossible	
$(d+1,1)$	$[(d+1)\omega_2] - 1 + (d+1)\lambda + \mu$					(1-3)
$(2d+1,2)$	$[(2d+1)\omega_2] - 1 + (2d+1)\lambda + [(2d+1)\omega_1]\mu$	$> \phi_6$				(1-4)
$(d-1,1)$	$[(d-1)\omega_2] - 1 + (d-1)\lambda$	impossible	impossible	impossible	impossible	
$(3d+2,3)$	$[(3d+2)\omega_2] - 1 + (3d+2)\lambda + [(3d+2)\omega_1]\mu$	$> \phi_6$	$> \phi_6$			(1-5)

Table 2. $(\mu > 0)$

(y,z)	$\psi_{2,y} = [\omega_2 y] - 1 + y\lambda + ([\omega_1 y] + 1)\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	No.
$(1,0)$	$[\omega_2] - 1 + \lambda + \mu$	impossible	impossible	
$(1,1)$	$[\omega_2] - 1 + \lambda + \mu$			(2-1)
$(1,-1)$	$[\omega_2] - 1 + \lambda + \mu$	impossible	impossible	
$(2,1)$	$[2\omega_2] - 1 + 2\lambda + ([2\omega_1] + 1)\mu$			(2-2)
$(d,1)$	$[d\omega_2] - 1 + d\lambda + \mu$			(2-3)
$(d+1,1)$	$[(d+1)\omega_2] - 1 + (d+1)\lambda + 2\mu$	impossible	impossible	
$(2d+1,2)$	$[(2d+1)\omega_2] - 1 + (2d+1)\lambda + ([(2d+1)\omega_1] + 1)\mu$	$> \phi_6$		(2-4)
$(d-1,1)$	$[(d-1)\omega_2] - 1 + (d-1)\lambda + \mu$			(2-5)
$(3d+2,3)$	$[(3d+2)\omega_2] - 1 + (3d+2)\lambda + ([(3d+2)\omega_1] + 1)\mu$	$> \phi_6$		(2-6)

Table 3

(y, z)	$\psi_{3,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] - 1)\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu < 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	$\mu < 0$ $\omega_2 < 1$	No.
$(1, 0)$	$[\omega_2] + \lambda - \mu$	impossible	impossible	impossible	impossible	
$(1, 1)$	$[\omega_2] + \lambda - \mu$	impossible	impossible	impossible	impossible	
$(1, -1)$	$[\omega_2] + \lambda - \mu$			< 1		(3-1)
$(2, 1)$	$[2\omega_2] + 2\lambda + ([2\omega_1] - 1)\mu$	impossible	impossible	impossible	impossible	
$(d, 1)$	$[d\omega_2] + d\lambda - \mu$	impossible	impossible	impossible	impossible	
$(d+1, 1)$	$[(d+1)\omega_2] + (d+1)\lambda$	impossible	impossible	impossible	impossible	
$(2d+1, 2)$	$[(2d+1)\omega_2] + (2d+1)\lambda + ([(2d+1)\omega_1] - 1)\mu$	impossible	impossible	impossible	impossible	
$(d-1, 1)$	$[(d-1)\omega_2] + (d-1)\lambda - \mu$	impossible	impossible	impossible	impossible	
$(3d+2, 3)$	$[(3d+2)\omega_2] + (3d+2)\lambda + ([(3d+2)\omega_1] - 1)\mu$	$> \phi_6$	$> \phi_6$			(3-2)

Table 4

(y, z)	$\psi_{4,y} = [\omega_2 y] + y\lambda + [\omega_1 y]\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu < 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	$\mu < 0$ $\omega_2 < 1$	No.
$(1, 0)$	$[\omega_2] + \lambda$			< 1	< 1	(4-1)
$(1, 1)$	$[\omega_2] + \lambda$	impossible	impossible	impossible	impossible	
$(1, -1)$	$[\omega_2] + \lambda$	impossible	impossible	impossible	impossible	
$(2, 1)$	$[2\omega_2] + 2\lambda + [2\omega_1]\mu$	$> \phi_6$				(4-2)
$(d, 1)$	$[d\omega_2] + d\lambda$	impossible	impossible	impossible	impossible	
$(d+1, 1)$	$[(d+1)\omega_2] + (d+1)\lambda + \mu$	$> \phi_6$				(4-3)
$(2d+1, 2)$	$[(2d+1)\omega_2] + (2d+1)\lambda + [(2d+1)\omega_1]\mu$	$> \phi_6$	$> \phi_6$			(4-4)
$(d-1, 1)$	$[(d-1)\omega_2] + (d-1)\lambda$	impossible	impossible	impossible	impossible	
$(3d+2, 3)$	$[(3d+2)\omega_2] + (3d+2)\lambda + [(3d+2)\omega_1]\mu$	$> \phi_6$	$> \phi_6$			(4-5)

Table 5

(y, z)	$\psi_{5,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] + 1)\mu$	$\mu > 0$ $\omega_2 > 1$	$\mu < 0$ $\omega_2 > 1$	$\mu > 0$ $\omega_2 < 1$	$\mu < 0$ $\omega_2 < 1$	No.
$(1, 0)$	$[\omega_2] + \lambda + \mu$	impossible	impossible	impossible	impossible	
$(1, 1)$	$[\omega_2] + \lambda + \mu$				< 1	(5-1)
$(1, -1)$	$[\omega_2] + \lambda + \mu$	impossible	impossible	impossible	impossible	
$(2, 1)$	$[2\omega_2] + 2\lambda + ([2\omega_1] + 1)\mu$	$> \phi_6$				(5-2)
$(d, 1)$	$[d\omega_2] + d\lambda + \mu$	$> \phi_6 (d \geq 2)$				(5-3)
$(d+1, 1)$	$[(d+1)\omega_2] + (d+1)\lambda + 2\mu$	impossible	impossible	impossible	impossible	
$(2d+1, 2)$	$[(2d+1)\omega_2] + (2d+1)\lambda + ([(2d+1)\omega_1] + 1)\mu$	$> \phi_6$	$> \phi_6$			(5-4)
$(d-1, 1)$	$[(d-1)\omega_2] + (d-1)\lambda + \mu$	$> \phi_6 (d \geq 3)$				(5-5)
$(3d+2, 3)$	$[(3d+2)\omega_2] + (3d+2)\lambda + ([(3d+2)\omega_1] + 1)\mu$	$> \phi_6$	$> \phi_6$			(5-6)

Table 6. $(\mu > 0)$

(y, z)	$\psi_{6,y} = [\omega_2 y] + y\lambda + ([\omega_1 y] + 2)\mu$	$\mu > 0$ $\omega_2 \lesseqgtr 1$	No.
$(1, 0)$	$[\omega_2] + \lambda + 2\mu$	impossible	
$(1, 1)$	$[\omega_2] + \lambda + 2\mu$	impossible	
$(1, -1)$	$[\omega_2] + \lambda + 2\mu$	impossible	
$(2, 1)$	$[2\omega_2] + 2\lambda + ([2\omega_1] + 2)\mu$	impossible	
$(d, 1)$	$[d\omega_2] + d\lambda + 2\mu$	impossible	
$(d+1, 1)$	$[(d+1)\omega_2] + (d+1)\lambda + 3\mu$	impossible	
$(2d+1, 2)$	$[(2d+1)\omega_2] + (2d+1)\lambda + ([(2d+1)\omega_1] + 2)\mu$	impossible	
$(d-1, 1)$	$[(d-1)\omega_2] + (d-1)\lambda + 2\mu$	impossible	
$(3d+2, 3)$	$[(3d+2)\omega_2] + (3d+2)\lambda + ([(3d+2)\omega_1] + 2)\mu$	impossible	

Table 7. $(\mu > 0)$

(y, z)	$\psi_{7,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] - 1)\mu$	$\mu > 0$ $\omega_2 \lesseqgtr 1$	No.
$(1, 0)$	$[\omega_2] + 1 + \lambda - \mu$	impossible	
$(1, 1)$	$[\omega_2] + 1 + \lambda - \mu$	impossible	
$(1, -1)$	$[\omega_2] + 1 + \lambda - \mu$		(7-1)
$(2, 1)$	$[2\omega_2] + 1 + 2\lambda + ([2\omega_1] - 1)\mu$	impossible	
$(d, 1)$	$[d\omega_2] + 1 + d\lambda - \mu$	impossible	
$(d + 1, 1)$	$[(d + 1)\omega_2] + 1 + (d + 1)\lambda$	impossible	
$(2d + 1, 2)$	$[(2d + 1)\omega_2] + 1 + (2d + 1)\lambda + ([(2d + 1)\omega_1] - 1)\mu$	impossible	
$(d - 1, 1)$	$[(d - 1)\omega_2] + 1 + (d - 1)\lambda - \mu$	impossible	
$(3d + 2, 3)$	$[(3d + 2)\omega_2] + 1 + (3d + 2)\lambda + ([(3d + 2)\omega_1] - 1)\mu$	$> \phi_6$	

Table 8

(y, z)	$\psi_{8,y} = [\omega_2 y] + 1 + y\lambda + [\omega_1 y]\mu$	$\mu \lesseqgtr 0$ $\omega_2 \lesseqgtr 1$	No.
$(1, 0)$	$[\omega_2] + 1 + \lambda$		(8-1)
$(1, 1)$	$[\omega_2] + 1 + \lambda$	impossible	
$(1, -1)$	$[\omega_2] + 1 + \lambda$	impossible	
$(2, 1)$	$[2\omega_2] + 1 + 2\lambda + [2\omega_1]\mu$	$> \phi_6$	
$(d, 1)$	$[d\omega_2] + 1 + d\lambda$	impossible	
$(d + 1, 1)$	$[(d + 1)\omega_2] + 1 + (d + 1)\lambda + \mu$	$> \phi_6$	
$(2d + 1, 2)$	$[(2d + 1)\omega_2] + 1 + (2d + 1)\lambda + [(2d + 1)\omega_1]\mu$	$> \phi_6$	
$(d - 1, 1)$	$[(d - 1)\omega_2] + 1 + (d - 1)\lambda$	impossible	
$(3d + 2, 3)$	$[(3d + 2)\omega_2] + 1 + (3d + 2)\lambda + [(3d + 2)\omega_1]\mu$	$> \phi_6$	

Table 9. ($\mu < 0$)

(y, z)	$\psi_{9,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] + 1)\mu$	$\mu < 0$ $\omega_2 \lesseqgtr 1$	No.
$(1, 0)$	$[\omega_2] + 1 + \lambda + \mu$	impossible	
$(1, 1)$	$[\omega_2] + 1 + \lambda + \mu$		(9-1)
$(1, -1)$	$[\omega_2] + 1 + \lambda + \mu$	impossible	
$(2, 1)$	$[2\omega_2] + 1 + 2\lambda + ([2\omega_1] + 1)\mu$	$> \phi_6$	
$(d, 1)$	$[d\omega_2] + 1 + d\lambda + \mu$	$> \phi_6 (d \geq 2)$	
$(d + 1, 1)$	$[(d + 1)\omega_2] + 1 + (d + 1)\lambda + 2\mu$	impossible	
$(2d + 1, 2)$	$[(2d + 1)\omega_2] + 1 + (2d + 1)\lambda + ([(2d + 1)\omega_1] + 1)\mu$	$> \phi_6$	
$(d - 1, 1)$	$[(d - 1)\omega_2] + 1 + (d - 1)\lambda + \mu$	$> \phi_6 (d \geq 3)$	
$(3d + 2, 3)$	$[(3d + 2)\omega_2] + 1 + (3d + 2)\lambda + ([(3d + 2)\omega_1] + 1)\mu$	$> \phi_6$	

Table 10. ($\mu < 0$)

(y, z)	$\psi_{10,y} = [\omega_2 y] + 1 + y\lambda + ([\omega_1 y] + 2)\mu$	$\mu < 0$ $\omega_2 \lesseqgtr 1$	No.
$(1, 0)$	$[\omega_2] + 1 + \lambda + 2\mu$	impossible	
$(1, 1)$	$[\omega_2] + 1 + \lambda + 2\mu$	impossible	
$(1, -1)$	$[\omega_2] + 1 + \lambda + 2\mu$	impossible	
$(2, 1)$	$[2\omega_2] + 1 + 2\lambda + ([2\omega_1] + 2)\mu$	impossible	
$(d, 1)$	$[d\omega_2] + 1 + d\lambda + 2\mu$	impossible	
$(d + 1, 1)$	$[(d + 1)\omega_2] + 1 + (d + 1)\lambda + 3\mu$	impossible	
$(2d + 1, 2)$	$[(2d + 1)\omega_2] + 1 + (2d + 1)\lambda + ([(2d + 1)\omega_1] + 2)\mu$	impossible	
$(d - 1, 1)$	$[(d - 1)\omega_2] + 1 + (d - 1)\lambda + 2\mu$	impossible	
$(3d + 2, 3)$	$[(3d + 2)\omega_2] + 1 + (3d + 2)\lambda + ([(3d + 2)\omega_1] + 2)\mu$	impossible	

Table 10 (continued)

(y, z)	$\psi_{12,y} = [\omega_2 y] + 2 + y\lambda + ([\omega_1 y] + 1)\mu$	$\mu < 0$ $\omega_2 \lesseqgtr 1$	No.
$(1, 0)$	$[\omega_2] + 2 + \lambda + \mu$	impossible	
$(1, 1)$	$[\omega_2] + 2 + \lambda + \mu$	$> \phi_6$	
$(1, -1)$	$[\omega_2] + 2 + \lambda + \mu$	impossible	
$(2, 1)$	$[2\omega_2] + 2 + 2\lambda + ([2\omega_1] + 1)\mu$	$> \phi_6$	
$(d, 1)$	$[d\omega_2] + 2 + d\lambda + \mu$	$> \phi_6$	
$(d+1, 1)$	$[(d+1)\omega_2] + 2 + (d+1)\lambda + 2\mu$	impossible	
$(2d+1, 2)$	$[(2d+1)\omega_2] + 2 + (2d+1)\lambda + ([(2d+1)\omega_1] + 1)\mu$	$> \phi_6$	
$(d-1, 1)$	$[(d-1)\omega_2] + 2 + (d-1)\lambda + \mu$	$> \phi_6 (d \geq 2)$	
$(3d+2, 3)$	$[(3d+2)\omega_2] + 2 + (3d+2)\lambda + ([(3d+2)\omega_1] + 1)\mu$	$> \phi_6$	

6. Main Theorems

THEOREM 6.1A. *Let $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that $0 < \lambda < 1,\ 0 < X_\mu < X_\lambda,\ 0 < \omega_1(\lambda, \mu) < 1,\ \omega_2(\lambda, \mu) > 0, a > 1,\ 2|b| < 1,\ 0 < \mu < 1,\ \phi_1 > 1$, where $a = F(\mu),\ b = Y_\mu$. Then*

(1) *If $F(\phi_1) < 1$:*

(i) *if $b < 0$, then the minimal point adjacent to 1 is ϕ_1, ϕ_3 or ϕ_4;*

(ii) *if $b > 0$, then the minimal point adjacent to 1 is ϕ_1 or ϕ_5.*

(2) *If $F(\phi_1) > 1,\ F(\phi_2) < 1$:*

(i) *if $b < 0$, then the minimal point adjacent to 1 is ϕ_2;*

(ii) *if $b > 0$, then the minimal point adjacent to 1 is ϕ_2 or ϕ_5.*

(3) *If $F(\phi_1) > 1,\ F(\phi_2) > 1,\ F(\phi_6) < 1$,*
then the minimal point adjacent to 1 is ϕ_6.

PROOF. Since $\phi_1 = [\omega_2] + \lambda > 1$, we have $[\omega_2] \geq 1$.

(1) was proved in [5], Theorem 2.1.

(2) We assume that $F(\psi_{4,1}) > 1,\ F(\psi_{5,1}) < 1$.

(i) the case $b < 0$, by Lemma 4.5,(4), we have $\phi_3 = \psi_{3,1} \neq \theta_g$. By Lemma 4.5,(8), we have $\phi_4 = \psi_{1,1} \neq \theta_g$. The others were proved in [5], Theorem 2.1;

(ii) The case $b > 0$. The case were all proved in [5], Theorem 2.1.

(3) We assume that $F(\psi_{4,1}) > 1,\ F(\psi_{5,1}) > 1,\ F(\psi_{8,1}) < 1$.

By Lemma 4.5,(1)(2) and Remark 4.4,(1), we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,1}\}$.

Also by Lemma 4.5,(10) we have $c_2 = [\omega_2] - \omega_2 < -1/2$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) from $\psi_{1,1} = \psi_{8,1} - 2$ and $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$.

(1-2) by Lemma 4.5,(12), $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu = 2[\omega_2] + 2\lambda + \mu$ or $2[\omega_2] - 1 + 2\lambda + \mu$. Since $c_2 < -1/2$, $\psi_{1,2} \neq 2[\omega_2] - 1 + 2\lambda + \mu$. Hence $\psi_{1,2} = 2[\omega_2] + 2\lambda + \mu > \psi_{8,1}$.

(1-3) $d \geq 2 \Rightarrow \psi_{1,d+1} > \psi_{8,1}$. If $d = 1$, then $\psi_{1,d+1} = \psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu$. This case is just the same as (1-2).

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) by Lemma 4.5,(4) $\phi_3 = \psi_{3,1} \neq \theta_g$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) by the assumption $\psi_{4,1} \neq \theta_g$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) by the assumption $\psi_{5,1} \neq \theta_g$.

As a result, $\psi_{8,1}$ remains.

(ii) The case $b > 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) by Lemma 4.5,(9), $\psi_{2,1} \neq \theta_g$.

(2-2) by Lemma 4.5,(12), $\psi_{2,2} = [2\omega_2] - 1 + 2\lambda + \mu = 2[\omega_2] + 2\lambda + \mu (> \psi_{8,1})$ or $2[\omega_2] - 1 + 2\lambda + \mu$.

The case $\psi_{2,2} = 2[\omega_2] - 1 + 2\lambda + \mu$. If $[\omega_2] \geq 2$, then we have $2[\omega_2] - 1 + 2\lambda + \mu > \psi_{8,1}$. If $[\omega_2] = 1$, then $\psi_{2,2} = 1 + 2\lambda + \mu$. We shall show that $F(1 + 2\lambda + \mu) > 1$. Since $F(\phi_6) = F(2 + \lambda) < 1$, we have $-1 < Y_{2+\lambda} < 1$, so $-3 < Y_\lambda < -1$. Suppose that $Y_\lambda > -3/2$. Then $Y_{2+\lambda} = 2 + Y_\lambda > 1/2$. From this, we have $1/4 + Z_{2+\lambda}^2 < Y_{2+\lambda}^2 + Z_{2+\lambda}^2 < 1$. Hence, $|Z_{2+\lambda}| < \sqrt{3}/2$. Since $Y_\lambda > -3/2$ and $Y_\lambda < -1$, we have $-1/2 < Y_{1+\lambda} < 0$. Hence, $F(1 + \lambda) = Y_{1+\lambda}^2 + Z_{1+\lambda}^2 = Y_{1+\lambda}^2 + Z_{2+\lambda}^2 < 1/4 + 3/4 = 1$. Since $F(\phi_1) = F(1 + \lambda) > 1$, we have reached a contradiction. Therefore, we have $Y_\lambda < -3/2$. From this, we have $Y_{1+2\lambda+\mu} = 1 + 2Y_\lambda + Y_\mu < 1 - 3 + Y_\mu < -3/2$. Hence, $F(1 + 2\lambda + \mu) > 1$.

(2-3) $d \geq 3 \Rightarrow \psi_{2,d} = [d\omega_2] - 1 + d\lambda + \mu > \psi_{8,1}$.

The case $d = 1, 2$ are just the same as (2-1) or (2-2).

(2-5) Similar to (2-3).

(b) In the case of $\psi_{4,y}$, based on Table 4, (4-1) by the assumption, $\psi_{4,1} \neq \theta_g$.

(c) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) by the assumption $\psi_{5,1} \neq \theta_g$.

(d) In the case of $\psi_{6,y}$, based on Table 6,
no case is included

(e) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) by Lemma 4.5,(5), $\psi_{7,1} \neq \theta_g$.

As a result, $\psi_{8,1}$ remains. □

Remark. From the proof in [5, Theorem 2.1], (1) and (2) don't require the assumption $0 < X_\mu < X_\lambda$. Moreover, in (1) and (2) (except for the part of ϕ_4), we can weaken the condition from $a > 1$, $2|b| < 1$ to $a > \max(1, 2b^2, 2|b|)$.

Theorem 6.2A. *Let $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $2|b| < 1$, $\mu > 1$, $\phi_1 > 1$, where $a = F(\mu)$, $b = Y_\mu$. Then*

(1) If $F(\phi_1) < 1$:

(i) if $b < 0$, then the minimal point adjacent to 1 is ϕ_1, ϕ_3 or ϕ_4;

(ii) if $b > 0$, then the minimal point adjacent to 1 is ϕ_1 or ϕ_7.

(2) If $F(\phi_1) > 1$, $F(\phi_6) < 1$:

(i) if $b < 0$, then the minimal point adjacent to 1 is ϕ_6;

(ii) if $b > 0$, then the minimal point adjacent to 1 is ϕ_5 or ϕ_6.

Proof. Since $\phi_1 = \psi_{4,1} = [\omega_2] + \lambda > 1$, we have $[\omega_2] \geq 1$.

(1) We assume that $F(\psi_{4,1}) < 1$.

By Lemma 4.5,(1)(2) and Remark 4.4,(2), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{7,y}, \psi_{4,1}\}$.

(i) The case $b < 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) $\psi_{1,1}$.

(1-2) $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu > \psi_{8,1}$.

(1-3) $\psi_{1,d+1} > \psi_{8,1} > \psi_{4,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) $\psi_{3,1}$.

As a result, $\psi_{4,1}$, $\psi_{3,1}$ and $\psi_{1,1}$ remain.

(ii) The case $b > 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{7,y}, \psi_{4,1}\}$.

(a) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) $\psi_{7,1}$.

As a result, $\psi_{4,1}$ and $\psi_{7,1}$ remain.

(2) We assume that $F(\psi_{4,1}) > 1$, $F(\psi_{8,1}) < 1$.

By Lemma 4.5,(1)(2) and Remark 4.4,(2), we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) from $\psi_{1,1} = \psi_{8,1} - 2$ and $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$.

(1-2) $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu > \psi_{8,1}$.

(1-3) $\psi_{1,d+1} > \psi_{8,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) by Lemma 4.5,(4) $\phi_3 = \psi_{3,1} \neq \theta_g$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) by the assumption $\psi_{4,1} \neq \theta_g$.

As a result, $\psi_{8,1}$ remains.

(ii) The case $b > 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) $\psi_{2,1} = [\omega_2] - 1 + \lambda + \mu (> \psi_{4,1})$.

(2-2) $\psi_{2,2} = [2\omega_2] - 1 + 2\lambda + \mu > \psi_{8,1}$.

(2-3) $d \geq 3 \Rightarrow \psi_{2,d} = [d\omega_2] - 1 + d\lambda + \mu > \psi_{8,1}$.

The cases $d = 1, 2$ are just the same as (2-1) or (2-2).

(2-5) Similar to (2-3).

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) by the assumption $\psi_{4,1} \neq \theta_g$.

(c) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) by Lemma 4.5,(5) $\psi_{7,1} \neq \theta_g$.

As a result, $\psi_{8,1}$ and $\psi_{2,1}$ remain. □

Theorem 6.3A. *Let $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $2|b| < 1$, $\mu < 0$, $\phi_1 > 1$, where $a = F(\mu)$, $b = Y_\mu$. Then*

(1) *If $F(\phi_1) < 1$:*

(i) *if $[\omega_2] \geq 2$, then the minimal point adjacent to 1 is ϕ_1, ϕ_2 or ϕ_4;*

(ii-a) *if $[\omega_2] = 1$, $\lambda + \mu < 0$, then the minimal point adjacent to 1 is ϕ_1 or $1 + \phi_9$,*

(ii-b) *if $[\omega_2] = 1$, $\lambda + \mu > 0$, then the minimal point adjacent to 1 is ϕ_1 or ϕ_2.*

(2) *If $F(\phi_1) > 1$, $F(\phi_6) < 1$, then the minimal point adjacent to 1 is ϕ_2, ϕ_6 or ϕ_8.*

Proof. Since $\mu < 0$ and $0 < X_\mu$, we have $b < 0$ and $-1/2 < \mu$.

From Table 10 and Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,y}, \psi_{9,y}\}$.

(1) We assume that $F(\psi_{4,1}) < 1$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) $\psi_{1,1}$.

(1-2) by Lemma 4.5,(12) $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu = 2[\omega_2] + 2\lambda + \mu (> \psi_{4,1})$ or $2[\omega_2] - 1 + 2\lambda + \mu$.

The case $\psi_{1,2} = 2[\omega_2] - 1 + 2\lambda + \mu$. If $[\omega_2] \geq 2$, then we have $\psi_{1,2} > \psi_{4,1}$. If $[\omega_2] = 1$, $\psi_{1,2} = 1 + 2\lambda + \mu$.

(1-3) $d \geq 2 \Rightarrow \psi_{1,d+1} \geq [3\omega_2] - 1 + 3\lambda + \mu > \psi_{4,1}$. The case $d = 1$ is just the same as (1-2).

(1-4) $\psi_{1,2d+1} > \psi_{4,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) $\psi_{3,1} = [\omega_2] + \lambda - \mu > [\omega_2] + \lambda = \psi_{4,1}$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) $\psi_{4,1}$.

(4-2) $\psi_{4,2} = [2\omega_2] + 2\lambda + \mu > \psi_{4,1}$.

(4-3) $\psi_{4,d+1} > \psi_{4,1}$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) $\psi_{5,1} = [\omega_2] + \lambda + \mu$.

(5-2) $\psi_{5,2} = [2\omega_2] + 2\lambda + \mu > \psi_{4,1}$.

(5-3) $d \geq 2 \Rightarrow \psi_{5,d} \geq [2\omega_2] + 2\lambda + \mu > \psi_{4,1}$.

The case $d = 1$ is just the same as (5-1).

(5-5) Similar to (5-3).

(e) In the case of $\psi_{8,y}$, based on Table 8,

(8-1) $\psi_{8,1} > \psi_{4,1}$.

(f) In the case of $\psi_{9,y}$, based on Table 9,

(9-1) $\psi_{9,1} = [\omega_2] + 1 + \lambda + \mu > \psi_{4,1}$.

As a result, $\psi_{4,1}$, $\psi_{5,1}$, $\psi_{1,1}$ and $1 + 2\lambda + \mu$ remain. Moreover, If $[\omega_2] \geq 2$, then we have $\theta_g \neq 1 + 2\lambda + \mu$. The case $[\omega_2] = 1$. Since $\phi_4 = \psi_{1,1} = [\omega_2] - 1 + \lambda = \lambda < 1$, we have $\theta_g \neq \psi_{1,1}$. If $\lambda + \mu < 0$, then we have $\phi_2 = 1 + \lambda + \mu < 1$. If $\lambda + \mu > 0$, then we have $1 + 2\lambda + \mu \neq \theta_g$, because $1 + 2\lambda + \mu = 1 + \lambda + (\lambda + \mu) > 1 + \lambda = \psi_{4,1}$.

(2) We assume that $F(\phi_1) > 1$, $F(\phi_6) < 1$.

We note that by Lemma 4.5,(10), we have $c_2 = [\omega_2] - \omega_2 < -1/2$. So by Lemma 4.5,(12), we have $[2\omega_2] = 2[\omega_2] + 1$.

(a) In the case of $\psi_{1,y}$, based on Table 1, (1-1) from $\psi_{1,1} = \psi_{8,1} - 2$ and $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$.

(1-2) $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + \mu = 2[\omega_2] + 2\lambda + \mu$. If such a $\psi_{1,2}$ exist, then by $[2\omega_1] = 1$, we have $c_1 < -1/2 (\Leftrightarrow [2\omega_1] = 1)$.

(i) The case $[\omega_2] \geq 2$. We have $\psi_{1,2} > \psi_{8,1}$.

(ii) The case $[\omega_2] = 1$. $\psi_{1,2} = 2 + 2\lambda + \mu > 2 + \lambda + \mu = \psi_{9,1}$.

From Lemma 4.5,(11), we have $F(\psi_{9,1}) < F(\psi_{8,1})$. So we have $F(\psi_{9,1}) < 1$. Therefore, $\psi_{1,2} = 2 + 2\lambda + \mu \neq \theta_g$.

(1-3) (i) The case $d \geq 2$. We have $\psi_{1,d+1} \geq [3\omega_2] - 1 + 3\lambda + \mu \geq [2\omega_2] + [\omega_2] - 1 + 3\lambda + \mu = 3[\omega_2] + 3\lambda + \mu > \psi_{8,1}$.

(ii) The case $d = 1$. Since $d = 1 \Leftrightarrow [2\omega_1] = 1$, this case is just the same as (1-2).

(1-4) $\psi_{1,2d+1} \geq [3\omega_2] - 1 + 3\lambda + 2\mu \geq [2\omega_2] + [\omega_2] - 1 + 3\lambda + 2\mu = 3[\omega_2] + 3\lambda + 2\mu > \psi_{8,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) by Lemma 4.5,(4) $\phi_3 = \psi_{3,1} \neq \theta_g$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-1) $F(\psi_{4,1}) > 1$.

(4-2) $\psi_{4,2} = [2\omega_2] + 2\lambda + \mu = 2[\omega_2] + 1 + 2\lambda + \mu > \psi_{8,1}$.

(4-3) $\psi_{4,d+1} \geq [2\omega_2] + 2\lambda + \mu > \psi_{8,1}$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) $\psi_{5,1} = [\omega_2] + \lambda + \mu$.

(5-2) $\psi_{5,2} = [2\omega_2] + 2\lambda + \mu = 2[\omega_2] + 1 + 2\lambda + \mu > \psi_{8,1}$.

(5-3) $d \geq 2 \Rightarrow \psi_{5,d} \geq [2\omega_2] + 2\lambda + \mu > \psi_{8,1}$.

The case $d = 1$ is just the same as (5-1).

(5-5) Similar to (5-3).

(e) In the case of $\psi_{8,y}$, based on Table 8,

(8-1) $F(\psi_{8,1}) < 1$.

(f) In the case of $\psi_{9,y}$, based on Table 9,

(9-1) $\psi_{9,1} = [\omega_2] + 1 + \lambda + \mu$.

As a result, $\psi_{8,1}, \psi_{5,1}$ and $\psi_{9,1}$ remain. □

Theorem 6.1B. *Let* $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda,\mu) < 1$, $\omega_2(\lambda,\mu) > 0, a > 1$, $2|b| < 1$, $0 < \mu < 1$, $\phi_1 < 1$, $F(\phi_6) < 1$, *where* $a = F(\mu)$, $b = Y_\mu$. *Then*

(1) *If* $F(\phi_2) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_2.

(2) *If* $\phi_2 > 1$, $F(\phi_2) > 1$, *then the minimal point adjacent to* 1 *is* ϕ_6.

(3) *If* $\phi_2 < 1$:

(i) *if* $b < 0$, *then the minimal point adjacent to* 1 *is* ϕ_6;

(ii-a) *if* $b > 0$, $2\lambda + \mu < 1$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* ϕ_{10},

(ii-b) *if* $b > 0$, $2\lambda + \mu > 1$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* ϕ_9.

Proof. From the assumption $\phi_1 < 1$, by Lemma 5.2,(1), we have $Y_\lambda < -1/2$. By Corollary 5.3, if $b < 0$, then we have $1 > \omega_2 > 1/2$.

(1) We assume that $F(\psi_{5,1}) < 1$. Since $\mathscr{R}$ is a reduced lattice, we have $\psi_{5,1} = [\omega_2] + \lambda + ([\omega_1] + 1)\mu = \lambda + \mu > 1$.

By Lemma 4.5,(1)(2) and Remark 4.4,(1) we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{7,y}, \psi_{5,1}\}$.

(i) The case $b < 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-2) since $[2\omega_2] = 1$, we have $\psi_{1,2} = 2\lambda + \mu > \psi_{8,1} > \psi_{5,1}$.

(1-3) $[(d+1)\omega_2] \geq 2 \Rightarrow \psi_{1,d+1} > \psi_{8,1} > \psi_{5,1}$. $[(d+1)\omega_2] = 1 \Rightarrow \psi_{1,d+1} = (d+1)\lambda + \mu \Rightarrow Y_{\psi_{1,d+1}} = (d+1)Y_\lambda + Y_\mu < -1$.

(1-4) $[(2d+1)\omega_2] \geq 2 \Rightarrow \psi_{1,2d+1} > \psi_{8,1} > \psi_{5,1}$. $[(2d+1)\omega_2] = 1 \Rightarrow \psi_{1,2d+1} = (2d+1)\lambda + 2\mu > \psi_{8,1} > \psi_{5,1}$.

(1-5) from $[(3d+2)\omega_2] \geq 2$, we have $\psi_{1,3d+2} \geq 1 + (3d+2)\lambda + 3\mu > \psi_{8,1} > \psi_{5,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-2) $\psi_{3,3d+2} > \psi_{8,1} > \psi_{5,1}$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) since $[2\omega_2] = 1$, we have $\psi_{4,2} = 1 + 2\lambda + \mu > \psi_{8,1} > \psi_{5,1}$.

(4-3) $\psi_{4,d+1} > \psi_{8,1} > \psi_{5,1}$.

(4-4) $\psi_{4,2d+1} > \psi_{8,1} > \psi_{5,1}$.

(4-5) $\psi_{4,3d+2} > \psi_{8,1} > \psi_{5,1}$.

(ii) The case $b > 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{7,y}, \psi_{5,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) $\psi_{2,1} = -1 + \lambda + \mu < 1$.

(2-2) $[2\omega_2] = 0 \Rightarrow \psi_{2,2} = -1 + 2\lambda + \mu \Rightarrow Y_{\psi_{2,2}} = -1 + 2Y_\lambda + Y_\mu < -1$. $[2\omega_2] = 1 \Rightarrow \psi_{2,2} = 2\lambda + \mu > \psi_{8,1} > \psi_{5,1}$.

(2-3) $[d\omega_2] \geq 2 \Rightarrow \psi_{2,d} > \psi_{8,1} > \psi_{5,1}$. $[d\omega_2] = 1 \Rightarrow$ Since $d \geq 2$, $\psi_{2,d} = d\lambda + \mu > \psi_{8,1} > \psi_{5,1}$. $[d\omega_2] = 0 \Rightarrow \psi_{2,d} = -1 + d\lambda + \mu \Rightarrow Y_{\psi_{2,d}} = -1 + dY_\lambda + Y_\mu < -1$.

(2-4) $[(2d+1)\omega_2] \geq 2 \Rightarrow \psi_{2,2d+1} > \psi_{8,1} > \psi_{5,1}$. $[(2d+1)\omega_2] = 1 \Rightarrow \psi_{2,2d+1} = (2d+1)\lambda + 2\mu > \psi_{8,1} > \psi_{5,1}$. $[(2d+1)\omega_2] = 0 \Rightarrow \psi_{2,2d+1} = -1 + (2d+1)\lambda + 2\mu \Rightarrow Y_{\psi_{2,2d+1}} = -1 + (2d+1)Y_\lambda + 2Y_\mu < -1$.

(2-5) Similar to (2-3).

(2-6) $[(3d+2)\omega_2] \geq 2 \Rightarrow \psi_{2,3d+2} > \psi_{8,1} > \psi_{5,1}$. $[(3d+2)\omega_2] = 1 \Rightarrow \psi_{2,3d+2} = (3d+2)\lambda + 3\mu > \psi_{8,1} > \psi_{5,1}$. $[(3d+2)\omega_2] = 0 \Rightarrow \psi_{2,3d+2} = -1 + (3d+2)\lambda + 3\mu \Rightarrow Y_{\psi_{2,3d+2}} = -1 + (3d+2)Y_\lambda + 3Y_\mu < -1$.

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $[2\omega_2] = 0 \Rightarrow \psi_{4,2} = 2\lambda + \mu > \psi_{8,1} > \psi_{5,1}$. $[2\omega_2] = 1 \Rightarrow \psi_{4,2} = 1 + 2\lambda + \mu > \psi_{8,1} > \psi_{5,1}$.

(4-3) $\psi_{4,d+1} > \psi_{8,1} > \psi_{5,1}$.

(4-4) $\psi_{4,2d+1} > \psi_{8,1} > \psi_{5,1}$.

(4-5) $\psi_{4,3d+2} > \psi_{8,1} > \psi_{5,1}$.

(c) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) by Lemma 4.5,(5) $\psi_{7,1} \neq \theta_g$.

As a result, $\psi_{5,1}$ remains.

(2) We assume that $\psi_{5,1} = \lambda + \mu > 1$, $F(\psi_{5,1}) > 1$.

By Lemma 4.5,(1)(2) and Remark 4.4,(1) we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,
similar to (1).

(b) In the case of $\psi_{3,y}$, based on Table 3,
similar to (1).

(c) In the case of $\psi_{4,y}$, based on Table 4,
similar to (1).

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) from the assumption, $F(\psi_{5,1}) > 1$.

(5-2) $\psi_{5,2} > \phi_6$. (5-3) $\psi_{5,d} > \phi_6 (d \geq 2)$.

(5-4) $\psi_{5,2d+1} > \phi_6$. (5-5) $\psi_{5,d-1} > \phi_6 (d \geq 3)$.

As a result, $\psi_{8,1}$ remains.

(ii) The case $b > 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,
similar to (1).

(b) In the case of $\psi_{4,y}$, based on Table 4,
similar to (1).

(c) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) from the assumption, $F(\psi_{5,1}) > 1$.

(5-2) $\psi_{5,2} > \phi_6$. (5-3) $\psi_{5,d} > \phi_6 (d \geq 2)$.

(5-4) $\psi_{5,2d+1} > \phi_6$. (5-5) $\psi_{5,d-1} > \phi_6 (d \geq 3)$.

(d) In the case of $\psi_{6,y}$, based on Table 6,
no case included

(e) In the case of $\psi_{7,y}$, based on Table 7,
similar to (1).

As a result, $\psi_{8,1}$ remains.

(3) We assume that $\psi_{5,1} < 1$.

By Lemma 4.5,(1)(2) and Remark 4.4,(1) we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-2) $\psi_{1,2} = 2\lambda + \mu$, $Y_{\psi_{1,2}} = 2Y_\lambda + Y_\mu < -1$.

(1-3) The case $d \geq 3$. $\psi_{1,d+1} > 1 + 4\lambda + \mu > \phi_6$. The case $d = 2$. $\psi_{1,d+1} = [3\omega_2] - 1 + 3\lambda + \mu$. $[3\omega_2] = 2 \Rightarrow \psi_{1,d+1} = 1 + 3\lambda + \mu > \phi_6$. $[3\omega_2] = 1 \Rightarrow \psi_{1,d+1} = 3\lambda + \mu$. $Y_{\psi_{1,d+1}} = 3Y_\lambda + Y_\mu < -1$.

(1-4) The case $d \geq 2$. $\psi_{1,2d+1} > \phi_6$.

The case $d = 1$. $\psi_{1,2d+1} = [3\omega_2] - 1 + 3\lambda + 2\mu$. $[3\omega_2] = 2 \Rightarrow \psi_{1,2d+1} = 1 + 3\lambda + 2\mu > \phi_6$. $[3\omega_2] = 1 \Rightarrow \psi_{1,2d+1} = 3\lambda + 2\mu$. $Y_{\psi_{1,2d+1}} = 3Y_\lambda + 2Y_\mu < -1$.

(1-5) $\psi_{1,3d+2} > \phi_6$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-2) $\psi_{3,3d+2} > \phi_6$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $\psi_{4,2} > \phi_6$. (4-3) $\psi_{4,d+1} > \phi_6$. (4-4) $\psi_{4,2d+1} > \phi_6$. (4-5) $\psi_{4,3d+2} > \phi_6$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) from the assumption, $\psi_{5,1} < 1$. (5-2) $\psi_{5,2} > \phi_6$.

(5-3) $\psi_{5,d} > \phi_6 (d \geq 2)$. (5-4) $\psi_{5,2d+1} > \phi_6$. (5-5) $\psi_{5,d-1} > \phi_6 (d \geq 3)$.

As a result, $\psi_{8,1}$ remains.

(ii) The case $b > 0$. by Lemma 4.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) $\psi_{2,1} = -1 + \lambda + \mu < 1$.

(2-2) $[2\omega_2] = 0 \Rightarrow \psi_{2,2} = -1 + 2\lambda + \mu < \lambda < 1$. $[2\omega_2] = 1 \Rightarrow \psi_{2,2} = 2\lambda + \mu$.

(2-3) The case $[d\omega_2] \geq 2$. $\psi_{2,d} > \psi_{8,1} > \psi_{5,1}$.

The case $[d\omega_2] = 1$. We have $d \geq 2 \Rightarrow \psi_{2,d} = d\lambda + \mu$. If $d \geq 3$, then we have $Y_{\psi_{2,d}} = dY_\lambda + Y_\mu < -1$. Hence, only when $d = 2$, it is possible to have $\theta_g = \psi_{2,d} = \psi_{2,2} = 2\lambda + \mu$. The case $[d\omega_2] = 0$. $\psi_{2,d} = -1 + d\lambda + \mu$. $Y_{\psi_{2,d}} = -1 + dY_\lambda + Y_\mu < -1$.

(2-4) The case $[(2d+1)\omega_2] \geq 2$. $\psi_{2,2d+1} > \psi_{8,1}$. The case $[(2d+1)\omega_2] = 1$. $\psi_{2,2d+1} = (2d+1)\lambda + 2\mu$. If $d \geq 2$, then we have $Y_{\psi_{2,2d+1}} = (2d+1)Y_\lambda + 2Y_\mu < -1$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{2,3} = 3\lambda + 2\mu$. The case $[(2d+1)\omega_2] = 0$. $\psi_{2,2d+1} = -1 + (2d+1)\lambda + 2\mu$. $Y_{\psi_{2,2d+1}} = -1 + (2d+1)Y_\lambda + 2Y_\mu < -1$.

(2-5) Similar to (2-3).

(2-6) The case $[(3d+2)\omega_2] \geq 2$. $\psi_{2,3d+2} > \psi_{8,1}$. The case $[(3d+2)\omega_2] = 1$. $\psi_{2,3d+2} = (3d+2)\lambda + 3\mu$. $Y_{\psi_{2,3d+2}} = (3d+2)Y_\lambda + 3Y_\mu < -1$. The case $[(3d+2)\omega_2] = 0$. $\psi_{2,3d+2} = -1 + (3d+2)\lambda + 3\mu$. $Y_{\psi_{2,3d+2}} = -1 + (3d+2)Y_\lambda + 3Y_\mu < -1$.

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $[2\omega_2] = 0 \Rightarrow \psi_{4,2} = 2\lambda + \mu$. $[2\omega_2] = 1 \Rightarrow \psi_{4,2} = 1 + 2\lambda + \mu > \psi_{8,1}$.

(4-3) The case $[(d+1)\omega_2] \geq 1$. $\psi_{4,d+1} > \psi_{8,1}$. The case $[(d+1)\omega_2] = 0$. $\psi_{4,d+1} = (d+1)\lambda + \mu$. If $d \geq 2$, then we have $Y_{\psi_{4,d+1}} = (d+1)Y_\lambda + Y_\mu < -1$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{4,2} = 2\lambda + \mu$.

(4-4) The case $[(2d+1)\omega_2] \geq 1$. $\psi_{4,2d+1} > \psi_{8,1}$. The case $[(2d+1)\omega_2] = 0$. $\psi_{4,2d+1} = (2d+1)\lambda + 2\mu$. If $d \geq 2$, then we have $Y_{\psi_{4,2d+1}} = (2d+1)Y_\lambda + 2Y_\mu < -1$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{4,3} = 3\lambda + 2\mu$.

(4-5) $[(3d+2)\omega_2] \geq 1 \Rightarrow \psi_{4,3d+2} > \psi_{8,1}$. $[(3d+2)\omega_2] = 0 \Rightarrow \psi_{4,3d+2} = (3d+2)\lambda + 3\mu$. $Y_{\psi_{4,3d+2}} = (3d+2)Y_\lambda + 3Y_\mu < -1$.

(c) In the case of $\psi_{5,y}$, based on Table 5,

(5-1) from the assumption, $F(\psi_{5,1}) > 1$.

(5-2) $[2\omega_2] = 0 \Rightarrow \psi_{5,2} = 2\lambda + \mu$. $[2\omega_2] = 1 \Rightarrow \psi_{5,2} = 1 + 2\lambda + \mu > \psi_{8,1}$.

(5-3) The case $[d\omega_2] \geq 1$. $\psi_{5,d} > \psi_{8,1}$.

The case $[d\omega_2] = 0$. $\psi_{5,d} = d\lambda + \mu$. If $d \geq 3$, then we have $Y_{\psi_{5,d}} = dY_\lambda + Y_\mu < -1$. Hence, only when $d = 2$, it is possible to have $\theta_g = \psi_{5,2} = 2\lambda + \mu$.

(5-4) The case $[(2d+1)\omega_2] \geq 1$. $\psi_{5,2d+1} > \psi_{8,1}$. The case $[(2d+1)\omega_2] = 0$. $\psi_{5,2d+1} = (2d+1)\lambda + 2\mu$. If $d \geq 2$, then we have $Y_{\psi_{5,2d+1}} = (2d+1)Y_\lambda + 2Y_\mu < -1$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{5,3} = 3\lambda + 2\mu$.

(5-5) The case $[(d-1)\omega_2] \geq 1$. $\psi_{5,d-1} > \psi_{8,1}$. The case $[(d-1)\omega_2] = 0$. $\psi_{5,d-1} = (d-1)\lambda + \mu$. If $d \geq 4$, then we have $Y_{\psi_{5,d-1}} = (d-1)Y_\lambda + Y_\mu < -1$. Hence, only when $d = 3$, it is possible to have $\theta_g = \psi_{5,2} = 2\lambda + \mu$.

(d) In the case of $\psi_{6,y}$. based on Table 6,
no case included

(e) In the case of $\psi_{7,y}$, based on Table 7,

(7-1) By Lemma 4.5,(5) $\psi_{7,1} \neq \theta_g$.

As a result, $2\lambda + \mu, 3\lambda + 2\mu$ and $\psi_{8,1}$ remain. If $2\lambda + \mu < 1$, then we have

$2\lambda+\mu \neq \theta_g$. If $2\lambda+\mu > 1$, then we have $3\lambda+2\mu \neq \theta_g$, because $3\lambda+2\mu = (2\lambda+\mu)+\lambda+\mu > 1+\lambda = \psi_{8,1}$. □

THEOREM 6.2B. *Let $\mathscr{R} = \langle 1, \lambda, \mu\rangle$ be a reduced lattice of K such that $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $2|b| < 1$, $\mu > 1$, $\phi_1 < 1$, $F(\phi_6) < 1$, where $a = F(\mu)$, $b = Y_\mu$. Then the minimal point adjacent to* 1 *is ϕ_6.*

PROOF. From the assumption $\phi_1 < 1$, by Lemma 5.2,(1), we have $Y_\lambda < -1/2$. By Corollary 5.3, if $b < 0$, then we have $\omega_2 > 1/2$.

By Lemma 4.5,(1)(2) and Remark 4.4,(2) we have $\theta_g \in \{\psi_{1,y}, \psi_{2,y}, \psi_{3,y}, \psi_{4,y}, \psi_{7,y}, \psi_{8,1}\}$.

(i) The case $b < 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-1) from $\psi_{1,1} = \psi_{8,1} - 2$ and $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$.

(1-2) $\psi_{1,2} = 2\lambda + \mu > \psi_{8,1}$. (1-3) $\psi_{1,d+1} > \psi_{8,1}$.

(1-4) $\psi_{1,2d+1} > \psi_{8,1}$. (1-5) $\psi_{1,3d+2} > \psi_{8,1}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-2) $\psi_{3,3d+2} > \psi_{8,1}$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $\psi_{4,2} > \psi_{8,1}$. (4-3) $\psi_{4,3} > \psi_{8,1}$.

(4-4) $\psi_{4,2d+1} > \psi_{8,1}$. (4-5) $\psi_{4,3d+2} > \psi_{8,1}$.

As a result $\psi_{8,1}$ remains.

(ii) The case $b > 0$. By Lemma 4.5,(3), we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{7,y}, \psi_{8,1}\}$.

(a) In the case of $\psi_{2,y}$, based on Table 2,

(2-1) $\psi_{2,1} = -1 + \lambda + \mu$. $Y_{\psi_{2,1}} = -1 + Y_\lambda + Y_\mu < -1$.

(2-2) $\psi_{2,2} = [2\omega_2] - 1 + 2\lambda + \mu$. $[2\omega_2] = 0 \Rightarrow \psi_{2,2} = -1 + 2\lambda + \mu$. $Y_{\psi_{2,2}} = -1 + 2Y_\lambda + Y_\mu < -1$. $[2\omega_2] = 1 \Rightarrow \psi_{2,2} = 2\lambda + \mu > \psi_{8,1}$.

(2-3) $[d\omega_2] \geq 1 \Rightarrow \psi_{2,d} = [d\omega_2] - 1 + d\lambda + \mu > \psi_{8,1}$. $[d\omega_2] = 0 \Rightarrow \psi_{2,d} = -1 + d\lambda + \mu$. $Y_{\psi_{2,d}} = -1 + dY_\lambda + Y_\mu < -1$.

(2-4) $\psi_{2,2d+1} > \psi_{8,1}$. (2-5) Similar to (2-3).

(2-6) $\psi_{2,3d+2} > \psi_{8,1}$.

(b) In the case of $\psi_{4,y}$, based on Table 4,

(4-2) $\psi_{4,2} > \psi_{8,1}$. (4-3) $\psi_{4,d+1} > \psi_{8,1}$.

(4-4) $\psi_{4,2d+1} > \psi_{8,1}$. (4-5) $\psi_{4,3d+2} > \psi_{8,1}$.

(c) In the case of $\psi_{7,y}$, based on Table 7,

$$\psi_{7,1} = 1 + \lambda - \mu < \lambda < 1.$$

As a result, $\psi_{8,1}$ remains. □

THEOREM 6.3B. *Let* $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $2|b| < 1$, $\mu < 0$, $\phi_1 < 1$, $F(\phi_6) < 1$, *where* $a = F(\mu)$, $b = Y_\mu$. *Then*

(1) *If* $F(\phi_8) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_8.

(2) *If* $F(\phi_8) > 1$:

(i) *if* $2\lambda + \mu < 0$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* $\phi_6 + \phi_9$;

(ii) *if* $2\lambda + \mu > 0$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* $1 + \phi_9$.

PROOF. From the assumption $\phi_1 < 1$, by Lemma 5.2,(1), we have $Y_\lambda < -1/2$. Since $\mu < 0$ and $0 < X_\mu$, we have $b < 0$. By Corollary 5.3, we have $\omega_2 > 1/2$. From Table 10 and Lemma 4.5,(3), we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,y}, \psi_{9,y}\}$.

(a) In the case of $\psi_{1,y}$, based on Table 1,

(1-2) $\psi_{1,2} = 2\lambda + \mu$. $Y_{\psi_{1,2}} = 2Y_\lambda + Y_\mu < -1$.

*(1-3) $d \geq 5 \Rightarrow \psi_{1,d+1} \geq [6\omega_2] - 1 + 6\lambda + \mu \geq 2 + 6\lambda + \mu > \psi_{8,1}$. $d = 1 \Rightarrow \psi_{1,d+1} = 2\lambda + \mu$. $Y_{\psi_{1,d+1}} = 2Y_\lambda + Y_\mu < -1$.

Hence, only when $2 \leq d \leq 4$, it is possible to have $\theta_g = \psi_{1,d+1}$.

*(1-4) $d \geq 3 \Rightarrow \psi_{1,2d+1} \geq [7\omega_2] - 1 + 7\lambda + 2\mu \geq 2 + 7\lambda + 2\mu > \psi_{8,1}$.

Hence, only when $1 \leq d \leq 2$, it is possible to have $\theta_g = \psi_{1,2d+1}$.

*(1-5) $d \geq 2 \Rightarrow \psi_{1,3d+2} \geq [8\omega_2] - 1 + 8\lambda + 3\mu \geq 3 + 8\lambda + 3\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{1,2d+1} = \psi_{1,5}$.

(b) In the case of $\psi_{3,y}$, based on Table 3,

(3-1) By Lemma 4.5,(4), $\phi_3 = \psi_{3,1} \neq \theta_g$.

*(3-2) $d \geq 2 \Rightarrow \psi_{3,3d+2} > \psi_{8,1}$. Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{3,3d+2} = \psi_{3,5}$.

(c) In the case of $\psi_{4,y}$, based on Table 4,

*(4-2) $\psi_{4,2} = 1 + 2\lambda + \mu$.

*(4-3) $d \geq 3 \Rightarrow \psi_{4,d+1} \geq [4\omega_2] + 4\lambda + \mu \geq 2 + 4\lambda + \mu > \psi_{8,1}$.

Hence, only when $1 \leq d \leq 2$, it is possible to have $\theta_g = \psi_{4,d+1}$.

*(4-4) $d \geq 2 \Rightarrow \psi_{4,2d+1} \geq [5\omega_2] + 5\lambda + 2\mu \geq 2 + 5\lambda + 2\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{4,2d+1}$.

*(4-5) $d \geq 2 \Rightarrow \psi_{4,3d+2} \geq [8\omega_2] + 8\lambda + 3\mu \geq 4 + 8\lambda + 3\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{4,3d+2}$.

(d) In the case of $\psi_{5,y}$, based on Table 5,

*(5-2) $\psi_{5,2} = 1 + 2\lambda + \mu$.

*(5-3) $d \geq 4 \Rightarrow \psi_{5,d} \geq [4\omega_2] + 4\lambda + \mu \geq 2 + 4\lambda + \mu > \psi_{8,1}$. $d = 1 \Rightarrow \psi_{5,d} = \lambda + \mu < 1$.

Hence, only when $2 \leq d \leq 3$, it is possible to have $\theta_g = \psi_{5,d}$.

*(5-4) $d \geq 2 \Rightarrow \psi_{5,2d+1} \geq [5\omega_2] + 5\lambda + 2\mu \geq 2 + 5\lambda + 2\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{5,2d+1}$.

*(5-5) $d \geq 5 \Rightarrow \psi_{5,d-1} \geq [4\omega_2] + 4\lambda + \mu \geq 2 + 4\lambda + \mu > \psi_{8,1}$. $d = 2 \Rightarrow \psi_{5,d} = \lambda + \mu < 1$.

Hence, only when $3 \leq d \leq 4$, it is possible to have $\theta_g = \psi_{5,d-1}$.

*(5-6) $d \geq 2 \Rightarrow \psi_{5,3d+2} \geq [8\omega_2] + 8\lambda + 3\mu \geq 4 + 8\lambda + 3\mu > \psi_{8,1}$.

Hence, only when $d = 1$, it is possible to have $\theta_g = \psi_{5,3d+2}$.

(e) In the case of $\psi_{8,y}$, based on Table 8,

*(8-1) From the assumption, $F(\psi_{8,1}) < 1$.

(f) In the case of $\psi_{9,y}$, based on Table 9,

*(9-1) $\psi_{9,1} = [\omega_2] + 1 + \lambda + \mu$.

From described above, we shall select all the elements in each part with asterisk (*), using $1 \leq [3\omega_2] \leq 2$, $2 \leq [4\omega_2] \leq 3$, $2 \leq [5\omega_2] \leq 4$. Then we have the following set

$$\begin{aligned}&\{1 + \lambda, 1 + \lambda + \mu, 1 + 2\lambda + \mu, j + 3\lambda + \mu (0 \leq j \leq 2),\\ &\quad j + 3\lambda + 2\mu (0 \leq j \leq 2), j + 4\lambda + \mu (1 \leq j \leq 2), j + 5\lambda + \mu (1 \leq j \leq 3),\\ &\quad j + 5\lambda + 2\mu (1 \leq j \leq 3), j + 5\lambda + 3\mu (1 \leq j \leq 4)\} = \Sigma.\end{aligned}$$

Here, we eliminate elements $\psi \in \Sigma$ such that $\psi > \phi_6$ or $Y_\psi < -1$. Then we have

$$\Sigma' = \{1 + \lambda, 1 + \lambda + \mu, 1 + 2\lambda + \mu, 1 + 3\lambda + \mu, 1 + 3\lambda + 2\mu, 2 + 5\lambda + 3\mu\}.$$

(1) We assume that $F(\phi_8) < 1$. Since $\mathscr{R}$ is a reduced lattice, we have $\phi_8 = \psi_{9,1} = 1 + \lambda + \mu > 1$. Hence, we have $\lambda + \mu > 0$. From this, we have $1 + \lambda + \mu < 1 + 2\lambda + \mu$, $1 + 3\lambda + \mu$, $1 + 3\lambda + 2\mu$, $2 + 5\lambda + 3\mu$. Therefore we conclude that $\theta_g = \phi_8 = 1 + \lambda + \mu$ because $\phi_8 < \phi_6 = 1 + \lambda$.

(2) We assume that $F(\phi_8) > 1$. We note that $d(\lambda, \mu) = 1 \Leftrightarrow 1/2 < \omega_1$. Hence, if $d = 1$, then by Lemma 4.5,(11), we have $F(\phi_8) < 1$. Therefore we have $d \geq 2$. So we have $\theta_g \neq 1 + 3\lambda + 2\mu$, $2 + 5\lambda + 3\mu$.

(i) The case $2\lambda + \mu < 0$. We have $\theta_g = 1 + \lambda$ or $1 + 3\lambda + \mu$.

(ii) The case $2\lambda + \mu > 0$. We have $\theta_g = 1 + \lambda$ or $1 + 2\lambda + \mu$. □

7. Examples

Voronoi-algorithm:

Let K be a cubic algebraic number field of negative discriminant and let $\mathscr{R}$ be a reduced lattice of K. We define the increasing chain of the minimal points of $\mathscr{R}$ by:

$$\theta_0 = 1, \quad \theta_{k+1} = \min\{\gamma \in \mathscr{R}; \theta_k < \gamma, F(\theta_k) > F(\gamma)\} \quad \text{if } k \geq 0.$$

Then θ_{k+1} is the minimal point adjacent to θ_k in $\mathscr{R}$.

Let $\mathscr{O}_K$ be the ring of integers in K and $\mathscr{R} = \mathscr{O}_K$. By Voronoi we know that the previous chain is of purely periodic form:

$$1 = \theta_0,\ \theta_1, \ldots, \theta_{\ell-1},\ \epsilon,\ \epsilon\theta_1, \ldots, \epsilon\theta_{\ell-1}, \ldots,$$

where ℓ denotes the period length and $\epsilon (> 1)$ is the fundamental unit of $\mathscr{O}_K$. To calculate such a sequence, it is sufficient to know how to find the minimal point adjacent to 1 in a lattice $\mathscr{R}$.

Indeed, let $\theta_g^{(1)}$ be the minimal point adjacent to 1 in $\mathscr{R}_1 = \mathscr{O}_K = \langle 1, \beta, \gamma \rangle$ and $\theta_1 = \theta_g^{(1)}$.

(i) We choose an appropriate point $\theta_h^{(1)}$ so that $\{1, \theta_g^{(1)}, \theta_h^{(1)}\}$ is a basis of $\mathscr{R}_1$.

(ii) Let $\mathscr{R}_2 = \dfrac{1}{\theta_g^{(1)}}\mathscr{R}_1$, then $\mathscr{R}_2$ is a reduced lattice. $\theta_g^{(2)}$ is the minimal point adjacent to 1 in $\mathscr{R}_2 = \dfrac{1}{\theta_g^{(1)}}\mathscr{R}_1 = \langle 1, 1/\theta_g^{(1)}, \theta_h^{(1)}/\theta_g^{(1)} \rangle$, is equivalent to $\theta_2 = \theta_1\theta_g^{(2)} = \theta_g^{(1)}\theta_g^{(2)}$ being the minimal point adjacent to θ_1 in $\mathscr{R}_1$.

This process can be continued by induction.

EXAMPLE 7.1. Let $K = \mathbf{Q}(\theta)$ be a cubic number field defined by $\theta^3 - 7\theta - 12 = 0$ ($\theta = 3.2669$). Then $\mathscr{R}_8 = \left\langle 1, -2 + \dfrac{1}{6}\theta + \dfrac{1}{6}\theta^2, 2 + \dfrac{2}{3}\theta - \dfrac{1}{3}\theta^2 \right\rangle = \langle 1, \lambda, \mu \rangle$.

It is easily seen that $0 < \lambda < 1$, $0 < \mu < 1$.

Since $\mathscr{R}_8$ is a reduced lattice, we have $a = F(\mu) > 1$.

$$Y_\theta = \frac{1}{2}(T_{K/\mathbf{Q}}\theta - \theta) = -\frac{1}{2}\theta, \quad Y_{\theta^2} = \frac{1}{2}(T_{K/\mathbf{Q}}\theta^2 - \theta^2) = \frac{1}{2}(14 - \theta^2).$$

$$X_\theta = \frac{1}{2}(3\theta - T_{K/\mathbf{Q}}\theta) = \frac{3}{2}\theta, \quad X_{\theta^2} = \frac{1}{2}(3\theta^2 - T_{K/\mathbf{Q}}\theta^2) = \frac{1}{2}(3\theta^2 - 14).$$

$$X_\mu = X_{2+(2/3)\theta-(1/3)\theta^2} = \frac{2}{3}X_\theta - \frac{1}{3}X_{\theta^2} = \frac{7}{3} + \theta - \frac{1}{2}\theta^2 > 0,$$

$$X_\lambda - X_\mu = -\frac{7}{2} - \frac{3}{4}\theta + \frac{3}{4}\theta^2 > 0.$$

$$Y_\mu = Y_{2+(2/3)\theta-(1/3)\theta^2} = 2 + \frac{2}{3}Y_\theta - \frac{1}{3}Y_{\theta^2} = \frac{1}{6}(-2 - 2\theta + \theta^2), \quad 0 < Y_\mu < \frac{1}{2}.$$

$$Y_\lambda = \frac{1}{12}(-10-\theta-\theta^2). \quad \omega_1(\lambda,\mu) = \frac{\theta-1}{2(\theta+2)}, \quad 0<\omega_1<1.$$

$$\omega_2(\lambda,\mu) = -\frac{1}{12}(-10-\theta-\theta^2) - \frac{\theta-1}{2(\theta+2)} \times \frac{1}{6}(-2-2\theta+\theta^2)$$

$$= \frac{1}{4}(\theta^2-3), \quad [\omega_2]=1.$$

$$F([\omega_2]+\lambda) = F(1+\lambda) = 1+\frac{1}{2}(\theta-3) > 1.$$

$$F([\omega_2]+\lambda+\mu) = F(1+\lambda+\mu) = 2-5\theta+\theta^2+\frac{50}{\theta} > 1.$$

$$F([\omega_2]+1+\lambda) = F(2+\lambda) = F\left(\frac{1}{6}\theta+\frac{1}{6}\theta^2\right) = \frac{1}{3\theta^2}(12+\theta-\theta^2) < 1.$$

Therefore, by Theorem 6.1A,(3), we have $\theta_g = [\omega_2]+1+\lambda = 2+\lambda$.

Example 7.2. Let $K = \mathbf{Q}(\theta)$ be a cubic number field defined by $\theta^3-2\theta-111=0$ $(\theta = 4.9445)$. Then

$$\mathscr{R}_7 = \langle 1, (-71+15\theta+\theta^2)/98, (-61-23\theta+5\theta^2)/196\rangle = \langle 1,\lambda,\mu\rangle.$$

It is easily seen that $0<\lambda<1$, $\mu<0$.

Since $\mathscr{R}_7$ is a reduced lattice, we have $a = F(\mu) > 1$.

$$X_\theta = \frac{3}{2}\theta, \quad X_{\theta^2} = \frac{1}{2}(3\theta^2-4).$$

$$X_\mu = \frac{1}{2c}(15\theta^2-69\theta-20) = 0.0141 > 0 \quad (c=196).$$

$$X_\lambda - X_\mu = \frac{1}{2c}(-9\theta^2+159\theta+12) = 1.4748 > 0.$$

$$Y_\mu = \frac{1}{2c}(-5\theta^2+23\theta-102) = -0.2819, \quad 0<|Y_\mu|<\frac{1}{2}.$$

$$Y_\lambda = \frac{1}{2\times 98}(-\theta^2-15\theta-138) = \frac{1}{c}(-\theta^2-15\theta-138) = -1.2072.$$

$$\omega_1(\lambda,\mu) = \frac{-2\theta+30}{5\theta+23} = 0.4214, \quad 0<\omega_1<1.$$

$$\omega_2(\lambda,\mu) = -Y_\lambda - \omega_1 Y_\mu = 1.2072 - 0.4214\times -0.2819, \quad [\omega_2]=1.$$

(1) $N_{K/\mathbf{Q}}(x + y\theta + z\theta^2) = x^3 + 2 \times 2x^2z - 2xy^2 - 3 \times 111xyz + 2^2xz^2 + 111y^3 - 2 \times 111yz^2 + 111^2z^3$.

(a) By (1),

$$F(\phi_1) = F([\omega_2] + \lambda) = F\left(\frac{1}{98}(27 + 15\theta + \theta^2)\right)$$

$$= \frac{1}{98^2} F(27 + 15\theta + \theta^2) = \frac{1}{98^2} \frac{N_{K/\mathbf{Q}}(27 + 15\theta + \theta^2)}{27 + 15\theta + \theta^2}$$

$$= \frac{1}{98^2} \frac{259308}{27 + 15\theta + \theta^2} = 0.2149 < 1.$$

(b) $\lambda + \mu = \frac{1}{c}(7\theta^2 + 7\theta - 203) = \frac{1}{c} \times 2.7480 > 0.$

(c) By (1),

$$F(\phi_2) = F([\omega_2] + \lambda + \mu) = F\left(\frac{1}{c}(-7 + 7\theta + 7\theta^2)\right)$$

$$= \frac{1}{c^2} F(-7 + 7\theta + 7\theta^2) = \frac{1}{c^2} \frac{N_{K/\mathbf{Q}}(-7 + 7\theta + 7\theta^2)}{-7 + 7\theta + 7\theta^2}$$

$$= \frac{1}{c^2} \frac{4302592}{-7 + 7\theta + 7\theta^2} = 0.5635 < 1.$$

Therefore, by Theorem 6.3A,(1),(ii-b), we have $\theta_g = \phi_2$.

EXAMPLE 7.3. Let $K = \mathbf{Q}(\theta)$ be a cubic number field defined by $\theta^3 - 77\theta - 513 = 0$ $(\theta = 11.1002)$. Then

$$\mathscr{R}_{39} = \langle 1, (-674 - 28\theta + 9\theta^2)/613, (1205 + 121\theta - 17\theta^2)/613 \rangle = \langle 1, \lambda, \mu \rangle.$$

It is easily seen that $0 < \lambda < 1$, $0 < \mu < 1$.

Since $\mathscr{R}_{39}$ is a reduced lattice, we have $a = F(\mu) > 1$.

$$X_\theta = \frac{3}{2}\theta, \quad X_{\theta^2} = \frac{1}{2}(3\theta^2 - 154).$$

$$X_\mu = \frac{1}{2c}(-51\theta^2 + 363\theta + 2618) = \frac{1}{2c} \times 363.4361 > 0 \quad (c = 613).$$

$$X_\lambda - X_\mu = \frac{1}{2c}(78\theta^2 - 457\theta - 4004) = \frac{1}{2c} \times 533.9349 > 0.$$

$$Y_\mu = \frac{1}{2c}(17\theta^2 - 121\theta - 208) = 0.4433, \quad 0 < Y_\mu < \frac{1}{2}.$$

$$Y_\lambda = \frac{1}{2c}(-9\theta^2 + 28\theta + 38) = -0.6200. \quad \omega_1(\lambda,\mu) = \frac{9\theta + 28}{17\theta + 121} = 0.4129,$$

$0 < \omega_1 < 1$. $\omega_2(\lambda,\mu) = -Y_\lambda - \omega_1 Y_\mu = 0.6200 - 0.4129 \times 0.4433$, $[\omega_2] = 0$.

(a) $\phi_2 = \lambda + \mu = \frac{1}{c}(-8\theta^2 + 93\theta + 521) = 0.9259 < 1$.

(b) $2\lambda + \mu = \frac{1}{c}(\theta^2 + 65\theta - 143) = 1.1447 > 1$.

(1) $N_{K/\mathbf{Q}}(x + y\theta + z\theta^2) = x^3 + 2 \times 77x^2z - 77xy^2 - 3 \times 513xyz + 77^2xz^2 + 513y^3 - 77 \times 513yz^2 + 513^2z^3$.

(c) By (1),

$$\begin{aligned} F(\phi_6) &= F([\omega_2] + 1 + \lambda) = F\left(\frac{1}{c}(-61 - 28\theta + 9\theta^2)\right) \\ &= \frac{1}{c^2}F(-61 - 28\theta + 9\theta^2) = \frac{1}{c^2}\frac{N_{K/\mathbf{Q}}(-61 - 28\theta + 9\theta^2)}{-61 - 28\theta + 9\theta^2} \\ &= \frac{1}{c^2}\frac{225837169}{-61 - 28\theta + 9\theta^2} = 0.8153 < 1. \end{aligned}$$

(d) By (1),

$$\begin{aligned} F(2\lambda + \mu) &= \frac{1}{c^2}F(\theta^2 + 65\theta - 143) = \frac{1}{c^2}\frac{N_{K/\mathbf{Q}}(\theta^2 + 65\theta - 143)}{\theta^2 + 65\theta - 143} \\ &= \frac{1}{c^2}\frac{198781801}{\theta^2 + 65\theta - 143} = 0.7538 < 1. \end{aligned}$$

Therefore, by Theorem 6.1B,(3),(ii-b), we have $\theta_g = 2\lambda + \mu$.

Example 7.4 (Williams and Dueck [8, p. 690]). Let $K = \mathbf{Q}(\theta)$ be a cubic number field defined by $\theta^3 - 68781 = 0$ $(\theta = 40.97221992)$. Then

$$\begin{aligned} \mathscr{R}_{2307} &= \langle 1, \phi, \psi \rangle \\ &= \langle 1, (-72036 + 18090\theta + 2\theta^2)/126539, (117574 - 26668\theta + 67\theta^2)/126539 \rangle \\ &= \langle 1, \phi, \psi - 1 \rangle = \langle 1, (-72036 + 18090\theta + 2\theta^2)/126539, \\ &\qquad (-8965 - 26668\theta + 67\theta^2)/126539 \rangle \\ &= \langle 1, \lambda, \mu \rangle. \quad 0 < \lambda < 1,\ \mu < 0. \quad 0 < X_\mu < X_\lambda. \end{aligned}$$

Since $\mathscr{R}_{2307}$ is a reduced lattice, we have $a = F(\mu) > 1$.

$$\omega_1(\lambda, \mu) = \frac{-2\theta + 1809}{67\theta + 2668}. \quad Y_\lambda = -\frac{1}{2c}(2\theta^2 + 1809\theta + 144072) \quad (c = 126539).$$

$$Y_\mu = -\frac{1}{2c}(67\theta^2 - 2668\theta + 17930).$$

$$\omega_1 = 0.31904891. \quad Y_\lambda = -0.87541450. \quad Y_\mu = -0.08333592.$$

$$\omega_2 = 0.90200274.$$

Hence $[\omega_2] = 0$, $\phi_1 = [\omega_2] + \lambda = \lambda < 1$.

(1) $N_{K/\mathbf{Q}}(x + y\theta + z\theta^2) = x^3 - 3 \times 68781xyz + 68781y^3 + 68781^2z^3$.

(a) By (1),

$$F(\phi_6) = F([\omega_2] + 1 + \lambda) = F(1 + \lambda) = F\left(\frac{1}{c}(54503 + 1809\theta + 2\theta^2)\right)$$

$$= \frac{1}{c^2}F(54503 + 1809\theta + 2\theta^2) = \frac{1}{c^2}\frac{N_{K/\mathbf{Q}}(54503 + 1809\theta + 2\theta^2)}{54503 + 1809\theta + 2\theta^2}$$

$$= \frac{1}{c^2}\frac{528431935430042}{54503 + 1809\theta + 2\theta^2} = 0.25005464 < 1.$$

(b) By (1),

$$F(1 + 2\lambda + \mu) = F\left(\frac{-26498 + 950\theta + 71\theta^2}{c}\right)$$

$$= \frac{1}{c^2}F(-26498 + 950\theta + 71\theta^2) = \frac{1}{c^2}\frac{N_{K/\mathbf{Q}}(-26498 + 950\theta + 71\theta^2)}{-26498 + 950\theta + 71\theta^2}$$

$$= \frac{1}{c^2}\frac{2102375149688779}{-26498 + 950\theta + 71\theta^2} = 0.99760062 < 1.$$

(c) By (1),

$$F(\phi_8) = F(1 + \lambda + \mu) = F\left(\frac{45538 - 859\theta + 69\theta^2}{c}\right)$$

$$= \frac{1}{c^2}F(45538 - 859\theta + 69\theta^2) = \frac{1}{c^2}\frac{N_{K/\mathbf{Q}}(45538 - 859\theta + 69\theta^2)}{45538 - 859\theta + 69\theta^2}$$

$$= \frac{1}{c^2}\frac{2161892194231336}{45538 - 859\theta + 69\theta^2} = 1.07007239 > 1.$$

(d) Since $-153037 + 950\theta + 71\theta^2 > 0$, $2\lambda + \mu = \dfrac{-153037 + 950\theta + 71\theta^2}{c} > 0$.

(e) Since $\lambda + \mu = \dfrac{-81001 - 859\theta + 69\theta^2}{c} < 0$, we have $1 + 2\lambda + \mu < 1 + \lambda$.

Therefore, by Theorem 6.3B,(2),(ii), we have $\theta_g = 1 + 2\lambda + \mu$.

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References

[1] B. Adam, Voronoi-algorithm expansion of two families with period length going to infinity, Math. Comp. **64** (1995), 1687–1704.

[2] B. N. Delone and D. K. Faddeev, The theory of irrationalities of the third degree, Transl. Math. Monographs, vol. 10, Amer. Math. Soc., Providence, RI, 1964.

[3] K. Kaneko, On the cubic fields $\mathbf{Q}(\theta)$ defined by $\theta^3 - 3\theta + b^3 = 0$, Sut J. Math. vol. **32**, No. 2 (1996), 141–147.

[4] K. Kaneko, Voronoi-algorithm expansion of a family with period length going to infinity, Sut J. Math. vol. **34**, No. 1 (1998), 49–62.

[5] O. Lahlou and A. Farhane, Sur les points extrémaux dans un ordre cubique, Bull. Belg. Math. Soc. **12** (2005), 449–459.

[6] H. C. Williams, G. Cormack and E. Seah, Calculation of the regulator of a pure cubic field, Math. Comp. **34** (1980), 567–611.

[7] H. C. Williams, G. W. Dueck and B. K. Schmid, A rapid method of evaluating the regulator and class number of a pure cubic field, Math. Comp. **41** (1983), 235–286.

[8] H. C. Williams and G. W. Dueck, An analogue of the nearest integer continued fraction for certain cubic irrationalities, Math. Comp. **42** (1984), 683–705.

[9] H. C. Williams, Continued fractions and number-theoretic computations, Rocky Mountain J. Math. **15** (1985), 621–655.

[10] H. C. Williams, The period length of Voronoi's algorithm for certain cubic orders, Publ, Math. Debrecen **37** (1990), 245–265.

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