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# DOUBLE POINTS OF THE SLOWNESS SURFACE OF THE SYSTEM OF CRYSTAL ACOUSTICS FOR TETRAGONAL CRYSTALS

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**Abstract.** The aim of this paper is to study the location and the geometrical properties of the double points of the slowness surface associated to the system of linear crystal-elasticity in three space dimensions for tetragonal crystals. It will turn out that compared with the case of cubic crystals, the location of these double points is more involved. Moreover, for some specific choices of the "so-called" stiffness constants, a new type of singularities namely "*biplanar*" ones, will appear.

## 1. Introduction

The aim of this paper is to study the double points of the slowness surface associated to the system of linear crystal acoustics for tetragonal crystals in $\mathbf{R}^3$. In particular, the position and the geometrical properties of the double points will be specified.

The paper provides the greater part of the algebraic results needed to understand the long time behavior of global solutions of the homogeneous system of crystal acoustic for tetragonal crystals. The complementary analytical part of the argument, and applications to nonlinear perturbations of the system will be given in a forthcoming paper. We recall that the system of crystal acoustics is a linear $3 \times 3$ hyperbolic system of second order partial differential equations. It is a special case of the time-dependent system of elasticity in three space variables. Specifically, we will deal here with the following system of linear partial differential equations of second order (cf. e.g., [4], [11],

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[8], [12]).

$$(1.1) \qquad \partial_t^2 u_i(t,x) = \sum_{k,l,j=1}^{3} c_{ijkl} \partial_{lj}^2 u_k(t,x), \quad i = 1,2,3, \quad \partial_t^2 = \frac{\partial^2}{\partial t^2}, \quad \partial_{lj}^2 = \frac{\partial^2}{\partial x_\ell \partial x_j},$$

where $u(t,x) \in \mathbf{R}^3$ is the displacement vector and $c_{ijkl} \in \mathbf{R}$ are the stiffness constants. Our final goal is to obtain decay properties of global solution $u(t,x)$ on $\mathbf{R}_x^3 \times \mathbf{R}_t$ of the system for $t \to +\infty$. By this we mean that we are interested in estimates of type $|u_i(t,x)| \leq c(1+|t|)^{-\chi}$, $\forall (x,t) \in \mathbf{R}_x^3 \times \mathbf{R}_t$ for some constant $\chi > 0$. Apart from their intrinsic interest, such estimates play an essential role when we want to study the long-time existence of small non-linear perturbations to the system. For a similar study in the case of isotropic wave type equations see e.g., [9].

The way by which it is possible to obtain results on the decay for solutions of the system is to represent them as parametric Fourier-type oscillatory integrals which live on the so called slowness surface (see definition 1.4 later on in this paper) of the system. The main difficulties in the study of these integrals come from the presence of isolated singularities in the characteristic surface associated to the system and of points where the curvature of the surface vanishes.

In order to justify the results of this paper, we now state a theorem on decay estimates for the solutions $u(t,x)$ of the system (1.1)

THEOREM 1.1. *Assume that the stiffness constants $c_{ij}$ satisfy some prescribed conditions such that the system* (1.1) *is hyperbolic and* "near" *the cubic case* (*i.e., we consider the tetragonal case as a small perturbation of the cubic one*). *Then, there are a constant $C_1$ and a natural number $k \geq 2$ such that*

$$(1.2) \qquad |u(t,x)| \leq C_1 (1+|t|)^{-1/2-1/k} \sum_{j=1}^{3} \sum_{|\alpha| \leq k} (\|\partial_x^\alpha f_j\|_1 + \|\partial_x^\alpha g_j\|_1),$$

*for all $(t,x) \in \mathbf{R}^4$, for any solution of the Cauchy problem of the system* (1.1), *with the initial data $u_j(0,x) = f_j(x)$, $\partial_t u_j(0,x) = g_j(x)$, $j = 1,2,3$, where $f_j$ and $g_j$ are smooth functions on $\mathbf{R}^3$ and have compact support.*

The details of the argument and specific information on the conditions on the stiffness constants will be given in a forthcoming paper. (Also see [10].) Similar estimates are valid for the solutions of the system of crystal acoustics for cubic crystals (see e.g. [8]). The general strategy for proving results of the type of theorem 1.1 is well established, see [5], [8]. Starting point is that we will write the solution of the Cauchy problem in terms of parametric integrals over the slowness surface of the system. Moreover, it is possible to separate the contributions coming from different parts of the slowness surface, depending on their geometrical features. In particular, we will have to use theorems about estimates for Fourier transform of surface carried densities for surfaces which have singular points or for which the curvature may degenerate in the smooth part. In the absence of biplanar singular points, the necessary results have essentially been established in [1], [7], [6] and [16]. However, when the slowness surface has biplanar double points, we also need the following theorem (see [10] and a forthcoming paper).

Theorem 1.2. *Assume that $S$ is a surface with a biplanar double point in the origin, defined by the equation $g(x, y) = z$, and let $F : S \to \mathbf{C}$ be a continuous function which is bounded in a neighborhood of the origin which is such that the function $f(x, y) = F(x, y, g(x, y))$ is $\mathscr{C}^1$ on $(x, y) \neq 0$ small, and for which there is a constant $c$ such that*

$$|\nabla_{(x,y)} F(x, y, g(x, y))| \leq c/\sqrt{\Delta(x, y)} \quad \text{for } 0 \neq |(x, y)| \leq \varepsilon.$$

*If $\kappa$ are small enough, we can find a constant $c'$, such that*

$$I(\xi, \eta, \tau) = \int_S \exp[i\xi x + i\eta y + i\tau z] F(x, y, z)\, d\sigma,$$

*satisfies the estimate*

$$|I(\xi, \eta, \tau)| \leq c'(1 + |(\xi, \eta, \tau)|)^{-1/2} \ln(1 + |(\xi, \eta, \tau)|), \tag{1.3}$$

*provided $F(x, y, g(x, y))$ vanishes for $|(x, y)| \geq \kappa$.*

The geometrical properties and the position of the double points of the slowness surface are well known in the case of cubic crystals: in the non degenerate case they are precisely 14 in number (cf. e.g., [7], and [11]). Of these, 6 lie on the coordinate axes, exactly one on each semi-axis, and are of uniplanar type. The remaining 8 double points lie on the space diagonals (i.e. the lines $|\xi_1| = |\xi_2| = |\xi_3|$), in each octant of $\mathbf{R}^3$ lying precisely one and are of conical type.

We will see that in the general tetragonal case the double points lie in different and in fact more complex configurations. In addition, a new type of singular points appears for tetragonal crystals in specific cases. They will be called singular points of *biplanar* type. The name *biplanar* comes from the fact that the best local second order affine approximation of the slowness surface at the respective singular points is the union of two transversal planes.

We next observe that if we take into account the symmetries inherent to the tetragonal crystal class, we can rewrite the system in the following form

$$\begin{aligned}
\partial_t^2 u_1 &= (c_{11}\partial_{11}^2 + c_{66}\partial_{22}^2 + c_{44}\partial_{33}^2)u_1 + (c_{12}+c_{66})\partial_{12}^2 u_2 + (c_{13}+c_{44})\partial_{13}^2 u_3,\\
\partial_t^2 u_2 &= (c_{12}+c_{66})\partial_{12}^2 u_1 + (c_{66}\partial_{11}^2 + c_{11}\partial_{22}^2 + c_{44}\partial_{33}^2)u_2 + (c_{13}+c_{44})\partial_{23}^2 u_3,\\
\partial_t^2 u_3 &= (c_{13}+c_{44})\partial_{13}^2 u_1 + (c_{13}+c_{44})\partial_{23}^2 u_2 + (c_{44}\partial_{11}^2 + c_{44}\partial_{22}^2 + c_{33}\partial_{33}^2)u_3,
\end{aligned} \tag{1.4}$$

where we used the two-index notation for the stiffness constants (cf. e.g., [11]). Moreover we assume several restrictions on the constants $c_{ij}$, which come from physical considerations, and in particular imply that the system (1.1) becomes hyperbolic. We will not write these conditions down explicitly here, but we will assume that the following implicit condition on the stiffness tensor holds: we suppose that the matrix

$$A(\xi) = \left(\sum_{j,l=1}^{3} c_{ijkl}\xi_j\xi_l\right)_{i,k=1,2,3} \tag{1.5}$$

is positive definite for all $\xi \in \mathbf{R}^3$ (cf. e.g. [4], [8] and [6]). We recall that the characteristic polynomial of the system is given by the determinant of $P(\tau,\xi)$, where $\tau \in \mathbf{R}$, $\xi \in \mathbf{R}^3$ and $P(\tau,\xi)$ is the following matrix:

$$\begin{pmatrix}
\tau^2 - c_{11}\xi_1^2 - c_{66}\xi_2^2 - c_{44}\xi_3^2 & -(c_{12}+c_{66})\xi_1\xi_2 & -(c_{13}+c_{44})\xi_1\xi_3\\
-(c_{12}+c_{66})\xi_1\xi_2 & \tau^2 - c_{66}\xi_1^2 - c_{11}\xi_2^2 - c_{44}\xi_3^2 & -(c_{13}+c_{44})\xi_2\xi_3\\
-(c_{13}+c_{44})\xi_1\xi_3 & -(c_{13}+c_{44})\xi_1\xi_3 & \tau^2 - c_{44}\xi_1^2 - c_{44}\xi_2^2 - c_{33}\xi_3^2
\end{pmatrix}.$$

Thus, the characteristic manifold associated with the system is

$$\{(\tau,\xi) \in \mathbf{R}^4; \det P(\tau,\xi) = 0\}.$$

An easy computation shows that the characteristic polynomial $p(\tau,\xi)$ has the form

$$\begin{aligned}
p(\tau,\xi) &= n_1(\xi)d_2(\tau,\xi)d_3(\tau,\xi) + n_2(\xi)d_3(\tau,\xi)d_1(\tau,\xi)\\
&\quad + n_3(\xi)d_1(\tau,\xi)d_2(\tau,\xi) - d_1(\tau,\xi)d_2(\tau,\xi)d_3(\xi),
\end{aligned}$$

where

$$n_1(\xi) = (c_{12}+c_{66})\xi_1^2, \quad n_2(\xi) = (c_{12}+c_{66})\xi_2^2, \quad n_3(\xi) = \frac{(c_{13}+c_{44})^2}{c_{12}+c_{66}}\xi_3^2,$$

and

$$d_1(\tau,\xi) = \tau^2 - d_1'(\xi), \quad d_1'(\xi) = c_{11}\xi_1^2 + c_{66}\xi_2^2 + c_{44}\xi_3^2 - (c_{12}+c_{66})\xi_1^2,$$
$$d_2(\tau,\xi) = \tau^2 - d_2'(\xi), \quad d_2'(\xi) = c_{66}\xi_1^2 + c_{11}\xi_2^2 + c_{44}\xi_3^2 - (c_{12}+c_{66})\xi_2^2,$$
$$d_3(\tau,\xi) = \tau^2 - d_3'(\xi), \quad d_3'(\xi) = c_{44}\xi_1^2 + c_{44}\xi_2^2 + c_{33}\xi_3^2 - \frac{(c_{13}+c_{44})^2}{c_{12}+c_{66}}\xi_3^2.$$

With these notations the characteristic surface is given by $p(\tau,\xi) = 0$.

This is often written in the so called "Kelvin's form" (see [4], [11], [7]):

$$\frac{n_1(\xi)}{d_1(\tau,\xi)} + \frac{n_2(\xi)}{d_2(\tau,\xi)} + \frac{n_3(\xi)}{d_3(\tau,\xi)} = 1. \tag{1.6}$$

Remark 1.3. It follows immediately that $p(\tau,\xi)$ is a homogeneous polynomial of degree six. Thus, the condition on hyperbolicity implies that for every fixed $\xi \in \mathbf{R}^3$ the equation $p(\tau,\xi) = 0$ has 6 real roots, if multiplicities are counted, and it is obvious that for every fixed $\xi \neq 0$ three of them are positive and three negative.

Definition 1.4. The surface $S$, defined by the condition $p(\xi) = 0$, where $p(\xi) = p(1,\xi)$, is called the slowness surface of the crystal.

Moreover, we define:

$$d_1(\xi) = d_1(1,\xi), \quad d_2(\xi) = d_2(1,\xi), \quad d_3(\xi) = d_3(1,\xi).$$

We observe that the slowness surface is essentially the intersection of the characteristic surface with the plane $\tau = 1$.

As in the case of the characteristic manifold we say that the equation which defines the slowness surface is written in Kelvin's form if the equation $p(\xi) = 0$ is in the form 1.6 (where the $d_i(\tau,\xi)$ are replaced by the $d_i(\xi)$). The equation of the slowness surface is thus

$$\frac{n_1(\xi)}{d_1(\xi)} + \frac{n_2(\xi)}{d_2(\xi)} + \frac{n_3(\xi)}{d_3(\xi)} = 1.$$

First of all we want to find the double roots of $p(\xi)$ and we want to give some conditions on the stiffness constant in order to avoid triple roots. If we assume $p(\xi) = 0$ in Kelvin's form, it is easy to see that, if $n_i(\xi) \geq 0$ and $d'(\xi) \geq 0$ for all $\xi \in \mathbf{R}^3$, then we can have a double root at $\tilde{\xi} \in \mathbf{R}^3$ only when

$$d_1'(\tilde{\xi}) = d_2'(\tilde{\xi}) = d_3'(\tilde{\xi}), \tag{1.7}$$

or when

$$n_1(\tilde{\xi})n_2(\tilde{\xi})n_3(\tilde{\xi}) = 0. \tag{1.8}$$

Note that, with our assumptions on the constants, the condition $n_1(\tilde{\xi})n_2(\tilde{\xi})n_3(\tilde{\xi}) = 0$ means that $\xi$ lies on a coordinate plane.

Now we observe that the points of intersection between the slowness surface and the axes are

$$\left(\pm\frac{1}{\sqrt{c_{66}}},0,0\right),\quad \left(\pm\frac{1}{\sqrt{c_{44}}},0,0\right),\quad \left(\pm\frac{1}{\sqrt{c_{11}}},0,0\right), \tag{1.9}$$

$$\left(0,\pm\frac{1}{\sqrt{c_{66}}},0\right),\quad \left(0,\pm\frac{1}{\sqrt{c_{44}}},0\right),\quad \left(0,\pm\frac{1}{\sqrt{c_{11}}},0\right), \tag{1.10}$$

$$\left(0,0,\pm\frac{1}{\sqrt{c_{44}}}\right),\quad \left(0,0,\pm\frac{1}{\sqrt{c_{44}}}\right),\quad \left(0,0,\pm\frac{1}{\sqrt{c_{33}}}\right). \tag{1.11}$$

Note that this gives a geometrical interpretation of the quantities $c_{ii}$.

Also note that it follows from these expressions that we always have double roots on the $\xi_3$-axis. On the other hand, we have double roots on the other two axes only when we have $c_{66} = c_{44}$, $c_{66} = c_{11}$, or $c_{11} = c_{44}$.

So, considering this and the condition on hyperbolicity of the system, we assume the following conditions on the stiffness constants (cf. [7])

$$c_{ii} > 0, \quad \text{for } i = 1,3,4,6, \quad c_{66} > c_{12}, \quad c_{44} \neq c_{13}, \tag{1.12}$$

$$c_{11} - c_{66} - c_{12} > 0, \quad c_{33} - \frac{(c_{13}+c_{44})^2}{c_{12}+c_{66}} > 0. \tag{1.13}$$

Moreover, in order to avoid triple roots on the axes, we assume

$$c_{33} \neq c_{44} \quad \text{and that the} \quad c_{11}, c_{66}, c_{44} \text{ are not all equal.} \tag{1.14}$$

Remark 1.5. Here we want to write down explicitly the relation between the stiffness constants in the cubic case and in the tetragonal case. The cubic case is when we have

$$c_{11} = c_{33}, \quad c_{44} = c_{66}, \quad c_{12} = c_{13}.$$

It follows from conditions (1.7) and (1.8) that $S$ has double points only when we can write the sixth degree polynomial $p(\xi)$ as the product of two homogeneous polynomials of degree two and four respectively. Indeed, if $n_i = 0$ for some $i \in \{1, 2, 3\}$, then

$$p(\xi) = d_i(n_{i+1}d_{i+2} + n_{i+2}d_{i+1} - d_{i+1}d_{i+2}), \tag{1.15}$$

whereas if $d_1 = d_2 = d_3$, then

$$p(\xi) = d_1^2(n_i + n_{i+1} + n_{i+2} - d_1), \tag{1.16}$$

where in the previous two equations the indices are counted modulo three. We will study the location of the double points of the sextic (1.15) in section 3 and the location of the double points of the sextic (1.16) in the section 4.

We conclude this introduction with some considerations concerning the quartics in the plane of the type which appears in the factorization of $p(\xi)$.

Definition 1.6. A bi-quadratic quartic in the plane will be called (following a suggestion of O. Liess) of "slowness type" if each ray starting from the origin has (when counted with multiplicities) exactly two intersection points with it.

Let

$$\tilde{q}(x, y) = a_1x^4 + a_2y^4 + a_3x^2y^2 + b_1x^2 + b_2y^2 + c_1, \tag{1.17}$$

for some constants $a_i$, $b_j$, $c_1$ with $a_1 > 0$, $a_2 > 0$, $c_1 \neq 0$, and

$$q(x, y) = (x^2 + y^2)^2 + ax^2y^2 + b(x^2 + y^2) + c, \tag{1.18}$$

with $a > 0$ and $c \neq 0$. The following proposition (suggested by O. Liess) is straightforward (for details see [10]).

Proposition 1.7. *Let $\tilde{q}(x, y)$ be a quartic of the form (1.17).*

- *If $\tilde{q}(x, y)$ is of the slowness type then the following conditions must be satisfied*

$$b_1 < 0, \quad b_2 < 0, \quad c_1 > 0,$$

$$b_1^2 - 4a_1c_1 \geq 0, \quad b_2^2 - 4a_2c_1 \geq 0.$$

- *$\tilde{q}(x, y)$ has double points if and only if it is the product of two factors of degree two.*
- *If $\tilde{q}(x, y)$ is of slowness type, and if it has double points, these must lie on the axis, or else we have the following conditions*

$$a_3^2 - 4a_1a_2 > 0,$$

$$2b_2a_1 - b_1a_3 \geq 0,$$

$$2b_1a_2 - b_2a_3 \geq 0,$$

$$(a_3b_1 - 2b_2a_1)^2 = (b_1^2 - 4c_1a_1)(a_3^2 - 4a_2a_1),$$

*with* $(2b_2a_1 - b_1a_3)(2b_1a_2 - b_2a_3) \neq 0$. *In this case* $\tilde{q}(x, y)$ *has the following form*:

$$\tilde{q}(x, y) = \left[a_1x^2 - \frac{1}{2}\left(-a_3y^2 - b_1 + \sqrt{a_3^2 - 4a_2a_1}\left(y^2 + \frac{a_3b_1 - 2a_1b_2}{a_3^2 - 4a_1a_2}\right)\right)\right]$$
$$\left[a_1x^2 - \frac{1}{2}\left(-a_3y^2 - b_1 - \sqrt{a_3^2 - 4a_2a_1}\left(y^2 + \frac{a_3b_1 - 2a_1b_2}{a_3^2 - 4a_1a_2}\right)\right)\right].$$

*In addition, let* $q(x, y)$ *be of the form* (1.18). *If* $q(x, y)$ *is of the slowness type, then the following conditions must hold*:

$$b \leq 0, \quad c > 0, \quad b^2 - 4c \geq 0,$$

$$a + 4 > 0, \quad b^2 - (a + 4)c \geq 0.$$

*Moreover* $q(x, y) = 0$ *has double points if and only if either* $b^2 - 4c = 0$ *or* $b^2 - (a + 4)c = 0$.

*If* $b^2 - 4c = 0$, *then* $q(x, y)$ *has one double point on each axis and it is possible to write it in the following form as the product of two ellipses*:

$$q(x, y) = \left(x^2 + y^2 + \frac{b}{2} - \sqrt{a}xy\right)\left(x^2 + y^2 + \frac{b}{2} + \sqrt{a}xy\right).$$

*If* $b^2 - (a + 4)c = 0$, *then* $q(x, y)$ *has one double point on each semi-diagonal and it is possible to write it in the following form as the product of two ellipses*:

$$q(x, y) = (\alpha x^2 + \beta y^2 - \gamma)(\beta x^2 + \alpha y^2 - \gamma),$$

*where* $(\alpha + \beta)^2 = a + 4$, $\gamma = -b/(\alpha + \beta)$ *and* $\gamma^2 = c$.

## 2. Remarks on the Hexagonal Case

For completeness we review in this section some results which are related to the case of hexagonal crystals. Our main reference is [13] although the results in itself were known much earlier. We are in the hexagonal case if the following condition on the stiffness constants hold:

$$c_{12} = c_{11} - 2c_{66}. \tag{2.1}$$

The main consequence of this assumption is that the sixth degree polynomial $p(\xi)$ factors into the product of two polynomials, one of degree two and one of degree four. Explicitly we have that (2.1) implies $d_2(\xi) = d_1(\xi)$ and then

$$p(\xi) = d_1(\xi)[d_3(\xi)(n_1(\xi) + n_2(\xi)) + d_1(\xi)(n_3(\xi) - d_3(\xi))].$$

The factor of degree two has an ellipsoid as wave surface and it is very simple ($d_1(\xi) = 1 - c_{66}(\xi_1^2 + \xi_2^2) + c_{44}\xi_3^2$). The fourth degree factor is also easy to study since the variables $\xi_1$ and $\xi_2$ appear always in the form $\xi_1^2 + \xi_2^2$ (note that $n_1(\xi) + n_2(\xi) = (c_{11} - c_{66})(\xi_1^2 + \xi_2^2)$), in accordance with the property of rotational symmetry with respect to an axis of hexagonal crystals. Therefore the wave surface is known explicitly. Thus it's sufficient to study the double points on the coordinate planes $\xi_1 = 0$ (or $\xi_2 = 0$) and $\xi_3 = 0$.

Moreover the singularities have the following form.

(i) If $c_{11} = c_{44}$ the quartic associated to the fourth degree factor has two double points on each $\xi_i$-axis, with $i = 1, 2$ (cfr. Proposition 3.2).

(ii) If $c_{11} = c_{66}$, the quartic associated to the fourth degree factor has two double points on each $\xi_i$-axis, with $i = 1, 2$ and four double points one on each diagonal of the form $\pm\xi_1 = \pm\xi_2$, $\xi_3 = 0$ (cfr. Proposition 3.3).

(iii) If $c_{44} = c_{66}$, the quartic associated to the fourth degree factor intersects the ellipsoid on each $\xi_i$-semiaxis, with $i = 1, 2$.

(iv) The quartic intersects the ellipsoid one time on each $\xi_3$-semiaxis.

(v) If some particular conditions on the stiffness constants hold (cfr. Proposition 3.4), the quartic intersects the ellipsoid in eight double points, one on each quadrant of coordinate planes $\xi_i = 0$, with $i = 1, 2$.

## 3. Double Points of the Slowness Surface in the Coordinate Planes

In this section we will study the location of the double points on the sextics which appear when we restrict the slowness surface of a tetragonal crystal to the coordinate planes. If we now restrict to the coordinate plane $\{\xi \in \mathbf{R}^3; \xi_i = 0\}$ for some $i \in \{1, 2, 3\}$, then the terms in $p(\xi)$ which contain $n_i(\xi)$ as a factor vanish, and we obtain the curve

$$\{\xi \in \mathbf{R}^3; \xi_i = 0, d_i = 0\} \cup \{\xi \in \mathbf{R}^3; \xi_i = 0, n_{i+1}d_{i+2} + n_{i+2}d_{i+1} - d_{i+1}d_{i+2} = 0\},$$

with indices calculated modulo 3. Our restriction is thus the union of an ellipse with a bounded quartic. Real double points can appear then in principle in two ways: if we intersect the ellipse with the quartic, or if the quartic itself has double points.

We will investigate this two cases in the next subsection. We will prove that the quartic has double points if and only if two of the $c_{ii}$, $i = 1, 4, 6$ assume the same value, whereas the ellipse can intersect the quartic in zero, two, six or eight points, according to the values of the stiffness constants. Figures 3.1 and 3.2 show restrictions of the slowness surface to the coordinate planes for different values of the stiffness constants.

We observe that $p(\xi)$ is symmetric in the variables $\xi_1$ and $\xi_2$. So, we will only study what happens in the planes $\xi_1 = 0$ and $\xi_3 = 0$.

Our first concern is to understand for which values of the constants $c_{ij}$ we can have double points on the quartic. In fact, as we have already seen, the quartic can have double points only if it is the union of two ellipses which intersect.

### 3.1. Double Points of the Quartic in the Coordinate Planes

Assume at first that $\xi_1 = 0$. Then the restriction of $p$ to $\xi_1 = 0$ factors into the form $d_1(n_2 d_3 + n_3 d_2 - d_2 d_3)$. This means that $\{(\xi_2, \xi_3); p(0, \xi_2, \xi_3) = 0\}$ is the union of the two curves $C_1 = \{(\xi_2, \xi_3); d_1(0, \xi_2, \xi_3) = 0\}$ and $C_2 = \{(\xi_2, \xi_3); (n_2 d_3 + n_3 d_2 - d_2 d_3)(0, \xi_2, \xi_3) = 0\}$. $C_1$ is the ellipse $s_1(\xi_2, \xi_3) = 0$, where

$$s_1(\xi_2, \xi_3) = 1 - c_{66}\xi_2^2 - c_{44}\xi_3^2, \tag{3.1}$$

whereas $C_2$ is the quartic given by $q_1(\xi_2, \xi_3) = 0$, where

$$\begin{aligned} q_1(\xi_2, \xi_3) = {} & c_{11}c_{44}\xi_2^4 + c_{33}c_{44}\xi_3^4 - (c_{13}^2 - c_{11}c_{33} + 2c_{13}c_{44})\xi_2^2\xi_3^2 \\ & - c_{44}\xi_2^2 - c_{11}\xi_2^2 - c_{33}\xi_3^2 - c_{44}\xi_3^2 + 1. \end{aligned} \tag{3.2}$$

We can write $q_1(\xi_2, \xi_3)$ as $X\xi_3^4 + Y(\xi_2)\xi_3^2 + Z(\xi_2)$, where

$$X = c_{44}c_{33},$$

$$Y(\xi_2) = (-2c_{13}c_{44} + c_{11}c_{33} - c_{13}^2)\xi_2^2 - c_{33} - c_{44},$$

$$Z(\xi_2) = c_{11}c_{44}\xi_2^4 - (c_{11} + c_{44})\xi_2^2 + 1.$$

We denote by $D(\xi_2)$ the quantity $Y(\xi_2)^2 - 4XZ(\xi_2)$. We have seen in proposition 1.7 that a necessary condition for the quartic (3.2) to have double points is that $D(\xi_2)$ have positive double roots and its leading coefficient be positive. After some calculations, we can write $D(\xi_2)$ as

$$D(\xi_2) = A\xi_2^4 + B\xi_2^2 + C$$

with

$$A = (c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2)^2 - 4c_{44}^2 c_{33}c_{11},$$
$$B = -2(c_{33} + c_{44})(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2) + 4c_{44}c_{33}(c_{44} + c_{11}),$$
$$C = c_{44}^2 + c_{33}^2 - 2c_{44}c_{33}.$$

We can therefore have double roots only if $D_1 = B^2 - 4AC = 0$. The expression for $D_1$ is quite long, but it factors conveniently to

$$D_1 = 16c_{44}c_{33}(c_{13} + c_{44})^2(c_{13}^2 + 2c_{13}c_{44} + c_{44}c_{33} - c_{33}c_{11} + c_{44}c_{11}),$$

and so we have $D_1 = 0$ if and only if

$$D_1 = (c_{13}^2 + 2c_{13}c_{44} + c_{44}c_{33} - c_{33}c_{11} + c_{44}c_{11}) = 0. \tag{3.3}$$

Note that $c_{33}c_{44}$ is strictly positive and $c_{13} + c_{44} \neq 0$ by conditions (1.12). Moreover the double root of $D(\xi_2)$ is positive if and only if

$$B = -2(c_{33} + c_{44})(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2) + 4c_{44}c_{33}(c_{44} + c_{11}) \leq 0. \tag{3.4}$$

A further condition for the quartic (3.2) to have double points is that $A > 0$, i.e.

$$(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2)^2 - 4c_{44}^2 c_{33}c_{11} > 0. \tag{3.5}$$

If we denote by $\bar{\xi}_2$ the double root of $D$, the last condition for the quartic (3.2) to have double points is that $Y(\bar{\xi}_2) \leq 0$, i.e.

$$(c_{11} + c_{44})(c_{11}c_{33} - 2c_{13}c_{44} - c_{13}^2) - 2(c_{33} + c_{44})c_{11}c_{44} \geq 0. \tag{3.6}$$

Remark 3.1. Here and in the following we assume another condition on the stiffness constants. This condition comes from physics and numerical examples of stiffness constants for tetragonal crystals agree with it. We assume that $c_{12}$ and $c_{13}$ are small when compared with $c_{ii}$, for $i = 1, 3, 4, 6$.

With the assumptions of remark 3.1 the conditions (3.3), (3.4), (3.5) and (3.6) reduce to the following

$$c_{44}c_{33} - c_{33}c_{11} + c_{44}c_{11} = 0, \tag{3.7}$$

$$c_{11}c_{33} > 4c_{44}^2, \tag{3.8}$$

$$c_{11}(c_{33} - c_{44}) > 2c_{44}^2, \tag{3.9}$$

$$c_{33}(c_{11} - c_{44}) > 2c_{44}^2. \tag{3.10}$$

From (3.7) it follows that $c_{11} = c_{33}c_{44}/(c_{33} - c_{44})$ and, taking into account this condition, the conditions (3.8), (3.9) and (3.10) yield $c_{33} < c_{44}$. But if $c_{33} < c_{44}$, then $c_{11}$ must be negative and therefore we can conclude that $q_1(\xi_2, \xi_3)$ can have double points only on the axes.

We can now understand whether or not the quartic $q_1 = 0$ can have double points on the axes. We recall from (1.11) that the points on the positive $\xi_3$-axis are

$$\left(0, 0, \frac{1}{\sqrt{c_{44}}}\right) \quad \text{and} \quad \left(0, 0, \frac{1}{\sqrt{c_{44}}}\right), \quad \left(0, 0, \frac{1}{\sqrt{c_{33}}}\right).$$

Since the first point here is a point on the ellipse (3.1), it follows that the double point on the positive $\xi_3$-axis is the result of the fact that the ellipse and the quartic touch. The points on the positive $\xi_2$-axis are

$$\left(0, \pm\frac{1}{\sqrt{c_{66}}}, 0\right), \quad \left(0, \pm\frac{1}{\sqrt{c_{44}}}, 0\right), \quad \left(0, \pm\frac{1}{\sqrt{c_{11}}}, 0\right).$$

Again, the first point lies on the ellipse (3.1), so it follows that the quartic has double points on the positive $\xi_2$-axis when $c_{11} = c_{44}$.

Thus, we have proved the following proposition.

Proposition 3.2. *Let $q_1(\xi_2, \xi_3) = 0$ be the quartic defined by* (3.2). *It has double points if and only if $c_{11} = c_{44}$. In this case $q_1(\xi_2, \xi_3) = 0$ has two double points of coordinates $(0, \pm 1/\sqrt{c_{44}}, 0)$, on the $\xi_2$-axis.*

We now deal with the restriction to $\xi_3 = 0$. Since the restriction of $p$ to this plane factors into $d_3(n_1 d_2 + n_2 d_1 - d_1 d_2) = 0$, we then have to look at the ellipse

$$s_3(\xi_1, \xi_2) = d_3(\xi_1, \xi_2, 0) = 0 \tag{3.11}$$

and the quartic $q_3(\xi_1, \xi_2) = 0$, where

$$\begin{aligned} q_3(\xi_1, \xi_2) = {} & c_{11}c_{66}(\xi_1^4 + \xi_2^4) + (c_{11}^2 - c_{12}^2 - 2c_{12}c_{66})\xi_1^2\xi_2^2 \\ & - (c_{11} + c_{66})(\xi_1^2 + \xi_2^2) + 1. \end{aligned} \tag{3.12}$$

We have seen in the previous section that such double points can only lie on the axes or on the diagonals. The double points on the axes are known from the relations (1.9) and will exist when $c_{11} = c_{66}$. The points on the positive principal diagonal $\xi_1 = \xi_2$, $\xi_1 \geq 0$ of the quartic are on the other hand

$$\left(\frac{1}{\sqrt{c_{11}-c_{12}}}, \frac{1}{\sqrt{c_{11}-c_{12}}}, 0\right), \quad \left(\frac{1}{\sqrt{c_{11}+c_{12}+2c_{66}}}, \frac{1}{\sqrt{c_{11}+c_{12}+2c_{66}}}, 0\right).$$

It follows from this that the quartic has double points only in the case when $c_{11} - c_{12} = c_{11} + c_{12} + 2c_{66}$, i.e., when $c_{12} + c_{66} = 0$. Since we assume that $c_{12}$ is small compared with $c_{66}$, there will thus be no double points on the quartic and the double points of $p(\xi_1, \xi_2, 0) = 0$ must come from the intersection of the ellipse with the quartic which we will now compute. Thus, we have proved the following proposition.

Proposition 3.3. *Let $q_3(\xi_1, \xi_2) = 0$ be the quartic defined by* (3.12).

*It has double points if and only if $c_{11} = c_{66}$. In this case $q_3(\xi_1, \xi_2) = 0$ has four double points, two on the $\xi_1$-axis, and two on the $\xi_2$-axis, of coordinates $(\pm 1/\sqrt{c_{66}}, 0, 0)$ and $(0, \pm 1/\sqrt{c_{66}}, 0)$ respectively.*

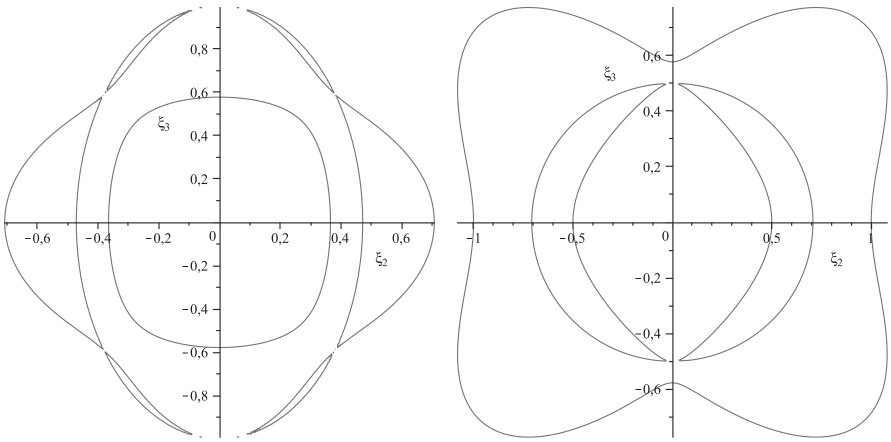


Figure 1: Restrictions of $S$ to the plane $\xi_1 = 0$ with $(c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13})$ equal to $(4, 3, 1, 2, -1/2, 1/5)$ and $(1, 3, 4, 2, -1/2, 1/5)$ respectively.

## 3.2. On the Intersection of the Ellipses with the Quartics in the Coordinate Planes

As before, we assume at first that $\xi_1 = 0$. We have the following proposition.

Proposition 3.4. *Let $s_1(\xi_2,\xi_3)$ and $q_1(\xi_2,\xi_3)$ be the polynomials defined in* (3.1) *and* (3.2) *respectively. We denote*

$$\tilde{\xi}_2^2 = \frac{(c_{13}+c_{44})^2 + (c_{44}-c_{33})(c_{11}-c_{66})}{c_{11}c_{44}^2 + 2c_{13}c_{44}c_{66} + c_{13}^2c_{66} - c_{11}c_{33}c_{66} + c_{33}c_{66}^2},$$

$$\tilde{\xi}_3^2 = \frac{(c_{44}-c_{66})(c_{11}-c_{66})}{c_{11}c_{44}^2 + 2c_{13}c_{44}c_{66} + c_{13}^2c_{66} - c_{11}c_{33}c_{66} + c_{33}c_{66}^2}.$$

*If the stiffness constants $c_{ij}$ are such that $\tilde{\xi}_2$ and $\tilde{\xi}_3$ are positive and $c_{11} \neq c_{66} \neq c_{44}$, then $s_1(\xi_2,\xi_3)=0$ intersects $q_1(\xi_2,\xi_3)=0$ in six points of coordinates*:

$$\left(0, 0, \pm\sqrt{\frac{1}{c_{44}}}\right), \quad (0, \pm\tilde{\xi}_2, \pm\tilde{\xi}_3).$$

*If the stiffness constants $c_{ij}$ are such that $\tilde{\xi}_2$ and $\tilde{\xi}_3$ are not positive, then $s_1(\xi_2,\xi_3)=0$ intersects $q_1(\xi_2,\xi_3)=0$ only in the points $(0,0,\pm\sqrt{1/c_{44}})$.*

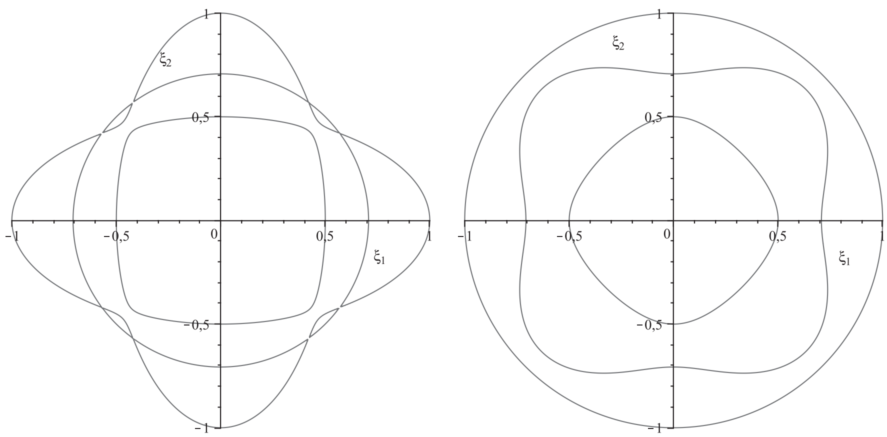


Figure 2: Restrictions of $S$ on the plane $\xi_3 = 0$ with $(c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13})$ equal to $(2, 3, 1, 4, -1/2, 1/5)$ and $(4, 3, 2, 1, -1/2, 1/5)$ respectively.

Proof. We denote $P = (0, \xi_2, \xi_3)$. $P' = (\xi_2, \xi_3)$ then corresponds to an intersection point of $s_1 = 0$ with $q_1 = 0$ if we have simultaneously

$$d_1(P) = 0, \quad (n_2d_3 + n_3d_2 - d_2d_3)(P) = 0,$$

with the usual notation. The condition $d_1(P) = 0$ means that $P' = (\xi_2, \xi_3)$ lies on the ellipse

$$1 - c_{66}\xi_2^2 - c_{44}\xi_3^2 = 0,$$

which gives

$$\xi_3^2 = g(\xi_2) = \frac{1 - c_{66}\xi_2^2}{c_{44}}. \tag{3.13}$$

We have to insert the value of $\xi_3^2$ given by (3.13) into the equation $(n_2d_3 + n_3d_2 - d_2d_3)(P) = 0$, and to solve the resulting equation for $\xi_2$. Calculations are simplified if we make the following preliminary remarks: the values of $d_2$, $n_2 - d_2$, and $d_3$, for $\xi_1 = 0$ and $\xi_3$ given by (3.13) are

$$d_2 = (-c_{11} + 2c_{66} + c_{12})\xi_2^2,$$

$$n_2 - d_2 = (c_{11} - c_{66})\xi_2^2,$$

$$d_3 = \frac{c_{12} + c_{66}}{c_{13} + c_{44}} - \frac{c_{12} + c_{66}}{c_{13} + c_{44}}\left(c_{44}\xi_2^2 + \frac{c_{33}}{c_{44}}(1 - c_{66}\xi_2^2)\right) + \frac{c_{13} + c_{44}}{c_{44}}(1 - c_{66}\xi_2^2).$$

After some calculations, it follows that $[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2))$, is divisible by $\xi_2^2$ and that we have

$$\begin{aligned}
&\frac{[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2))}{\xi_2^2} \\
&\quad = \frac{(c_{13} + c_{44})(1 - c_{66}\xi_2^2)(-c_{11} + 2c_{66} + c_{12})}{c_{44}} \\
&\qquad + \frac{c_{12} + c_{66}}{c_{13} + c_{44}}\frac{1}{c_{44}}(c_{44} - (c_{44}^2\xi_2^2 + c_{33}(1 - c_{66}\xi_2^2)) \\
&\qquad\qquad + (c_{13} + c_{44})(1 - c_{66}\xi_2^2))(c_{11} - c_{66}).
\end{aligned}$$

In particular, we see that $\xi_2 = 0$ is a solution of $[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2)) = 0$ with multiplicity 2. When $\xi_1 = \xi_2 = 0$, the value of $\xi_3^2$ for which we have intersection is $1/c_{44}$. Thus, the first part of the proposition is proved.

The other solutions of $[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2)) = 0$ are also easy to calculate, since $[n_3d_2 + d_3(n_2 - d_2)](0, \xi_2, g(\xi_2))/\xi_2^2$ is linear in the variable $s = \xi_2^2$. We obtain precisely $\xi_2^2 = \tilde{\xi}_2^2$. Inserting this into $s_1(\xi_2, \xi_3)$, we obtain the value $\xi_3 = \tilde{\xi}_3$ corresponding to $\xi_2 = \tilde{\xi}_2$. □

We now deal with the restriction to $\xi_3 = 0$.

PROPOSITION 3.5. *Let* $s_3(\xi_1, \xi_2)$ *and* $q_3(\xi_1, \xi_2)$ *be the polynomials defined in* (3.11) *and* (3.12) *respectively. We denote*

$$R = \frac{(c_{12} + 2c_{44} - c_{11})(c_{12} + 2c_{66} - 2c_{44} + c_{11})}{(c_{12} + c_{11})(c_{12} - c_{11} + 2c_{66})}.$$

*If the stiffness constants* $c_{ij}$ *are such that* $0 < R < 1$ *and* $c_{11} \neq c_{44} \neq c_{66}$, *then* $s_3(\xi_1, \xi_2)$ *intersects* $q_3(\xi_1, \xi_2) = 0$ *in eight points of coordinates*

$$\left(\pm\left(\frac{1+\sqrt{R}}{2c_{44}}\right)\right)^{1/2}, \pm\left(\frac{1-\sqrt{R}}{2c_{44}}\right)^{1/2}, 0\Bigg),$$

$$\left(\pm\left(\frac{1-\sqrt{R}}{2c_{44}}\right)\right)^{1/2}, \pm\left(\frac{1+\sqrt{R}}{2c_{44}}\right)^{1/2}, 0\Bigg).$$

PROOF. We proceed as in the proof of proposition (3.4). The condition $d_3(P) = 0$ gives $\xi_2^2 = c_{44}^{-1} - \xi_1^2$. Inserting this value of $\xi_2^2$ into the equation $(n_1 d_2 + n_2 d_1 - d_1 d_2)(P) = 0$, we obtain

$$\xi_1^2 = \frac{B \pm \sqrt{D}}{2A},$$

where

$$A = (c_{12} + c_{11})(c_{12} - c_{11} + 2c_{66})c_{44},$$

$$B = (c_{12} + c_{11})(c_{12} - c_{11} + 2c_{66}),$$

$$D = (c_{12} + c_{11})(c_{12} + 2c_{44} - c_{11})(c_{12} - c_{11} + 2c_{66})(c_{12} + 2c_{66} - 2c_{44} + c_{11}).$$

So we find the requested values of $\xi_1$ and consequently $\xi_2$. □

We conclude this section noting that, if $c_{44} = c_{66}$, the quartics $q_1(\xi_2, \xi_3) = 0$ and $q_3(\xi_1, \xi_2) = 0$ do not have double points, but the ellipses $s_1(\xi_2, \xi_3) = 0$ and $s_3(\xi_1, \xi_2) = 0$ intersect the quartics $q_1(\xi_2, \xi_3) = 0$ and $q_3(\xi_1, \xi_2) = 0$ respectively, on the $\xi_1$-axis and on the $\xi_3$-axis.

## 4. Double Points of the Slowness Surface Near the Diagonal

In this section we will study the location of double points of the slowness surface of a tetragonal crystal, which occur when $d_1(\tilde{\xi}) = d_2(\tilde{\xi}) = d_3(\tilde{\xi})$. Now, suppose that this condition holds and that $d_i(\tilde{\xi}) \neq 0$. Then $\tilde{\xi}$ must be a double point of $(n_1 + n_2 + n_3 - d_1)(\tilde{\xi}) = 0$, but this is absurd because, given our assumptions on $c_{ij}$, $(n_1 + n_2 + n_3 - d_1)(\tilde{\xi}) = 0$ is an ellipse in $\mathbf{R}^3$. Conversely, if we know that

$$d_1(\tilde{\xi}) = d_2(\tilde{\xi}) = d_3(\tilde{\xi}) = 0,$$

for some point $\tilde{\xi}$, then $\tilde{\xi}$ is a double point of $S$. Thus we have the following lemma.

Lemma 4.1. *Singular points $\tilde{\xi}$ of the slowness surface, which do not lie on the coordinate planes, can occur if and only if $d_1(\tilde{\xi}) = d_2(\tilde{\xi}) = d_3(\tilde{\xi}) = 0$.*

Our next remark is that $d_1(\xi) = d_2(\xi)$ implies $\xi_1^2 = \xi_2^2$. Inserting this information into $d_1(\xi) = d_3(\xi)$ shows that $\xi_1^2$ and $\xi_3^2$ must be related by the condition

$$\xi_3^2 = \frac{(-c_{11} + c_{12} + 2c_{44})(c_{12} + c_{66})}{(c_{13} + c_{44})^2 + (c_{12} + c_{66})(c_{44} - c_{33})} \xi_1^2. \tag{4.1}$$

Using $\xi_1^2 = \xi_2^2$ and (4.1), $p(\xi) = 0$ reduces to a third-degree polynomial in $t = \xi_1^2$, which will have a double root.

Solving this equation we then obtain the following value for $\xi_1^2$

$$\xi_1^2 = -\frac{(c_{13} + c_{44})^2 + (c_{12} + c_{66})(c_{44} - c_{33})}{(c_{13} + c_{44})^2 (c_{12} - c_{11}) + (c_{12} + c_{66})(c_{33}c_{11} - c_{12}c_{33} - 2c_{44}^2)}. \tag{4.2}$$

This gives

$$\xi_3^2 = -\frac{(2c_{44} + c_{12} - c_{11})(c_{12} + c_{66})}{(c_{13} + c_{44})^2 (c_{12} - c_{11}) + (c_{12} + c_{66})(c_{33}c_{11} - c_{12}c_{33} - 2c_{44}^2)}.$$

Thus we have the following proposition.

Proposition 4.2. *Let $S$ be the slowness surface for the tetragonal crystal system. We denote*

$$\tilde{\xi}_1^2 = -\frac{(c_{13} + c_{44})^2 + (c_{12} + c_{66})(c_{44} - c_{33})}{(c_{13} + c_{44})^2 (c_{12} - c_{11}) + (c_{12} + c_{66})(c_{33}c_{11} - c_{12}c_{33} - 2c_{44}^2)},$$

$$\tilde{\xi}_3^2 = -\frac{(2c_{44} + c_{12} - c_{11})(c_{12} + c_{66})}{(c_{13} + c_{44})^2 (c_{12} - c_{11}) + (c_{12} + c_{66})(c_{33}c_{11} - c_{12}c_{33} - 2c_{44}^2)}.$$

*If the stiffness constants $c_{ij}$ are such that $\tilde{\xi}_1^2$ and $\tilde{\xi}_3^2$ are positive, then $S$ has eight double points, four on each plane $\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_1 = \pm\xi_2\}$, of coordinates*:

$$(\tilde{\xi}_1, \tilde{\xi}_1, \pm\tilde{\xi}_3), \quad (-\tilde{\xi}_1, -\tilde{\xi}_1, \pm\tilde{\xi}_3).$$
$$(\tilde{\xi}_1, -\tilde{\xi}_1, \pm\tilde{\xi}_3), \quad (-\tilde{\xi}_1, \tilde{\xi}_1, \pm\tilde{\xi}_3).$$

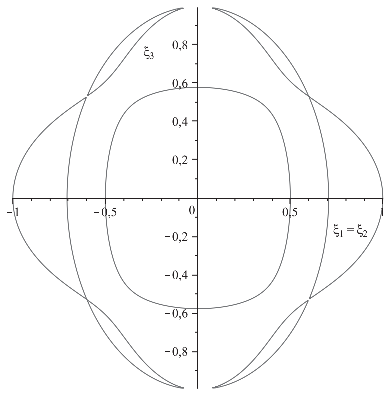


Figure 3: Restriction of $S$ on the plane $\xi_1 = \xi_2$ with $(c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13})$ equal to $(4, 3, 1, 2, -1/2, 1/5)$.

Remark 4.3. In the case of cubic crystals the condition $d_1 = d_2 = d_3$ implies that we must have $\xi_1^2 = \xi_2^2 = \xi_3^2$. So, if we call the eight lines defined by these conditions the space diagonals, in the cubic case we have eight double points, one on each space diagonal. Denote $F = (-c_{11} + c_{12} + 2c_{44})(c_{12} + c_{66}) - (c_{13} + c_{44})^2 - (c_{12} + c_{66})(c_{44} - c_{33})$. Then, in the tetragonal case, we have double points on the space diagonals if $F = 0$. Further, we can decompose $F$ as

$$F = (-c_{11} + c_{12} + c_{44} + c_{33})(c_{12} + c_{66} - c_{13} - c_{44})$$
$$+ (c_{13} + c_{44})(-c_{13} - c_{11} + c_{12} + c_{33}).$$

It follows in particular that $F = 0$ if $c_{12} + c_{66} - c_{13} - c_{44} = 0$ and $-c_{13} - c_{11} + c_{12} + c_{33} = 0$ simultaneously. To put the conditions into a symmetric form we can also write them as

$$c_{66} - c_{44} = c_{13} - c_{12} = c_{33} - c_{11}. \tag{4.3}$$

Note however that these conditions are only sufficient to guarantee that the double points lie on the diagonals. The nice thing about the conditions in (4.3) is that the three quantities $c_{66} - c_{44}$, $c_{13} - c_{12}$, $c_{33} - c_{11}$ measure the "distance" to the cubic case. These conditions therefore say that the three quantities which determine this distance are equal, but do not necessarily vanish. Thus, if we are near the cubic case, we can expect the double points of tetragonal crystal on the planes $\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_1 = \pm\xi_2\}$ to be near the space diagonal.

## 5. Tetragonal Crystals When $c_{11} = c_{66}$

Here and in the remainder of this section we will assume that the stiffness constants $c_{11}$ and $c_{66}$ are equal. In this case not only do we have some simplifications in the calculations, but also a particular type of double point appears on the slowness surface. We will call it a "biplanar" double point (see definition 5.3). It is a type of double point which does not appear in the cubic case, and it is the main reason why we need a theorem of the type of theorem 1.2.

We begin the study of tetragonal crystal when we have $c_{11} = c_{66}$ with the description of where the double points of the slowness surface are located. The results of the previous two sections yield the following proposition.

Proposition 5.1. *Assume $c_{11} = c_{66}$ and let $c_{ij}$ be such that the conditions* (1.12), (1.13), (1.14) *are satisfied and $c_{ij}$, with $i \neq j$, is small compared with $c_{ii}$. Moreover, let $S$ be the slowness surface for the tetragonal crystal system. Then $S$ has six double points, one on each semi-axis, of coordinates* (*see figure 4, left*)

$$\left(\pm\frac{1}{\sqrt{c_{66}}}, 0, 0\right), \quad \left(0, \pm\frac{1}{\sqrt{c_{66}}}, 0\right), \quad \left(0, 0, \pm\frac{1}{\sqrt{c_{44}}}\right).$$

*In addition, if $(c_{12} + 2c_{44} - c_{66})(c_{12} + 3c_{66} - 2c_{44}) > 0$, then $S$ has eight double points on the plane $\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_3 = 0\}$* (*see figure 4, right*), *of coordinates*

$$\left(\pm\left(\frac{1+\sqrt{R}}{2c_{44}}\right)\right)^{1/2}, \pm\left(\frac{1-\sqrt{R}}{2c_{44}}\right)^{1/2}, 0\Big),$$

$$\left(\pm\left(\frac{1-\sqrt{R}}{2c_{44}}\right)\right)^{1/2}, \pm\left(\frac{1+\sqrt{R}}{2c_{44}}\right)^{1/2}, 0\Big),$$

*where all combinations of signs are allowed and*

$$R = \frac{(c_{12} + 2c_{44} - c_{66})(c_{12} + 3c_{66} - 2c_{44})}{(c_{12} + c_{66})^2}.$$

*Finally, if* $\tilde{\xi}_1 > 0$ *and* $\tilde{\xi}_3 > 0$, *then* $S$ *has four double points on each of the planes* $\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_1^2 = \xi_2^2\}$ *of coordinates*

$$(\tilde{\xi}_1, \tilde{\xi}_1, \pm\tilde{\xi}_3), \quad (-\tilde{\xi}_1, -\tilde{\xi}_1, \pm\tilde{\xi}_3),$$
$$(\tilde{\xi}_1, -\tilde{\xi}_1, \pm\tilde{\xi}_3), \quad (-\tilde{\xi}_1, \tilde{\xi}_1, \pm\tilde{\xi}_3),$$

*where*

$$\tilde{\xi}_1^2 = -\frac{(c_{13}+c_{44})^2 + (c_{12}+c_{66})(c_{44}-c_{33})}{(c_{13}+c_{44})^2(c_{12}-c_{66}) + (c_{12}+c_{66})(c_{33}c_{66} - c_{12}c_{33} - 2c_{44}^2)},$$

$$\tilde{\xi}_3^2 = -\frac{(2c_{44}+c_{12}-c_{66})(c_{12}+c_{66})}{(c_{13}+c_{44})^2(c_{12}-c_{66}) + (c_{12}+c_{66})(c_{33}c_{66} - c_{12}c_{33} - 2c_{44}^2)}.$$

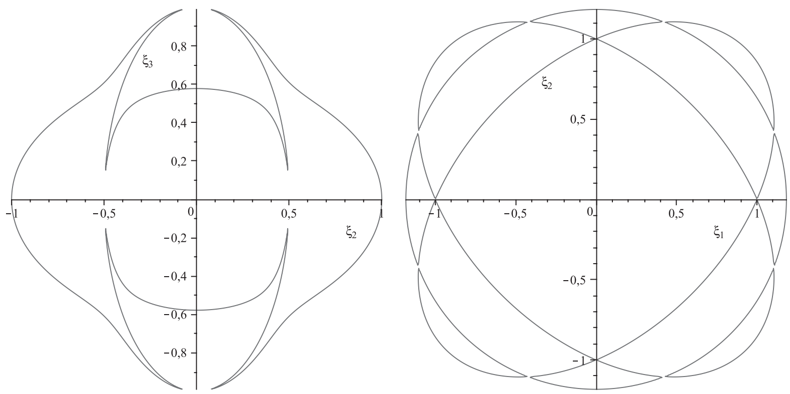


Figure 4: Restrictions of $S$ on the plane $\xi_1 = 0$ and $\xi_3 = 0$, with $(c_{11}, c_{33}, c_{44}, c_{66}, c_{12}, c_{13})$ equal to $(4, 3, 1, 4, -1/2, 1/5)$ and $(1, 2, 5/7, 1, -1/7, 1/2)$ respectively.

Remark 5.2. We observe that there exist admissible values of the stiffness constants such that the conditions $\tilde{\xi}_i > 0$, with $i = 1, 3$, and $R > 0$ of the proposition 5.1 can be either both satisfied or both not satisfied or one satisfied and the other not satisfied.

Now we want to classify the double points of $S$ into three different types, depending on their geometrical properties. To do so, we need the following definitions.

Definition 5.3. Let $S$ be a surface in $\mathbf{R}^3$ on which the coordinates are denoted by $\xi = (\xi_1, \xi_2, \xi_3)$. We assume that $P \in S$ and that in a neighborhood

$\mathscr{U}$ of $P$, $S$ is defined by an equation of form $f(\xi) = 0$, with $\xi \in \mathscr{U}$, for some function $f \in C^\infty(\mathscr{U})$. We assume that $\nabla f(\xi) = 0$ precisely when $\xi = P$ and denote by $J_k f(\xi) = \sum_{|\alpha|=k} (1/\alpha!) \partial_\xi^\alpha f(P) \xi^\alpha$ the homogeneous part of degree $k$ in the Taylor expansion of $f$ at $P$.

- We say that $P$ is a conical singularity if for some suitable choice of linear coordinates $J_2 f$ has the form $J_2 f(\xi) = \xi_1^2 - \xi_2^2 - \xi_3^2$.
- We say that $P$ is a uniplanar singularity if it is possible to find linear coordinates for which $J_2 f(\xi) = \xi_3^2$ and if $f = 0$ is locally equivalent to

  $$\xi_3^2 + A(\xi_1, \xi_2)\xi_3 + B(\xi_1, \xi_2) = 0$$

  with $A(P_1, P_2) = 0$, $B(P_1, P_2) = 0$, $\nabla A(P_1, P_2) = 0$, for some smooth function $A$, $B$.

  Moreover, we assume that if we denote by $\Delta$ the quantity $\Delta = A^2 - 4B$, then we have $\Delta(\xi_1, \xi_2) = \mathcal{O}(|\xi_1, \xi_2|^4)$ for $(\xi_1, \xi_2) \to (P_1, P_2)$.
- We say that $P$ is a biplanar singularity if the following happens: for some suitable choice of linear coordinates $J_2 f(\xi) = \xi_1^2 - \xi_2^2$ (see fig. 5).

In the next three subsections, we will prove the following proposition about the nature of the singular points of the slowness surface for the tetragonal crystal system, when we have $c_{11} = c_{66}$.

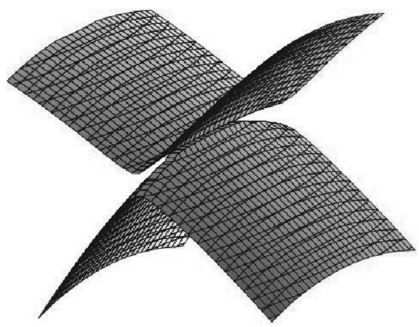

Figure 5: The biplanar double point at the origin of the surface defined by the equation $z^2 - (1/2)x^2 + 2yz^2 - 2zx^2 + x^4 + 2x^2y^2 + (1/2)y^4 = 0$.

Proposition 5.4. *Let $S$ be the slowness surface for the tetragonal crystal system.*

*The double points of $S$ on the $\xi_3$-axis are uniplanar singularities.*

*The double points of $S$ on the $\xi_1$-axis and $\xi_2$-axis are biplanar singularities.*

*If $S$ has double points on the plane $\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_3 = 0\}$, then they are conical singularities. If $S$ has double points on the planes $\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_1^2 = \xi_2^2\}$, and $c_{66} - c_{44} = c_{13} - c_{12} = c_{33} - c_{11}$, then they are conical singularities.*

### 5.1. Hessians at the Singular Points on the Axes

In this subsections we suppose that the assumptions on the stiffness constants made in the proposition 5.1 hold. We begin with the proof of the fact that the points $(0, 0, \pm 1/\sqrt{c_{44}}) \in S$ are uniplanar singularities.

We denote $P = (0, 0, 1/\sqrt{c_{44}})$. We will prove that $\nabla p(P) = 0$, $(\partial/\partial\xi_3)^2 p(P) \neq 0$, $(\partial^2/\partial\xi_i\partial\xi_j)^2 p(P) = 0$ if $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$, $(\partial/\partial\xi)^\alpha p(P) = 0$ if $|\alpha| = 3$ and the order of derivations in $(\xi_1, \xi_2)$ is odd.

The gradient of $p$ at $P$ vanishes since $P$ is a double point of the slowness surface.

The following remarks help us simplify the calculations of second order derivatives:

- The factors $n_1$, $n_2$ vanish twice at $P$.
- The expressions $d_i$ vanish at $P$ for $i = 1, 2$ (but not necessarily for $i = 3$).
- When we derivate one of the $d_i$, $i = 1, 2, 3$, in one of the variables $\xi_j$, $j = 1$ or 2, then we obtain a factor $\xi_j$, and therefore this derivative will vanish at $P$.
- $(\partial/\partial\xi_i)n_3 = 0$, $(\partial^2/\partial\xi_i\partial\xi_j)n_3 = 0$ for $i = 1, 2$, whatever $j$ is.

We conclude from these remarks, that the terms $n_1 d_2 d_3$, $n_2 d_3 d_1$ vanish of order 3 at $P$. Therefore, they will not contribute to the Hessian of $p$ at $P$. Moreover, when we calculate second order derivatives of type $(\partial^2/\partial\xi_i\partial\xi_j)$ of $(n_3 - d_3)d_1 d_2$, then, in order to have a nontrivial contribution, we must derivate each one of the factors $d_1$ and $d_2$, since these factors vanish at $P$. However, first order derivatives of $d_1, d_2$ again vanish at $P$, so we do not have enough derivations to obtain a nontrivial contribution. In a similar way we conclude that derivatives of form $(\partial/\partial\xi_i)(\partial/\partial\xi_3)^k p(P)$ vanish when $i \in \{1, 2\}$, $k \geq 2$.

We next calculate $(\partial/\partial\xi_3)^2 p(P)$. Again, only $(n_3 - d_3)d_1 d_2$ can give a nontrivial contribution. We must of course derivate each of the factors $d_1$ and $d_2$ once, to get a nontrivial contribution. Therefore

$$(\partial/\partial\xi_3)^2 p(P) = (n_3 - d_3)(P)(\partial/\partial\xi_1)d_1(P)(\partial/\partial\xi_1)d_2(P).$$

After some calculations we obtain

$$\frac{\partial^2}{\partial\xi_3^2} p(P) = \frac{8(c_{33} - c_{44})(c_{12} + c_{66})}{c_{13} + c_{44}}.$$

By assumption on the stiffness constants this is non vanishing.

We still have to say something about third order derivatives. If we derivate once in $\xi_3$ and the remaining derivatives are in the variables $\xi_1$, $\xi_2$, then the result may be non vanishing.

We now turn to the case of the $\xi_1$-axis and of the $\xi_2$-axis. The two cases are of course symmetric. We will prove that the points $(\pm 1/\sqrt{c_{66}}, 0, 0) \in S$ and $(0, \pm 1/\sqrt{c_{66}}, 0) \in S$ are biplanar singularities.

Now, we denote $P = (1/\sqrt{c_{66}}, 0, 0)$. As before we can simplify the calculations with some preliminary remarks. We now have:

- $n_2$, $n_3$ vanish of order 2 at $P$.
- $d_2$ vanishes at $P$.
- The first order derivatives of the $d_i$, $i = 1, 2, 3$, in the variables $\xi_2, \xi_3$ vanish at $P$.
- $n_1 - d_1$ vanishes at $P$.

We first prove that

$$\frac{\partial^2 p(P)}{\partial \xi_1^2} = \frac{8(c_{12} + c_{66})(c_{44} - c_{66})}{c_{13} + c_{44}},$$

$$\frac{\partial^2 p(P)}{\partial \xi_2^2} = -\frac{2(c_{12} + c_{66})^3 (c_{44} - c_{66})}{c_{66}^2 (c_{13} + c_{44})},$$

$$\frac{\partial^2 p(P)}{\partial \xi_3^2} = 0.$$

It follows, by the assumptions on the stiffness constants, that $(\partial/\partial\xi_1)^2 p(P)$ and $(\partial/\partial\xi_2)^2 p(P)$ have opposite signs.

To calculate the second derivatives in $\xi_1$, we notice that the terms containing $n_1$, $n_2$ will not give any contribution: they contain factors of type $\xi_2^2$, $\xi_3^2$ and these factors are like constants if we derivate them in $\xi_1$. Since $d_2$ and $n_1 - d_1$ vanish at $P$, we have

$$\frac{\partial^2}{\partial \xi_1^2} p(P) = \frac{\partial}{\partial \xi_1} d_2(P) \frac{\partial}{\partial \xi_1} (n_1 d_3 - d_1 d_3)(P).$$

After some calculations, we obtain the desired result referring to $(\partial/\partial\xi_1)^2 p(P)$.

When we calculate $(\partial/\partial\xi_2)^2 p(P)$, the term $n_3 d_1 d_2$ gives no contribution due to the factor $n_3$, which behaves like a constant under derivations in $\xi_2$. We may thus write that

$$\frac{\partial^2}{\partial \xi_2^2} p(P) = \frac{\partial^2}{\partial \xi_2^2} [(n_1 - d_1) d_2 d_3](P) + d_1(P) d_3(P) \frac{\partial^2}{\partial \xi_2^2} n_2(P).$$

Since $(n_1 - d_1)$ and $d_2$ both vanish at $P$, we have

$$\frac{\partial^2}{\partial \xi_2^2}[(n_1 - d_1)d_2 d_3](P) = d_3(P)\frac{\partial}{\partial \xi_2}(n_1 - d_1)(P)\frac{\partial}{\partial \xi_2}d_2(P).$$

However, $(\partial/\partial\xi_2)d_2(P) = 0$. Therefore,

$$\frac{\partial^2}{\partial \xi_2^2}p(P) = d_1(P)d_3(P)\frac{\partial^2}{\partial \xi_2^2}n_2(P).$$

It follows after some calculations that $(\partial/\partial\xi_2)^2 p(P)$ is as stated in the lemma.

To calculate $(\partial/\partial\xi_3)^2 p(P)$ we note that the term containing $n_2$ will give no contribution. The same is true for the term $n_3 d_1 d_2$: here we use the fact that $n_3 d_2$ vanishes of order 3 at $P$. We are left with

$$\frac{\partial^2}{\partial \xi_3^2}[(n_1 - d_1)d_2 d_3](P).$$

Since $(n_1 - d_1)$, $d_2$ both vanish at $P$, we must have that

$$\left[\frac{\partial^2}{\partial \xi_3^2}(n_1 - d_1)d_2 d_3\right](P) = d_3(P)\left[\frac{\partial}{\partial \xi_3}(n_1 - d_1)(P)\right]\left[\frac{\partial}{\partial \xi_3}d_2(P)\right].$$

We use again that $(\partial/\partial\xi_3)d_2(P) = 0$ and, in the end, we obtain 0.

Now we prove that

$$\frac{\partial^2}{\partial \xi_1 \partial \xi_2}p(P) = 0, \quad \frac{\partial^2}{\partial \xi_1 \partial \xi_3}p(P) = 0, \quad \frac{\partial^2}{\partial \xi_2 \partial \xi_3}p(P) = 0.$$

To calculate $(\partial^2/\partial\xi_1\partial\xi_2)p(P)$, we note that the terms with $n_2$ and $n_3$ give no contribution. Thus,

$$\frac{\partial^2}{\partial \xi_1 \partial \xi_2}p(P) = \frac{\partial^2}{\partial \xi_1 \partial \xi_2}[(n_1 - d_1)d_2 d_3](P).$$

We can now argue as above. The evaluation of $(\partial^2/\partial\xi_1\partial\xi_3)p(P)$, $(\partial^2/\partial\xi_3\partial\xi_3)p(P)$ is done in exactly the same way.

REMARK 5.5. The localization polynomial of $S$ at $P = (1/\sqrt{c_{66}}, 0, 0)$ is thus

$$\frac{8(c_{12} + c_{66})(c_{44} - c_{66})}{c_{13} + c_{44}}\xi_1^2 - \frac{2(c_{12} + c_{66})^3(c_{44} - c_{66})}{c_{66}^2(c_{13} + c_{44})}\xi_2^2.$$

The tangent set at $S$ in $P$ is then given by

$$\left\{ v \in \mathbf{R}^3; \frac{8(c_{12}+c_{66})(c_{44}-c_{66})}{c_{13}+c_{44}} v_1^2 - \frac{2(c_{12}+c_{66})^3(c_{44}-c_{66})}{c_{66}^2(c_{13}+c_{44})} v_2^2 = 0, v_3 \in \mathbf{R} \right\}.$$

We have two planes and this is the reason to call them "biplanar".

### 5.2. Hessians at the Singular Points in the Coordinate Planes, but not on the Axes

We have to consider the singular points on the plane $\xi_3 = 0$. Let $P$ be one of these points, e.g., $\left(\left(\frac{1+\sqrt{R}}{2c_{44}}\right)\right)^{1/2}, \left(\frac{1-\sqrt{R}}{2c_{44}}\right)^{1/2}, 0\right)$. Here we want to calculate the determinant of the Hessian of $p(\xi)$ in $P$. Recall that $P$ satisfies simultaneously

$$d_3(P) = 0, \quad f_3(P) = 0, \tag{5.1}$$

where we denote $f_3 = n_1 d_2 + n_2 d_1 - d_1 d_2$. Relation (5.1) can be used to simplify the calculation of the Hessians. Indeed, $(\partial^2/\partial\xi_3^2)p(P)$ is very easy to calculate. This is based on the following remarks:

- in view of (5.1)

$$\frac{\partial^2}{\partial\xi_3^2}[d_3(n_1 d_2 + n_2 d_1 - d_1 d_2)](P) = 2\frac{\partial}{\partial\xi_3} d_3(P) \frac{\partial}{\partial\xi_3}(n_1 d_2 + n_2 d_1 - d_1 d_2)(P).$$

- If we derivate $d_3$, $n_3$ or $n_1 d_2 + n_2 d_1 - d_1 d_2$ just once in $\xi_3$, then the expression which we obtain will be a multiple of $\xi_3$ and will therefore vanish on our coordinate plane.

It follows that

$$\frac{\partial^2}{\partial\xi_3^2}[d_3(P)(n_1 d_2 + n_2 d_1 - d_1 d_2)(P)] = 0.$$

It is then clear that

$$\begin{aligned} \frac{\partial^2}{\partial\xi_3^2} p(P) &= \frac{\partial^2 n_3}{\partial\xi_3^2}(P) d_1(P) d_2(P) \\ &= 2\frac{(c_{13}+c_{44})^2(c_{44}-c_{66})(c_{12}+2c_{44}-c_{66})}{c_{44}^2(c_{12}+c_{66})}. \end{aligned} \tag{5.2}$$

Remark 5.6. Mixed derivatives of $p$ which contain just one derivation in $\xi_3$ will vanish. This is proved with an argument similar to the one just used for the calculation of $(\partial/\partial\xi_3)^2 p(P)$.

We are now left with derivatives of form $(\partial^2/\partial\xi_i\partial\xi_j)$ where $i, j \in \{1,2\}$. It is obvious that $(\partial^2/\partial\xi_i\partial\xi_j)(n_3 d_1 d_2)(P) = 0$. Now, we can again argue as above and conclude that

$$\frac{\partial^2}{\partial\xi_i\partial\xi_j}(d_3 f_3)(P) = \frac{\partial}{\partial\xi_i}d_3(P)\frac{\partial}{\partial\xi_j}f_3(P) + \frac{\partial}{\partial\xi_j}d_3(P)\frac{\partial}{\partial\xi_i}f_3(P). \tag{5.3}$$

The situation is further simplified by noting that $(\partial/\partial\xi_i)d_3$ is divisible by $\xi_i$ and $(\partial/\partial\xi_j)(n_1 d_2 + n_2 d_1 - d_1 d_2)$ is divisible by $\xi_j$. In (5.3) we can therefore divide out a factor $\xi_i\xi_j$. If we also take into account that

$$\frac{1}{\xi_i}\frac{\partial d_3}{\partial\xi_i} = \frac{1}{\xi_j}\frac{\partial d_3}{\partial\xi_j} = -2c_{44}$$

then we obtain that

$$\frac{\partial^2}{\partial\xi_i\partial\xi_j}(d_3 f_3)(P) = -2c_{44}\left[\xi_i\xi_j\left(\frac{1}{\xi_i}\frac{\partial f_3}{\partial\xi_i} + \frac{1}{\xi_j}\frac{\partial f_3}{\partial\xi_j}\right)\right](P).$$

An elementary calculation gives

$$\frac{1}{\xi_1}\frac{\partial}{\partial\xi_1}f_3(\xi) = -4c_{66}^2\xi_1^2 + 2(c_{12}^2 - c_{66}^2 + 2c_{12}c_{66})\xi_2^2 - 4c_{66}c_{44}\xi_3^2 + 4c_{66},$$
$$\frac{1}{\xi_2}\frac{\partial}{\partial\xi_2}f_3(\xi) = -4c_{66}^2\xi_2^2 + 2(c_{12}^2 - c_{66}^2 + 2c_{12}c_{66})\xi_1^2 - 4c_{66}c_{44}\xi_3^2 + 4c_{66}.$$

It also follows from this that the determinant of the Hessian in the variables $\xi_1$, $\xi_2$ is

$$\begin{aligned} &4\xi_1^2\xi_2^2c_{44}^2\left(4\frac{1}{\xi_1}\frac{\partial f_3}{\partial\xi_1}\frac{1}{\xi_2}\frac{\partial f_3}{\partial\xi_2} - \left(\frac{1}{\xi_2}\frac{\partial f_3}{\partial\xi_2} + \frac{1}{\xi_1}\frac{\partial f_3}{\partial\xi_1}\right)^2\right) \\ &\quad = -4\xi_1^2\xi_2^2c_{44}^2\left(\frac{1}{\xi_2}\frac{\partial f_3}{\partial\xi_2} - \frac{1}{\xi_1}\frac{\partial f_3}{\partial\xi_1}\right)^2. \end{aligned} \tag{5.4}$$

From (5.4), (5.2) and remark 5.6, it follows that the Hessian of $p = 0$ in $P$ has the form

$$R = \begin{pmatrix} A & 0 \\ 0 & \dfrac{\partial^2 p}{\partial\xi_3^2}(P) \end{pmatrix},$$

where the determinant of $A$ is negative. Thus, $P$ is a conical singularity. For the other singular points on the plane $\xi_3 = 0$ we can argue as above.

### 5.3. Hessians at the Singular Points in the Planes $\xi_1^2 = \xi_2^2$

In principle it is not difficult to calculate the Hessian of $p$ at these points. For explicit numerical constants $c_{ij}$ this is a simple arithmetic calculation, but for general constants the expressions which one obtains are not as easy to understand. We now begin our discussion recalling that, if $P$ is a double point on the planes $\xi_1^2 = \xi_2^2$, then $d_i(P) = 0$ for $i = 1, 2, 3$. Then, the part $d_1 d_2 d_3$ of $p$ will vanish of third order at $P$, and will not contribute to the Hessian at $P$. It is also clear that we have

$$\frac{\partial^2 (n_j d_{j+1} d_{j+2})(P)}{\partial \xi_i \partial \xi_l}$$
$$= n_j(P)\left(\frac{\partial}{\partial \xi_i} d_{j+1}(P) \frac{\partial}{\partial \xi_l} d_{j+2}(P) + \frac{\partial}{\partial \xi_l} d_j(P) \frac{\partial}{\partial \xi_i} d_{j+2}(P)\right),$$

where the indices are calculated modulo 3, since $d_{j+1} d_{j+2}$ vanishes of order two at $P$. Calculations are quite complex and we will only discuss what happens under the additional assumption $c_{66} - c_{44} = c_{13} - c_{12} = c_{33} - c_{11}$ (see remark 4.3). In this case, if we set $c_{13} + c_{44} = c_{12} + c_{66}$ and $c_{33} = 2c_{66} - c_{44}$, then the double points lie on the space diagonal. In particular they have the following coordinates

$$(\pm(c_{44} + c_{66} - c_{12})^{-1/2}, \pm(c_{44} + c_{66} - c_{12})^{-1/2}, \pm(c_{44} + c_{66} - c_{12})^{-1/2}).$$

Moreover the $n_i(P)$ have the same values and

$$d_1(\xi) = 1 - c_{66}\xi_1^2 - c_{66}\xi_2^2 - c_{44}\xi_3^2 + (c_{12} + c_{66})\xi_1^2,$$
$$d_2(\xi) = 1 - c_{66}\xi_1^2 - c_{66}\xi_2^2 - c_{44}\xi_3^2 + (c_{12} + c_{66})\xi_2^2,$$
$$d_3(\xi) = 1 - c_{44}\xi_1^2 - c_{44}\xi_2^2 - (2c_{66} - c_{44})\xi_3^2 + (c_{12} + c_{66})\xi_3^2.$$

Denoting $q = d_1 d_2 + d_2 d_3 + d_3 d_1$, it follows that

$$\frac{1}{n_1(P)\xi_i \xi_j} \frac{\partial^2 p(P)}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 q(P)}{\partial \xi_i \partial \xi_j}, \quad i = 1, 2, 3.$$

The Hessian of $p$ at $P$ is then proportional to the matrix

$$A = \begin{pmatrix} \dfrac{\partial^2 q}{\partial \xi_1^2} & \dfrac{\partial^2 q}{\partial \xi_1 \partial \xi_2} & \dfrac{\partial^2 q}{\partial \xi_1 \partial \xi_3} \\ \dfrac{\partial^2 q}{\partial \xi_2 \partial \xi_1} & \dfrac{\partial^2 q}{\partial \xi_2^2} & \dfrac{\partial^2 q}{\partial \xi_2 \partial \xi_3} \\ \dfrac{\partial^2 q}{\partial \xi_3 \partial \xi_1} & \dfrac{\partial^2 q}{\partial \xi_3 \partial \xi_2} & \dfrac{\partial^2 q}{\partial \xi_3^2} \end{pmatrix}.$$

We can easily obtain

$$\frac{1}{\xi_1}\frac{\partial d_1(P)}{\partial \xi_1} = 2c_{12}, \quad \frac{1}{\xi_2}\frac{\partial d_2(P)}{\partial \xi_2} = 2c_{12}, \quad \frac{1}{\xi_3}\frac{\partial d_3(P)}{\partial \xi_3} = 2c_{44} + 2c_{12} - 2c_{66},$$

and

$$\frac{1}{\xi_2}\frac{\partial d_1(P)}{\partial \xi_2} = -2c_{66}, \quad \frac{1}{\xi_3}\frac{\partial d_1(P)}{\partial \xi_3} = -2c_{44}, \quad \frac{1}{\xi_1}\frac{\partial d_2(P)}{\partial \xi_1} = -2c_{66},$$
$$\frac{1}{\xi_3}\frac{\partial d_2(P)}{\partial \xi_3} = -2c_{44}, \quad \frac{1}{\xi_1}\frac{\partial d_3(P)}{\partial \xi_1} = -2c_{44}, \quad \frac{1}{\xi_2}\frac{\partial d_3(P)}{\partial \xi_2} = -2c_{44},$$

Finally, explicit calculations give

$$(c_{44} + c_{66} - c_{12})\frac{\partial^2 q}{\partial \xi_i^2}(P) = 8(-c_{12}c_{66} + c_{44}c_{66} - c_{44}c_{12}), \quad i = 1, 2,$$

$$(c_{44} + c_{66} - c_{12})\frac{\partial^2 q}{\partial \xi_3^2}(P) = 8c_{44}(2c_{66} - 2c_{12} - c_{44}),$$

$$(c_{44} + c_{66} - c_{12})\frac{\partial^2 q}{\partial \xi_1 \partial \xi_2}(P) = 4((c_{12} + c_{66})^2 + 2(-c_{12}c_{66} + c_{44}c_{66} - c_{44}c_{12})),$$

$$(c_{44} + c_{66} - c_{12})\frac{\partial^2 q}{\partial \xi_i \partial \xi_3}(P) = 4((c_{12} - c_{66}^2) + 2c_{44}^2), \quad i = 1, 2.$$

Thus, the eigenvalues of the Hessian of $p$ at $P$ are

$$4(c_{12} + c_{66})^2, \quad -8(c_{44} + c_{66} - c_{12})^2, \quad 4(2c_{44} - c_{66} + c_{12})^2. \tag{5.5}$$

Moreover, the determinant of the Hessian is equal to

$$128(c_{12} + c_{66})^2(c_{44} + c_{66} - c_{12})^2(2c_{44} - c_{66} + c_{12})^2. \tag{5.6}$$

Thus, if we assume $2c_{44} - c_{66} + c_{12} \neq 0$, using the symmetries between $\xi_1$ and $\xi_2$, we have proved that the double points of the slowness surface on the planes $\{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 : \xi_1^2 = \xi_2^2\}$ lie on the space diagonals and are conical singularities.

## 6. Hessians of Singular Points When $c_{11} \neq c_{66}$

In the previous section we studied the nature of the singular points which appear on the slowness surface, in the case when we have $c_{11} = c_{66}$. Using exactly the same arguments it is possible to study the nature of the singular points in the case when $c_{11} \neq c_{66}$. We observe that the main difference between these two cases is that, if $c_{11} = c_{66}$, then the slowness surface has four biplanar singular points, one on each $\xi_i$-semi-axis, with $i \in \{1,2\}$, whereas if $c_{11} \neq c_{66}$, then the slowness surface does not have biplanar singular points, but it may have eight more singular points on the planes $\xi_i = 0$, with $i \in \{1,2\}$, as proposition 3.4 shows. Thus, with similar calculations, even though a little bit more involved, it is possible to prove the following proposition.

Proposition 6.1. *Let $S$ be the slowness surface associated with the tetragonal crystal system. Assume that the stiffness constants $c_{ij}$ satisfy the assumptions we made and, in addition, suppose $c_{11} \neq c_{66}$. Then*:

- *The double points of $S$ on the $\xi_3$-axes are uniplanar singularities.*
- *The double points of $S$ on the plane $\xi_3 = 0$, which do not lie on the coordinate axes are conical singularities.*
- *If, in addition, we assume that $c_{11} - c_{33} = c_{12} - c_{13} = c_{44} - c_{66}$, then the double points of $S$ on the planes $\xi_1^2 = \xi_2^2$ are conical singularities.*

Remark 6.2. We observe that the proposition 6.1 holds for generic tetragonal stiffness constants $c_{ij}$. It is not difficult to show that, if we consider the tetragonal system in the nearly cubic case, i.e. if we assume $c_{11} - c_{33} = c_{12} - c_{13} = c_{44} - c_{66} = e$, with $|e|$ sufficiently small, then the proposition still remains valid.

Now, we want to investigate the case when $c_{11} \neq c_{66}$, but $c_{44} = c_{66}$. We have seen that, in this case, the slowness surface has four double points, one on each semi-axis of the coordinate plane $\xi_3 = 0$. We have the following results about the nature of these double points.

Proposition 6.3. *Let $S$ be the slowness surface associated with the tetragonal crystal system. Assume that the stiffness constants $c_{ij}$ satisfy the assumption made in the previous sections and, in addition, suppose $c_{11} \neq c_{66}$, but $c_{44} = c_{66}$. Then the four double points of coordinate $(\pm 1/\sqrt{c_{44}}, 0, 0)$ and $(0, \pm 1/\sqrt{c_{44}}, 0)$ are uniplanar singularities.*

Proof. We denote $P = (1/\sqrt{c_{44}}, 0, 0)$. We will prove that $\nabla p(P) = 0$, $(\partial/\partial\xi_1)^2 p(P) \neq 0$, $(\partial^2/\partial\xi_i\partial\xi_j)^2 p(P) = 0$ if $i \in \{2,3\}$, $j \in \{1,2,3\}$, $(\partial/\partial\xi)^\alpha p(P) = 0$ if $|\alpha| = 3$ and the order of derivations in $(\xi_2, \xi_3)$ is odd.

The gradient of $p$ at $P$ vanishes since $P$ is a double point of the slowness surface.

The following remarks help us simplify the calculations of second order derivatives:

- The factors $n_2$, $n_3$ vanish twice at $P$.
- The expressions $d_i$ vanish at $P$ for $i = 2, 3$ (but not necessarily for $i = 1$).
- When we derivate one of the $d_i$, $i = 1, 2, 3$, in one of the variables $\xi_j$, $j = 2$ or 3, then we obtain a factor $\xi_j$, and therefore this derivative will vanish at $P$.
- $(\partial/\partial\xi_i)n_1 = 0$, $(\partial^2/\partial\xi_i\partial\xi_j)n_1 = 0$ for $i = 2, 3$, whatever $j$ is.

We can argue precisely as in the section (5.1) and conclude that derivatives of form $(\partial/\partial\xi_i\partial\xi_j)^2 p(P)$, and $(\partial/\partial\xi_i)(\partial/\partial\xi_1)^k p(P)$ vanish when $i, j \in \{2, 3\}$, $k \geq 2$. Moreover, after some calculations we obtain

$$\frac{\partial^2}{\partial\xi_1^2} p(P) = 8(c_{11} - c_{44}).$$

By assumption on the stiffness constants this is non vanishing.

We still have to say something about third order derivatives. If we derivate once in $\xi_3$ and the remaining derivatives are in the variables $\xi_1$, $\xi_2$, then the result may be non vanishing. □

We conclude this section with a proposition about the nature of the singular points on the plane $\xi_1^2 = \xi_2^2$ in the case when they are near the diagonal. Indeed, as we have observed, it seems difficult to calculate the Hessian of $p$ at these points in the general situation, whereas if $c_{11} - c_{33} = c_{12} - c_{13} = c_{44} - c_{66}$, the singular points lie on the space diagonals and it is easy to establish the conical nature of these singularities. Now, we will prove that, if these singular points remain near the space diagonals, then they still remain of conical type. In particular we will assume $c_{44} = c_{66}$, $c_{33} - c_{11} = e_1$ and $c_{13} - c_{12} = e_3$, with $|e_i|$, $i = 1, 3$ small. The choice of this assumption will become easy to understand when we will discuss the curvature properties of the slowness surface (cf. [7] and [10]).

Proposition 6.4. *Let $S$ be the slowness surface associated with the tetragonal crystal system. Assume that the stiffness constants $c_{ij}$ satisfy the assumptions made in the previous sections and, in addition, suppose $c_{11} \neq c_{66}$, either $c_{12} - c_{13} = c_{44} - c_{66} = 0$ and $c_{11} - c_{33} = e$, or $c_{44} = c_{66}$, $c_{33} - c_{11} = e_1$ and $c_{13} - c_{12} = e_3$. Then, if $|e_i|$, $i = 1, 3$ and $|e|$ are sufficiently small, the double points of $S$ on the planes $\xi_1^2 = \xi_2^2$ are conical singularities.*

PROOF. We prove the proposition with the assumption $c_{12} - c_{13} = c_{44} - c_{66} = 0$ and $c_{11} - c_{33} = e$. The proof in the case when $c_{44} = c_{66}$, $c_{33} - c_{11} = e_1$ and $c_{13} - c_{12} = e_3$ is exactly the same, with the expressions of $f_2$ and $f_3$ a little bit more involved.

We recall that, if $c_{12} - c_{13} = c_{44} - c_{66} = c_{11} - c_{33}$, then the double points have the following coordinates

$$(\pm(c_{44} + c_{11} - c_{12})^{-1/2}, \pm(c_{44} + c_{11} - c_{12})^{-1/2}, \pm(c_{44} + c_{11} - c_{12})^{-1/2}).$$

Now, if we assume $c_{12} - c_{13} = c_{44} - c_{66} = 0$ and $c_{11} - c_{33} = e$, it is not difficult to see that the double points have coordinates

$$(\pm\tilde{\xi}_2, \pm\tilde{\xi}_2, \pm\tilde{\xi}_3) \tag{6.1}$$

where

$$\tilde{\xi}_2^2 = \frac{1}{c_{44} + c_{11} - c_{12}} + ef_2(c_{ij}) \quad \tilde{\xi}_3^2 = \frac{1}{c_{44} + c_{11} - c_{12}} + ef_3(c_{ij})$$

and

$$f_2(c_{ij}) = \frac{c_{44} + 2c_{11} - 2c_{12}}{(c_{44} + c_{11} - c_{12})((c_{11} - c_{12})^2 - (c_{11} - c_{12})(c_{44} - e) - 2c_{44}^2)}$$

$$f_3(c_{ij}) = \frac{c_{11} - c_{12}}{(c_{44} + c_{11} - c_{12})((c_{11} - c_{12})^2 - (c_{11} - c_{12})(c_{44} - e) - 2c_{44}^2)}.$$

Thus, as above, it is possible to write the quantities $(\partial^2 q/\partial\xi_i\partial\xi_j)(P)$, with $i = 1, 2, 3$, in terms of the same quantities used in the case when $c_{12} - c_{13} = c_{44} - c_{66} = c_{11} - c_{33}$ plus $e$ times a rational function of the stiffness constants $c_{ij}$ (here and in the remainder of the proof, $P$ will be one of the singular points in (6.1)). So, we can argue as above and prove that, if $e$ is small enough, the signs of the eigenvalues of the Hessian of $p$ at $P$ and of the determinant of the Hessians when we assume $c_{12} - c_{13} = c_{44} - c_{66} = c_{11} - c_{33}$, do not change if we only have $c_{12} - c_{13} = c_{44} - c_{66} = 0$ and $c_{11} - c_{33} = e$. This concludes the proof. □

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