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# HOCHSCHILD COHOMOLOGY RING OF THE GENERALIZED QUATERNION ALGEBRAS

By

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**Abstract.** We will give an efficient bimodule projective resolution of the generalized quaternion $\mathbf{Z}$ algebra $\Gamma$. As a main result, we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ by calculating the Yoneda products using this resolution.

## Introduction

Let $R$ be a commutative ring and $\Lambda$ an $R$-algebra which is a finitely generated projective $R$-module. If $M$ is a $\Lambda$-bimodule (i.e., a $\Lambda^e = \Lambda \otimes_R \Lambda^{op}$-module), then the $n$th Hochschild cohomology of $\Lambda$ with coefficients in $M$ is defined by $H^n(\Lambda, M) := \mathrm{Ext}^n_{\Lambda^e}(\Lambda, M)$. If $M = \Lambda$, we set $HH^n(\Lambda) = H^n(\Lambda, \Lambda)$. The Yoneda product gives $HH^*(\Lambda) := \bigoplus_{n \geq 0} HH^n(\Lambda)$ a graded ring structure with $1 \in Z\Lambda \simeq HH^0(\Lambda)$ where $Z\Lambda$ denotes the center of $\Lambda$. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of $\Lambda$ (see [4], [1], [5]). The Hochschild cohomology ring $HH^*(\Lambda)$ is a graded-commutative algebra, that is, $\alpha\beta = (-1)^{pq}\beta\alpha$ holds for $\alpha \in HH^p(\Lambda)$ and $\beta \in HH^q(\Lambda)$. The Hochschild cohomology is an important invariant of algebras, however the Hochschild cohomology ring is difficult to compute in general.

Suppose that $a$ and $b$ are any nonzero rational integers. We consider the generalized quaternion $\mathbf{Z}$ algebra $\Gamma := \mathbf{Z} \oplus \mathbf{Z}i \oplus \mathbf{Z}j \oplus \mathbf{Z}ij$ with the relations $i^2 = a$, $j^2 = b$, $ij = -ji$. In the case $a = -1$ and $b = -1$, the ring structure of the Hochschild cohomology of the ordinary quaternion algebra $\Gamma$ is already known by Sanada [6, Section 3.4].

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Theorem (Sanada [6]). *Let* $\Gamma$ *be the ordinary quaternion algebra over* $\mathbf{Z}$. *Then the Hochschild cohomology ring of* $\Gamma$ *is as follows*:

$$HH^*(\Gamma) = \mathbf{Z}[\lambda, \mu, \nu]/(2\lambda, 2\mu, 2\nu, \lambda^2 + \mu^2 + \nu^2),$$

*where* $\deg \lambda = \deg \mu = \deg \nu = 1$.

In [2], the author reproves and generalizes this result. Thus it is natural question to investigate the ring structure of the Hochschild cohomology of the generalized quaternion algebra. In this article, we will give an efficient bimodule projective resolution of the generalized quaternion algebra $\Gamma$. Moreover by using this resolution, we determine the ring structure of the Hochschild cohomology ring of the generalized quaternion algebra $\Gamma$. This is a method similar to [2] or [3].

In Section 1, we state an efficient bimodule projective resolution of $\Gamma$ (Theorem 1.1). In Section 2, we use the resolution to describe the module structure of $HH^*(\Gamma)$, giving explicit generators (Theorem 2.1). In Section 3, we compute the Yoneda products of the generators. Then, as a main result of this article, we give a complete description of the Hochschild cohomology ring $HH^*(\Gamma)$ (Theorem 3.8). The result is more complicated than the known result for the ordinary quaternion algebra.

## 1 Bimodule Projective Resolution for the Generalized Quaternion Algebras

Suppose that $a$ and $b$ are any nonzero rational integers. Let

$$\Gamma := \mathbf{Z} \oplus \mathbf{Z}i \oplus \mathbf{Z}j \oplus \mathbf{Z}ij$$

be the generalized quaternion algebra over $\mathbf{Z}$ with the relations $i^2 = a$, $j^2 = b$, $ij = -ji$.

In the following, we give an efficient bimodule projective resolution of $\Gamma$. For each integer $q \geq 0$, let $Y_q$ be the direct sum of $q+1$ copies of $\Gamma \otimes \Gamma$. We define elements of $Y_q$ by

$$c_q^s = \begin{cases} (0, \dots, 0, 1 \overset{\overset{s}{\vee}}{\otimes} 1, 0, \dots, 0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have that $Y_q = \bigoplus_{k=1}^{q+1} \Gamma c_q^k \Gamma$.

Theorem 1.1. *There exists the following bimodule projective resolution of* $\Gamma$:

$$(Y, \delta) : \cdots \to Y_3 \xrightarrow{\delta_3} Y_2 \xrightarrow{\delta_2} Y_1 \xrightarrow{\delta_1} Y_0 \xrightarrow{\delta_0} \Gamma \to 0,$$

*where* $\delta_0 : Y_0 \to \Gamma$ *is the multiplication map, and for integer* $q > 0$, $\delta_q : Y_q \to Y_{q-1}$ *is a* $\Gamma^{\mathrm{e}}$*-homomorphism given by*

$$\delta_q(c_q^s) = \begin{cases} ic_{q-1}^s - c_{q-1}^s i + jc_{q-1}^{s-1} - c_{q-1}^{s-1} j & \text{for } q \text{ odd}, \\ ic_{q-1}^s + c_{q-1}^s i + jc_{q-1}^{s-1} + c_{q-1}^{s-1} j & \text{for } q \text{ even}. \end{cases}$$

Proof. By direct computations, we have that $\delta_q \cdot \delta_{q+1} = 0$ for $q \geq 0$. For example, if $q (\geq 1)$ is odd, then we have

$$\begin{aligned} \delta_q \cdot \delta_{q+1}(c_{q+1}^s) &= \delta_q(ic_q^s + c_q^s i + jc_q^{s-1} + c_q^{s-1} j) \\ &= ac_{q-1}^s - ic_{q-1}^s i + ijc_{q-1}^{s-1} - ic_{q-1}^{s-1} j \\ &\quad + ic_{q-1}^s i - c_{q-1}^s a + jc_{q-1}^{s-1} i - c_{q-1}^{s-1} ji \\ &\quad + jic_{q-1}^{s-1} - jc_{q-1}^{s-1} i + bc_{q-1}^{s-2} - jc_{q-1}^{s-2} j \\ &\quad + ic_{q-1}^{s-1} j - c_{q-1}^{s-1} ij + jc_{q-1}^{s-2} j - c_{q-1}^{s-2} b \\ &= 0. \end{aligned}$$

Thus we have $\operatorname{Im} \delta_{q+1} \subset \operatorname{Ker} \delta_q$ for $q \geq 0$.

Next we prove the reverse inclusion. We give a contracting homotopy. We define right $\Gamma$-homomorphisms $T_{-1} : \Gamma \to Y_0$ and $T_q : Y_q \to Y_{q+1}$ $(q \geq 0)$ as follows:

$$T_{-1}(\gamma) = c_0^1 \gamma \quad (\forall \gamma \in \Gamma),$$

$$T_q(i^m j^n c_q^s) = \begin{cases} mc_{q+1}^1 & (q \geq 0,\ s = 1,\ m = 0, 1,\ n = 0), \\ (-1)^q mc_{q+1}^1 j + i^m c_{q+1}^2 & (q \geq 0,\ s = 1,\ m = 0, 1,\ n = 1), \\ 0 & (q \geq 0,\ s \geq 2,\ m = 0, 1,\ n = 0), \\ i^m c_{q+1}^{s+1} & (q \geq 0,\ s \geq 2,\ m = 0, 1,\ n = 1). \end{cases}$$

We may see that $T_q : Y_q \to Y_{q+1}$ $(q \geq -1)$ is a contracting homotopy. Thus we must check that the equation

$$(\delta_{q+1} T_q + T_{q-1} \delta_q)(i^m j^n c_q^s) = i^m j^n c_q^s$$

holds for $q \geq 0$; $1 \leq s \leq q + 1$; $m = 0, 1$; $n = 0, 1$.

If $q = 0$, we have

$$T_{-1} \delta_0(i^m j^n c_0^1) = T_{-1}(i^m j^n) = c_0^1 i^m j^n,$$

where $m = 0, 1$ and $n = 0, 1$. On the other hand, if $n = 0$, we have

$$\delta_1 T_0(i^m c_0^1) = \delta_1(mc_1^1) = m(ic_0^1 - c_0^1 i),$$

where $m = 0, 1$. If $n = 1$, we have

$$\begin{aligned}\delta_1 T_0(i^m jc_0^1) &= \delta_1(mc_1^1 j + i^m c_1^2) \\ &= m(ic_0^1 - c_0^1 i)j + i^m(jc_0^1 - c_0^1 j) \\ &= i^m jc_0^1 - c_0^1 i^m j,\end{aligned}$$

where $m = 0, 1$. Thus we have $\delta_1 T_0 + T_{-1}\delta_0 = \mathrm{id}_{Y_0}$.

If $q(\geq 1)$ is odd, then for $s = 1$ and $n = 0$, we have

$$\delta_{q+1} T_q(i^m c_q^1) = \delta_{q+1}(mc_{q+1}^1) = m(ic_q^1 + c_q^1 i) \quad (m = 0, 1),$$

$$T_{q-1}\delta_q(i^m c_q^1) = T_{q-1}(i^{m+1} c_{q-1}^1 - i^m c_{q-1}^1 i) = \begin{cases} c_q^1 & (m = 0), \\ -c_q^1 i & (m = 1). \end{cases}$$

If $q(\geq 1)$ is odd, then for $s = 1$ and $n = 1$, we have

$$\begin{aligned}\delta_{q+1} T_q(i^m jc_q^1) &= \delta_{q+1}(-mc_{q+1}^1 j + i^m c_{q+1}^2) \\ &= \begin{cases} jc_q^1 + c_q^1 j + ic_q^2 + c_q^2 i & (m = 0), \\ ijc_q^1 - c_q^1 ij + ac_q^2 + ic_q^2 i & (m = 1), \end{cases}\end{aligned}$$

$$\begin{aligned}T_{q-1}\delta_q(i^m jc_q^1) &= \begin{cases} T_{q-1}(-ijc_{q-1}^1 - jc_{q-1}^1 i) & (m = 0), \\ T_{q-1}(-ajc_{q-1}^1 - ijc_{q-1}^1 i) & (m = 1) \end{cases} \\ &= \begin{cases} -c_q^1 j - ic_q^2 - c_q^2 i & (m = 0), \\ c_q^1 ij - ac_q^2 - ic_q^2 i & (m = 1). \end{cases}\end{aligned}$$

If $q(\geq 1)$ is odd, then for $s \geq 2$, $m = 0, 1$, and $n = 0$, we have

$$\begin{aligned}&\delta_{q+1} T_q(i^m c_q^s) = 0, \\ &T_{q-1}\delta_q(i^m c_q^s) = T_{q-1}(i^{m+1} c_{q-1}^s - i^m c_{q-1}^s i + i^m jc_{q-1}^{s-1} - i^m c_{q-1}^{s-1} j) \\ &\qquad = i^m c_q^s.\end{aligned}$$

If $q(\geq 1)$ is odd, then for $s \geq 2$ and $n = 1$, we have

$$\begin{aligned}\delta_{q+1} T_q(i^m jc_q^s) &= \delta_{q+1}(i^m c_{q+1}^{s+1}) \\ &= i^{m+1} c_q^{s+1} + i^m c_q^{s+1} i + i^m jc_q^s + i^m c_q^s j \quad (m = 0, 1),\end{aligned}$$

$$T_{q-1}\delta_q(i^m jc_q^s) = \begin{cases} T_{q-1}(-ijc_{q-1}^s - jc_{q-1}^s i + bc_{q-1}^{s-1} - jc_{q-1}^{s-1} j) & (m=0), \\ T_{q-1}(-ajc_{q-1}^s - ijc_{q-1}^s i + bic_{q-1}^{s-1} - ijc_{q-1}^{s-1} j) & (m=1) \end{cases}$$

$$= \begin{cases} -ic_q^{s+1} - c_q^{s+1} i - c_q^s j & (m=0), \\ -ac_q^{s+1} - ic_q^{s+1} i - ic_q^s j & (m=1). \end{cases}$$

Thus we have $\delta_{q+1}T_q + T_{q-1}\delta_q = \mathrm{id}_{Y_q}$ for $q(\geq 1)$ odd. In the case $q(\geq 2)$ even, the computations are similar. Hence $\mathrm{Im}\,\delta_{q+1} \supset \mathrm{Ker}\,\delta_q$ is proved for $q \geq 0$. □

Remark 1.2. If $a = -1$ and $b = -1$, an efficient bimodule projective resolution of the ordinary quaternion algebra $\Gamma$ is given in [2]. Even in that special situation, the differential of the resolution in Theorem 1.1 is different from that of [2, Theorem 1.1].

## 2 Module Structure

We keep the notations in Section 1. In this section, we calculate the Hochschild cohomology group $HH^*(\Gamma)$.

Let $M^q$ denote the direct sum of $q$ copies of a module $M$ for any integer $q > 0$. As elements of $\Gamma^{q+1}$, we set

$$\iota_q^s = \begin{cases} (0, \dots, 0, \overset{s}{\check{1}}, 0, \dots, 0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have $\Gamma^{q+1} = \bigoplus_{k=1}^{q+1} \Gamma \iota_q^k$.

Applying the functor $\mathrm{Hom}_{\Gamma^e}(-, \Gamma)$ to the resolution $(Y, \delta)$, we have the following complex, where we identify $\mathrm{Hom}_{\Gamma^e}(Y_q, \Gamma)$ with $\Gamma^{q+1}$ using an isomorphism $\mathrm{Hom}_{\Gamma^e}(Y_q, \Gamma) \to \Gamma^{q+1}$; $f \mapsto \sum_{k=1}^{q+1} f(c_q^k)\iota_q^k$:

$$(\mathrm{Hom}_{\Gamma^e}(Y, \Gamma), \delta^\#) : 0 \longrightarrow \Gamma \xrightarrow{\delta_1^\#} \Gamma^2 \xrightarrow{\delta_2^\#} \Gamma^3 \xrightarrow{\delta_3^\#} \Gamma^4 \xrightarrow{\delta_4^\#} \Gamma^5 \longrightarrow \cdots,$$

$$\delta_{q+1}^\#(\gamma\iota_q^s) = \begin{cases} (i\gamma - \gamma i)\iota_{q+1}^s + (j\gamma - \gamma j)\iota_{q+1}^{s+1} & \text{for } q \text{ odd}, \\ (i\gamma + \gamma i)\iota_{q+1}^s + (j\gamma + \gamma j)\iota_{q+1}^{s+1} & \text{for } q \text{ even}. \end{cases}$$

In the above, note that

$$\gamma\iota_q^s = \begin{cases} (0, \dots, 0, \overset{s}{\check{\gamma}}, 0, \dots, 0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}), \end{cases}$$

for $\gamma \in \Gamma$, and so on.

In the following we set

$$d = gcd(a, b), \quad a' = \frac{a}{d}, \quad b' = \frac{b}{d}.$$

Then we have the following theorem.

Theorem 2.1. *The Hochschild cohomology group of* $\Gamma$ *is as follows*:

$$HH^n(\Gamma) = \begin{cases} \mathbf{Z} & (n = 0), \\ (\mathbf{Z}/2\mathbf{Z})^{n+1} \oplus (\mathbf{Z}/2d\mathbf{Z})^n & (n \ odd), \\ \mathbf{Z}/2a\mathbf{Z} \oplus (\mathbf{Z}/2d\mathbf{Z})^{n-1} \oplus \mathbf{Z}/2b\mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})^n & (n(\neq 0) \ even). \end{cases}$$

*Furthermore, module generators of* $HH^n(\Gamma)$ *are given as follows*:

(i) *If* $n(\geq 1)$ *is odd, the* $(\mathbf{Z}/2\mathbf{Z})^{n+1}$ *summands are generated by* $ij\iota_n^k$ $(k = 1, 2, \dots, n+1)$ *and the* $(\mathbf{Z}/2d\mathbf{Z})^n$ *summands are generated by* $a'j\iota_n^k - b'i\iota_n^{k+1}$ $(k = 1, 2, \dots, n)$.

(ii) *If* $n(\geq 2)$ *is even, the* $\mathbf{Z}/2a\mathbf{Z}$ *summand is generated by* $\iota_n^1$, *the* $(\mathbf{Z}/2d\mathbf{Z})^{n-1}$ *summands are generated by* $\iota_n^k$ $(k = 2, 3, \dots, n)$, *the* $\mathbf{Z}/2b\mathbf{Z}$ *summand is generated by* $\iota_n^{n+1}$, *and the* $(\mathbf{Z}/2\mathbf{Z})^n$ *summands are generated by* $i\iota_n^k + j\iota_n^{k+1}$ $(k = 1, 2, \dots, n)$.

Proof. For any element $\gamma = x + yi + zj + wij$ $(x, y, z, w \in \mathbf{Z})$ in $\Gamma$, we have

$$i\gamma = xi + ay + zij + awj, \quad \gamma i = xi + ay - zij - awj,$$

$$j\gamma = xj - yij + bz - bwi, \quad \gamma j = xj + yij + bz + bwi.$$

We prove the case $n(> 0)$ even only. Let $(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$ be any element in $\Gamma^{n+1}$ where we set $\gamma_k = x_k + y_k i + z_k j + w_k ij$ $(x_k, y_k, z_k, w_k \in \mathbf{Z})$. Since

$$\begin{aligned} &(\gamma_1, \gamma_2, \dots, \gamma_{n+1}) \in \operatorname{Ker} \delta_{n+1}^{\#} \\ &\Leftrightarrow \begin{cases} 2z_1 ij + 2aw_1 j = 0, \\ -2y_k ij - 2bw_k i + 2z_{k+1} ij + 2aw_{k+1} j = 0 & (k = 1, 2, \dots, n), \\ -2y_{n+1} ij - 2bw_{n+1} i = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} z_1 = y_{n+1} = 0, \\ w_k = 0 & (k = 1, 2, \dots, n+1), \\ y_k = z_{k+1} & (k = 1, 2, \dots, n) \end{cases} \\ &\Leftrightarrow (\gamma_1, \gamma_2, \dots, \gamma_{n+1}) = \sum_{k=1}^{n+1} x_k \iota_n^k + \sum_{k=1}^{n} y_k (i\iota_n^k + j\iota_n^{k+1}), \end{aligned}$$

we have

$$\operatorname{Ker} \delta_{n+1}^{\#} = \bigoplus_{k=1}^{n+1} \mathbf{Z}\iota_n^k \oplus \bigoplus_{k=1}^{n} \mathbf{Z}(i\iota_n^k + j\iota_n^{k+1}).$$

Next we show

$$\operatorname{Im} \delta_n^{\#} = 2a\mathbf{Z}\iota_n^1 \oplus \bigoplus_{k=2}^{n} 2d\mathbf{Z}\iota_n^k \oplus 2b\mathbf{Z}\iota_n^{n+1} \oplus \bigoplus_{k=1}^{n} 2\mathbf{Z}(i\iota_n^k + j\iota_n^{k+1}).$$

Let $(\gamma_1, \gamma_2, \dots, \gamma_n)$ be any element in $\Gamma^n$ where we set $\gamma_k = x_k + y_k i + z_k j + w_k ij$ $(x_k, y_k, z_k, w_k \in \mathbf{Z})$. Then we have

$$\begin{aligned} &\delta_n^{\#}(\gamma_1, \gamma_2, \dots, \gamma_n) \\ &\quad = 2ay_1\iota_n^1 + 2\sum_{k=2}^{n}(ay_k + bz_{k-1})\iota_n^k + 2bz_n\iota_n^{n+1} + 2\sum_{k=1}^{n} x_k(i\iota_n^k + j\iota_n^{k+1}) \\ &\quad = 2ay_1\iota_n^1 + 2d\sum_{k=2}^{n}(a'y_k + b'z_{k-1})\iota_n^k + 2bz_n\iota_n^{n+1} + 2\sum_{k=1}^{n} x_k(i\iota_n^k + j\iota_n^{k+1}). \end{aligned}$$

Note that $a'y_k + b'z_{k-1}$ is to be any element of $\mathbf{Z}$ by choosing $y_k$ and $z_{k-1}$ properly. Hence we have

$$HH^n(\Gamma) = (\mathbf{Z}/2a\mathbf{Z})\iota_n^1 \oplus \bigoplus_{k=2}^{n}(\mathbf{Z}/2d\mathbf{Z})\iota_n^k \oplus (\mathbf{Z}/2b\mathbf{Z})\iota_n^{n+1} \oplus \bigoplus_{k=1}^{n}(\mathbf{Z}/2\mathbf{Z})(i\iota_n^k + j\iota_n^{k+1}).$$

The other cases are similar. □

Remark 2.2. In particular if $a = \pm 1$, $b = \pm 1$, then we have that

$$HH^n(\Gamma) = \begin{cases} \mathbf{Z} & (n = 0), \\ (\mathbf{Z}/2\mathbf{Z})^{2n+1} & (n \geq 1). \end{cases}$$

## 3 Ring Structure

We maintain the notations in Sections 1 and 2. In this section, we determine the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$.

Recall the Yoneda product in $HH^*(\Gamma)$. Let $\alpha \in HH^n(\Gamma)$ and $\beta \in HH^m(\Gamma)$, where $\alpha$ and $\beta$ are represented by cocycles $f_\alpha : Y_n \to \Gamma$ and $f_\beta : Y_m \to \Gamma$, respectively. We have the commutative diagram of $\Gamma^e$-modules with exact rows:

$$
\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\delta_{n+m+1}} & Y_{n+m} & \xrightarrow{\delta_{n+m}} & \cdots & \xrightarrow{\delta_{m+2}} & Y_{m+1} & \xrightarrow{\delta_{m+1}} & Y_m & \xrightarrow{f_\beta} & \Gamma & & \\
 & & \downarrow{\scriptstyle u_n} & & & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \| & & \\
\cdots & \xrightarrow[\delta_{n+1}]{} & Y_n & \xrightarrow[\delta_n]{} & \cdots & \xrightarrow[\delta_2]{} & Y_1 & \xrightarrow[\delta_1]{} & Y_0 & \xrightarrow[\delta_0]{} & \Gamma & \longrightarrow & 0,
\end{array}
$$

where $u_l$ $(0 \le l \le n)$ are liftings of $f_\beta$. We define the product $\alpha \cdot \beta \in HH^{n+m}(\Gamma)$ by the cohomology class of $f_\alpha u_n$. This product is independent of the choice of representatives $f_\alpha$ and $f_\beta$, and liftings $u_l$ $(0 \le l \le n)$.

By Theorem 2.1, we take generators of $HH^1(\Gamma)$ as follows:

$$\lambda_1 = ij\iota_1^1, \quad \mu_1 = ij\iota_1^2, \quad \nu_1 = a'j\iota_1^1 - b'i\iota_1^2.$$

Then we have $2\lambda_1 = 2\mu_1 = 2d\nu_1 = 0$, and $\lambda_1$, $\mu_1$, $\nu_1$ are represented by the following $\Gamma^{\mathrm{e}}$-homomorphisms, respectively:

$$\hat{\lambda}_1 : Y_1 \to \Gamma; \quad c_1^1 \mapsto ij, \quad c_1^2 \mapsto 0,$$

$$\hat{\mu}_1 : Y_1 \to \Gamma; \quad c_1^1 \mapsto 0, \quad c_1^2 \mapsto ij,$$

$$\hat{\nu}_1 : Y_1 \to \Gamma; \quad c_1^1 \mapsto a'j, \quad c_1^2 \mapsto -b'i.$$

We state an initial part of liftings of these cocycles.

Lemma 3.1. (i) *An initial part of a lifting* $u_n : Y_{n+1} \to Y_n$ *of* $\hat{\lambda}_1$ *is as follows*:

$$u_0(c_1^1) = ijc_0^1, \quad u_0(c_1^2) = 0;$$

$$u_1(c_2^1) = -ijc_1^1, \quad u_1(c_2^2) = -ijc_1^2, \quad u_1(c_2^3) = 0.$$

(ii) *An initial part of a lifting* $v_n : Y_{n+1} \to Y_n$ *of* $\hat{\mu}_1$ *is as follows*:

$$v_0(c_1^1) = 0, \quad v_0(c_1^2) = ijc_0^1;$$

$$v_1(c_2^1) = 0, \quad v_1(c_2^2) = -ijc_1^1, \quad v_1(c_2^3) = -ijc_1^2.$$

(iii) *An initial part of a lifting* $w_n : Y_{n+1} \to Y_n$ *of* $\hat{\nu}_1$ *is as follows*:

$$w_0(c_1^1) = a'jc_0^1, \quad w_0(c_1^2) = -b'ic_0^1;$$

$$w_1(c_2^1) = -a'jc_1^1, \quad w_1(c_2^2) = b'ic_1^1 - a'jc_1^2, \quad w_1(c_2^3) = b'ic_1^2.$$

Proof. (i) Clearly $\hat{\lambda}_1 = \delta_0 u_0$ holds. Since

$$u_0\delta_2(c_2^1) = ajc_0^1 + ijc_0^1 i = -ij(ic_0^1 - c_0^1 i) = \delta_1 u_1(c_2^1),$$

$$u_0\delta_2(c_2^2) = -bic_0^1 + ijc_0^1 j = -ij(jc_0^1 - c_0^1 j) = \delta_1 u_1(c_2^2),$$

$$u_0\delta_2(c_2^3) = 0 = \delta_1 u_1(c_2^3),$$

we have $u_0\delta_2 = \delta_1 u_1$. (ii) and (iii) are similar. □

Proposition 3.2. *The following equations hold in* $HH^2(\Gamma)$:

$$\lambda_1^2 = ab\iota_2^1, \quad \lambda_1\mu_1 = ab\iota_2^2, \quad \mu_1^2 = ab\iota_2^3,$$

$$\lambda_1\nu_1 = a'b(i\iota_2^1 + j\iota_2^2), \quad \mu_1\nu_1 = a'b(i\iota_2^2 + j\iota_2^3), \quad a'\lambda_1^2 + b'\mu_1^2 + d\nu_1^2 = 0.$$

*In particular, in the case* $a = \pm 1$ *and* $b = \pm 1$, $HH^2(\Gamma)$ *is generated by the products of* $\lambda_1$, $\mu_1$, *and* $\nu_1$.

Proof. We calculate $\lambda_1\nu_1$ as an example. Since

$$\hat{\lambda}_1 w_1(c_2^1) = \hat{\lambda}_1(-a'jc_1^1) = a'bi,$$

$$\hat{\lambda}_1 w_1(c_2^2) = \hat{\lambda}_1(b'ic_1^1 - a'jc_1^2) = ab'j = a'bj,$$

$$\hat{\lambda}_1 w_1(c_2^3) = \hat{\lambda}_1(b'ic_1^2) = 0,$$

it follows that $\lambda_1\nu_1 = a'b(i\iota_2^1 + j\iota_2^2)$ holds. Other computations are similar. □

In the following we take generators of $HH^2(\Gamma)$ as follows:

$$\tau_2 = \iota_2^1, \quad \nu_2 = \iota_2^2, \quad \xi_2 = \iota_2^3,$$

$$\lambda_2 = i\iota_2^1 + j\iota_2^2, \quad \mu_2 = i\iota_2^2 + j\iota_2^3.$$

Then $\tau_2$, $\nu_2$, $\xi_2$, $\lambda_2$, and $\mu_2$ are represented by the following $\Gamma^e$-homomorphisms, respectively:

$$\hat{\tau}_2 : Y_2 \to \Gamma; \quad c_2^1 \mapsto 1, \quad c_2^2 \mapsto 0, \quad c_2^3 \mapsto 0;$$

$$\hat{\nu}_2 : Y_2 \to \Gamma; \quad c_2^1 \mapsto 0, \quad c_2^2 \mapsto 1, \quad c_2^3 \mapsto 0;$$

$$\hat{\xi}_2 : Y_2 \to \Gamma; \quad c_2^1 \mapsto 0, \quad c_2^2 \mapsto 0, \quad c_2^3 \mapsto 1;$$

$$\hat{\lambda}_2 : Y_2 \to \Gamma; \quad c_2^1 \mapsto i, \quad c_2^2 \mapsto j, \quad c_2^3 \mapsto 0;$$

$$\hat{\mu}_2 : Y_2 \to \Gamma; \quad c_2^1 \mapsto 0, \quad c_2^2 \mapsto i, \quad c_2^3 \mapsto j.$$

REMARK 3.3. By Theorem 2.1 and Proposition 3.2, we have the following:

$$2a\tau_2 = 2b\xi_2 = 2\lambda_2 = 2\mu_2 = 2d\nu_2 = 0,$$

$$\lambda_1^2 = ab\tau_2, \quad \lambda_1\mu_1 = ab\nu_2, \quad \mu_1^2 = ab\xi_2, \quad \lambda_1\nu_1 = a'b\lambda_2, \quad \mu_1\nu_1 = a'b\mu_2.$$

Note that, since $HH^*(\Gamma)$ is graded-commutative and $2ab\nu_2 = 0$, it follows that $\mu_1\lambda_1 = -\lambda_1\mu_1 = -ab\nu_2 = ab\nu_2 = \lambda_1\mu_1$ hold. Similarly we have $\nu_1\lambda_1 = \lambda_1\nu_1$ and $\nu_1\mu_1 = \mu_1\nu_1$.

Next we state liftings of $\tau_2$, $\xi_2$, $\nu_2$, $\lambda_2$, and $\mu_2$.

LEMMA 3.4. (i) *A lifting* $f_n : Y_{n+2} \to Y_n$ *of* $\hat{\tau}_2$ *is given by* $f_n(c_{n+2}^k) = c_n^k$ *for* $n \geq 0$.

(ii) *A lifting* $g_n : Y_{n+2} \to Y_n$ *of* $\hat{\xi}_2$ *is given by* $g_n(c_{n+2}^k) = c_n^{k-2}$ *for* $n \geq 0$.

(iii) *A lifting* $h_n : Y_{n+2} \to Y_n$ *of* $\hat{\nu}_2$ *is given by* $h_n(c_{n+2}^k) = c_n^{k-1}$ *for* $n \geq 0$.

(iv) *A lifting* $r_n : Y_{n+2} \to Y_n$ *of* $\hat{\lambda}_2$ *is given by* $r_n(c_{n+2}^k) = jc_n^{k-1} + ic_n^k$ *for* $n \geq 0$.

(v) *A lifting* $t_n : Y_{n+2} \to Y_n$ *of* $\hat{\mu}_2$ *is given by* $t_n(c_{n+2}^k) = jc_n^{k-2} + ic_n^{k-1}$ *for* $n \geq 0$.

PROOF. (i) Clearly $\hat{\tau}_2 = \delta_0 f_0$ holds. If $n(\geq 1)$ is odd, then

$$\begin{aligned} f_n\delta_{n+3}(c_{n+3}^k) &= f_n(ic_{n+2}^k + c_{n+2}^k i + jc_{n+2}^{k-1} + c_{n+2}^{k-1} j) \\ &= ic_n^k + c_n^k i + jc_n^{k-1} + c_n^{k-1} j = \delta_n f_{n+1}(c_{n+3}^k). \end{aligned}$$

If $n(\geq 2)$ is even, then

$$\begin{aligned} f_n\delta_{n+3}(c_{n+3}^k) &= f_n(ic_{n+2}^k - c_{n+2}^k i + jc_{n+2}^{k-1} - c_{n+2}^{k-1} j) \\ &= ic_n^k - c_n^k i + jc_n^{k-1} - c_n^{k-1} j = \delta_n f_{n+1}(c_{n+3}^k). \end{aligned}$$

Thus (i) is proved. Other computations are similar. □

REMARK 3.5. Let $\alpha = \sum_{k=1}^{n+1} \gamma_k \iota_n^k$ $(\gamma_k \in \Gamma)$ be any element in $HH^n(\Lambda)$ for $n \geq 1$. Then by Lemma 3.4 we have

$$\alpha\tau_2 = \sum_{k=1}^{n+1} \gamma_k \iota_{n+2}^k, \quad \alpha\xi_2 = \sum_{k=1}^{n+1} \gamma_k \iota_{n+2}^{k+2}, \quad \alpha\nu_2 = \sum_{k=1}^{n+1} \gamma_k \iota_{n+2}^{k+1}.$$

By using this remark, it is shown that $HH^3(\Gamma)$ is generated by products of $\lambda_1$, $\mu_1$, $\nu_1$, $\tau_2$, $\nu_2$, and $\xi_2$:

$$\lambda_1\tau_2 = ij\iota_3^1, \quad \mu_1\tau_2(= \lambda_1 \nu_2) = ij\iota_3^2, \quad \lambda_1\xi_2(= \mu_1\nu_2) = ij\iota_3^3, \quad \mu_1\xi_2 = ij\iota_3^4,$$
$$\nu_1\tau_2 = a'j\iota_3^1 - b'i\iota_3^2, \quad \nu_1\nu_2 = a'j\iota_3^2 - b'i\iota_3^3, \quad \nu_1\xi_2 = a'j\iota_3^3 - b'i\iota_3^4.$$

We state the relations in degree 3, which are given by using Lemma 3.4.

Proposition 3.6. *The following relations hold in* $HH^3(\Gamma)$:

$$\mu_1\tau_2 = \lambda_1\nu_2, \quad \lambda_1\xi_2 = \mu_1\nu_2, \quad \lambda_1\mu_2 = \mu_1\lambda_2 = d\nu_1\nu_2, \quad \lambda_1\lambda_2 = d\nu_1\tau_2,$$
$$\mu_1\mu_2 = d\nu_1\xi_2, \quad \nu_1\lambda_2 = a'\lambda_1\tau_2 + b'\lambda_1\xi_2, \quad \nu_1\mu_2 = a'\mu_1\tau_2 + b'\mu_1\xi_2.$$

Likewise, we may show that $HH^4(\Gamma)$ is generated by products of $\tau_2$, $\nu_2$, $\xi_2$, $\lambda_2$, $\mu_2$:

$$\tau_2^2 = \iota_4^1, \quad \tau_2\nu_2 = \iota_4^2, \quad \tau_2\xi_2(= \nu_2^2) = \iota_4^3, \quad \nu_2\xi_2 = \iota_4^4, \quad \xi_2^2 = \iota_4^5,$$
$$\lambda_2\tau_2 = i\iota_4^1 + j\iota_4^2, \quad \mu_2\tau_2(= \lambda_2\nu_2) = i\iota_4^2 + j\iota_4^3,$$
$$\lambda_2\xi_2(= \mu_2\nu_2) = i\iota_4^3 + j\iota_4^4, \quad \mu_2\xi_2 = i\iota_4^4 + j\iota_4^5.$$

We state the relations in degree 4, which are given by using Lemma 3.4.

Proposition 3.7. *The following relations hold in* $HH^4(\Gamma)$:

$$\tau_2\xi_2 = \nu_2^2, \quad \mu_2\tau_2 = \lambda_2\nu_2, \quad \lambda_2\xi_2 = \mu_2\nu_2,$$
$$\lambda_2^2 = a\tau_2^2 + b\tau_2\xi_2, \quad \lambda_2\mu_2 = a\tau_2\nu_2 + b\nu_2\xi_2, \quad \mu_2^2 = a\tau_2\xi_2 + b\xi_2^2.$$

Similarly, by using Remark 3.5, it is not hard to see that $HH^n(\Gamma)$ for $n \geq 5$ is multiplicatively generated by products of $\lambda_1$, $\mu_1$, $\nu_1$, $\tau_2$, $\nu_2$, $\xi_2$, $\lambda_2$, and $\mu_2$.

Now suppose that $\mathscr{A} = \mathbf{Z}[X_{1,1}, X_{1,2}, X_{1,3}, X_{2,1}, X_{2,2}, X_{2,3}, X_{2,4}, X_{2,5}]$ is a graded algebra with $\deg X_{k,\ell} = k$ for $k = 1, 2$. We consider the algebra homomorphism $\Phi : \mathscr{A} \to HH^*(\Gamma)$ induced by $X_{k,1} \mapsto \lambda_k$, $X_{k,2} \mapsto \mu_k$, $X_{k,3} \mapsto \nu_k$, $X_{2,4} \mapsto \tau_2$, and $X_{2,5} \mapsto \xi_2$ where $k = 1, 2$. Let $\mathscr{I}$ denote the set of the relations $2\lambda_1 = 2\mu_1 = 2d\nu_1 = 0$ and the relations given by Propositions 3.2, 3.6, and 3.7 and Remark 3.3. We rewrite $\mathscr{I}$ by the correspondence $\theta$ which is defined by $\lambda_k \mapsto X_{k,1}$, $\mu_k \mapsto X_{k,2}$, $\nu_k \mapsto X_{k,3}$, $\tau_2 \mapsto X_{2,4}$, and $\xi_2 \mapsto X_{2,5}$ where $k = 1, 2$, and denote it by $\theta(\mathscr{I})$. The algebra homomorphism $\Phi$ induces a surjective algebra homomorphism $\tilde{\Phi} : \mathscr{B} = \mathscr{A}/\theta(\mathscr{I}) \to HH^*(\Gamma)$. Let $\mathscr{B}_n = \{z \in \mathscr{B} \mid \deg z = n\}$ for $n > 0$.

If $n = 1$, $\mathscr{B}_1$ is additively generated by $X_{1,1}$, $X_{1,2}$, $X_{1,3}$. Then $X_{1,1}$ and $X_{1,2}$ have order dividing 2, and $X_{1,3}$ has order dividing $2d$. Thus the order of $\mathscr{B}_1$ is at most $2^2 \cdot 2d$.

If $n = 2$, $\mathscr{B}_2$ is additively generated by $X_{2,1}$, $X_{2,2}$, $X_{2,3}$, $X_{2,4}$, and $X_{2,5}$. Then $X_{2,1}$, $X_{2,2}$, $X_{2,3}$, $X_{2,4}$, $X_{2,5}$ have order dividing 2, 2, $2d$, $2a$, $2b$, respectively. Thus the order of $\mathscr{B}_2$ is at most $2^2 \cdot 2d \cdot 2a \cdot 2b$.

If $n = 2k + 1$ $(k \neq 0)$, then $\mathscr{B}_n$ is additively generated by

$$X_{1,\ell} X_{2,4}^{k-s} X_{2,5}^{s} \quad (\ell = 1, 2, 3,\ 0 \leq s \leq k), \quad X_{1,3} X_{2,3} X_{2,4}^{k-s} X_{2,5}^{s-1} \quad (1 \leq s \leq k).$$

Thus $X_{1,\ell} X_{2,4}^{k-s} X_{2,5}^{s}$ $(\ell = 1, 2,\ 0 \leq s \leq k)$ have order dividing 2, and the other generators have order dividing $2d$. Thus the order of $\mathscr{B}_n$ is at most $2^{n+1} \cdot (2d)^n$.

If $n = 2k$ $(k \geq 2)$, then $\mathscr{B}_n$ is additively generated by

$$X_{2,4}^{k-s} X_{2,5}^{s} \quad (0 \leq s \leq k), \quad X_{2,\ell} X_{2,4}^{k-s} X_{2,5}^{s-1} \quad (\ell = 1, 2, 3,\ 1 \leq s \leq k).$$

Then $X_{2,4}^{k}$ has order dividing $2a$, $X_{2,5}^{k}$ has order dividing $2b$, $X_{2,\ell} X_{2,4}^{k-s} X_{2,5}^{s-1}$ $(\ell = 1, 2,\ 1 \leq s \leq k)$ have order dividing 2, and the other generators have order dividing $2d$. Note that for $1 \leq s \leq k-1$, $X_{2,4}^{k-s} X_{2,5}^{s}$ has order dividing $2d$, because $X_{2,3}^2 = X_{2,4} X_{2,5}$ and $X_{2,3}$ has order dividing $2d$. Thus the order of $\mathscr{B}_n$ is at most $2^n \cdot 2a \cdot 2b \cdot (2d)^{n-1}$.

Hence the order of $\mathscr{B}_n$ is at most the order of $HH^n(\Gamma)$ for $n > 0$ (see Theorem 2.1). Therefore $\tilde{\Phi} : \mathscr{B} \to HH^*(\Gamma)$ is also injective.

Finally, we state the ring structure of $HH^*(\Gamma)$.

THEOREM 3.8. *The Hochschild cohomology ring $HH^*(\Gamma)$ is the commutative graded ring which is generated by the elements*

$$\lambda_1, \mu_1, \nu_1 \in HH^1(\Gamma), \quad \tau_2, \xi_2, \nu_2, \lambda_2, \mu_2 \in HH^2(\Gamma),$$

*and is defined by the following relations*:

(i) *degree*-1 *relations*

$$2\lambda_1 = 2\mu_1 = 2d\nu_1 = 0.$$

(ii) *degree*-2 *relations*

$$2a\tau_2 = 2b\xi_2 = 2\lambda_2 = 2\mu_2 = 2d\nu_2 = 0, \quad a'\lambda_1^2 + b'\mu_1^2 + d\nu_1^2 = 0,$$

$$\lambda_1^2 = ab\tau_2, \quad \lambda_1\mu_1 = ab\nu_2, \quad \mu_1^2 = ab\xi_2, \quad \lambda_1\nu_1 = a'b\lambda_2, \quad \mu_1\nu_1 = a'b\mu_2.$$

(iii) *degree*-3 *relations*

$$\mu_1\tau_2 = \lambda_1\nu_2, \quad \lambda_1\xi_2 = \mu_1\nu_2, \quad \lambda_1\mu_2 = \mu_1\lambda_2 = d\nu_1\nu_2, \quad \lambda_1\lambda_2 = d\nu_1\tau_2,$$

$$\mu_1\mu_2 = d\nu_1\xi_2, \quad \nu_1\lambda_2 = a'\lambda_1\tau_2 + b'\lambda_1\xi_2, \quad \nu_1\mu_2 = a'\mu_1\tau_2 + b'\mu_1\xi_2.$$

(iv) *degree*-4 *relations*

$$\tau_2\xi_2 = \nu_2^2, \quad \mu_2\tau_2 = \lambda_2\nu_2, \quad \lambda_2\xi_2 = \mu_2\nu_2,$$

$$\lambda_2^2 = a\tau_2^2 + b\tau_2\xi_2, \quad \lambda_2\mu_2 = a\tau_2\nu_2 + b\nu_2\xi_2, \quad \mu_2^2 = a\tau_2\xi_2 + b\xi_2^2.$$

REMARK 3.9. The result of Sanada [6, Section 3.4] follows from Theorem 3.8:

If $a = \pm 1$ and $b = \pm 1$, then $\tau_2$, $\xi_2$, $\nu_2$, $\lambda_2$, $\mu_2$ are generated by the products of $\lambda_1$, $\mu_1$, and $\nu_1$. Hence $HH^*(\Gamma)$ is the commutative graded ring which is generated by $\lambda_1, \mu_1, \nu_1 \in HH^1(\Gamma)$, and is defined by the following relations:

$$2\lambda_1 = 2\mu_1 = 2\nu_1 = 0, \quad \lambda_1^2 + \mu_1^2 + \nu_1^2 = 0.$$

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