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ASYMPTOTIC DIMENSION AND BOUNDARY DIMENSION OF PROPER CAT(0) SPACES

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Abstract. In this paper, we investigate asymptotic dimension of proper CAT(0) spaces and we show that for a proper cocompact CAT(0) space (X, d), the asymptotic dimension $\operatorname{asdim}(X, d)$ is greater than the covering dimension $\dim \partial X$ of the boundary of X.

1. Introduction and Preliminaries

In this paper, we study asymptotic dimension $\operatorname{asdim}(X, d)$ of a proper CAT(0) space (X, d) and the covering dimension $\dim \partial X$ of the boundary ∂X of X. Details of proper CAT(0) spaces and their boundaries are found in [2].

Asymptotic dimension was introduced by Gromov as an invariant of a finitely generated group [9]. Asymptotic dimension of groups relates to the Novikov conjecture and there are some interesting recent research on asymptotic dimension (cf. [3], [6], [9], [13]). Asymptotic dimension of CAT(0) groups and CAT(0) spaces are unknown in general. A group G is called a CAT(0) *group* if G acts geometrically (i.e. properly and cocompactly by isometries) on some proper CAT(0) space (X, d).

Let $\overline{X}^d$ be the Higson compactification of a proper metric space (X, d) and $\nu_d X = \overline{X}^d \setminus X$ the Higson corona of (X, d). Details of the Higson compactification and the Higson corona are found in [10]. We note that $\operatorname{asdim}(X, d) \geq \dim \nu_d X$ [6] and also that if $\operatorname{asdim}(X, d) < \infty$ then $\operatorname{asdim}(X, d) = \dim \nu_d X$ [5].

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let (X, d) be a proper* CAT(0) *space. Suppose that there exists an isometry $\psi : (X, d) \to (X, d)$ such that some orbit $\{\psi^i(x) : i \in \mathbf{Z}\}$ is unbounded.*

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Then

$$\operatorname{asdim}(X,d) \geq \dim \nu_d X \geq \sup\{n : \check{H}^n(\partial X) \neq 0\} + 1,$$

where $\check{H}^n(\partial X)$ *is the reduced Čech cohomology of the boundary* ∂X *of* X.

It is known that if (X,d) is a proper *cocompact* CAT(0) space, then the boundary ∂X has finite covering dimension [12] and $\dim \partial X = \max\{n : \check{H}^n(\partial X) \neq 0\}$ [8]. Thus we obtain the following corollary.

COROLLARY 1.2. *If* (X,d) *is a proper cocompact* CAT(0) *space then*

$$\operatorname{asdim}(X,d) \geq \dim \nu_d X \geq \dim \partial X + 1.$$

For a CAT(0) group G which acts geometrically on a proper CAT(0) space (X,d), Corollary 1.2 implies that $\operatorname{asdim} G \geq \dim \partial X + 1$, i.e., $\operatorname{asdim} G \geq \dim \partial G + 1$.

If the unbounded condition of some orbit of some isometry in Theorem 1.1 is dropped, we can prove the following theorem.

THEOREM 1.3. *If* (X,d) *is a noncompact proper* CAT(0) *space, then*

$$\operatorname{asdim}(X,d) \geq \dim \nu_d X \geq \dim \partial X.$$

2. Proper CAT(0) Spaces and Their Boundaries

We first give notation used in this paper.

NOTATION 2.1. Let the set of all natural number and real number denote by $\mathbf{N}$ and $\mathbf{R}$, respectively. Set $\mathbf{B}^n = \{x \in \mathbf{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$ and $\mathbf{S}^n = \{x \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$. Let (X,d) be a metric space. Also set $B(x,r) = \{y \in X : d(x,y) \leq r\}$ and $S(x,r) = \{y \in X : d(x,y) = r\}$.

DEFINITION 2.2. Let X be a normal space. A map $f : X \to \mathbf{B}^n$ is said to be *essential*, if there exists no map $\tilde{f} : X \to \mathbf{S}^{n-1}$ such that $\tilde{f}|_{f^{-1}(\mathbf{S}^{n-1})} = f|_{f^{-1}(\mathbf{S}^{n-1})}$. A map $f : X \to \mathbf{B}^n$ is said to be *inessential*, if there exists such a map above $\tilde{f} : X \to \mathbf{S}^{n-1}$. It is known that $\dim X \geq n$ if and only if there exists an essential map $f : X \to \mathbf{B}^n$ (cf. [7]).

DEFINITION 2.3. Let (X,d) be a proper CAT(0) space, $x \in X$, and $s \in \mathbf{R}_+$. For every $z \in X$, let $\xi_{x,z} : [0, d(x,z)] \to X$ be the geodesic segment from x to z in

(X, d). Then we define a map $p_s^x : X \to X$ as follows:

$$p_s^x(z) = \begin{cases} z & \text{if } d(x, z) \le s \\ \xi_{x,z}(s) & \text{if } s \le d(x, z). \end{cases}$$

For simplicity of notation, we write p_s instead of p_s^x.

The following remark and proposition are known.

Remark 2.4. Let (X, d) be a proper CAT(0) space, $x \in X$, and $k \in \mathbf{N}$.

(1) $X \cup \partial X$ is a compactification of X which is homeomorphic to $\varprojlim\{B(x,k), p_k|_{B(x,k+1)}\}$ such that ∂X is homeomorphic to $\varprojlim\{S(x,k), p_k|_{S(x,k+1)}\}$ and $\check{H}^n(\partial X)$ is isomorphic to $\varinjlim\{H^n(S(x,k)), (p_k|_{S(x,k+1)})^*\}$.

(2) For every compact subset D of $X \backslash B(x,k)$, we have

$$\operatorname{diam} p_1(D) < \frac{\operatorname{diam} D}{k}.$$

Proposition 2.5 ([10, Proposition 1]). *Let (X, d) be a noncompact proper metric space, let $\overline{X}^d$ be the Higson compactification of (X, d), let (Y, σ) be a compact metric space, and let $f : (X, d) \to (Y, \sigma)$ be a map. Then, f has an extension to $\overline{X}^d$ if and only if f has property $(*)_d$: for every $r > 0$ and every $\varepsilon > 0$, there exists a compact set K of X such that* $\operatorname{diam} f(B(x, r)) < \varepsilon$ *for all $x \in X \backslash K$.*

3. Proofs

We prove the main theorems. We first show Theorem 1.1.

Proof of Theorem 1.1. Let (X, d) be a proper CAT(0) space. Suppose that there exists an isometry $\psi : (X, d) \to (X, d)$ such that some orbit $\{\psi^i(x) : i \in \mathbf{Z}\}$ is unbounded.

Let $n \in \mathbf{N}$ such that $\check{H}^n(\partial X) \neq 0$. Fix $x_0 \in X$.

Since $B(x_0, k)$ is contractible for each $k \in \mathbf{N}$, $\check{H}^n(\partial X)(\cong \check{H}^{n+1}(X \cup \partial X, \partial X))$ is isomorphic to $\varinjlim\{H^{n+1}(B(x_0,k), S(x_0,k)), (p_k|_{B(x_0,k+1)})^*\}$. Since $\check{H}^n(\partial X) \neq 0$, for every $k \in \mathbf{N}$ there exists a map $a_k : (B(x_0,k), S(x_0,k)) \to (\mathbf{B}^{n+1}, \mathbf{S}^n)$ such that $0 \neq [a_k] \in H^{n+1}(B(x_0,k), S(x_0,k))$ and $[a_{k+1}] = [a_k \circ p_k]$.

Since $\{\psi^i(x) : i \in \mathbf{Z}\}$ is unbounded, there exists a sequence $i_1, i_2, \ldots$ of $\mathbf{N}$ such that $\psi^{i_k}(B(x_0,k)) \cap \psi^{i_{k'}}(B(x_0,k')) = \varnothing$ whenever $k \neq k'$ and $\{\psi^{i_k}(B(x_0,k)) : k \in \mathbf{Z}\}$ is unbounded.

Here we consider $f_k = a_1 \circ p_1 \circ \psi^{-i_k} : \psi^{i_k}(B(x_0, k)) \to \mathbf{B}^{n+1}$ for each $k \in \mathbf{N}$, $Y = \bigcup_{k \in \mathbf{N}} \psi^{i_k}(B(x_0, k))$, $\rho = d|_Y$, and $f = \bigcup_{k \in \mathbf{N}} f_k : Y \to \mathbf{B}^{n+1}$. Since $[a_k] \neq 0$ in $H^{n+1}(B(x_0, k), S(x_0, k))$, we note that f_k is essential for each $k \in \mathbf{N}$. Then f satisfies $(*)_\rho$ by Remark 2.4 (2). Hence there exists an extension $\bar{f} : \overline{Y}^\rho \to \mathbf{B}^{n+1}$ of f.

Now we show that $g = \bar{f}|_{\nu_\rho Y} : \nu_\rho Y \to \mathbf{B}^{n+1}$ is essential. On the contrary, suppose that g is inessential, i.e., there exists an extension $\bar{g} : \nu_\rho Y \to \mathbf{S}^n$ of $g|_{g^{-1}(\mathbf{S}^n)} : g^{-1}(\mathbf{S}^n) \to \mathbf{S}^n$. Since $\mathbf{S}^n$ is an ANR, there exist an open subset U of $\overline{Y}^\rho$ containing $\nu_\rho Y \cup \bar{f}^{-1}(\mathbf{S}^n)$ and an extension $\tilde{g} : U \to \mathbf{S}^n$ of $\bar{g} \cup \bar{f}|_{\bar{f}^{-1}(\mathbf{S}^n)} : \nu_\rho Y \cup \bar{f}^{-1}(\mathbf{S}^n) \to \mathbf{S}^n$. Here there exists $k \in \mathbf{N}$ such that $\psi^{i_k}(B(x_0, k)) \subset U$. Then $\tilde{g}|_{\psi^{i_k}(B(x_0,k))} : \psi^{i_k}(B(x_0, k)) \to \mathbf{S}^n$ is an extension of $f_k|_{f_k^{-1}(\mathbf{S}^n)} : \psi^{i_k}(B(x_0, k)) \to \mathbf{S}^n$, which is a contradiction because f_k is essential.

Therefore $\dim \nu_d X \geq n + 1$. □

We can check the following lemma which is used in the proof of Theorem 1.3.

LEMMA 3.1. *Let X be a compact space and let Y and A be closed subsets of X. Suppose that $Z = (Y \cap A) \times [0,1] \cup A \times \{0\} \cup Y \times \{1\}$ and $H : Y \times [0,1] \cup X \times \{0\} \to \mathbf{B}^n$ satisfies that $H(Z) \subset \mathbf{S}^{n-1}$. Then, there exist a closed neighborhood V of $Y \times [0,1]$ in $X \times [0,1]$ and a map $\overline{H} : V \to \mathbf{B}^n$ such that $\overline{H}|_{V \cap (Y \times [0,1] \cup X \times \{0\})} = H|_{V \cap (Y \times [0,1] \cup X \times \{0\})}$ and $\overline{H}(((V \cap A) \times [0,1]) \cup (V \cap (X \times \{1\}))) \subset \mathbf{S}^{n-1}$.*

We prove Theorem 1.3.

PROOF OF THEOREM 1.3. Let (X, d) be a noncompact proper CAT(0) space. Suppose that $\check{H}^n(\partial X, A) \neq 0$ for some $n \in \mathbf{N}$ and some closed subset A of ∂X.

Let $A_k = P_k(A)$ for each $k \in \mathbf{N}$, where $P_k : \partial X \to S(x_0, k)$ is the projection. Since $\check{H}^n(\partial X, A)$ is isomorphic to $\varinjlim\{H^n(S(x_0, k), A_k), (p_k|_{S(x_0,k+1)})^*\}$, there exists $[a_k] \in H^n(S(x_0, k), A_k) \setminus \{0\}$ such that $[a_{k+1}] = [a_k \circ p_k|_{S(x_0,k+1)}]$ for each $k \in \mathbf{N}$.

Let $b_k = a_1 \circ p_1|_{S(x_0,k)} : (S(x_0, k), A_k) \to (\mathbf{B}^n, \mathbf{S}^{n-1})$, $B = \bigcup_{k \in \mathbf{N}} A_k$ and $f = \bigcup_{k \in \mathbf{N}} b_k : \bigcup_{k \in \mathbf{N}} S(x_0, k) \to \mathbf{B}^n$. Here we note that $[b_k] \neq 0$ for each $k \in \mathbf{N}$.

Let $X' = \mathrm{Cl}_{\overline{X}^d} \bigcup_{k \in \mathbf{N}} S(x_0, k)$. Then f satisfies $(*)_d$ by Remark 2.4 (2) and there exists an extension $\bar{f} : (X', \mathrm{Cl}_{\overline{X}^d} B) \to (\mathbf{B}^n, \mathbf{S}^{n-1})$ of f. We note that $\nu_d X \subset X'$ and $\bar{f}(\mathrm{tr}_{\overline{X}^d} B) \subset \mathbf{S}^{n-1}$, where $\mathrm{tr}_{\overline{X}^d} B = \mathrm{Cl}_{\overline{X}^d} B \setminus B$.

Let $g = \bar{f}|_{\nu_d X} : (\nu_d X, \mathrm{tr}_{\overline{X}^d} B) \to (\mathbf{B}^n, \mathbf{S}^{n-1})$. Then we show that $[g] \neq 0$ in $H^n(\nu_d X, \mathrm{tr}_{\overline{X}^d} B)$. Suppose that $[g] = 0 \in H^n(\nu_d X, \mathrm{tr}_{\overline{X}^d} B)$. Then there exists

$H : (v_d X \times [0,1], \mathrm{tr}_{\bar{X}^d} B \times [0,1]) \to (\mathbf{B}^n, \mathbf{S}^{n-1})$ such that $H_0 = g$ and $H_1(v_d X) \subset \mathbf{S}^{n-1}$. By Lemma 3.1, there exist a closed neighborhood V of $v_d X \times [0,1]$ in $X' \times [0,1]$, a map $\overline{H} : V \to \mathbf{B}^n$ and $N_0 \in \mathbf{N}$ such that $\overline{H}(x,0) = \bar{f}(x)$ for all $(x,0) \in V \cap (X' \times \{0\})$, $\overline{H}|_{v_d X \times [0,1]} = H$ and $\overline{H}((\bigcup_{k \geq N_0} A_k) \times [0,1]) \cup \overline{H}(V \cap (X' \times \{1\})) \subset \mathbf{S}^{n-1}$. Since there exists $N_1 \in \mathbf{N}$ with $N_1 \geq N_0$ such that $S(x_0,k) \times [0,1] \subset V$ for all $k \geq N_1$, we have that $[b_k] = 0$ for some k, which is a contradiction.

Therefore, we obtain that $\dim v_d X \geq \dim_{\mathbf{Z}} v_d X \geq n$. □

4. Applications

We introduce some applications. We first note that if (X,d) is a noncompact proper CAT(0) space, it follows easily that $\mathrm{asdim}(X,d) \geq 1$.

Corollary 4.1. *If (X,d) is a noncompact cocompact proper* CAT(0) *space, then* $\mathrm{asdim}(X,d) \geq \dim v_d X \geq \dim \partial X + 1$.

Proof. If (X,d) is a proper *cocompact* CAT(0) space, then $\dim \partial X = \max\{n : \check{H}^n(\partial X) \neq 0\}$ by [8]. Hence Theorem 1.1 implies the desired conclusion. □

Remark 4.2. If (X,d_X) and (Y,d_Y) are metric spaces, then it is known that

$$\mathrm{asdim}(X \times Y, d_{X\times Y}) \leq \mathrm{asdim}(X,d_X) + \mathrm{asdim}(Y,d_Y)$$

(cf. [3]).

Let (X,d) be a noncompact proper geodesic space. If $\mathrm{asdim}(X,d) = 1$, then by Remark 4.2, $\mathrm{asdim}(X \times \mathbf{R}, d_{X\times\mathbf{R}}) = 2$, because $(X \times \mathbf{R}, d_{X\times\mathbf{R}})$ contains $(\mathbf{R}^2_+, d_{\mathbf{R}^2_+})$.

Corollary 4.3. *Let (X,d) be a noncompact proper* CAT(0) *space. Then* $\mathrm{asdim}(X \times \mathbf{R}, d_{X\times\mathbf{R}}) \geq \max\{n : \check{H}^n(\partial X) \neq 0\} + 2$. *In particular, if X is a noncompact proper* CAT(0) *space that is homeomorphic to $\mathbf{R}^2$, then* $\mathrm{asdim}(X \times \mathbf{R}, d_{X\times\mathbf{R}}) = 3$.

Proof. By Theorem 1.1, $\mathrm{asdim}(X \times \mathbf{R}, d_{X\times\mathbf{R}}) \geq \max\{n : \check{H}^n(\partial(X \times \mathbf{R})) \neq 0\} + 1$. Since $\partial(X \times \mathbf{R})$ is homeomorphic to the suspension of ∂X, we have that

$$\max\{n : \check{H}^n(\partial(X \times \mathbf{R})) \neq 0\} = \max\{n : \check{H}^n(\partial X) \neq 0\} + 1,$$

which implies our assertion.

If X is homeomorphic to $\mathbf{R}^2$, then by [4], ∂X is homeomorphic to the circle and $\operatorname{asdim}(X, d) = 2$. Hence, by Remark 4.2,

$$\begin{aligned} 3 = \max\{n : \check{H}^n(\partial X) \neq 0\} + 2 &\leq \operatorname{asdim}(X \times \mathbf{R}, d_{X \times \mathbf{R}}) \\ &\leq \operatorname{asdim}(X, d) + 1 = 3. \end{aligned}$$

Thus $\operatorname{asdim}(X \times \mathbf{R}, d_{X \times \mathbf{R}}) = 3$. □

COROLLARY 4.4. *Let (X, d) be a noncompact proper cocompact* CAT(0) *space satisfying that* $\operatorname{asdim}(X, d) = \dim \partial X + 1$. *Then*

$$\operatorname{asdim}(X \times \mathbf{R}, d_{X \times \mathbf{R}}) = \operatorname{asdim}(X, d) + 1.$$

PROOF. By [8], we have that

$$\operatorname{asdim}(X, d) + 1 = \dim \partial X + 2 = \max\{n : \check{H}^n(\partial X) \neq 0\} + 2.$$

Hence, by Corollary 4.3 and Remark 4.2, we obtain

$$\operatorname{asdim}(X \times \mathbf{R}, d_{X \times \mathbf{R}}) = \operatorname{asdim}(X, d) + 1.$$ □

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