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APERIODIC HOMEOMORPHISMS APPROXIMATE CHAIN MIXING ENDOMORPHISMS ON THE CANTOR SET

By

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Abstract. Let f be a chain mixing continuous onto mapping from the Cantor set onto itself. Let g be an aperiodic homeomorphism on the Cantor set. We show that homeomorphisms that are topologically conjugate to g approximate f in the topology of uniform convergence if a trivial necessary condition on periodic points is satisfied. In particular, let f be a chain mixing continuous onto mapping from the Cantor set onto itself with a fixed point and g, an aperiodic homeomorphism on the Cantor set. Then, homeomorphisms that are topologically conjugate to g approximate f.

1. Introduction

Let (X, d) be a compact metric space. Let $\mathcal{H}^+(X)$ be the set of all continuous mappings from X onto itself. In this manuscript, the pair (X, f) ($f \in \mathcal{H}^+(X)$) is called a *topological dynamical system*. We mainly consider the case in which X is homeomorphic to the Cantor set, denoted by C. For any $f, g \in \mathcal{H}^+(X)$, we define $d(f, g) := \sup_{x \in X} d(f(x), g(x))$. Then, $(\mathcal{H}^+(X), d)$ is a metric space of uniform convergence. Let $\mathcal{H}(X)$ be the set of all homeomorphisms from X onto itself. Let X be homeomorphic to C. $\mathrm{SFT}(X)$ denotes the set of all $f \in \mathcal{H}(X)$ that are topologically conjugate to some two-sided subshift of finite type. T. Kimura [3, Theorem 1] and I [4] have shown that elements in $\mathcal{H}(C)$ are approximated by expansive homeomorphisms with the pseudo-orbit tracing property. $\mathrm{SFT}(C)$ coincides with the set of all expansive $f \in \mathcal{H}(C)$

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with the pseudo-orbit tracing property (P. Walters [6, Theorem 1]). Therefore, $\mathrm{SFT}(C)$ is dense in $\mathcal{H}(C)$. Fix $f \in \mathcal{H}(C)$. Homeomorphisms that are topologically conjugate to f will approximate some other homeomorphisms. Let (X, f) be a topological dynamical system. $x \in X$ is called a periodic point of period n if $f^n(x) = x$. Let $\mathrm{Per}(X, f) := \{n \in \mathbf{Z}_+ \mid f^n(x) = x \text{ for some } x \in X\}$, where $\mathbf{Z}_+$ denotes the set of all positive integers. Let (X, f) and (Y, g) be topological dynamical systems. In this manuscript, a continuous mapping $\phi : Y \to X$ is said to be *commuting* if $\phi \circ g = f \circ \phi$ holds. We write $(Y, g) \rhd (X, f)$ if there exists a sequence of homeomorphisms $\{\psi_k\}_{k \in \mathbf{Z}_+}$ from Y onto X such that $\psi_k \circ g \circ \psi_k^{-1} \to f$ as $k \to \infty$. Suppose that $(Y, g) \rhd (X, f)$ and that g^n has a fixed point for some positive integer n. Then, f^n must also have a fixed point. Therefore, we get $\mathrm{Per}(Y, g) \subseteq \mathrm{Per}(X, f)$. Let $\delta > 0$. A sequence $\{x_i\}_{i=0,1,\dots,l}$ of elements of X is a δ *chain* from x_0 to x_l if $d(f(x_i), x_{i+1}) < \delta$ for all $i = 0, 1, \dots, l-1$. Then, l is called the length of the chain. A topological dynamical system (X, f) is *chain mixing* if for every $\delta > 0$ and for every pair $x, y \in X$, there exists a positive integer N such that for all $n \geq N$, there exists a δ chain from x to y of length n. Let (Λ, σ) be a two-sided subshift such that Λ is homeomorphic to C. Let X be homeomorphic to C and f, a chain mixing element of $\mathcal{H}^+(X)$. In a previous paper [5, Theorem 1.1], it was shown that the following conditions are equivalent:

(1) $\mathrm{Per}(\Lambda, \sigma) \subseteq \mathrm{Per}(X, f)$;
(2) $(\Lambda, \sigma) \rhd (X, f)$.

Let (Y, g) be a topological dynamical system and $n \in \mathbf{Z}_+$. In this manuscript, we say that g is *periodic* of period n if $g^n = id_Y$, where id_Y denotes the identity mapping on Y. We say that g is *aperiodic* if g is not periodic. Suppose that $g \in \mathcal{H}(Y)$ is periodic of period n and that $(Y, g) \rhd (X, f)$ for some $f \in \mathcal{H}^+(X)$. Then, it is easy to check that f is also periodic of period n. Note that even if g is aperiodic, all the orbits of g may be periodic. This may happen if g has periodic points of least period n for infinitely many $n \in \mathbf{Z}_+$. In this manuscript, we shall show the following:

THEOREM 1.1. *Let X and Y be homeomorphic to C; $f \in \mathcal{H}^+(X)$, chain mixing; and $g \in \mathcal{H}(Y)$, aperiodic. Then, the following conditions are equivalent*:

(1) $\mathrm{Per}(Y, g) \subseteq \mathrm{Per}(X, f)$;
(2) $(Y, g) \rhd (X, f)$.

In the previous theorem, suppose that f has a fixed point. Then, $\mathrm{Per}(X, f) = \mathbf{Z}_+$. Therefore, the following corollary is obtained:

Corollary 1.2. *Let X and Y be homeomorphic to C; $f \in \mathcal{H}^+(X)$, chain mixing; and $g \in \mathcal{H}(Y)$, aperiodic. Suppose that f has a fixed point. Then, $(Y, g) \rhd (X, f)$.*

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2. Preliminaries

Although many lemmas in this manuscript are listed in [5], we show the proof here for completeness. A compact metrizable totally disconnected perfect space is homeomorphic to C. Therefore, any non-empty closed and open subset of C is homeomorphic to C. Let $\mathbf{Z}$ denote the set of all integers. Let $V = \{v_1, v_2, \dots, v_n\}$ be a finite set of n symbols with discrete topology. Let $\Sigma(V) := V^{\mathbf{Z}}$ with the product topology. Then, $\Sigma(V)$ is a compact metrizable totally disconnected perfect space; hence, it is homeomorphic to C. We define a homeomorphism $\sigma : \Sigma(V) \to \Sigma(V)$ as

$$(\sigma(t))(i) = t(i+1) \quad \text{for all } i \in \mathbf{Z}, \text{ where } t = (t(i))_{i \in \mathbf{Z}} \in \Sigma(V).$$

The pair $(\Sigma(V), \sigma)$ is known as a *two-sided full shift* of n symbols. If a closed set $\Lambda \subseteq \Sigma(V)$ is invariant under σ, i.e. $\sigma(\Lambda) = \Lambda$, then $(\Lambda, \sigma|_\Lambda)$ is known as a *two-sided subshift*. In this manuscript, $\sigma|_\Lambda$ is abbreviated to σ. A finite sequence $u_1 u_2 \cdots u_l$ of elements of V is called a *word* of length l. For a word u of length l and $m \in \mathbf{Z}$, we define the cylinder $C_m(u) \subseteq \Lambda$ as

$$C_m(u) := \{t \in \Lambda \mid t(m + j - 1) = u_j \text{ for all } 1 \le j \le l\}.$$

Let (X, f) be a topological dynamical system such that X is homeomorphic to C. Let $\mathcal{U}$ be a finite partition of X by non-empty closed and open subsets. In this manuscript, we consider partitions that are not trivial, i.e., they consist of more than one element. We define a directed graph $G = G(f, \mathcal{U})$ as follows:

(1) G has the set of vertices $V(f, \mathcal{U}) = \mathcal{U}$

(2) G has the set of directed edges $E(f, \mathcal{U}) \subseteq \mathcal{U} \times \mathcal{U}$ such that

$$(U, U') \in E(f, \mathcal{U}) \quad \text{if and only if} \quad f(U) \cap U' \neq \varnothing.$$

Note that all elements of $V(f, \mathcal{U})$ have at least one outdegree and at least one indegree. Let $G = (V, E)$ be a directed graph, where V is a finite set of vertices and $E \subseteq V \times V$ is a set of directed edges. $\Sigma(G)$ denotes the two-sided subshift defined as

$$\Sigma(G) := \{t \in V^{\mathbf{Z}} \mid (t(i), t(i+1)) \in E \text{ for all } i \in \mathbf{Z}\}.$$

A two-sided subshift is said to be of *finite type* if it is topologically conjugate to $(\Sigma(G), \sigma)$ for some directed graph G. Throughout this manuscript, unless otherwise stated, we assume that all the vertices appear in some element of $\Sigma(G)$, i.e., all the vertices of G have at least one outdegree and at least one indegree. For the sake of conciseness, we write $(\Sigma(f, \mathscr{U}), \sigma)$ instead of $(\Sigma(G(f, \mathscr{U})), \sigma)$. The next lemma follows:

LEMMA 2.1. *Let (X, f) be a topological dynamical system such that X is homeomorphic to C. Let $\mathscr{U}$ be a partition of X by non-empty closed and open subsets of X. Then,* $\mathrm{Per}(X, f) \subseteq \mathrm{Per}(\Sigma(f, \mathscr{U}), \sigma)$.

PROOF. Let $x \in X$ be a periodic point of period n under f. Then, there exists a sequence $\{U_i\}_{i=0,1,\dots,n}$ of elements of $\mathscr{U}$ such that $U_n = U_0$ and $f(U_i) \cap U_{i+1} \neq \varnothing$ for all $i = 0, 1, \dots, n-1$. Thus, $(\Sigma(f, \mathscr{U}), \sigma)$ has a periodic point of period n. □

LEMMA 2.2 (Lemma 1.3 of R. Bowen [1]). *Let $G = (V, E)$ be a directed graph. Suppose that every vertex of V has at least one outdegree and at least one indegree. Then, $\Sigma(G)$ is topologically mixing if and only if there exists an $N \in \mathbf{Z}_+$ such that for any pair of vertices u and v of V, there exists a path from u to v of length $n \geq N$.*

PROOF. See Lemma 1.3 of R. Bowen [1]. □

Let $K \subseteq X$. The diameter of K is defined as $\mathrm{diam}(K) := \sup\{d(x, y) \mid x, y \in K\}$. We define $\mathrm{mesh}(\mathscr{U}) := \max\{\mathrm{diam}(U) \mid U \in \mathscr{U}\}$.

LEMMA 2.3. *Let (X, d) be a compact metric space and $f : X \to X$, a continuous mapping. Then, for any $\varepsilon > 0$, there exists $\delta = \delta(f, \varepsilon) > 0$ such that*

$$\delta < \frac{\varepsilon}{2};$$

$$\textit{if } d(x, y) \leq \delta, \textit{ then } d(f(x), f(y)) < \frac{\varepsilon}{2} \textit{ for all } x, y \in X.$$

Proof. This lemma directly follows from the uniform continuity of f. $\square$

For two directed graphs $G = (V, E)$ and $G' = (V', E')$, G is said to be a *subgraph* of G' if $V \subseteq V'$ and $E \subseteq E'$.

Lemma 2.4. *Let (X, d) be a compact metric space; $f : X \to X$, a continuous mapping; and $\varepsilon > 0$. Let $\delta = \delta(f, \varepsilon)$ be as in lemma 2.3 and $\mathcal{U}$, a finite covering of X such that $\mathrm{mesh}(\mathcal{U}) < \delta$. Let $g : X \to X$ be a mapping such that $G(g, \mathcal{U})$ is a subgraph of $G(f, \mathcal{U})$. Then, $d(f, g) < \varepsilon$.*

Proof. Let $x \in X$. Then, $x \in U$ and $g(x) \in U'$ for some $U, U' \in \mathcal{U}$. Because $G(g, \mathcal{U})$ is a subgraph of $G(f, \mathcal{U})$, there exists a $y \in U$ such that $f(y) \in U'$. Therefore, from lemma 2.3, it follows that

$$d(f(x), g(x)) \le d(f(x), f(y)) + d(f(y), g(x)) < \frac{\varepsilon}{2} + \mathrm{diam}(U') < \varepsilon. \qquad \square$$

From this lemma, the next lemma follows directly.

Lemma 2.5. *Let (X, d) be a compact metric space; $f : X \to X$, a continuous mapping; and $\{\mathcal{U}_k\}_{k \in \mathbf{Z}_+}$, a sequence of coverings of X such that $\mathrm{mesh}(\mathcal{U}_k) \to 0$ as $k \to \infty$. Let $\{g_k\}_{k \in \mathbf{Z}_+}$ be a sequence of mappings from X to X such that $G(g_k, \mathcal{U}_k)$ is a subgraph of $G(f, \mathcal{U}_k)$ for all k. Then, $g_k \to f$ as $k \to \infty$.*

Lemma 2.6. *Let (X_1, f_1) and (X_2, f_2) be topological dynamical systems such that both X_1 and X_2 are homeomorphic to C. Let $\{\mathcal{U}_k\}_{k \in \mathbf{Z}_+}$ be a sequence of finite partitions by non-empty closed and open subsets of X_1 such that $\mathrm{mesh}(\mathcal{U}_k) \to 0$ as $k \to \infty$. Let $\{\pi_k\}_{k \in \mathbf{Z}_+}$ be a sequence of continuous commuting mappings from X_2 to X_1. Suppose that for all $k \in \mathbf{Z}_+$, $\pi_k(X_2) \cap U \neq \emptyset$ for all $U \in \mathcal{U}_k$. Then, $(X_2, f_2) \rhd (X_1, f_1)$.*

Proof. Let $k \in \mathbf{Z}_+$. Let $U \in \mathcal{U}_k$. Because $\pi_k(X_2) \cap U \neq \emptyset$, $\pi_k^{-1}(U)$ is a non-empty closed and open subset of X_2. Both $\pi_k^{-1}(U)$ and U are homeomorphic to C. Therefore, there exists a homeomorphism $\psi_k : X_2 \to X_1$ such that $\psi_k(\pi_k^{-1}(U)) = U$ for all $U \in \mathcal{U}_k$. Because π_k is commuting, $\pi_k(f_2(\pi_k^{-1}(U))) \cap U' \neq \emptyset$ only if $f_1(U) \cap U' \neq \emptyset$. Let $g_k = \psi_k \circ f_2 \circ \psi_k^{-1}$. Then, from the construction of ψ_k, $G(g_k, \mathcal{U}_k)$ is a subgraph of $G(f_1, \mathcal{U}_k)$. Because $k \in \mathbf{Z}_+$ is arbitrary, from lemma 2.5, we get the result. $\square$

LEMMA 2.7. *Let* (X_1, f_1) *and* (X_2, f_2) *be topological dynamical systems. Let* (Y_k, g_k) $(k \in \mathbf{Z}_+)$ *be a sequence of topological dynamical systems. Suppose that there exists a sequence of homeomorphisms* $\psi_k : Y_k \to X_1$ *such that* $\psi_k \circ g_k \circ \psi_k^{-1} \to f_1$ *as* $k \to \infty$ *and that* $(X_2, f_2) \rhd (Y_k, g_k)$ *for all* $k \in \mathbf{Z}_+$. *Then,* $(X_2, f_2) \rhd (X_1, f_1)$.

PROOF. Let $\varepsilon > 0$. There exists an $N \in \mathbf{Z}_+$ such that $d(\psi_k \circ g_k \circ \psi_k^{-1}, f_1) < \varepsilon/2$ for all $k > N$. Fix $k > N$. Let $\delta > 0$ be such that if $d(y, y') < \delta$, then $d(\psi_k(y), \psi_k(y')) < \varepsilon/2$. Because $(X_2, f_2) \rhd (Y_k, g_k)$, there exists a homeomorphism $\psi' : X_2 \to Y_k$ such that $d(\psi' \circ f_2 \circ \psi'^{-1}, g_k) < \delta$. Then, we find that $d((\psi_k \circ \psi') \circ f_2 \circ (\psi_k \circ \psi')^{-1}, f_1) < d(\psi_k \circ (\psi' \circ f_2 \circ \psi'^{-1}) \circ \psi_k^{-1}, \psi_k \circ g_k \circ \psi_k^{-1}) + d(\psi_k \circ g_k \circ \psi_k^{-1}, f_1) < \varepsilon$. □

LEMMA 2.8. *Let* $G = (V, E)$ *be a directed graph. Suppose that every vertex of* G *has at least one outdegree and at least one indegree. Suppose that* $\Sigma(G)$ *is topologically mixing and that* $\Sigma(G)$ *is not a single point. Then,* $\Sigma(G)$ *is homeomorphic to* C.

PROOF. Suppose that $\Sigma(G)$ is topologically mixing. Then, by lemma 2.2, there exists an $N \in \mathbf{Z}_+$ such that for any pair u and v of vertices of G, there exists a path from u to v of length n for all $n \geq N$. Then, it is easy to check that every point $t \in \Sigma(G)$ is not isolated. Hence, $\Sigma(G)$ is homeomorphic to C. □

LEMMA 2.9 (Krieger's Marker Lemma, (2.2) of M. Boyle [2]). *Let* (Λ, σ) *be a two-sided subshift. Given* $k > N > 1$, *there exists a closed and open set* F *such that*

(1) *the sets* $\sigma^l(F)$, $0 \leq l < N$, *are disjoint, and*
(2) *if* $t \in \Lambda$ *and* $t_{-k} \cdots t_k$ *is not a* j*-periodic word for any* $j < N$, *then*

$$t \in \bigcup_{-N < l < N} \sigma^l(F).$$

PROOF. See M. Boyle [2, (2.2)]. □

The next lemma is essentially a part of the proof of the extension lemma given by M. Boyle [2, (2.4)]. Although the next lemma is slightly strengthened from Lemma 3.4 in [5], the proof is quite similar. In spite of the similarlity of the proof, we show all of the proof here again for completeness.

Lemma 2.10. *Let* (Σ, σ) *be a mixing two-sided subshift of finite type. Let* W *be a finite set of words that appear in some elements of* Σ. *Then, there exists an* $M \in \mathbf{Z}_+$ *that satisfies the following condition*:

- *if* (Λ, σ) *is a two-sided subshift such that* $\mathrm{Per}(\Lambda, \sigma) \subseteq \mathrm{Per}(\Sigma, \sigma)$ *and* Λ *has either a non-periodic orbit or a periodic orbit of least period greater than* M, *then there exists a continuous shift-commuting mapping* $\pi : \Lambda \to \Sigma$ *such that there exists a* $t \in \pi(\Lambda)$ *in which all words in* W *appear as segments of* t.

Proof. Σ is isomorphic to $\Sigma(G)$ for some directed graph $G = (V, E)$. Therefore, without loss of generality, we assume that $\Sigma = \Sigma(G)$. Because $(\Sigma(G), \sigma)$ is a mixing subshift of finite type, there exists an $n > 0$ such that for every pair of elements $v, v' \in V$ and every $m \geq n$, there exists a word of the form $v \cdots v'$ of length m. In addition, there exists an element $\bar{t} \in \Sigma(G)$ such that $\bar{t}$ contains all words of W as segments. Let w_0 be a segment of $\bar{t}$ that contains all words of W. Let n_0 be the length of the word w_0. Let $N = 2n + n_0$. If $v, v' \in V$ and $m \geq N$, then there exists a word of the form $v \cdots w_0 \cdots v'$ of length m. Let $k > 2N$. Let $M > N$. Note that N depends only on $\Sigma(G)$ and W. Therefore, M also depends only on $\Sigma(G)$ and W. Let Λ be a two-sided subshift such that $\mathrm{Per}(\Lambda, \sigma) \subseteq \mathrm{Per}(\Sigma, \sigma)$ and Λ has either a non-periodic orbit or a periodic orbit of least period greater than M. Using Krieger's marker lemma, there exists a closed and open subset $F \subseteq \Lambda$ such that the following conditions hold:

(1) the sets $\sigma^l(F)$, $0 \leq l < N$, are disjoint;

(2) if $t \in \Lambda$ and $t \notin \bigcup_{-N<l<N} \sigma^l(F)$, then $t(-k) \cdots t(k)$ is a j-periodic word for some $j < N$;

(3) the number k is large enough to ensure that if j is less than N and a j-periodic word of length $2k + 1$ occurs in some element of Λ, then that word defines a j-periodic orbit that actually occurs in Λ.

The existence of k follows from the compactness of Λ. Let $t \in \Lambda$. If $\sigma^i(t) \in F$, then we *mark* t at position i. There exists a large number $L > 0$ such that whether $\sigma^i(t) \in F$ is determined only by the $2L + 1$ block $t(i - L) \cdots t(i + L)$. If t is marked at position i, then t is unmarked for position l with $i < l < i + N$. Suppose that $t(i) \cdots t(i')$ is a segment of t such that t is marked at i and i' and t is unmarked at l for all $i < l < i'$. Then, $i' - i \geq N$. If $t \in \bigcup_{-N<l<N} \sigma^l(F)$, then t is marked at some i where $-N < i < N$. Suppose that $t(-N + 1) \cdots t(N - 1)$ is an unmarked segment. Then, $t \notin \bigcup_{-N<l<N} \sigma^l(F)$, and according to condition (2), $t(-k) \cdots t(k)$ is a j-periodic word for some $j < N$. Suppose that $t(i) \cdots t(i')$ is an unmarked segment of length at least $2N-1$, i.e., $i'-i \geq 2N-2$. Then, for each l with $i+N-1 \leq l \leq i'-N+1$, $t(l-k)\cdots t(l+k)$ is a j-periodic word for some $j<N$. Therefore, it is easy to check that $t(i+N-1-k)\cdots t(i'-N+1+k)$ is a j-periodic word for some $j<N$. In this proof, we call a maximal unmarked segment an *interval*. Let $t \in \Lambda$. Let $\cdots t(i)$ be a left infinite interval. Then, it is j-periodic for some $j<N$. Similarly, a right infinite interval $t(i)\cdots$ is j-periodic for some $j<N$. If t itself is an interval, then it is a periodic point with period $j<N$. If an interval is finite, then it has a length of at least $N-1$. We call intervals of length less than $2N-1$ as *short* intervals. We call intervals of length greater than or equal to $2N-1$ as *long* intervals. If t has a long interval $t(i)\cdots t(i')$, then $t(i+N-1-k)\cdots t(i'-N+1+k)$ is j-periodic for some $j<N$. We have to construct a shift-commuting mapping $\phi : \Lambda \to \Sigma$. Let V' be the set of symbols of Λ. Let $\Phi : V' \to V$ be an arbitrary mapping. Let $t \in \Lambda$. Suppose that t is marked at i. Then, we let $(\phi(t))(i)$ be $\Phi(t(i))$. We map periodic points of period $j<N$ to periodic points of Σ. Then, we construct a coding of $\phi(t)$ in three parts. For any $(v, v', l) \in V \times V \times \{N-1, N, N+1, \ldots, 2N-2\}$, we choose a word $\Psi(v, v', l)$ in G of length l such that the word of the form $v\Psi(v, v', l)v'$ is a path in G.

(A) *Coding for short interval*: Let $t(i)\cdots t(i')$ be a short interval. Then, t is marked at $i-1$ and $i'+1$. We have already defined a code for positions $i-1$ and $i'+1$ as $\Phi(t(i-1))$ and $\Phi(t(i'+1))$, respectively. The coding for $\{i, i+1, i+2, \ldots, i'\}$ is defined by the path $\Psi(\Phi(t(i-1)), \Phi(t(i'+1)), i'-i+1)$.

(B) *Coding for periodic segment*: For an infinite or long interval, there exists a corresponding periodic point of Λ. The periodic points of Λ are already mapped to periodic points of Σ. Therefore, an infinite or long periodic segment can be mapped to a naturally corresponding periodic segment.

(C) *Coding for transition part*: To consider a transition segment, let $t(i)\cdots t(i')$ be a long interval. Then, $t(i-1)$ has already been mapped to $\Phi(t(i-1))$, and $t(i+N-1)$ is mapped according to periodic points. Assume that $t(i+N-1)$ is mapped to v_0. The segment $t(i-1)\cdots t(i+N-1)$ has length $N+1$. We map the segment $t(i)\cdots t(i+N-2)$ to $\Psi(\Phi(t(i-1)), v_0, N-1)$. In the same manner, the transition coding of the right-hand side of a long interval is defined. Similarly, the transition coding of the left or right infinite interval is defined.

It is easy to check that there exists a large number $L'>0$ such that the coding of $(\phi(t))(i)$ is determined only by the block $t(i-L')\cdots t(i+L')$. Therefore, $\phi : \Lambda \to \Sigma$ is continuous. Because Λ has either a $t \in \Lambda$, which is not a periodic point, or a $t' \in \Lambda$, which is a periodic point of least period greater than

M, there appears a short interval or transition segment in some elements of Λ. In the above coding, we can take Ψ such that both short intervals and transition segments are mapped to words that involve w_0. $\square$

3. Proof of the Main Result

Lemma 3.1. *Let X be homeomorphic to C and f, a chain mixing element of $\mathcal{H}^+(X)$. Let $\{\mathcal{W}_k\}_{k\in\mathbf{Z}_+}$ be a sequence of non-trivial finite partitions by non-empty closed and open subsets of X such that* $\mathrm{mesh}(\mathcal{W}_k) \to 0$ *as $k \to \infty$. Then, there exists a sequence $\{\psi_k\}_{k\in\mathbf{Z}_+}$ of homeomorphisms from $\Sigma(f, \mathcal{W}_k)$ to X such that $\psi_k \circ \sigma \circ \psi_k^{-1} \to f$ as $k \to \infty$. Furthermore, if f is chain mixing, then all $(\Sigma(f, \mathcal{W}_k), \sigma)$ $(k \in \mathbf{Z}_+)$ are mixing.*

Proof. Consider a sequence $\{\mathcal{W}_k\}_{k\in\mathbf{Z}_+}$ of non-trivial partitions of X by non-empty closed and open subsets such that $\mathrm{mesh}(\mathcal{W}_k) \to 0$ as $k \to \infty$. Assume that $k \in \mathbf{Z}_+$. Let $G_k = G(f, \mathcal{W}_k)$. Let $\delta > 0$ be such that if $x, x' \in X$ satisfy $d(x, x') < \delta$, then both x and x' are contained in the same element of $\mathcal{W}_k$. Let $\{x_0, x_1\}$ be a δ chain. Let $U, U' \in \mathcal{W}_k$ be such that $x_0 \in U$ and $x_1 \in U'$. Then, $f(U) \cap U' \neq \emptyset$. Therefore, (U, U') is an edge of G_k. Let $U, V \in \mathcal{W}_k$. Let $x \in U$ and $y \in V$. Because f is chain mixing, there exists an $N > 0$ such that for every $n \geq N$, there exists a δ chain from x to y of length n. Therefore, for every $n \geq N$, there exists a path in G_k from U to V of length n. From lemma 2.2, $(\Sigma(G_k), \sigma)$ is topologically mixing. By lemma 2.8, $\Sigma(G_k)$ is homeomorphic to C. Therefore, there exists a homeomorphism $\psi_k : \Sigma(G_k) \to X$ such that for any vertex u of G_k, $\psi_k(C_0(u)) = u$. Let $g_k = \psi_k \circ \sigma \circ \psi_k^{-1}$. Then, by construction, we obtain $G(g_k, \mathcal{U}_k) = G(f, \mathcal{U}_k)$. Because $\mathrm{mesh}(\mathcal{U}_k) \to 0$ as $k \to \infty$, we conclude that $g_k \to f$ as $k \to \infty$ by lemma 2.5. $\square$

Proof of Theorem 1.1

Proof. Let X and Y be homeomorphic to C. First, suppose that $(Y, g) \rhd (X, f)$. Then, it is easy to see that $\mathrm{Per}(Y, g) \subseteq \mathrm{Per}(X, f)$. Conversely, suppose that $f \in \mathcal{H}^+(X)$ is chain mixing; $g \in \mathcal{H}(Y)$, aperiodic; and that $\mathrm{Per}(Y, g) \subseteq \mathrm{Per}(X, f)$. Let $\{\mathcal{W}_i\}_{i\in\mathbf{Z}_+}$ be a sequence of non-trivial finite partitions by non-empty closed and open subsets of X such that $\mathrm{mesh}(\mathcal{W}_i) \to 0$ as $i \to \infty$. By lemma 3.1, there exists a sequence of homeomorphisms $\psi_i : \Sigma(f, \mathcal{W}_i) \to X$ such that $\psi_i \circ \sigma \circ \psi_i^{-1} \to f$ as $i \to \infty$ and that all $(\Sigma(f, \mathcal{W}_i), \sigma)$ $(i \in \mathbf{Z}_+)$ are mixing. Fix $i \in \mathbf{Z}_+$. Let $\Sigma = \Sigma(f, \mathcal{W}_i)$. Let $\{\mathcal{U}_k\}_{k\in\mathbf{Z}_+}$ be a sequence of finite partitions of Σ by non-empty closed and open subsets. Let $\mathscr{U}_k = \{U_{k,j} : 1 \le j \le n_k\}$ for $k \in \mathbf{Z}_+$. Then, there exists a sequence $u_{k,j}$ $(k \in \mathbf{Z}_+, 1 \le j \le n_k)$ of words and a sequence $m(k,j)$ $(1 \le j \le n_k)$ of integers such that the following condition is satisfied:

$$C_{m(k,j)}(u_{k,j}) \subseteq U_{k,j} \quad (k \in \mathbf{Z}_+, 1 \le j \le n_k).$$

Fix $k \in \mathbf{Z}_+$. Let $W = \{u_{k,j} \mid 1 \le j \le n_k\}$. We shall show the following:

(1) there exists a continuous commuting mapping $\bar{\phi}_k : Y \to \Sigma$ such that $\bar{\phi}_k(Y)$ contains an element $t \in \Sigma$ that contains all words of W.

Then, $\bar{\phi}_k(Y) \cap U \neq \emptyset$ for all $U \in \mathscr{U}_k$. Because $k \in \mathbf{Z}_+$ is arbitrary, we conclude that $(Y,g) \rhd \Sigma$ by lemma 2.6. Then, by lemma 3.1 and lemma 2.7, we can conclude that $(Y,g) \rhd (X,f)$.

Let M be a positive integer that satisfies the condition in lemma 2.10. Let $\mathscr{V}$ be a partition of Y by non-empty closed and open subsets. Then, for each $y \in Y$, there exists a unique $t_y \in \Sigma(g,\mathscr{V})$ such that $g^l(y) \in t_y(l) \in \mathscr{V}$ for all $l \in \mathbf{Z}$. Therefore, there exists a commuting mapping $\phi_{\mathscr{V}} : Y \to \Sigma(g,\mathscr{V})$ such that $\phi_{\mathscr{V}}(y) = t_y$ for all $y \in Y$. Because all elements of $\mathscr{V}$ are open, it is easy to see that $\phi_{\mathscr{V}}$ is continuous. Let $\Lambda = \phi_{\mathscr{V}}(Y)$. Then, Λ is a two-sided subshift. Because Σ is mixing, there exists an $m \in \mathbf{Z}_+$ such that for all integer $n \ge m$, there exists a periodic point $t_n \in \Sigma$ of period n. If $\mathscr{V}$ is sufficiently fine, then the period $n \in \mathrm{Per}(\Sigma(g,\mathscr{V}),\sigma)$, where $n < m$, has a real periodic point of (Y,g) of period n. Therefore, because $\mathrm{Per}(Y,g) \subseteq \mathrm{Per}(X,f)$, we get $\mathrm{Per}(\Sigma(g,\mathscr{V}),\sigma) \subseteq \mathrm{Per}(\Sigma,\sigma)$ for all sufficiently fine $\mathscr{V}$. Let $\overline{M} > \max\{m, M\}$ be an arbitrary positive integer. Because g is aperiodic, if $\mathscr{V}$ is sufficiently fine, then Λ is not a set of periodic points of period less than $\overline{M}$. Therefore, by lemma 2.10, there exists a continuous commuting mapping $\pi_k : \Lambda \to \Sigma$ such that $\pi_k(\Lambda)$ contains an element that contains all words of W. Finally, let $\bar{\phi}_k = \pi_k \circ \phi_{\mathscr{V}}$; this concludes the proof. □

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