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GALOIS–TUKEY CONNECTION INVOLVING SETS OF METRICS

By

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Abstract. Kada proved in a previous paper (Topology Appl., 2009) that the collection of compatible metrics on a locally compact separable metrizable space has the same cofinal type, in the sense of Tukey relation, as the set of functions from ω to ω with respect to eventually dominating order. By generalizing this result, we characterize the order structure of the collection of compatible metrics on a separable metrizable space in terms of generalized Galois–Tukey connection.

1. Introduction

Tukey relation between directed sets is defined as follows. For directed sets $(D, \leq_D)$ and $(E, \leq_E)$, we write $(D, \leq_D) \leq_T (E, \leq_E)$ if there is a mapping from E to D which maps every cofinal subset of E to a cofinal subset of D. We write $D \leq_T E$ if referred order relations on D and E are clear from the context. Clearly the relation $\leq_T$ is transitive. We write $D \equiv_T E$ if $D \leq_T E$ and $E \leq_T D$. See [7] for details.

We also consider the notion of generalized Galois–Tukey connections introduced by Vojtáš [8]. We follow the formulation and terminology of Blass [1]. We deal with triples of the form $\mathbf{A} = (A_-, A_+, A)$, where A_- and A_+ are nonempty sets and A is a binary relation between A_- and A_+ (in other words, $A \subseteq A_- \times A_+$). For $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$, a *morphism* from $\mathbf{A}$ to $\mathbf{B}$ is a pair $\varphi = (\varphi_-, \varphi_+)$ of mappings such that $\varphi_- : B_- \to A_-$, $\varphi_+ : A_+ \to B_+$ and,

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for $b \in B_-$ and $a \in A_+$ if $\varphi_-(b)\, A\, a$ then $b\, B\, \varphi_+(a)$. We write $\mathbf{A} \to \mathbf{B}$ if there is a morphism from $\mathbf{A}$ to $\mathbf{B}$. Clearly, if $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{C}$ then $\mathbf{A} \to \mathbf{C}$.

The generalized Galois–Tukey connections can be seen as a generalization of the Tukey relation. The following lemma is easy to check:

LEMMA 1.1. *For directed sets* $(D, \leq_D)$ *and* $(E, \leq_E)$, $D \leq_T E$ *holds if and only if we have* $(E, E, \leq_E) \to (D, D, \leq_D)$.

For $f, g \in \omega^\omega$, we write $f \leq g$ if $f(n) \leq g(n)$ for all $n < \omega$, and $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Let $\omega^{\uparrow\omega}$ denote the set of all strictly increasing functions in ω^ω. Since there are morphisms between $(\omega^\omega, \omega^\omega, \leq)$ and $(\omega^{\uparrow\omega}, \omega^{\uparrow\omega}, \leq)$ in both directions, we will often identify these two triples.

We use the following notational convention: for two ordered sets $(D, \leq_D)$ and $(E, \leq_E)$, $\leq_D \times \leq_E$ denotes the usual product order on $D \times E$, that is,

$$(d_1, e_1)\,(\leq_D \times \leq_E)\,(d_2, e_2) \quad \text{if and only if } d_1 \leq_D d_2 \text{ and } e_1 \leq_E e_2.$$

For a metrizable space X, let $\mathrm{M}(X)$ denote the set of all metrics on X which are compatible with the topology on X. For $d_1, d_2 \in \mathrm{M}(X)$, we write $d_1 \preceq d_2$ if the identity mapping on X is uniformly continuous as a function from (X, d_2) to (X, d_1).

We will often regard a separable metrizable space X as a subspace of the Hilbert cube $\mathbf{H} = [0,1]^\omega$. We fix a metric function μ on $\mathbf{H}$ throughout this paper. For a subspace X of $\mathbf{H}$, let $X^* = \mathrm{cl}_{\mathbf{H}}\, X \setminus X$, and $\mathscr{K}(X^*)$ denotes the set of all compact subsets of X^*. If X is a locally compact separable metrizable space, X^* is compact since X is then open in $\mathrm{cl}_{\mathbf{H}}(X)$.

Todorčević asked the authors (in private communication) the following question about the order structure of $(\mathrm{M}(X), \preceq)$ for a separable metrizable space X. $X^{(1)}$ denotes the first Cantor–Bendixson derivative of X, that is, the subspace of X which consists of all nonisolated points of X.

QUESTION 1.2. *For a separable metrizable space* X *such that* $X^{(1)}$ *is non-compact, does* $(\mathrm{M}(X), \preceq) \equiv_T (\omega^\omega \times \mathscr{K}(X^*), \leq^* \times \subseteq)$ *hold*?

Here we briefly review the background of this question. See Remark 2 at the end of Section 4 for more about the origin of this question.

For a completely regular Hausdorff space X, let $\mathsf{Cpt}(X)$ denote the class of compactifications of X. For $\alpha X, \gamma X \in \mathsf{Cpt}(X)$, we let $\alpha X \leq \gamma X$ if there is a continuous surjection $f : \gamma X \to \alpha X$ such that $f \restriction X$ is the identity map on X. If such an f can be chosen to be a homeomorphism, we write $\alpha X \simeq \gamma X$. When we identify $\simeq$-equivalent compactifications, the ordered set $(\mathsf{Cpt}(X), \leq)$ is a complete upper semilattice whose largest element is the Stone–Čech compactification βX.

There have been many studies about approximating βX by simple subclasses of $\mathsf{Cpt}(X)$, in the sense that βX is obtained as the supremum (taken in $(\mathsf{Cpt}(X), \leq)$) of each such class. The following theorem, which is due to Woods, is one of those results. The *Smirnov compactification* of a metric space (X, d), denoted by $u_d X$, is the unique compactification characterized by the following property: A bounded continuous function f from X to $\mathbf{R}$ is continuously extended over $u_d X$ if and only if f is uniformly continuous with respect to the metric d. It is easy to see that, for $d_1, d_2 \in \mathrm{M}(X)$, $u_{d_1} X \leq u_{d_2} X$ if and only if $d_1 \preceq d_2$.

Theorem 1.3 [9, Theorem 2.11]. *For a metrizable space X, we have $\beta X \simeq \sup\{u_d X : d \in \mathrm{M}(X)\}$.*

The studies on approximation of βX as in the theorem above may be seen in the context of the investigation of the order structure of $(\mathsf{Cpt}(X), \leq)$. From this perspective the theorem above may be understood as saying that $(\mathrm{M}(X), \preceq)$ is nicely embedded into $(\mathsf{Cpt}(X), \leq)$. The positive answer to Question 1.2 would further underline this close connection of $(\mathrm{M}(X), \preceq)$ to $(\mathsf{Cpt}(X), \leq)$.

Unfortunately, Question 1.2 is unanswered so far. As a partial answer, Kada [3] proved the following theorem.

Theorem 1.4 [3, Theorem 3.1]. *For a locally compact separable metrizable space X such that $X^{(1)}$ is noncompact, $(\mathrm{M}(X), \preceq) \equiv_T (\omega^\omega, \leq^*)$ holds.*

Note that Theorem 1.4 answers Question 1.2 in a case when X is locally compact, since X^* is then compact and $(\mathscr{K}(X^*), \subseteq)$ has the largest element X^*.

While attempting to find an answer to Question 1.2, we noticed that the above theorem is nicely refined by involving yet another set $\mathrm{PC}(X)$ and using generalized Galois–Tukey connection. For a metrizable space X, let $\mathrm{PC}(X)$ denote the set of all pairs of disjoint closed sets of X, and for $(A, B) \in \mathrm{PC}(X)$ we write $(A, B) \mathsf{Sep}\, d$ if $d(A, B) > 0$. The proof of Theorem 1.3 [9, Theorem 2.11] actually claims that for any $(A, B) \in \mathrm{PC}(X)$ there is $d \in \mathrm{M}(X)$ such that $d(A, B) > 0$ (see Lemma 4.8), which is one of the reason why the structure $\mathrm{PC}(X)$ and the relation Sep fit in the present context.

Using $\mathrm{PC}(X)$ and Sep, Theorem 1.4 is refined to the following form.

THEOREM 1.5. *For a locally compact separable metrizable space X such that $X^{(1)}$ is noncompact, the following cycle of morphisms exists*:

$$(\omega^\omega, \omega^\omega, \leq^*) \to (\mathrm{M}(X), \mathrm{M}(X), \preceq) \to (\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}) \to (\omega^\omega, \omega^\omega, \leq^*).$$

So it seems natural to ask the following question, instead of Question 1.2.

QUESTION 1.6. *For a separable metrizable space X such that $X^{(1)}$ is noncompact, does the following cycle of morphisms exist?*

$$(\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq^* \times \subseteq) \to (\mathrm{M}(X), \mathrm{M}(X), \preceq) \to (\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}) \to (\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq^* \times \subseteq).$$

The Tukey equivalence in Question 1.2 would follow from this cycle by Lemma 1.1.

Although we do not have an answer to Question 1.6, we can construct a cycle of morphisms of a slightly modified form. For $(A, B) \in \mathrm{PC}(X)$, $d \in \mathrm{M}(X)$ and $\varepsilon > 0$, we write $(A, B)\, \mathsf{Sep}_\varepsilon\, d$ if $d(A, B) \geq \varepsilon$. For $d_1, d_2 \in \mathrm{M}(X)$, $d_1 \preceq_\varepsilon d_2$ if and only if, for $p, q \in X$, $d_1(p, q) \geq \varepsilon$ implies $d_2(p, q) \geq \varepsilon$. We replace Sep in Question 1.6 by Sep_1, $\preceq$ by $\preceq_1$ and $\leq^*$ by $\leq$.

THEOREM 1.7. *For a separable metrizable space X such that $X^{(1)}$ is noncompact, the following cycle of morphisms exists*:

$$(\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq) \to (\mathrm{M}(X), \mathrm{M}(X), \preceq_1) \to (\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1) \to (\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq).$$

The following corollary shows that Tukey equivalence quite similar to the one in Question 1.2 holds. The corollary follows immediately from Theorem 1.7 by Lemma 1.1.

COROLLARY 1.8. *For a separable metrizable space X such that $X^{(1)}$ is noncompact, the Tukey equivalence $(\mathrm{M}(X), \preceq_1) \equiv_T (\omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq)$ holds.*

The main purpose of this paper is to prove Theorem 1.7. In Section 2 we observe how Theorem 1.4 is refined to Theorem 1.5, and in Section 3 we further extend this result to establish Theorem 1.7.

In Section 4 we discuss cardinality questions about approximating the Stone–Čech compactification by Smirnov compactifications, which have been studied in the preceding paper [6].

2. $\mathrm{M}(X)$ for a Locally Compact Separable X

Let X be a locally compact separable metrizable space such that $X^{(1)}$ is noncompact. In this section, we review the proof of Theorem 1.4 (presented in [3]) and observe how it is refined to the construction of the following cycle of morphisms (Theorem 1.5).

$$(\omega^\omega, \omega^\omega, \le^*) \to (\mathrm{M}(X), \mathrm{M}(X), \preceq) \to (\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}) \to (\omega^\omega, \omega^\omega, \le^*).$$

In Section 3, we extend the results in this section to obtain the main theorem (Theorem 1.7).

We will frequently use the following lemma. It is derived from Theorems 4.5 and 4.6, however, one can easily find a direct proof.

Lemma 2.1. *For a metrizable space X and $d_1, d_2 \in \mathrm{M}(X)$, the following are equivalent.*

(1) $d_1 \preceq d_2$.
(2) *For* $(A, B) \in \mathrm{PC}(X)$, *if* $d_1(A, B) > 0$ *then* $d_2(A, B) > 0$.

Remark 1. It is obvious that, for a metrizable space X, $d_1, d_2 \in \mathrm{M}(X)$ and $\varepsilon > 0$, the following are equivalent.

(1) $d_1 \preceq_\varepsilon d_2$ (that is, for $p, q \in X$, if $d_1(p, q) \ge \varepsilon$ then $d_2(p, q) \ge \varepsilon$).
(2) For $(A, B) \in \mathrm{PC}(X)$, if $d_1(A, B) \ge \varepsilon$ then $d_2(A, B) \ge \varepsilon$.

In this sense the relations $\preceq$ and $\preceq_\varepsilon$ look alike, though there is no obvious implication between them.

The second morphism in the sequence is easily obtained. The first and third morphisms are obtained by refining the proof of [3, Theorem 3.1].

Lemma 2.2. *For a metrizable space X, there is a morphism from $(\mathrm{M}(X), \mathrm{M}(X), \preceq)$ to $(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep})$.*

Proof. In the proof of [9, Theorem 2.11] Woods proved the following fact: for every $(A, B) \in \mathrm{PC}(X)$ there is a metric $d \in \mathrm{M}(X)$ such that $d(A, B) \ge 1$ holds.

Let φ_- be the correspondence from (A,B) to d in this fact, and φ_+ the identity mapping on $\mathrm{M}(X)$. It is straightforward to check that $\varphi = (\varphi_-, \varphi_+)$ is a desired morphism. □

LEMMA 2.3. *For a locally compact subspace X of $\mathbf{H}$ such that $X^{(1)}$ is noncompact, there is a morphism from $(\omega^\omega, \omega^\omega, \le^*)$ to $(\mathrm{M}(X), \mathrm{M}(X), \preceq)$.*

PROOF. We will use the following lemma, which was originally established by Kada, Tomoyasu and Yoshinobu [6, Lemma 2.8]. For a function γ from X to $\mathbf{R}$, we write $\gamma(x) \to \infty$ *as* $x \to \infty$ if, for any $M \in \mathbf{R}$ there is a compact subset K of X such that $\gamma(x) > M$ holds for all $x \in X \backslash K$.

LEMMA 2.4 [3, Lemma 3.2]. *Suppose that X is a locally compact separable metrizable space, $d \in \mathrm{M}(X)$, $\mathrm{diam}_d(X)$ is finite, and γ is a continuous function from X to $[0,\infty)$ such that $\gamma(x) \to \infty$ as $x \to \infty$. For $n \in \omega$, let $K_n = \{x \in X : \gamma(x) \le \mathrm{diam}_d(X) + n\}$. Then we can define a mapping from $\omega^{\uparrow\omega}$ to $\mathrm{M}(X)$, which maps g to d_g, with the following properties.*

(1) *If $x, y \in X \backslash K_n$, then $d_g(x,y) \ge g(n) \cdot d(x,y)$.*
(2) *For $x, y \in X$, $d_g(x,y) \ge |\gamma(x) - \gamma(y)|$.*
(3) *For $g_1, g_2 \in \omega^{\uparrow\omega}$, $g_1 \le^* g_2$ implies $d_{g_1} \preceq d_{g_2}$, and $g_1 \le g_2$ implies $d_{g_1} \le d_{g_2}$.*[1]

We apply the above lemma to (X,μ) by letting $\gamma(p) = 1/\mu(p, X^*)$ for $p \in X$. Let φ_+ be the mapping obtained by the lemma, which maps $g \in \omega^{\uparrow\omega}$ to $\mu_g \in \mathrm{M}(X)$. For $n < \omega$, let K_n be as in the above lemma. Define φ_- by letting, for $\rho \in \mathrm{M}(X)$, $\varphi_-(\rho) = h_\rho \in \omega^{\uparrow\omega}$ be a function recursively defined by $h_\rho(0) = 0$ and

$$h_\rho(n) = \min\{l : l > h_\rho(n-1) \text{ and } \forall p, q \in K_{n+2}\ (\rho(p,q) \ge 1/n \to \mu(p,q) \ge 1/l)\}$$

for $n \ge 1$. We verify that $\varphi = (\varphi_-, \varphi_+)$ is a morphism from $(\omega^{\uparrow\omega}, \omega^{\uparrow\omega}, \le^*)$ to $(\mathrm{M}(X), \mathrm{M}(X), \preceq)$. Fix $\rho \in \mathrm{M}(X)$, $g \in \omega^{\uparrow\omega}$ and assume $h_\rho \le^* g$. To see $\rho \preceq \mu_g$, fix $(A,B) \in \mathrm{PC}(X)$ with $\rho(A,B) > 0$, and we shall show $\mu_g(A,B) > 0$. Take $k \in \omega$ so that $\rho(A,B) > 1/k$ and $g(n) \ge h_\rho(n)$ for all $n \ge k$. By the definition of h_ρ, for all $n \ge k$ we have $\mu(A \cap (K_{n+2} \backslash K_n), B \cap (K_{n+2} \backslash K_n)) \ge 1/h_\rho(n)$. Since $g(n) \ge h_\rho(n)$ for $n \ge k$ and by the property of μ_g, we have $\mu_g(A \cap (K_{n+2} \backslash K_n), B \cap (K_{n+2} \backslash K_n)) \ge 1$ for all $n \ge k$. Also, since $\mu_g(X \backslash K_{m+1}, K_m) \ge 1$ for all $m \in \omega$, we can conclude that $\mu_g(A,B) \ge \min\{1, \mu_g(A \cap K_{k+1}, B \cap K_{k+1})\} > 0$. □

[1] In [3, Lemma 3.2], the corresponding clause does not have "$g_1 \le g_2$ implies $d_{g_1} \le d_{g_2}$" part. To make the proof work for the modified statement, we slightly modified the definition of K_n's.

Lemma 2.5. *For a locally compact subspace X of $\mathbf{H}$ such that $X^{(1)}$ is noncompact, there is a morphism from* $(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep})$ *to* $(\omega^\omega, \omega^\omega, \le^*)$.

Proof. Fix a sequence $\langle a_n : n < \omega \rangle$ in $X^{(1)}$ converging to some $a \in X^*$. Such a sequence exists because $X^{(1)}$ is noncompact. Note that the set $\{a_n : n < \omega\}$ is closed discrete in X. For each n, fix a sequence $\langle b_{n,j} : j < \omega \rangle$ in X converging to a_n. We may assume that a_n's and $b_{n,j}$'s are all distinct, and for each n, for all j we have $\mu(a_n, b_{n,j}) < 2^{-n}$.

We define a mapping φ_- from ω^ω to $\mathrm{PC}(X)$ in a simple way. For $g \in \omega^\omega$, just let $A = \{a_n : n < \omega\}$, $B_g = \{b_{n,g(n)} : n < \omega\}$ and $\varphi_-(g) = (A, B_g)$.

Now we define a mapping φ_+ from $\mathrm{M}(X)$ to ω^ω. For $\rho \in \mathrm{M}(X)$ we define $\varphi_+(\rho) = H_\rho \in \omega^\omega$ by letting

$$H_\rho(n) = \min\{i : \forall j > i \ \ (\rho(a_n, b_{n,j}) \le 2^{-n})\}$$

for each n.

Suppose that $g \in \omega^\omega$, $\rho \in \mathrm{M}(X)$ and $\rho(A, B_g) = \varepsilon > 0$. Then for all but finitely many n we have $\rho(a_n, b_{n,g(n)}) \ge \varepsilon > 2^{-n}$, and by the definition of H_ρ, we have $H_\rho(n) \ge g(n)$. This means that $\varphi = (\varphi_-, \varphi_+)$ is a desired morphism. □

Now we can check that we may replace $\le^*$ with $\le$, $\preceq$ with $\preceq_1$, and Sep with Sep_1 in the cycle of morphisms, which produces the following cycle.

Theorem 2.6. *For a locally compact separable metrizable space X such that $X^{(1)}$ is noncompact, the following cycle of morphisms exists*:

$$(\omega^\omega, \omega^\omega, \le) \to (\mathrm{M}(X), \mathrm{M}(X), \preceq_1) \to (\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1) \to (\omega^\omega, \omega^\omega, \le).$$

For the second morphism, the pair $\varphi = (\varphi_-, \varphi_+)$ in Lemma 2.2 works.

Lemma 2.7. *For a metrizable space X, there is a morphism from* $(\mathrm{M}(X), \mathrm{M}(X), \preceq_1)$ *to* $(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1)$.

The first and third morphisms are obtained by slightly modifying the proofs of Lemmas 2.3 and 2.5 respectively, which we leave to the readers.

Lemma 2.8. *For a locally compact subspace X of $\mathbf{H}$ such that $X^{(1)}$ is noncompact, there is a morphism from* $(\omega^\omega, \omega^\omega, \le)$ *to* $(\mathrm{M}(X), \mathrm{M}(X), \preceq_1)$.

LEMMA 2.9. *For a locally compact subspace X of* $\mathbf{H}$ *such that* $X^{(1)}$ *is noncompact, there is a morphism from* $(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1)$ *to* $(\omega^\omega, \omega^\omega, \leq)$.

3. The Main Result

This section is devoted to the proof of the main theorem (Theorem 1.7). For a separable metrizable space X such that $X^{(1)}$ is noncompact, we shall provide the following cycle of morphisms:

$$\begin{aligned}(\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq) &\to (\mathrm{M}(X), \mathrm{M}(X), \preceq_1)\\ &\to (\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1)\\ &\to (\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq).\end{aligned}$$

The second morphism is already provided by Lemma 2.7.

We will use the following lemma for the construction of both the first and the third morphisms.

LEMMA 3.1 [5, Lemma 4.4]. *Suppose that X is a subspace of* $\mathbf{H}$ *such that $X^{(1)}$ is noncompact, $d \in \mathrm{M}(X)$ and $\varepsilon > 0$. Then there is a compact subset $Y_{d,\varepsilon}$ of X^* with the following properties*:

(1) *For two sequences $\langle p_n : n \in \omega \rangle$, $\langle q_n : n \in \omega \rangle$ in X, if $d(p_n, q_n) \geq \varepsilon$ for all $n \in \omega$ and both sequences converge to $r \in \mathrm{cl}_{\mathbf{H}} X$, then $r \in Y_{d,\varepsilon}$.*
(2) *For disjoint closed subsets A, B of X, if $d(A, B) \geq \varepsilon$ then $\mathrm{cl}_{\mathbf{H}} A \cap \mathrm{cl}_{\mathbf{H}} B \subseteq Y_{d,\varepsilon}$.*

PROOF. For each $x \in X$, consider an open ball $B_d(x, \varepsilon/3)$ with center x and radius $\varepsilon/3$ in the metric space (X, d). Since X is a dense subspace of $\mathrm{cl}_{\mathbf{H}} X$ and $B_d(x, \varepsilon/3)$ is open in X, we can choose an open subset U_x of $\mathrm{cl}_{\mathbf{H}} X$ so that $U_x \cap X = B_d(x, \varepsilon/3)$ holds. Let $U = \bigcup\{U_x : x \in X\}$ and $Y = Y_{d,\varepsilon} = \mathrm{cl}_{\mathbf{H}} X \setminus U$. Since U is open in $\mathrm{cl}_{\mathbf{H}} X$ and covers X, Y is closed in $\mathrm{cl}_{\mathbf{H}} X$ and $Y \subseteq X^*$, and hence $Y \in \mathscr{K}(X^*)$.

We prove that Y satisfies the property (1). To prove this by contradiction, suppose that there are sequences $\langle p_n : n \in \omega \rangle$, $\langle q_n : n \in \omega \rangle$ in X such that $d(p_n, q_n) \geq \varepsilon$ for all $n \in \omega$ and both sequences converge to some $r \in \mathrm{cl}_{\mathbf{H}} X \setminus Y = U$. Find $x \in X$ such that $r \in U_x$. Since U_x is an open neighborhood of r and both $\langle p_n : n \in \omega \rangle$ and $\langle q_n : n \in \omega \rangle$ converge to r, we can pick $n \in \omega$ so that $p_n \in U_x$ and $q_n \in U_x$. Note that the points x, p_n, q_n are all from X. Since $U_x \cap X =$

$B_d(x, \varepsilon/3)$, we have $d(p_n, q_n) \le d(x, p_n) + d(x, q_n) < 2\varepsilon/3$. This contradicts the assumption that $d(p_n, q_n) \ge \varepsilon$.

The property (2) follows from (1). $\square$

For the construction of the first morphism, we will use Lemma 2.4 in an even stronger form. The following lemma is easily checked by reviewing the proof of [3, Lemma 3.2] and hence we omit the proof.

Lemma 3.2. *Let X be a subspace of $\mathbf{H}$ such that $X^{(1)}$ is noncompact. Suppose that $L_1, L_2 \in \mathscr{K}(X^*)$ and $L_1 \subseteq L_2$. For $i \in \{1,2\}$, let $X_i = \mathrm{cl}_{\mathbf{H}}\, X \setminus L_i$, $\gamma_i(p) = 1/\mu(p, L_i)$ for $p \in X_i$, $\bar{\mu}_g^i \in \mathrm{M}(X_i)$ the one obtained by applying Lemma 2.4 to (X_i, μ), γ_i and $g \in \omega^{\uparrow\omega}$, and μ_g^i the restriction of $\bar{\mu}_g^i$ to X, that is, $\mu_g^i = \bar{\mu}_g^i \restriction (X \times X)$. Then for every $g \in \omega^{\uparrow\omega}$ we have $\mu_g^1 \le \mu_g^2$.*

Theorem 3.3. *For a subspace X of $\mathbf{H}$ such that $X^{(1)}$ is noncompact, there is a morphism from $(\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \le \times \subseteq)$ to $(\mathrm{M}(X), \mathrm{M}(X), \preceq_1)$.*

Proof. First we define a mapping φ_- from $\mathrm{M}(X)$ to $\omega^{\uparrow\omega} \times \mathscr{K}(X^*)$. Fix $d \in \mathrm{M}(X)$. Let $Y = Y_{d,1}$ be the one in Lemma 3.1 applied to X, d and $\varepsilon = 1$, and $X_Y = \mathrm{cl}_{\mathbf{H}}\, X \setminus Y$. X_Y is a locally compact subspace of $\mathbf{H}$ and contains X as a subspace. We will define $h_d \in \omega^{\uparrow\omega}$ in a similar way as in the proof of Lemma 2.3. For $p \in X_Y$ let $\gamma(p) = 1/\mu(p, Y)$, and for $n < \omega$ let $K_n = \{x \in X_K : \gamma(x) \le \mathrm{diam}_\mu(X) + n\}$. Define $h_d \in \omega^{\uparrow\omega}$ recursively by letting $h_d(0) = 0$ and

$$h_d(n) = \min\{l : l > h_d(n-1) \text{ and } \forall p, q \in K_{n+2} \cap X \ \ (d(p,q) \ge 1 \to \mu(p,q) \ge 1/l)\}$$

for $n \ge 1$. The minimum in the right-hand side exists by the following reason. Suppose not. Then there are two sequences $\langle p_n : n \in \omega \rangle$, $\langle q_n : n \in \omega \rangle$ in $K_{n+2} \cap X$ such that $d(p_n, q_n) \ge 1$ for all $n \in \omega$ and $\mu(p_n, q_n) \to 0$ as $n \to \infty$. We may assume that both sequences converge, and then they must converge to the same point, say r. By Lemma 3.1, $r \in Y_{d,1} = Y$. But it is impossible because $\mu(r, K_{n+2} \cap X) \ge \mu(Y, K_{n+2}) > 0$. Now define $\varphi_-(d)$ by letting $\varphi_-(d) = (h_d, Y)$.

We turn to the definition of φ_+ from $\omega^{\uparrow\omega} \times \mathscr{K}(X^*)$ to $\mathrm{M}(X)$. Fix $g \in \omega^{\uparrow\omega}$ and $L \in \mathscr{K}(X^*)$. Let $X_L = \mathrm{cl}_{\mathbf{H}}\, X \setminus L$, $\bar{\rho} = \mu_g \in \mathrm{M}(X_L)$ as in Lemma 2.3, applied to the space X_L, the metric μ, $\gamma(p) = 1/\mu(p, L)$ for $p \in X_L$, and g. Let $\rho \in \mathrm{M}(X)$ be the restriction of $\bar{\rho}$ to X. Define $\varphi_+((g, L))$ by letting $\varphi_+((g, L)) = \rho$.

Now we are going to check that $\varphi = (\varphi_-, \varphi_+)$ is a desired morphism. Suppose that $d \in \mathrm{M}(X)$, $g \in \omega^{\uparrow\omega}$, $L \in \mathscr{K}(X^*)$, $\varphi_-(d) = (h_d, Y)$, $h_d \le g$ and $Y \subseteq L$. Let $\rho = \varphi_+(g, L)$. We will show that $d \preceq_1 \rho$. Fix $p, q \in X$. If $p, q \in K_{n+2} \setminus K_n$ for some

$n \in \omega$, then by the definition of h_d we have $\mu(p,q) \geq 1/h_d(n)$. By the assumption that $h_d \geq g$, $Y \subseteq L$, and Lemma 3.2, we have $\rho(p,q) \geq 1$. If it is not the case, we may assume that $p \in X \backslash K_{m+1}$ and $q \in K_m$ for some $m \in \omega$. By the property of μ_g shown in Lemma 2.4, we have $\rho(X \backslash K_{m+1}, K_m) \geq 1$ and hence $\rho(p,q) \geq 1$. □

THEOREM 3.4. *For a subspace X of $\mathbf{H}$ such that $X^{(1)}$ is noncompact, there is a morphism from* $(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1)$ *to* $(\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq)$.

PROOF. We define a mapping φ_- from $\omega^\omega \times \mathscr{K}(X^*)$ to $\mathrm{PC}(X)$. Fix $f \in \omega^\omega$ and $K \in \mathscr{K}(X^*)$. We will construct a pair $(A,B) = \varphi_-(f,K)$ of disjoint closed subsets of X so that $K \subseteq \mathrm{cl}_{\mathbf{H}} A \cap \mathrm{cl}_{\mathbf{H}} B$ and the information of f is "embedded" into the pair (A,B).

Fix a sequence $\langle a_n : n < \omega \rangle$ in $X^{(1)}$ converging to some $a \in X^*$. Such a sequence exists because $X^{(1)}$ is noncompact. For each n, fix a sequence $\langle b_{n,j} : j < \omega \rangle$ in X converging to a_n. We may assume that a_n's and $b_{n,j}$'s are all distinct, and for each n, for all j we have $\mu(a_n, b_{n,j}) < 2^{-n}$.

We will construct two closed subsets A, B of X from f and K in ω steps. We are going to define two increasing sequences of finite subsets of X, $A_0 \subseteq A_1 \subseteq \cdots$ and $B_0 \subseteq B_1 \subseteq \cdots$, and let $A = \bigcup_{n<\omega} A_n$, $B = \bigcup_{n<\omega} B_n$. For notational convention, let $A_{-1} = B_{-1} = \varnothing$.

Note that, since X is totally bounded with respect to μ and dense in $\mathrm{cl}_{\mathbf{H}} X$, for any $\varepsilon > 0$ there is a finite subset F of X such that $\bigcup\{B_\mu(x,\varepsilon) : x \in F\}$ covers K, where $B_\mu(x,\varepsilon)$ denotes the open ball with center x and radius ε in the metric space $(\mathrm{cl}_{\mathbf{H}} X, \mu)$.

We describe the construction in the step n below.

First, let $A'_n = A_{n-1} \cup \{a_n\}$ and $B'_n = B_{n-1} \cup \{b_{n,i}\}$, where $i = \min\{j : j \geq f(n)$ and $b_{n,j} \notin A_{n-1} \cup B_{n-1}\}$.

Let $r_n = \mu(A'_n \cup B'_n, K)/2$. Find a finite subset E_n of X such that $\bigcup\{B_\mu(x,r_n) : x \in E_n\}$ covers K and $B_\mu(x,r_n) \cap K \neq \varnothing$ (in other words, $\mu(x,K) < r_n$) for every $x \in E_n$. Note that E_n and $A'_n \cup B'_n$ never intersect. Let $A_n = A'_n \cup E_n$.

Let $s_n = \mu(A_n \cup B'_n, K)/2$. Find a finite subset F_n of X such that $\bigcup\{B_\mu(x,s_n) : x \in F_n\}$ covers K and $B_\mu(x,s_n) \cap K \neq \varnothing$ (in other words, $\mu(x,K) < s_n$) for every $x \in F_n$. It may happen that F_n contains a_k for some $k < \omega$. In such a case, we replace a_k by $b_{k,i}$ where $i = \min\{j : b_{k,j} \notin A_n \cup B'_n$ and $\mu(a_k, b_{k,j}) < s_n/2\}$, for each such k (to ensure that B and the set $\{a_n : n < \omega\}$ never intersect). Note that F_n and $A_n \cup B'_n$ do not intersect, and $\bigcup\{B_\mu(x, 3s_n/2) : x \in F_n\}$ covers K. Let $B_n = B'_n \cup F_n$.

This completes the construction in the step n.

It is easy to see that A and B are disjoint, closed in X, and satisfy $K \subseteq K \cup \{a\} \subseteq \mathrm{cl}_{\mathbf{H}} A \cap \mathrm{cl}_{\mathbf{H}} B$. We let $\varphi_-(f, K) = (A, B)$.

We turn to the definition of φ_+ from $\mathrm{M}(X)$ to $\omega^\omega \times \mathscr{K}(X^*)$. Fix $d \in \mathrm{M}(X)$. We define $g_d \in \omega^\omega$ by letting $g_d(n) = \max(\{j : d(a_n, b_{n,j}) \geq 1\} \cup \{0\})$ for each n, and let $\varphi_+(d) = (g_d, Y_{d,1})$, where $Y_{d,1}$ is the one obtained by Lemma 3.1 applied to X, d and $\varepsilon = 1$.

Now we check that $\varphi = (\varphi_-, \varphi_+)$ is a morphism from $(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1)$ to $(\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq)$. Suppose that $f \in \omega^\omega$, $K \in \mathscr{K}(X^*)$, $d \in \mathrm{M}(X)$, $(A, B) = \varphi_-(f, K)$, $(g_d, Y_{d,1}) = \varphi_+(d)$, and $d(A, B) \geq 1$ holds. We have to check that $f \leq g_d$ and $K \subset Y_{d,1}$.

First we show that $f \leq g_d$. Fix $n < \omega$. By the construction of (A, B) and g_d, A contains a_n and B contains $b_{n,i}$ for some i with $i \geq f(n)$. For such an i, since $d(A, B) \geq 1$, we have $d(a_n, b_{n,i}) \geq 1$, and by the definition of $g_d(n)$ we have $f(n) \leq i \leq g_d(n)$.

Next we show that $K \subseteq Y_{d,1}$. By the assumption that $d(A, B) \geq 1$ and the property of $Y_{d,1}$, we have $K \subseteq K \cup \{a\} \subseteq \mathrm{cl}_{\mathbf{H}} A \cap \mathrm{cl}_{\mathbf{H}} B \subseteq Y_{d,1}$. □

This concludes the proof of Theorem 1.7.

4. Applications to the Cardinal Function $\mathfrak{sa}(X)$

In a preceding paper [4] the following cardinal function was introduced.

Definition 4.1 [4, Definition 2.2]. For a metrizable space X, let $\mathfrak{sa}(X) = \min\{|D| : D \subseteq \mathrm{M}(X) \text{ and } \beta X \simeq \sup\{u_d X : d \in D\}\}$.

It is known that $\mathfrak{sa}(X) = 1$ holds (that is, $\beta X \simeq u_d X$ for some $d \in \mathrm{M}(X)$) if and only if $X^{(1)}$ is compact [9, Corollary 3.5].

Kada, Tomoyasu and Yoshinobu [5] proved the following theorem.

Theorem 4.2 [5, Corollary 4.6]. *For a separable metrizable space X such that $X^{(1)}$ is noncompact, $\mathfrak{sa}(X) = \mathfrak{d} \cdot \mathsf{cof}((\mathscr{K}(X^*), \subseteq))$ holds.*

Corollary 4.3 [6, Theorem 2.10]. *For a locally compact separable metrizable space X such that $X^{(1)}$ is noncompact, $\mathfrak{sa}(X) = \mathfrak{d}$ holds.*

Proof. Since X is locally compact and separable, X^* is compact and hence $\mathsf{cof}((\mathscr{K}(X^*), \subseteq)) = 1$ holds. □

In this section, we observe the relationship between the cardinal $\mathfrak{sa}(X)$ and generalized Galois–Tukey connection involving $\mathrm{M}(X)$.

We will use the following basic facts about the order relation $\leq$ on $\mathsf{Cpt}(X)$ and Smirnov compactifications. For a compactification αX of X and $(A,B) \in \mathrm{PC}(X)$, we write $A \| B\ (\alpha X)$ if $\mathrm{cl}_{\alpha X} A \cap \mathrm{cl}_{\alpha X} B = \emptyset$.

THEOREM 4.4 [2, Theorem 6.5]. *For a compactification αX of a normal space X, $\alpha X \simeq \beta X$ if and only if $A \| B\ (\alpha X)$ for every $(A,B) \in \mathrm{PC}(X)$.*

THEOREM 4.5 [9, Theorem 2.2]. *For compactifications αX, γX of a completely regular Hausdorff space X, the following are equivalent.*

(1) $\alpha X \leq \gamma X$.
(2) *For $(A,B) \in \mathrm{PC}(X)$, if $A \| B\ (\alpha X)$ then $A \| B\ (\gamma X)$.*

THEOREM 4.6 [9, Theorem 2.5]. *For a compactification αX of a metric space (X,d), the following are equivalent.*

(1) $\alpha X \simeq u_d X$.
(2) *for $(A,B) \in \mathrm{PC}(X)$, $A \| B\ (\alpha X)$ if and only if $d(A,B) > 0$.*

LEMMA 4.7 [4, Lemma 1.2]. *Suppose that $\mathscr{C} \subseteq \mathsf{Cpt}(X)$. For $(A,B) \in \mathrm{PC}(X)$, the following are equivalent.*

(1) $A \| B\ (\sup \mathscr{C})$.
(2) $A \| B\ (\sup \mathscr{F})$ *for some nonempty finite subset $\mathscr{F}$ of $\mathscr{C}$.*

For a directed set $(D, \leq_D)$, $\mathsf{cof}((D, \leq_D))$ denotes the smallest cardinality of a cofinal set of D with respect to the order relation $\leq_D$. We write $\mathsf{cof}(D)$ if the referred order relation on D is clear from the context. It is easy to see that $D \leq_T E$ implies $\mathsf{cof}(D) \leq \mathsf{cof}(E)$.

The *dominating number* $\mathfrak{d}$ is the cardinal defined by $\mathfrak{d} = \mathsf{cof}((\omega^\omega, \leq)) = \mathsf{cof}((\omega^\omega, \leq^*))$.

The *norm* $\|\mathbf{A}\|$ of a triple $\mathbf{A} = (A_-, A_+, A)$ is the smallest cardinality of a set $Y \subseteq A_+$ such that for any $x \in A_-$ there is a $y \in Y$ with $x\,A\,y$. It is easy to see that $\mathbf{A} \to \mathbf{B}$ implies $\|\mathbf{B}\| \leq \|\mathbf{A}\|$. For a directed set $(D, \leq_D)$, $\mathsf{cof}((D, \leq_D))$ is also described as $\|(D, D, \leq_D)\|$.

Using generalized Galois–Tukey connection, we can redefine $\mathfrak{sa}(X)$ in the following way.

Lemma 4.8. *Let X be a metrizable space.*

(1) *For a subset D of* $\mathrm{M}(X)$, *if for each* $(A,B) \in \mathrm{PC}(X)$ *there is* $d \in D$ *such that* $d(A,B) > 0$, *then* $\sup\{u_d X : d \in D\} \simeq \beta X$.
(2) *For a subset D of* $\mathrm{M}(X)$ *with* $|D| = 1$ *or* $|D| \geq \aleph_0$, *if* $\sup\{u_d X : d \in D\} \simeq \beta X$, *then there is a subset* D' *of* $\mathrm{M}(X)$ *such that* $|D'| = |D|$ *and for each* $(A,B) \in \mathrm{PC}(X)$ *there is* $d \in D'$ *such that* $d(A,B) > 0$.

Proof. (1) Follows from Theorems 4.4, 4.5 and 4.6.

(2) Note that $\mathrm{M}(X)$ is closed under pointwise addition as functions from $X \times X$ to $\mathbf{R}$. It is easy to see that, for $\rho_0, \ldots, \rho_{n-1} \in \mathrm{M}(X)$ and $\sigma = \rho_0 + \cdots + \rho_{n-1}$, we have $\sup\{u_{\rho_i} X : i < n\} \leq u_\sigma X$. Given D as in the assumption of (2), let D' be the closure of D under finite sums. Using Theorem 4.4 and Lemma 4.7 one can check that this D' works. □

Theorem 4.9. *For a metrizable space X,* $\mathfrak{sa}(X) = \|(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep})\|$.

Proof. Follows from Lemma 4.8. Note that the argument in the proof of Lemma 4.8 also shows that $\mathfrak{sa}(X)$ is either 1 or infinite. □

Corollary 4.10. *For a metrizable space X such that $X^{(1)}$ is noncompact,* $\mathfrak{sa}(X) = \|(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1)\|$.

Proof. Modify the proof of Lemma 4.8 so that D' is also closed under multiplications by positive integers. □

Let X be a separable metrizable space such that $X^{(1)}$ is noncompact. By Theorem 1.7 and Corollary 4.10, we have

$$\begin{aligned}
\mathfrak{sa}(X) &= \|(\mathrm{PC}(X), \mathrm{M}(X), \mathsf{Sep}_1)\| \\
&= \|(\omega^\omega \times \mathscr{K}(X^*), \omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq)\| \\
&= \mathsf{cof}((\omega^\omega \times \mathscr{K}(X^*), \leq \times \subseteq)) \\
&= \mathfrak{d} \cdot \mathsf{cof}((\mathscr{K}(X^*), \subseteq)),
\end{aligned}$$

which gives an alternate proof of Theorem 4.2.

Remark 2. After hearing the statement of Theorem 4.2 [5, Corollary 4.6] and its original proof, Todorčević suspected that $\mathfrak{d} \cdot \mathsf{cof}((\mathscr{K}(X^*), \subseteq))$ might be resulted from the cofinal structure of the ordered set $(\omega^\omega \times \mathscr{K}(X^*), \le^* \times \subseteq)$, and told the authors that the equality of cardinalities should reflect some relationship between the order structure of $(\omega^\omega \times \mathscr{K}(X^*), \le^* \times \subseteq)$ and some structure of the set $\mathrm{M}(X)$. That was the origin of Question 1.2 and our investigation into the structure of $\mathrm{M}(X)$.

References

[1] A. Blass, Combinatorial cardinal characteristics of the continuum, Handbook of Set Theory (M. Foreman and A. Kanamori, eds.), Springer, 2010, pp. 395–489.
[2] L. Gillman and M. Jerison, Rings of continuous functions, Van Nostrand, 1960.
[3] M. Kada, How many miles to βX? II—Approximations to βX versus cofinal types of sets of metrics, Topology Appl. **157** (2010), 1460–1464.
[4] M. Kada, K. Tomoyasu, and Y. Yoshinobu, How many miles to $\beta\omega$?—Approximating $\beta\omega$ by metric-dependent compactifications, Topology Appl. **145** (2004), 277–292.
[5] ———, How many miles to $\beta\omega$? II, Set Theoretic and Geometric Topology and Its Applications, RIMS Kokyuroku, no. 1419, Research Institute for Mathematical Sciences, Kyoto University, 2005, (Japanese), pp. 105–125.
[6] ———, How many miles to βX?—$\mathfrak{d}$ miles, or just one foot, Topology Appl. **153** (2006), 3313–3319.
[7] S. Solecki and S. Todorčević, Cofinal types of topological directed orders, Ann. Inst. Fourier (Grenoble) **54** (2004), 1877–1911.
[8] P. Vojtaš, Generalized Galois–Tukey connections between explicit relations on classical objects of real analysis, Set Theory of the Reals (H. Judah, ed.), Israel Mathematical Conference Proceedings, vol. 6, 1993, pp. 619–643.
[9] R. G. Woods, The minimum uniform compactification of a metric space, Fund. Math. **147** (1995), 39–59.

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