TSUKUBA J. MATH.
Vol. 35 No. 1 (2011), 103–113

AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF THE SUPREMUM OF A MARKOV-MODULATED RANDOM WALK*

By

Mikhail Sgibnev

Abstract. We obtain an asymptotic expansion for the distribution of the supremum of a Markov-modulated random walk, which takes into account the influence of the roots of the characteristic equation. An estimate is given for the remainder term by means of submultiplicative weight functions.

1. Introduction

Let $\{\kappa_n\}_{n=0}^{\infty}$ be an irreducible aperiodic Markov chain with finite state space $\mathcal{N} = \{1, \dots, N\}$ and transition matrix $\mathbf{P} = (p_{ij})$, where $p_{ij} = \mathsf{P}(\kappa_n = j \mid \kappa_{n-1} = i)$, $i, j \in \mathcal{N}$, $n = 1, 2, \dots$. Let $\boldsymbol{\pi} = (\pi_1, \dots, \pi_N)$ denote the stationary distribution of the chain. In our case, $\pi_i > 0$, $i \in \mathcal{N}$. Let $\{X_m(i, j)\}_{m=1}^{\infty}$ be a sequence of independent identically distributed random variables with distribution F_{ij}. Assume that the sequences of random variables $\{X_m(i, j)\}_{m=1}^{\infty}$, $(i, j) \in \mathcal{N} \times \mathcal{N}$, and $\{\kappa_n\}_{n=0}^{\infty}$ are mutually independent. Write $S_0 = 0$ and $S_n = S_{n-1} + X_n(\kappa_{n-1}, \kappa_n)$ for $n \geq 1$. Suppose that $M_\infty := \sup_{n \geq 0} S_n < \infty$ a.s. for every initial state of the chain. This is the case when the expectation of a one-step increment of the random walk $\{S_n\}$ is negative under the stationary distribution $\boldsymbol{\pi}$ of the chain: $\mathsf{E}_{\boldsymbol{\pi}} S_1 := \sum_{i,j=1}^{N} \pi_i p_{ij} \mathsf{E} X_1(i, j) < 0$, which will be assumed without loss of generality in the context of the present paper.

Let $\eta(x) := \min\{n \geq 1 : S_n > x\}$ and $\eta(x) := \infty$ on the event $\{M_\infty \leq x\}$. Clearly, $\{M_\infty > x\} = \bigcup_{j=1}^{N} \{\kappa_{\eta(x)} = j\}$. Denote by $\mathbf{A}$ the $N \times N$ matrix $(p_{ij} F_{ij})$

---

*2010 Mathematics Subject Classification: 60J05, 60J10, 60G50.

Key words and phrases: Supremum; Markov-modulated random walk; Finite Markov chain; Submultiplicative function; Characteristic equation; Asymptotic expansion; Banach algebra.
Received October 28, 2010.

and by $\mathbf{W}$ the $N \times N$ matrix (W_{ij}), where W_{ij} is the measure defined on $\mathscr{B}$ by the relations

$$W_{ij}((x,\infty)) := \mathsf{P}(\kappa_{\eta(x)} = j \,|\, \kappa_0 = i), \quad x > 0,$$

$W_{ij}((-\infty, 0)) := 0$, i, $j \in \mathscr{N}$, and $W_{ij}(\{0\}) := \delta_{ij} - \mathsf{P}(\kappa_{\eta(0)} = j \,|\, \kappa_0 = i)$, where δ_{ij} is the Kronecker delta (the reason for this definition will be clear from (4) below).

The asymptotic behaviour of $\mathsf{P}(M_\infty > x \,|\, \kappa_0 = i)$ has already been studied by K. Arndt [2], P. R. Jelenković and A. A. Lazar [5], G. Alsmeyer and M. Sgibnev [1]. The present paper is a continuation of [12]. We shall obtain an asymptotic expansion (see Theorem 4 and (6)) for the matrix measure $\mathbf{W}$ which takes into account the influence of roots of the characteristic equation (see (3) below). The integral estimate $\int_0^\infty \varphi(x)|\mathbf{\Delta}|(dx) < \infty$ is given for the remainder term $\mathbf{\Delta}$ by means of a submultiplicative weight function $\varphi(x)$.

2. Preliminaries

Let $\varphi(x)$, $x \in \mathbf{R}$, be a submultiplicative function, i.e., $\varphi(x)$ is a finite, positive, Borel measurable function with the following properties:

$$\varphi(0) = 1, \quad \varphi(x+y) \le \varphi(x)\varphi(y) \quad \text{for all } x, y \in \mathbf{R}.$$

It is well known [3, Section 7.6] that

$$-\infty < r_-(\varphi) := \lim_{x\to-\infty} \frac{\log \varphi(x)}{x} = \sup_{x<0} \frac{\log \varphi(x)}{x}$$

$$\le \inf_{x>0} \frac{\log \varphi(x)}{x} = \lim_{x\to\infty} \frac{\log \varphi(x)}{x} =: r_+(\varphi) < \infty. \tag{1}$$

Consider the collection $S(\varphi)$ of all complex-valued measures κ defined on the σ-algebra $\mathscr{B}$ of Borel subsets of $\mathbf{R}$ and such that

$$\|\kappa\|_\varphi := \int_{\mathbf{R}} \varphi(x)|\kappa|(dx) < \infty.$$

here $|\kappa|$ stands for the total variation of κ. The collection $S(\varphi)$ is a Banach algebra with norm $\|\kappa\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and κ of $S(\varphi)$ is defined as their convolution $\nu * \kappa$ [3, Section 4.16]. The unit element of $S(\varphi)$ is the Dirac measure δ_0, i.e., the atomic measure of unit mass at the origin. Relation (1) implies that the Laplace transform $\hat{\kappa}(s) = \int_{\mathbf{R}} \exp(sx)\kappa(dx)$ of an element $\kappa \in S(\varphi)$ converges absolutely with respect to $|\kappa|$ for all s in the strip

$$\Pi(\varphi) = \{s \in \mathbf{C} : r_-(\varphi) \le \Re s \le r_+(\varphi)\}.$$

The following theorem of [9] describes the structure of homomorphisms of $S(\varphi)$ onto $\mathbf{C}$.

Theorem 1. *Let* $m : S(\varphi) \to \mathbf{C}$ *be an arbitrary homomorphism. Then the following representation holds*:

$$m(v) = \int \chi(x, v) \exp(\beta x) v(dx), \quad v \in S(\varphi),$$

where β *is a real number such that* $r_-(\varphi) \leq \beta \leq r_+(\varphi)$ *and the function* $\chi(x, v)$ *of the two variables* $x \in \mathbf{R}$ *and* $v \in S(\varphi)$ *is a generalized character.*

We shall not give a complete definition of a generalized character here; in what follows only one property of a generalized character will be used:

$$v - \operatorname*{ess\,sup}_{x \in \mathbf{R}} |\chi(x, v)| \leq 1.$$

We shall need the following two theorems [10, Theorems 2 and 3].

Theorem 2. *Let* $\varphi(x)$, $x \in \mathbf{R}$, *be a submultiplicative function such that* $r_-(\varphi) < r_+(\varphi)$. *Suppose the function* $\varphi(x)/\exp[r_+(\varphi)x]$, $x \geq 0$, *is nondecreasing and the function* $\varphi(x)/\exp[r_-(\varphi)x]$, $x \leq 0$, *is nonincreasing. Assume* $v \in S(\varphi)$ *and let* s_0 *be an interior point of* $\Pi(\varphi)$. *Then the function* $[\hat{v}(s) - \hat{v}(s_0)]/(s - s_0)$, $s \in \Pi(\varphi)$, *is the Laplace transform of some measure, say* $T(s_0)v \in S(\varphi)$.

If s_0 lies on the boundary of the strip $\Pi(\varphi)$, the situation becomes more involved. Nevertheless, the following theorem holds (for the sake of definiteness we consider the case $\Re s_0 = r_+(\varphi)$).

Theorem 3. *Let* $\varphi(x)$, $x \in \mathbf{R}$, *be a submultiplicative function. Suppose the function* $\varphi(x)/\exp[r_+(\varphi)x]$, $x \geq 0$, *is nondecreasing and the function* $\varphi(x)/\exp[r_-(\varphi)x]$, $x \leq 0$, *is nonincreasing. Assume that*

$$\int_0^\infty (1 + x)\varphi(x)|v|(dx) < \infty \quad \text{or} \quad \int_{\mathbf{R}} (1 + |x|)\varphi(x)|v|(dx) < \infty,$$

depending on whether $r_-(\varphi) < r_+(\varphi)$ *or* $r_-(\varphi) = r_+(\varphi)$. *Let* $\Re s_0 = r_+(\varphi)$. *Then the function* $s \in \Pi(\varphi)$, $\gamma' \leq \Re s \leq \gamma$, *is the Laplace transform of some measure* $T(s_0)v \in S(\varphi)$.

The absolutely continuous component with respect to Lebesgue measure of an arbitrary distribution F will be denoted by F_c and its singular component, by F_s: $F_s = F - F_c$, i.e. $F_s = F_{\mathfrak{d}} + F_{\mathfrak{s}}$, where $F_{\mathfrak{d}}$ is the discrete component of F

and F_s is the singular component of F in the usual sense. We denote by $\mathbf{0}$ the zero matrix whose size will be determined by the context. We agree that all the operations with matrices and vectors are carried out elementwise. Suppose a matrix, say $\mathbf{B} = (B_{ij})$, is made up of elements of $S(\varphi)$. Then we shall denote by $\hat{\mathbf{B}}(s)$ the matrix whose elements are the Laplace transforms of the elements of $\mathbf{B}$, i.e. $\hat{\mathbf{B}}(s) := (\hat{B}_{ij}(s))$. In this case we shall also write $\mathbf{B} \in S(\varphi)$. A similar convention also applies to inequalities involving matrices or vectors.

Let $\mathbf{B}$ be a scalar $N \times N$-matrix and $\sigma(\mathbf{B})$ the set of all its eigenvalues. The number $\varrho(\mathbf{B}) := \max\{|\lambda| : \lambda \in \sigma(\mathbf{B})\}$ is called *the spectral radius* of $\mathbf{B}$. It is well known that if $\mathbf{B} \geq \mathbf{0}$, then $\varrho(\mathbf{B}) \in \sigma(\mathbf{B})$ and there exists a nonnegative vector $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Bx} = \varrho(\mathbf{B})\mathbf{x}$ [4, Theorem 8.3.1]. By Perron-Frobenius theorem [4, Theorem 8.4.4], each nonnegative irreducible matrix $\mathbf{B}$ has a positive eigenvalue of multiplicity 1 equal to $\varrho(\mathbf{B})$ and there exist positive left and right eigenvectors corresponding to this eigenvalue.

Define the convolution $\mathbf{A} * \mathbf{B}$ of two matrix measures $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$ as follows: $(\mathbf{A} * \mathbf{B})_{ij} := \sum_{k=1}^{N} A_{ik} * B_{kj}$. By $\mathbf{A}^{k*}$ we shall denote the k-fold convolution of the matrix measure $\mathbf{A}$, i.e. $\mathbf{A}^{1*} := \mathbf{A}$, $\mathbf{A}^{k*} := \mathbf{A} * \mathbf{A}^{(k-1)*}$, $k \geq 1$. Let $\bar{s}_n = \max_{1 \leq m \leq n} S_m$, $\chi(x) = S_{\eta(x)} - x$ and $\mathsf{P}_i(\cdot) = \mathsf{P}(\cdot \,|\, \kappa_0 = i)$, $i \in \mathcal{N}$.

Let $\hat{\mathbf{A}}(r) < \infty$, $r > 0$, and let $\mathbf{I}$ be the unit matrix. Choose $r' \in (0, r)$. By Arndt [2, Proposition 1], the matrix $\mathbf{I} - \hat{\mathbf{A}}(s)$ admits the factorization $\mathbf{I} - \hat{\mathbf{A}}(s) = \hat{\mathbf{A}}_-(s)\hat{\mathbf{A}}_+(s)$, $r' \leq \Re s \leq r$, with

$$\hat{\mathbf{A}}_-(s) = \mathbf{I} - \left(\sum_{n=1}^{\infty} \int_{-\infty}^{0} e^{sx} \mathsf{P}_i(\bar{s}_{n-1} < S_n \in dx, \kappa_n = j) \right),$$

$$\hat{\mathbf{A}}_+(s) = \mathbf{I} - \left(\int_{0}^{\infty} e^{sx} \mathsf{P}_i(\chi(0) \in dx, \kappa_{\eta(0)} = j) \right), \tag{2}$$

where $\mathbf{A}_-, \mathbf{A}_+ \in S(r', r) := S(\varphi)$ with $\varphi(x) := \max\{e^{r'x}, e^{rx}\}$. Moreover, the matrix measure $\mathbf{A}_-$ is invertible in $S(r', r)$, i.e. there exists a matrix measure $\mathbf{A}_-^{-1} \in S(r', r)$ such that $\mathbf{A}_- * \mathbf{A}_-^{-1} = \mathbf{A}_-^{-1} * \mathbf{A}_- = \delta_0 \mathbf{I}$. Notice that $\mathbf{A}_-$ may not be invertible in $S(0, r)$, which is one of the reasons why we deal with $S(r', r)$, where $r' > 0$.

3. Main Result

Let $\hat{\mathbf{A}}(r) < \infty$. Consider the *characteristic equation*

$$\det(\mathbf{I} - \hat{\mathbf{A}}(s)) = 0. \tag{3}$$

Assume that the set, say $\mathcal{Z}$, of the nonzero roots of (3) lying in the strip $\{s \in \mathbf{C} : 0 \leq \Re s \leq r\}$ is finite. Denote the elements of $\mathcal{Z}$ by $s_1, s_2, \ldots, s_l$. We do not exclude the case $\mathscr{Z} = \varnothing$. We then put $l = 0$ and use the following conventions: $\sum_{j=1}^{l} := 0$ and $\prod_{j=1}^{l} := 1$. Let n_j be the multiplicity of the root s_j. This means that $\det(\mathbf{I} - \hat{\mathbf{A}}(s)) = (s - s_j)^{n_j} f_j(s)$, where $f_j(s_j) \neq 0$. If $s \in \mathscr{Z}$, then $\bar{s} \in \mathscr{Z}$ and the root $\bar{s}$ has the same multiplicity as s.

Lemma 1. *Suppose that $\hat{\mathbf{A}}(r) < \infty$ for some $r > 0$. Let $\mathscr{Z} = \{s_1, \dots, s_l\}$ be the finite set of the nonzero roots of* (3) *lying in the strip $\{s \in \mathbf{C} : 0 \le \Re s \le r\}$. Then there exists one real root $q \in \mathscr{Z}$ of multiplicity* 1 *such that $\Re s_j > q$ for all $s_j \neq q$.*

Proof. Put $\lambda(\xi) := \varrho[\hat{\mathbf{A}}(\xi)]$, $\xi \in [0, r]$. First, let us prove that $\Re s_j > 0$ for all j. Suppose the contrary, i.e. that there exists $s_j \in \mathscr{Z}$ such that $\Re s_j = 0$. Since $\hat{\mathbf{A}}(0) \ge (|\hat{A}_{kl}(s_j)|)$, it follows by [4, Theorem 8.1.18] that $1 = \lambda(0) = \varrho[\hat{\mathbf{A}}(0)] \ge \varrho[\hat{\mathbf{A}}(s_j)] \ge 1$, and hence $\varrho[\hat{\mathbf{A}}(s_j)] = 1$. Applying [4, Theorem 8.4.5], we arrive at the following conclusion. There exist real numbers $\theta_1, \dots, \theta_N$ such that $\hat{\mathbf{A}}(s_j) = \mathbf{D}\hat{\mathbf{A}}(0)\mathbf{D}^{-1}$, where $\mathbf{D} = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_N})$ (diagonal matrix). We have $\hat{A}_{kl}(s_j) = e^{i(\theta_k - \theta_l)}\hat{A}_{kl}(0)$, which means that the measure A_{kl} is concentrated on the set $(\theta_k - \theta_l)/\Im s_j + (2\pi/\Im s_j)\mathbf{Z}$, $k, l = 1, \dots, N$ [7, Section 2.1]. Hence $\det(\mathbf{I} - \hat{\mathbf{A}}(mi\Im s_j)) = 0$, $m \in \mathbf{Z}$, i.e. $mi\Im s_j \in \mathscr{Z}$ for all $m \in \mathbf{Z}$, which contradicts the assumption that $\mathscr{Z}$ is a finite set. Thus $\Re s_j > 0$ for all j.

Further, suppose that $\mathsf{E}_{\boldsymbol{\pi}} S_1 < 0$ is finite. Then $\lambda'(0) = \mathsf{E}_{\boldsymbol{\pi}} S_1 < 0$ [8] and hence $\lambda(\xi) < 1$ for sufficiently small $\xi > 0$. Let $s_k \in \mathscr{Z}$. Then $(|\hat{A}_{ij}(s_k)|) \le \hat{\mathbf{A}}(\Re s_k)$ and hence $\lambda(\Re s_k) \ge 1$ [4, Theorem 8.1.18]. By continuity, there exists $q \in [0, \Re s_k]$ such that $\lambda(q) = 1$. The function $\lambda(\xi)$ is strictly convex [8, Theorem 2], which implies the uniqueness of q. To prove that the multiplicity of q is equal to 1, assume the contrary, i.e. $\det(\mathbf{I} - \hat{\mathbf{A}}(s)) = (s - q)g(s)$, where $g(q) = 0$. Choose positive left and right eigenvectors $\mathbf{l} = (l_1, \dots, l_N)$ and $\mathbf{r} = (r_1, \dots, r_N)^T$ corresponding to the eigenvalue 1 of $\hat{\mathbf{A}}(q)$ in such a way that $\mathbf{lr} = 1$; the superscript T denotes transposition of matrices. We have $\lambda'(q) = \mathbf{l}\hat{\mathbf{A}}'(q)\mathbf{r}$; this is essentially the same as $\lambda'(0) = \mathsf{E}_{\boldsymbol{\pi}} S_1 = \boldsymbol{\pi}\hat{\mathbf{A}}'(0)\mathbf{1}$ with $\mathbf{1} := (1, \dots, 1)^T$ in [8]. Also, we have $0 = \det(\mathbf{I} - \hat{\mathbf{A}}(s))'|_{s=q} = -c\mathbf{l}\hat{\mathbf{A}}'(q)\mathbf{r}$, where $c > 0$ [11, the proof of Lemma 9]. It follows that $\lambda'(q) = 0$ and hence the strictly convex function $\lambda(\xi)$ attains its minimum $\lambda(q) = 1$ at $\xi = q$, which contradicts the existence of $\xi > 0$ such that $\lambda(\xi) < 1$. Hence the multiplicity of q must be equal to 1.

Now suppose that $\mathsf{E}_{\boldsymbol{\pi}} S_1 = -\infty$. Let $Y_m(i,j) := X_m(i,j)$ if $X_m(i,j) > a$ and $Y_m(i,j) := a$ if $X_m(i,j) \le a$, where $a \in (-\infty, 0)$. We have $\mathsf{E}_{\boldsymbol{\pi}} Y_1(\kappa_0, \kappa_1)$ is finite and negative for sufficiently large $|a|$ and

$$\mathsf{E}\exp[\xi Y_1(i,j)] \ge \mathsf{E}\exp[\xi X_1(i,j)] \quad \text{for all } \xi \in (0, r).$$

Let $\lambda_1(\xi)$ be the spectral radius of the matrix $(p_{ij}\mathsf{E}\exp[\xi Y_1(i,j)])$. Then $\lambda_1(\xi) \geq \lambda(\xi)$ for all $\xi > 0$. By the above, $\lambda_1(\xi) < 1$ for sufficiently small $\xi > 0$. It follows that $\lambda(\xi) < 1$ for sufficiently small $\xi > 0$ and, by the above arguments, there exists a unique real root $q \in \mathscr{L}$ of multiplicity 1. For the sake of definiteness, we put $s_1 := q$.

Finally, repeating the reasoning at the beginning of the proof, we establish that $\Re s_j > q$ for all $j \geq 2$. The proof of Lemma 1 is complete.

We have (see Arndt [2])

$$\mathbf{I} - (\mathsf{P}_i(\kappa_{\eta(0)} = j)) + \left(\int_{0+}^{\infty} e^{sx} d_x \mathsf{P}_i(\kappa_{\eta(x)} = j)\right) = \hat{\mathbf{W}}(s) = [\hat{\mathbf{A}}_+(s)]^{-1}\hat{\mathbf{A}}_+(0). \quad (4)$$

In other terms,

$$\hat{\mathbf{W}}(s) = \{[\hat{\mathbf{A}}_-(s)]^{-1}[\mathbf{I} - \hat{\mathbf{A}}(s)]\}^{-1}\hat{\mathbf{A}}_+(0) = [\mathbf{I} - \hat{\mathbf{A}}(s)]^{-1}\hat{\mathbf{A}}_-(s)\hat{\mathbf{A}}_+(0).$$

Let the coefficients $\mathbf{B}_{jk}$, $k = 1, \ldots, n_j$, be defined by the asymptotic expansion

$$\hat{\mathbf{W}}(s) := \sum_{k=1}^{n_j} \frac{(-1)^k \mathbf{B}_{jk}}{(s - s_j)^k} + o\left(\frac{1}{s - s_j}\right) \quad \text{as } s \to s_j, \quad (5)$$

provided $\int_{\mathbf{R}} |x|^{n_j} e^{\Re s_j x}\mathbf{A}(dx) < \infty$. This inequality is automatically fulfilled if $\Re s_j < r$. Denote by $\mathscr{E}_j$ the complex-valued measure with density $\mathbf{1}_{(0,\infty)}(x)e^{-s_j x}$, $\mathbf{1}_A(x)$ being the indicator of A. Its Laplace transform is equal to $1/(s_j - s)$, $\Re(s - s_j) < 0$. The desired expansion for $\mathbf{W}$ will be of the form

$$\mathbf{W} = \sum_{j=1}^{l} \sum_{k=1}^{n_j} \mathbf{B}_{jk}\mathscr{E}_j^{k*} + \mathbf{\Delta}, \quad (6)$$

where the remainder $\mathbf{\Delta}$ will possess, roughly speaking, the same moments as the underlying matrix $\mathbf{A}$. If $\mathscr{L} \neq \varnothing$, then the main contribution to the asymptotics of $\mathbf{W}$ will be given by the term $\mathbf{B}_{11}\mathscr{E}_1$, corresponding to the root $s_1 = q$ of (3) since, by Lemma 1, $\Re s_j > q$, $j > 1$. Therefore, it is appropriate to calculate the matrix $\mathbf{B}_{11}$ in explicit form.

LEMMA 2. *Let* $\det(\mathbf{I} - \hat{\mathbf{A}}(q)) = 0$. *Choose positive left and right eigenvectors* $\mathbf{l} = (l_1, \ldots, l_N)$ *and* $\mathbf{r} = (r_1, \ldots, r_n)^T$ *corresponding to the eigenvalue* 1 *of* $\hat{\mathbf{A}}(q)$ *in such a way that* $\mathbf{lr} = 1$. *Then*

$$\mathbf{B}_{11} = \frac{\mathbf{rl}\hat{\mathbf{A}}_-(q)\hat{\mathbf{A}}_+(0)}{\mathbf{l}\hat{\mathbf{A}}'(q)\mathbf{r}}.$$

Proof. The function $\det(\mathbf{I} - \hat{\mathbf{A}}(s))$ is a linear combination of products of N factors. These factors are the Laplace transforms of elements of the matrix $\delta_0\mathbf{I} - \mathbf{A} \in S(r', r)$, where $r' \in (0, q)$. Consequently, $\det(\mathbf{I} - \hat{\mathbf{A}}(s))$ is the Laplace transform, say $\hat{\alpha}(s)$, of some measure α in $S(r', r)$. As $s \to q$, we have ($\hat{\mathbf{M}}(s)$ being the adjugate matrix of $\mathbf{I} - \hat{\mathbf{A}}(s)$)

$$[\mathbf{I} - \hat{\mathbf{A}}(s)]^{-1} = \frac{\hat{\mathbf{M}}(s)}{\hat{\alpha}(s)/(s-q)} \frac{1}{s-q} = \frac{\hat{\mathbf{M}}(q)}{\hat{\alpha}'(q)} \frac{1}{s-q} + o\left(\frac{1}{s-q}\right),$$

whence

$$[\mathbf{I} - \hat{\mathbf{A}}(s)]^{-1}\hat{\mathbf{A}}_-(s)\hat{\mathbf{A}}_+(0) = \frac{\hat{\mathbf{M}}(q)\hat{\mathbf{A}}_-(q)\hat{\mathbf{A}}_+(0)}{\hat{\alpha}'(q)} \frac{1}{s-q} + o\left(\frac{1}{s-q}\right).$$

Thus, $\mathbf{B}_{11} = -\hat{\mathbf{M}}(q)\hat{\mathbf{A}}_-(q)\hat{\mathbf{A}}_+(0)/\hat{\alpha}'(q)$. By [11, Lemma 9],

$$\frac{\hat{\mathbf{M}}(q)}{\hat{\alpha}'(q)} = -\frac{\mathbf{rl}}{\mathbf{l}\hat{\mathbf{A}}'(q)\mathbf{r}},$$

which completes the proof of the lemma.

Theorem 4. *Let $\varphi(x)$, $x \in \mathbf{R}$, be a submultiplicative function such that $\varphi(x) \equiv 1$ for $x < 0$, $r := r_+(\varphi) > 0$ and the function $\varphi(x)/\exp(rx)$, $x \geq 0$, is non-decreasing. Suppose that $\hat{\mathbf{A}}(r) < \infty$. Assume that the spectral radius of the matrix $(\mathbf{A}^{m*})_s^{\wedge}(r)$ is less than 1 for some integer $m \geq 1$. Let $\mathscr{Z} = \{s_1, \ldots, s_l\}$ be the set of the roots of* (3) *lying in the strip $\{s \in \mathbf{C} : 0 \leq \Re s \leq r\}$ and having multiplicities n_j, $j = 1, \ldots, l$. Denote by $\mathfrak{N}$ the maximal multiplicity of those roots which lie on $\{\Re s = r\}$ ($\mathfrak{N} = 0$ means that there are no such roots on this line). Suppose that $\int_0^\infty (1+x)^{2\mathfrak{N}}\varphi(x)\mathbf{A}(dx) < \infty$. Then the matrix $\mathbf{W}$ admits the representation* (6), *where the remainder $\boldsymbol{\Delta}$ satisfies the inequality $\int_0^\infty \varphi(x)|\boldsymbol{\Delta}|(dx) < \infty$.*

Proof. We form the following submultiplicative functions $\varphi_k(x)$: $\varphi_k(x) := (1+x)^k\varphi(x)$ for $x \geq 0$ and $\varphi_k(x) := \exp(r'x)$ for $x < 0$, where $r' \in (0, q)$ and $0 \leq k \leq 2\mathfrak{N}$. Obviously, $r_+(\varphi_k) = r$ and $r_-(\varphi_k) = r'$ for all $k = 0, \ldots, 2\mathfrak{N}$. Moreover, $S(\varphi_k) \subset S(\varphi_{k-1})$, $k \geq 1$.

Choose $a > r$ and put $p = \sum_{j=1}^l n_j$. Consider the function

$$d(s) := \frac{(s-a)^p \det(\mathbf{I} - \hat{\mathbf{A}}(s))}{\prod_{j=1}^l (s-s_j)^{n_j}} = \frac{(s-a)^p\hat{\alpha}(s)}{\prod_{j=1}^l (s-s_j)^{n_j}}.$$

Lemma 3. *Under the assumptions of Theorem* 4, *the function $d(s)$ is the Laplace transform of some measure $D \in S(\varphi_{\mathfrak{N}})$.*

PROOF OF LEMMA 3. The function $\det(\mathbf{I} - \hat{\mathbf{A}}(s))$ is a linear combination of products of N factors. These factors are the Laplace transforms of elements of the matrix $\delta_0 \mathbf{I} - \mathbf{A} \in S(\varphi_{2\mathfrak{N}})$. Consequently, $\det(\mathbf{I} - \hat{\mathbf{A}}(s))$ is the Laplace transform $\hat{\alpha}(s)$ of some measure $\alpha \in S(\varphi_{2\mathfrak{N}})$. Decomposing rational function into partial fractions, we have

$$d(s) = \left[1 + \sum_{j=1}^{l} \sum_{k=1}^{n_j} \frac{C_{jk}}{(s - s_j)^k}\right] \hat{\alpha}(s), \tag{7}$$

where C_{jk} are constants. Consider the functions $f_{jk}(s) := \hat{\alpha}(s)/(s - s_j)^k$, $k = 1, \dots, n_j$, $j = 1, \dots, l$. We shall establish that if $\Re s_j < r$, then $f_{jk}(s)$ is the Laplace transform of some measure belonging to $S(\varphi_{2\mathfrak{N}})$, and if $\Re s_j = r$, then $f_{jk}(s)$ is the Laplace transform of some measure belonging to $S(\varphi_{2\mathfrak{N}-k})$.

Let $v \in S(\varphi_m)$. If $\Re s_j < r$, then by Theorem 2 $T(s_j)v \in S(\varphi_m)$, and if $\Re s_j = r$ and $m > 0$, then by Theorem 3 $T(s_j)v \in S(\varphi_{m-1})$. Therefore, $f_{jk}(s) = [T(s_j)^k \alpha]^\wedge(s)$, $k = 1, \dots, n_j$, $j = 1, \dots, l$, are the Laplace transforms of some measures belonging to $S(\varphi_{2\mathfrak{N}})$ or to $S(\varphi_{2\mathfrak{N}-k})$, depending upon whether $\Re s_j$ is less than or equal to r. Thus, by (7), $\alpha \in S(\varphi_{\mathfrak{N}})$. The proof of Lemma 3 is complete.

LEMMA 4. *Let the conditions of Theorem* 4 *be satisfied. Then the element* $D \in S(\varphi_{\mathfrak{N}})$ *is invertible in* $S(\varphi_{\mathfrak{N}})$.

PROOF OF LEMMA 4. Let $\mathscr{M}$ be the space of maximal ideals of the Banach algebra $S(\varphi_{\mathfrak{N}})$. Each $M \in \mathscr{M}$ induces a homomorphism $h : S(\varphi_{\mathfrak{N}}) \to \mathbf{C}$ and M is the kernel of h. Denote by $v(M)$ the value of h at $v \in S(\varphi_{\mathfrak{N}})$, i.e. $v(M) := h(v)$, not the value of the measure v on the set M. An element $v \in S(\varphi_{\mathfrak{N}})$ has an inverse if and only if v does not belong to any maximal ideal $M \in \mathscr{M}$. In other words, v is invertible if and only if $v(M) \neq 0$ for all $M \in \mathscr{M}$.

The space $\mathscr{M}$ is split into two sets: $\mathscr{M}_1$ is the set of those maximal ideals which do not contain the collection $L(r', r)$ of all absolutely continuous measures from $S(\varphi_{\mathfrak{N}})$, and $\mathscr{M}_2 = \mathscr{M} \backslash \mathscr{M}_1$. If $M \in \mathscr{M}_1$, then the homomorphism induced by M is of the form $h(v) = \hat{v}(s_0)$, where $r' \le \Re s_0 \le r$. In this case, $M = \{v \in S(\varphi_{\mathfrak{N}}) : \hat{v}(s_0) = 0\}$ [3, Chapter IV, Section 4]. If $M \in \mathscr{M}_2$, then $v(M) = 0$ for all $v \in L(r', r)$.

We now show that $D(M) \neq 0$ for each $M \in \mathscr{M}$, thus establishing the existence of $D^{-1} \in S(\varphi_{\mathfrak{N}})$. Actually, if $M \in \mathscr{M}_1$, then, for some $s_0 \in \Pi(r', r)$, we have $D(M) = \hat{D}(s_0) \neq 0$. Now let $M \in \mathscr{M}_2$. By the multiplicative property of the functional $v \mapsto v(M)$, $v \in S(\varphi_{\Re})$, we have $\mathbf{A}(M)^m = \mathbf{A}^{m*}(M) = (\mathbf{A}^{m*})_s(M)$. Let $\mathbf{\Theta} = (\Theta_{ij}) := (\mathbf{A}^{m*})_s$. By Theorem 1,

$$|\Theta_{ij}(M)| = \left|\int_{\mathbf{R}} \chi(x, \Theta_{ij}) \exp(\beta x)\Theta_{ij}(dx)\right| \le \int_{\mathbf{R}} \exp(\beta x)\Theta_{ij}(dx)$$

for some $\beta \in [r', r]$. It follows that the spectral radius of $\mathbf{\Theta}(M)$ does not exceed that of $\hat{\mathbf{\Theta}}(\beta) = (\mathbf{A}^{m*})_s^{\wedge}(\beta)$. By [6, Corollaries 1 and 2], the function $\varrho[\hat{\mathbf{\Theta}}(t)]$, $t \in [0, r]$, is convex. By assumption, $\varrho[\hat{\mathbf{\Theta}}(r)] < 1$, Moreover, $\varrho[\hat{\mathbf{\Theta}}(0)] \le 1$ which is implied by $\hat{\mathbf{\Theta}}(0) \le \hat{\mathbf{A}}(0)^m$ and by the fact that $\hat{\mathbf{A}}(0)^m$ is stochastic (whence $\varrho[\hat{\mathbf{A}}(0)^m] = 1$). Consequently, $\varrho[\hat{\mathbf{\Theta}}(\beta)] < 1$. Thus the spectral radius of $\mathbf{A}(M)^m$ is less than 1 and the spectral radius of $\mathbf{A}(M)$, being equal to the m-th root of that of $\mathbf{A}(M)^m$, is also less than 1. Since $T(s_j)^k\alpha \in L(r', r)$ for all j, k, (7) implies

$$D(M) = \alpha(M) = \det(\mathbf{I} - \mathbf{A}(M)) \ne 0.$$

So $D(M) \ne 0$ for all $M \in \mathscr{M}$. This means that there exists $D^{-1} \in S(\varphi_{\Re})$ and the function $1/d(s)$, $r' \le \Re s \le r$, is the Laplace transform of D^{-1}. The proof of Lemma 4 is complete.

Consider the matrix

$$\mathbf{q}(s) := \frac{\prod_{j=1}^{l}(s - s_j)^{n_j}}{(s - a)^p}[\mathbf{I} - \hat{\mathbf{A}}(s)]^{-1}, \quad s \in \Pi(r', r)\backslash\mathscr{Z}.$$

Lemma 5. *Let the conditions of Theorem* 4 *be satisfied. Then* $\mathbf{q}(s)$ *is the Laplace transform of some matrix* $\mathbf{Q} \in S(\varphi_{\Re})$.

Proof of Lemma 5. Denote by $\hat{\mathbf{M}}(s)$ the adjugate matrix of $\mathbf{I} - \hat{\mathbf{A}}(s)$. Then $\mathbf{q}(s) = [1/d(s)]\hat{\mathbf{M}}(s)$. Consequently, by Lemma 4, $\mathbf{q}(s) = \hat{\mathbf{Q}}(s)$, where $\mathbf{Q} = D^{-1} * \mathbf{M} \in S(\varphi_{\Re})$ (the elements of $\mathbf{Q}$ are the convolutions of D^{-1} with the corresponding elements of $\mathbf{M}$). The proof of Lemma 5 is complete.

We return to the proof of Theorem 4. We have

$$\hat{\mathbf{W}}(s) = [\mathbf{I} - \hat{\mathbf{A}}(s)]^{-1}\hat{\mathbf{A}}_-(s)\hat{\mathbf{A}}_+(0) = \frac{(s - a)^p}{\prod_{j=1}^{l}(s - s_j)^{n_j}}\hat{\mathbf{Q}}(s)\hat{\mathbf{A}}_-(s)\hat{\mathbf{A}}_+(0)$$

$$= \left[1 + \sum_{j=1}^{l}\sum_{k=1}^{n_j}\frac{C_{jk}}{(s - s_j)^k}\right]\hat{\mathbf{Q}}(s)\hat{\mathbf{A}}_-(s)\hat{\mathbf{A}}_+(0). \tag{8}$$

Define $\mathbf{V} := \mathbf{Q} * \mathbf{A}_- \hat{\mathbf{A}}_+(0)$. Since the elements of $\mathbf{A}_-$ are finite measures concentrated on $(-\infty, 0]$, $\mathbf{A}_- \in S(\varphi_{\mathfrak{R}})$ and hence $\mathbf{V} \in S(\varphi_{\mathfrak{R}})$. Perform the following calculations:

$$\frac{\hat{\mathbf{V}}(s)}{(s-s_j)^k} = \frac{\hat{\mathbf{V}}(s_j)}{(s-s_j)^k} + \frac{\hat{\mathbf{V}}(s) - \hat{\mathbf{V}}(s_j)}{(s-s_j)^k} = \sum_{i=0}^{k-1} \frac{\mathbf{v}_{i,j}(s_j)}{(s-s_j)^{k-i}} + \mathbf{v}_{k,j}(s), \tag{9}$$

where

$$\mathbf{v}_{0,j}(s) := \hat{\mathbf{V}}(s), \quad \mathbf{v}_{i,j}(s) := \frac{\mathbf{v}_{i-1,j}(s) - \mathbf{v}_{i-1,j}(s_j)}{s - s_j}, \quad i = 1, \dots, k.$$

As before, applying step by step either Theorem 2 or Theorem 3, we establish that the matrix measure $\mathbf{V}_{i,j} := T(s_j)^i \mathbf{V}$ with Laplace transform $\mathbf{v}_{i,j}(s)$ belongs to $S(\varphi_{\mathfrak{R}})$ or $S(\varphi_{\mathfrak{R}-k})$, depending on whether $\Re s_j$ is less than or equal to r. Substituting (9) into (8) and collecting similar terms, we obtain, by the uniqueness of the expansion (5), that

$$\left[1 + \sum_{j=1}^{l} \sum_{k=1}^{n_j} \frac{C_{jk}}{(s-s_j)^k}\right] \hat{\mathbf{V}}(s) = \hat{\mathbf{V}}(s) + \sum_{j=1}^{l} \sum_{k=1}^{n_j} \frac{(-1)^k \mathbf{B}_{jk}}{(s-s_j)^k} + \sum_{j=1}^{l} \sum_{k=1}^{n_j} C_{jk} \mathbf{v}_{k,j}(s).$$

Put $\boldsymbol{\Delta} := \mathbf{V} + \sum_{j=1}^{l} \sum_{k=1}^{n_j} C_{jk} T(s_j)^k \mathbf{V}$. Then $\boldsymbol{\Delta} \in S(\varphi)$ and

$$\hat{\mathbf{W}}(s) = \sum_{j=1}^{l} \sum_{k=1}^{n_j} \frac{(-1)^k \mathbf{B}_{jk}}{(s-s_j)^k} + \hat{\boldsymbol{\Delta}}(s), \quad s \in \Pi(r', r) \backslash \mathscr{Z}.$$

Passing over in this equality from the Laplace transforms to the corresponding measures, we obtain the representation (6). Theorem 4 is proved.

Let $\mathbf{1}$ denote the $N \times 1$ column vector with unit elements. It follows from (2) that $\hat{\mathbf{A}}_+(0)\mathbf{1} = (\mathsf{P}_i(M_\infty = 0))$. Summing over $j \in \mathscr{N}$ the probabilities $\mathsf{P}_i(\kappa_{\eta(x)} = j)$, we obtain the following result about the asymptotic behaviour of the $\mathsf{P}_i(M_\infty > x)$.

Theorem 5. *Under the assumptions of Theorem* 4, *we have*

$$\left(\frac{\mathsf{P}_i(M_\infty > x)}{\tau(x)}\right) = \sum_{j=1}^{l} \sum_{k=1}^{n_j} \mathbf{B}_{jk} \mathbf{1} \mathscr{E}_j^{k*}((x, \infty)) + \boldsymbol{\Delta}((x, \infty))\mathbf{1},$$

where $|\boldsymbol{\Delta}((x,\infty))\mathbf{1}| \le |\boldsymbol{\Delta}|((x,\infty))\mathbf{1} = o(1/\varphi(x))\mathbf{1}$ *as* $x \to \infty$.

If $\mathscr{Z} \neq \varnothing$, then there is no need to use Theorems 2 and 3 in the proof of Theorem 4. It follows that, in this case, the conditions $\varphi(x)/\exp(rx)\uparrow$ and

$\varrho[\mathbf{A}^{m*})_s^{\wedge}(r)] < 1$ become superfluous. Thus, Theorem 4 of the present paper generalizes the sufficiency part of [12, Theorem 5].

References

[1] Alsmeyer, G. and Sgibnev, M., On the tail behaviour of the supremum of a random walk defined on a Markov chain, Yokohama Math. J. **46** (1999), 139–159.

[2] Arndt, K., Asymptotic properties of the distribution of the supremum of a random walk on a Markov chain, Theory Probab. Appl., **25** (1980), 309–324.

[3] Hille, E. and Phillips, R. S., Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ. **31**, American Mathematical Society, Providence, RI, 1957.

[4] Horn, R. A. and Johnson, C. R., Matrix Analysis, Cambridge University Press, Cambridge, 1986.

[5] Jelenković, P. R. and Lazar, A. A., Subexponential asymptotics of a Markov-modulated random walk with queueing applications, J. Appl. Probab., **35** (1998), 325–347.

[6] Kingman, J. F. C., A convexity property of positive matrices, Quart. J. Math., Oxford Series 2, **12** (1961), 283–284.

[7] Lukacs, E., Characteristic functions, Griffin, London, 1970.

[8] Miller, H. D., A convexity property in the theory of random variables defined on a finite Markov chain, Ann. Math. Statist., **32** (1961), 1260–1270.

[9] Rogozin, B. A. and Sgibnev, M. S., Banach algebras of measures on the line, Siberian Math. J., **21** (1980), 265–273.

[10] Sgibnev, M. S., An asymptotic expansion for the distribution of the supremum of a random walk, Studia Mathematica, **140** (2000), 41–55.

[11] Sgibnev, M. S., Systems of renewal equations on the line, J. Math. Sci. Univ. Tokyo, **10** (2003), 495–517.

[12] Sgibnev, M. S., Submultiplicative moments of the supremum of a Markov modulated random walk, Yokohama Math. J., **53** (2006), 1–8.

Sobolev Institute of Mathematics
Siberian Branch of the Russian Academy of Sciences
Novosibirsk 90, 630090 Russia
E-mail: sgibnev@math.nsc.ru