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# BEHAVIOR OF SOLUTIONS TO LINEAR AND SEMILINEAR PARABOLIC PSEUDO-DIFFERENTIAL EQUATIONS

By

Tomoyuki Kakehi and Kensuke Sakai

## 1. Introduction

In this paper, we investigate detailed properties of nonnegative solutions to the semilinear parabolic pseudo-differential equation

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha} u = u^p, & t \in (0, \infty),\ x \in \mathbf{R}^n, \\ u(0, x) = u_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$ and $1 < p$. Here $(-\Delta)^{\alpha}$ is defined by

$$(-\Delta)^{\alpha} v(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\xi \cdot x} |\xi|^{2\alpha} \hat{v}(\xi)\, d\xi. \tag{1.2}$$

Let us first recall Fujita's result [Fu] on the semilinear heat equation

$$\begin{cases} \partial_t u - \Delta u = u^p, & t \in (0, \infty),\ x \in \mathbf{R}^n \\ u(0, x) = u_0(x) \geq 0, & x \in \mathbf{R}^n. \end{cases} \tag{1.3}$$

In [Fu], he shows that the nonnegative solution to (1.3) blows up in finite time if $1 < p < 1 + \frac{2}{n}$ while there exists a time global solution if $1 + \frac{2}{n} < p$ and if the initial data $u_0$ is sufficiently small. In addition, the blow-up for the critical case $(p = 1 + \frac{2}{n})$ is shown by several papers (see for example, Weissler [We]).

On the other hand, Sugitani [Su] studies the blowing-up of a solution to the integral equation arising from (1.1). According to [Su], if $1 < p \leq 1 + \frac{2\alpha}{n}$, then the solution for non-negative and nontrivial initial data blows up in finite time. However, he does not mention the relationship between (1.1) and the corresponding integral equation. Moreover, equation (1.1) has not been studied intensively since Sugitani's work. This is mainly due to the fact that the fun-

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damental solution of the linearized equation $\partial_t u + (-\Delta)^\alpha u = 0$ is not given explicitly.

Taking into consideration the above situation, in this paper we will consider the following three problems.

(A) The behavior of the fundamental solution of the linear equation $\partial_t u + (-\Delta)^\alpha u = 0$.

(B) The regularity of the solution to (1.1).

(C) The life span of the solution.

First, let us discuss the problem (A). Obviously, for the investigation of a nonlinear equation, it is quite important to understand properties of the fundamental solution of the corresponding linearized equation. In fact, since the fundamental solution of the heat equation is explicitly given by the Gauss kernel, a lot of results on nonlinear heat equations are obtained by the study of the Gauss kernel. However, in the case of (1.1) it is quite difficult to give an explicit expression of the corresponding fundamental solution. Instead, in this paper, we will provide its asymptotic expansion formula.

Let $W^{(\alpha)}(t,x)$ be the fundamental solution to the linear parabolic pseudo-differential equation

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0. \tag{1.4}$$

Then, our first main theorem is stated as follows.

THEOREM A. *For each fixed* $t > 0$, $W^{(\alpha)}(t,x)$ *has the asymptotic expansion*

$$W^{(\alpha)}(t,x) \sim \sum_{j=1}^{\infty} a_j t^j |x|^{-n-2j\alpha}, \quad as\ |x| \to +\infty, \tag{1.5}$$

*where the coefficients* $a_j$ $(j = 1, 2, \dots,)$ *are given by*

$$a_j = \frac{(-1)^{j-1} 2^{2\alpha j}}{j! \pi^{n/2+1}} \sin(\pi\alpha j) \Gamma(1+\alpha j) \Gamma\left(\frac{n}{2} + \alpha j\right). \tag{1.6}$$

As is well known, the fundamental solution of the heat equation decays exponentially. Therefore, we have to stress here the fact that the fundamental solution $W^{(\alpha)}(t,x)$ to the equation (1.4) decays polynomially. As we see later, this difference affects the behavior of the solution to (1.1). We also remark that from the point of view of probability theory it is important to give an explicit asymptotic expansion formula of $W^{(\alpha)}(t,x)$. (See Remark 2.1 in Section 2.)

Next, let us go into the problem (B). The starting point is the integral equation arising from (1.1)

$$u(t,x) = W^{(\alpha)}(t,\cdot) * u_0(x) + \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s, x-y) u(s,y)^p \, dyds. \tag{1.7}$$

It is relatively easy to prove by contraction argument that (1.7) has a unique solution $u \in C([0,T), L^1 \cap L^\infty)$ for sufficiently small $T > 0$ if $u_0 \in L^1 \cap L^\infty$. But the problem is "In what sense does the above solution satisfy the original pseudo-differential equation (1.1)?" In other words, to what extent does the solution to (1.7) gain its regularity? We will answer this question by the following.

Theorem B. *We assume that* $\frac{1}{2} < \alpha < 1$. *Let* $u \in C([0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$ *satisfy the integral equation* (1.7) *in* $[0,T) \times \mathbf{R}^n$. *Then, the following* (i), (ii), *and* (iii) *hold.*

(i) *u is of class* $C^1$ *with respect to* $t \in (0,T)$ *and of class* $C^2$ *with respect to* $x \in \mathbf{R}^n$. *Moreover,* $u \in C((0,T), H_1^2(\mathbf{R}^n) \cap H_\infty^2(\mathbf{R}^n)) \cap C^1((0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$.

(ii) $(-\Delta)^\alpha u \in C((0,T) \times \mathbf{R}^n)$ *and* $(-\Delta)^\alpha u \in C((0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$.

(iii) *u satisfies the semilinear parabolic pseudo-differential equation*

$$\partial_t u + (-\Delta)^\alpha u = u^p,$$

*as an equality in* $C((0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$.

Here we define a Sobolev space $H_q^s = H_q^s(\mathbf{R}^n)$ ($s \geq 0$ and $1 \leq q \leq \infty$)) by

$$H_q^s = H_q^s(\mathbf{R}^n) = \{ f \in L^q(\mathbf{R}^n) \mid (1-\Delta)^{s/2} f \in L^q(\mathbf{R}^n) \}. \tag{1.8}$$

Finally, let us deal with the problem (C). We first explain the case of semilinear heat equations. Let $T(\lambda)$ be the life span of the nonnegative solution to the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u = u^p, & t > 0, \ x \in \mathbf{R}^n, \\ u(0,x) = \lambda \psi(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.9}$$

where $\psi(x) \geq 0$ and $\lambda > 0$ is a small parameter. Then, we see easily that $T(\lambda) \to \infty$ as $\lambda \to 0$. However, the growth order of $T(\lambda)$ depends on the decay rate of $\psi$ as $|x| \to \infty$. For example, the following is a part of the results shown by Lee and Ni [LN].

Theorem (Lee and Ni [LN]). *Let* $a > 0$. *Assume that there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1|x|^{-a} \leq \psi(x) \leq C_2|x|^{-a}, \quad \text{for sufficiently large } |x|. \tag{1.10}$$

*If* $1 < p < p_* = 1 + \frac{2}{n}$ *and* $a \neq n$, *then*

$$T(\lambda) \sim \lambda^{-1/(1/(p-1)-(1/2)\min\{a,n\})}, \quad \text{as } \lambda \to 0. \tag{1.11}$$

*The third purpose of this paper is to calculate the growth order of the life span* $T(\lambda)$ *of the solution to the following Cauchy problem.*

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = u^p, & t > 0, x \in \mathbf{R}^n, \\ u(0, x) = \lambda\psi(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.12}$$

*where* $\psi(x) \geq 0$ *and* $\lambda > 0$ *is a small parameter.*

*Our result is stated as follows.*

Theorem C. *Let* $p_*(\alpha) = 1 + \frac{2\alpha}{n}$. *We assume that* $\frac{1}{2} < \alpha < 1$. *In addition, we assume that for some constant* $c > 0$

$$0 \leq \psi(x) \leq c(1 + |x|)^{-n-2\alpha}. \tag{1.13}$$

(i) *If* $1 < p < p_*(\alpha)$, *then*

$$T(\lambda) \sim \lambda^{-2\alpha(p-1)/n(p_*(\alpha)-p)}, \quad \text{as } \lambda \to 0. \tag{1.14}$$

(ii) *If* $p = p_*(\alpha)$, *then*

$$\log T(\lambda) \sim \lambda^{-2\alpha/n}, \quad \text{as } \lambda \to 0. \tag{1.15}$$

(iii) *If* $p_*(\alpha) < p$, *then* $T(\lambda) = \infty$ *for sufficiently small* $\lambda > 0$.

*If we put* $\alpha = 1$ *in* (1.14), *then the order of the life span coincides with that in* (1.11). *In this sense, Theorem C is considered to be a generalization of the theorem by Lee and Ni.*

Our paper is organized as follows. In Section 2, we will give the asymptotic expansion of the fundamental solution $W^{(\alpha)}(t, x)$. The key idea in this section is to use the method of oscillatory integrals. Section 3 is devoted to prove the positivity of $W^{(\alpha)}(t, x)$. In this section, we also prove the comparison theorem. Then, as an application of the results in Section 2, we will show Theorem B (regularity theorem) in Section 4. We will deal with the problem (C) (the problem of life span) in both Section 5 and 6. In Section 5, we will give lower bounds of the life span, and in Section 6, we will give its upper bounds. Finally in Section 7, we give some generalization of our results.

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## 2. Asymptotic Property of $W^{(\alpha)}(t,x)$

In this section, we study asymptotic properties of the fundamental solution $W^{(\alpha)}(t,x)$ of the parabolic pseudodifferential equation

$$\partial_t u(t,x) + (-\Delta)^{\alpha} u(t,x) = 0, \tag{2.1}$$

with the initial data $u(0,x) = u_0(x)$. We see easily that $W^{(\alpha)}(t,x)$ is given by the Fourier integral

$$W^{(\alpha)}(t,x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{-t|\xi|^{2\alpha}} e^{i\xi\cdot x}\, d\xi. \tag{2.2}$$

Let

$$w^{(\alpha)}(x) = W^{\alpha}(1,x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{-|\xi|^{2\alpha}} e^{i\xi\cdot x}\, d\xi. \tag{2.3}$$

Then, we have

$$W^{(\alpha)}(t,x) = t^{-n/2\alpha} w^{(\alpha)}(t^{-1/2\alpha} x) \tag{2.4}$$

So, it suffices to calculate the asymptotic expansion of $w^{(\alpha)}(x)$ as $|x| \to \infty$.

The main purpose of this section is to prove the following.

Theorem 2.1.

$$w^{(\alpha)}(x) \sim \sum_{j=1}^{\infty} a_j |x|^{-n-2j\alpha}, \quad (|x| \to +\infty) \tag{2.5}$$

*where the constants* $a_j$ $(j = 1, 2, \dots,)$ *are given by*

$$a_j = \frac{(-1)^{j-1} 2^{2\alpha j}}{j!\pi^{n/2+1}} \sin(\pi\alpha j)\Gamma(1+\alpha j)\Gamma\left(\frac{n}{2} + \alpha j\right). \tag{2.6}$$

Let

$$w_\varepsilon^\alpha(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{-|\xi|^{2\alpha}} \widehat{\chi_\varepsilon}(\xi)\, d\xi, \tag{2.7}$$

where

$$\chi_\varepsilon(x) = \varepsilon^{-n} \chi\left(\frac{x}{\varepsilon}\right). \tag{2.8}$$

Here we take a compactly supported smooth function $\chi$ on $\mathbf{R}^n$ such that

$$\begin{gathered} \chi(x) = \chi(|x|) = \begin{cases} const. & (|x| \leq 1) \\ 0 & (|x| \geq 2) \end{cases} \\ \chi(x) \geq 0 \\ \int_{\mathbf{R}^n} \chi(x)\, dx = 1. \end{gathered} \tag{2.9}$$

Let us rewrite $w_\varepsilon^\alpha$ using the Taylor series expansion of $e^\lambda$ $(\lambda = -|\xi|^{2\alpha})$ as follows.

$$w_\varepsilon^\alpha(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \widehat{\chi_\varepsilon}(\xi) \left( \sum_{j=0}^{N} \frac{(-1)^j}{j!} |\xi|^{2\alpha j} + |\xi|^{2\alpha(N+1)} g_N(|\xi|^{2\alpha}) \right) d\xi, \tag{2.10}$$

$$g_N(\lambda) = \frac{(-1)^{N+1}}{N!} \int_0^1 (1-s)^N e^{-\lambda s}\, ds. \tag{2.11}$$

Here $N$ is an arbitrary fixed positive integer.

Then, we have

$$w_\varepsilon^\alpha(x) = \sum_{j=0}^{N} \frac{(-1)^j}{j!} (2\pi)^{-n} \int_{\mathbf{R}^n} |\xi|^{2\alpha j} \mathscr{F}_y[\chi_\varepsilon(y+x)](\xi)\, d\xi + R_\varepsilon^{(N)}(x), \tag{2.12}$$

where $\mathscr{F}_y[\chi_\varepsilon(y+x)]$ denotes the Fourier transform of $\chi_\varepsilon(y+x)$ with respect to $y$ and where the remainder term $R_\varepsilon^{(N)}$ is given by

$$R_\varepsilon^{(N)}(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \widehat{\chi_\varepsilon}(\xi) |\xi|^{2\alpha(N+1)} g_N(|\xi|^{2\alpha})\, d\xi \tag{2.13}$$

The following lemma plays an essential role in the computation of the asymptotic expansion of $w_\varepsilon^\alpha$.

Lemma 2.1. *Let $\varphi$ be a Schwartz class function on $\mathbf{R}^n$ which vanishes near the origin. Then, we have*

$$\int_{\mathbf{R}^n} |\xi|^{\beta} \hat{\varphi}(\xi)\, d\xi = C_\beta \int_{\mathbf{R}^n} |x|^{-n-\beta} \varphi(x)\, dx, \quad \textit{for } \beta > 0, \tag{2.14}$$

*where the above constant $C_\beta$ is given by*

$$C_\beta = -\pi^{n/2-1} 2^{n+\beta} \sin\left(\frac{\pi\beta}{2}\right) \Gamma\left(\frac{2+\beta}{2}\right) \Gamma\left(\frac{n+\beta}{2}\right) \tag{2.15}$$

(For the proof, see for example Helgason [Hel] page 134, Chapter I, Section 2, Lemma 2.34.)

Let us now assume that $x$ is away from the origin. We note that under the above asumption, $\chi_\varepsilon(y+x)$ vanishes near the origin as a function of $y$ if $2\varepsilon < |x|$. So, by the above lemma,

$$\int_{\mathbf{R}^n} |\xi|^{2\alpha j} \mathscr{F}_y[\chi_\varepsilon(y+x)](\xi)\, d\xi = C_{2\alpha j} \int_{\mathbf{R}^n} |y|^{-n-2\alpha j} \chi_\varepsilon(y+x)\, dy, \tag{2.16}$$

where $C_{2\alpha j}$ is the constant given in Lemma 2.1.

Next, let us admit the following lemma.

Lemma 2.2. *The limit $\lim_{\varepsilon\to 0} R_\varepsilon^{(N)}(x)$ exists for each $x(\neq 0)$. Moreover, there exists a constant $A_N$ such that*

$$\left| \lim_{\varepsilon\to 0} R_\varepsilon^{(N)}(x) \right| \leq A_N |x|^{-2\alpha(N+1)+2}, \quad \textit{for } |x| \gg 0. \tag{2.17}$$

We will prove this lemma later.

If we take the limit $\varepsilon \to 0$ in (2.12), then by making use of (2.16), we have

$$\lim_{\varepsilon\to 0} w_\varepsilon^\alpha(x) = \sum_{j=1}^{N} \frac{(-1)^j}{j!(2\pi)^n} C_{2\alpha j} \lim_{\varepsilon\to 0} \int_{\mathbf{R}^n} |y|^{-n-2\alpha j} \chi_\varepsilon(y+x)\, dy + \lim_{\varepsilon\to 0} R_\varepsilon^{(N)}(x) \tag{2.18}$$

$$= \sum_{j=1}^{N} \frac{(-1)^j}{j!(2\pi)^n} C_{2\alpha j} |x|^{-n-2\alpha j} + \lim_{\varepsilon\to 0} R_\varepsilon^{(N)}(x). \tag{2.19}$$

In the above computation, note that the right hand side of (2.16) equals 0 if $j = 0$. On the other hand, by (2.7), $\lim_{\varepsilon\to 0} w_\varepsilon^\alpha(x) = w^{(\alpha)}(x)$. Therefore, by

Lemma 2.2 we obtain

$$(2.20) \quad w^{(\alpha)}(x) = \sum_{j=1}^{N} \frac{(-1)^j}{j!(2\pi)^n} C_{2\alpha j}|x|^{-n-2\alpha j} + O(|x|^{-2\alpha(N+1)+2}), \quad (|x| \to +\infty),$$

which proves the assertion of Theorem 2.1.

Theorem 2.1 yields the following theorem on the asymptotic expansions of derivatives of $w^{(\alpha)}$.

THEOREM 2.2. *For each multi-index* $\gamma$, *the derivative* $\partial_x^\gamma w^{(\alpha)}$ *of* $w^{(\alpha)}$ *has the asymptotic expansion.*

$$(2.21) \qquad \partial_x^\gamma w^{(\alpha)}(x) \sim \sum_{j=1}^{\infty} a_j \partial_x^\gamma(|x|^{-n-2\alpha j}), \quad (|x| \to +\infty).$$

*Here* $a_j$ *in* (2.21) *is the constant given by* (2.6) *in Theorem* 2.1.

PROOF. It is easily seen from the expression (2.3) that $\partial_x^\gamma w^{(\alpha)}$ is a bounded function for each $\gamma$. So the above theorem is proved by applying the following lemma with respect to each variable repeatedly. □

LEMMA 2.3. *Let* $f$ *be a* $C^2$ *class function on* $\mathbf{R}$ *with the asymptotic expansion*

$$f(t) \sim \sum_{j=1}^{\infty} a_j t^{m-\alpha j}, \quad \text{as } |t| \to \infty,$$

*where* $\alpha > 0$ *and* $m \le 0$. *We assume that* $f''(t)$ *is a bounded function. Then* $f'(t)$ *has the asymptotic expansion*

$$f'(t) \sim \sum_{j=1}^{\infty} (m - \alpha j) a_j t^{m-\alpha j-1}, \quad \text{as } |t| \to \infty.$$

The above lemma is easily verified. So we omit the proof.

Similarly as in Theorem 2.1, we can prove the following.

THEOREM 2.3. *For* $\beta > 0$, $(-\Delta)^\beta w^{(\alpha)}$ *has the asymptotic expansion.*

$$(2.22) \qquad (-\Delta)^\beta w^{(\alpha)}(x) \sim \sum_{j=0}^{\infty} b_j |x|^{-n-2\alpha j-2\beta} \quad (|x| \to +\infty).$$

*Here the constant* $b_j$ *in* (2.22) *is given by*

$$b_j = \frac{(-1)^{j-1} 2^{2\alpha j+2\beta}}{j!\pi^{n/2+1}} \sin(\pi\alpha j + \pi\beta)\Gamma(1+\alpha j+\beta)\Gamma\left(\frac{n}{2}+\alpha j+\beta\right). \tag{2.23}$$

Summarizing the above results, we obtain

Corollary 2.1. *The fundamental solution* $W^{(\alpha)}(t,x)$ *and its derivatives have the following asymptotic expansions.*

$$\begin{aligned} W^{(\alpha)}(t,x) &\sim \sum_{j=1}^{\infty} a_j t^j |x|^{-n-2j\alpha}, \quad \text{as } |x| \to +\infty, \\ \partial_x^{\gamma} W^{(\alpha)}(t,x) &\sim \sum_{j=1}^{\infty} a_j t^j (\partial_x^{\gamma} |x|^{-n-2j\alpha}), \quad \text{as } |x| \to +\infty, \\ (-\Delta)^{\beta} W^{(\alpha)}(t,x) &\sim \sum_{j=0}^{\infty} b_j t^j |x|^{-n-2\alpha j-2\beta}, \quad (|x| \to +\infty), \end{aligned} \tag{2.24}$$

*where the coefficients* $a_j$ $(j=1,2,\dots,)$ *and* $b_j$ $(j=0,1,\dots,)$ *are given respectively by* (2.6) *and* (2.23).

Corollary 2.2.

$$\begin{aligned} w^{(\alpha)}(x) &= O(|x|^{-n-2\alpha}), \\ \partial_{x_j} w^{(\alpha)}(x) &= O(|x|^{-n-1-2\alpha}), \quad \text{for } j \ (1 \le j \le n), \\ \partial_{x_j}\partial_{x_k} w^{(\alpha)}(x) &= O(|x|^{-n-2-2\alpha}), \quad \text{for } j,k \ (1 \le j,k \le n), \\ (-\Delta)^{\beta} w^{(\alpha)}(x) &= O(|x|^{-n-2\beta}), \quad \text{for } \beta > 0, \text{ as } |x| \to \infty. \end{aligned} \tag{2.25}$$

*In particular,* $w^{(\alpha)}$, $\partial_{x_j} w^{(\alpha)}$, $\partial_{x_j}\partial_{x_k} w^{(\alpha)}$, *and* $(-\Delta)^{\beta} w^{(\alpha)}$ *are all integrable on* $\mathbf{R}^n$.

We will use this corollary to study the regularity of the solution to (1.1). Finally, we will prove Lemma 2.2 in the rest part of this section.

Proof of Lemma 2.2. Throughout the proof, we assume that $x$ is fixed and away from the origin, say for example $|x| \ge 1$.

The existence of the limit $\lim_{\varepsilon\to 0} R_\varepsilon^{(N)}(x)$ is obvious. (Take the limit $\varepsilon \to 0$ in both sides of (2.12).)

Let $\psi_1$ and $\psi_2$ be smooth functions on $\mathbf{R}^n$ such that

$$\psi_1(\xi) = \begin{cases} 1 & (|\xi| \le 1) \\ 0 & (|\xi| \ge 2), \end{cases}$$
$$0 \le \psi_1(\xi) \le 1, \tag{2.26}$$
$$\psi_2(\xi) = 1 - \psi_1(\xi).$$

Using above $\psi_1$ and $\psi_2$, we write $R_\varepsilon^{(N)}(x)$ as

$$\begin{aligned} R_\varepsilon^{(N)}(x) &= \int_{\mathbf{R}^n} e^{ix\cdot\xi}\widehat{\chi_\varepsilon}(\xi)|\xi|^{2\alpha(N+1)} g_N(|\xi|^{2\alpha})\psi_1(\xi)\,d\xi \\ &\quad + \int_{\mathbf{R}^n} e^{ix\cdot\xi}\widehat{\chi_\varepsilon}(\xi)|\xi|^{2\alpha(N+1)} g_N(|\xi|^{2\alpha})\psi_2(\xi)\,d\xi \\ &= I_\varepsilon^{(1)}(x;N) + I_\varepsilon^{(2)}(x;N). \end{aligned} \tag{2.27}$$

Step 1. First, we give the estimate of $I_\varepsilon^{(1)}(x;N)$. Let $l$ be the integer such that

$$2l \le 2\alpha(N+1) < 2l+2. \tag{2.28}$$

Then $|\xi|^{2\alpha(N+1)} g_N(|\xi|^{2\alpha})\psi_1(\xi)$ in the integrand of $I_\varepsilon^{(1)}(x;N)$ is a compactly supported function of class $C^{2l}$. Using the equality

$$\begin{aligned} e^{ix\cdot\xi}\widehat{\chi_\varepsilon}(\xi) &= (-\Delta_\xi)^l \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} \frac{\chi_\varepsilon(y)}{|x-y|^{2l}}\,dy \\ &= (-\Delta_\xi)^l \mathscr{F}_y\left[\frac{\chi_\varepsilon(x+y)}{|y|^{2l}}\right](\xi), \end{aligned} \tag{2.29}$$

we rewrite $I_\varepsilon^{(1)}(x;N)$ as

$$I_\varepsilon^{(1)}(x;N) = \int_{\mathbf{R}^n} \mathscr{F}_y\left[\frac{\chi_\varepsilon(x+y)}{|y|^{2l}}\right](\xi)(-\Delta_\xi)^l\{|\xi|^{2\alpha(N+1)} g_N(|\xi|^{2\alpha})\psi_1(\xi)\}\,d\xi. \tag{2.30}$$

Let us assume that $0 < \varepsilon < \frac{1}{2}|x|$, then

$$\begin{aligned} \left|\mathscr{F}_y\left[\frac{\chi_\varepsilon(x+y)}{|y|^{2l}}\right](\xi)\right| &\le \int_{\mathbf{R}^n} \frac{\chi_\varepsilon(x+y)}{|y|^{2l}}\,dy \\ &\to \frac{1}{|x|^{2l}}, \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{2.31}$$

So we can apply the covergence theorem of Lebesgue to $I_\varepsilon^{(1)}(x;N)$. Therefore, the limit $\lim_{\varepsilon\to 0} I_\varepsilon^{(1)}(x;N)$ exists and we have

$$\left|\lim_{\varepsilon\to 0} I_\varepsilon^{(1)}(x;N)\right| \le \frac{1}{|x|^{2l}}\int_{\mathbf{R}^n} |(-\Delta_\xi)^l\{|\xi|^{2\alpha(N+1)}g_N(|\xi|^{2\alpha})\psi_1(\xi)\}|\,d\xi \tag{2.32}$$

$$= \text{Const.}\frac{1}{|x|^{2l}} \le \text{Const.}\frac{1}{|x|^{2\alpha(N+1)-2}}.$$

Step 2. Next, we consider the estimate of $I_\varepsilon^{(2)}(x;N)$. Let

$$f_N(\xi) = |\xi|^{2\alpha(N+1)}g_N(|\xi|^{2\alpha})\psi_2(\xi). \tag{2.33}$$

Then, since $|\xi|^{2\alpha(N+1)}g_N(|\xi|^{2\alpha})$ is the $N$-th remainder term of the Taylor expansion of $e^\lambda$ ($\lambda = -|\xi|^{2\alpha}$),

$$f_N(\xi) = \left\{e^{-|\xi|^{2\alpha}} - \sum_{j=0}^{N}\frac{(-1)^j}{j!}|\xi|^{2\alpha j}\right\}\psi_2(\xi). \tag{2.34}$$

Note that $\psi_2$ vanishes near the origin. So $f_N$ is a smooth function with the following property. For each multi-index $\gamma$, there exists a constant $C_\gamma$ such that

$$|\partial_\xi^\gamma f_N(\xi)| \le C_\gamma(1+|\xi|)^{2\alpha N-|\gamma|}, \quad \text{for } \xi\in\mathbf{R}^n. \tag{2.35}$$

In other words, $f_N$ belongs to the symbol class $S_{1,0}^{2\alpha N}$. In particular, for some constant $C_m$ we have

$$|(-\Delta_\xi)^m f_N(\xi)| \le C_m(1+|\xi|)^{2\alpha N-2m}. \tag{2.36}$$

Let us take the integer $m$ such that

$$-n-3 < 2\alpha N - 2m \le -n-1. \tag{2.37}$$

Then, similarly as in (2.30), we have

$$I_\varepsilon^{(2)}(x;N) = \int_{\mathbf{R}^n}\mathscr{F}_y\left[\frac{\chi_\varepsilon(x+y)}{|y|^{2m}}\right](\xi)(-\Delta_\xi)^m f_N(\xi)\,d\xi. \tag{2.38}$$

Thus, by the same arguement as in Step 1, we see that the limit $\lim_{\varepsilon\to 0} I_\varepsilon^{(2)}(x;N)$ exists and

$$(2.39)\qquad \left|\lim_{\varepsilon\to 0} I_{\varepsilon}^{(2)}(x;N)\right| \le \frac{1}{|x|^{2m}}\int_{\mathbf{R}^n} |(-\Delta_{\xi})^m f_N(\xi)|\, d\xi$$
$$\le \frac{1}{|x|^{2m}} C_m \int_{\mathbf{R}^n} (1+|\xi|)^{2\alpha N-2m}\, d\xi = \text{Const.}\frac{1}{|x|^{2m}}$$
$$\le \text{Const.}\frac{1}{|x|^{2\alpha N+n+1}}.$$

Both (2.32) and (2.39) prove the assertion of Lemma 2.2.

Remark 2.1. As is well known in probability theory, the fundamental solution $W^{(\alpha)}(t,x)$ of $\partial_t u + (-\Delta)^{\alpha} u = 0$ is the density of the semigroup of $n$-dimensional symmetric stable process with index $2\alpha$. Moreover, its asymptotic expansion formula is known in one dimensional case. (See Zolotarev [Zo], Chapter 2, Section 2.5.) However, the proof depends on some probability theoretical argument. Therefore, it should be remarked that our asymptotic expansion formula is given in $n$-dimensional case and the proof is done by the method of Fourier analysis.

## 3. Positivity of $W^{(\alpha)}(t,x)$ and Comparison Theorem

In this section, we will show the positivity of the fundamental solution $W^{(\alpha)}(t,x)$ of the parabolic pseudo-differential equation $\partial_t u + (-\Delta)^{\alpha} u = 0$. Next, we will apply it to the comparison theorem.

We start with the definition of *a completely monotone function*.

Definition 3.1. A function $\varphi$ on $(0,\infty)$ is completely monotone if $\varphi$ is smooth and

$$(3.1)\qquad (-1)^m \frac{d^m}{d\lambda^m}\varphi(\lambda) \ge 0, \quad m = 0,1,2,\dots,\ \lambda > 0.$$

Example.

(i) $\varphi(\lambda) = \lambda^{-\sigma}$ is completely monotone if $\sigma > 0$.

(ii) If $\psi$ is a positive function with a completely monotone derivative, then $e^{-\psi}$ is completely monotone. (It is easily proved by induction.)

(iii) By the above two, $e^{-\lambda^{\alpha}}$ is completely monotone if $0 < \alpha < 1$.

(For the details of completely monotone functions, see [Fe] Chap. XIII, Section 4.)

Surprisingly, a completely monotone function is real analytic, namely, a smooth function on $(0, \infty)$ satisfying the condition (3.1) automatically becomes real analytic on $(0, \infty)$. Farthermore, such a function is written as the Laplace transform of a probability distribution. More precisely, we have the following theorem.

THEOREM 3.1 ([Fe], Chap. XIII, Section 4, Theorem 1). *A function $\varphi$ on $[0, \infty)$ is the Laplace transform of a probability distribution $\mu$, if and only if $\varphi$ is completely monotone and $\varphi(0) = 1$.*

Now let us prove the positivity of $W^{(\alpha)}(t, x)$.

It follows from the above example and Theorem 3.1 that if $0 < \alpha < 1$ there exists a probability distribution $\mu$ such that

$$e^{-\lambda^{\alpha}} = \int_0^{\infty} e^{-\lambda\rho}\mu(d\rho). \tag{3.2}$$

Substituting $\lambda = |\xi|^2$ in (3.2), we have

$$e^{-|\xi|^{2\alpha}} = \int_0^{\infty} e^{-\rho|\xi|^2}\mu(d\rho). \tag{3.3}$$

Hence,

$$\begin{aligned} w^{(\alpha)}(x) &= (2\pi)^{-n}\int_{\mathbf{R}^n} e^{-|\xi|^{2\alpha}}e^{i\xi\cdot x}\,d\xi \\ &= (2\pi)^{-n}\int_{\mathbf{R}^n}\int_0^{\infty} e^{i\xi\cdot x}e^{-\rho|\xi|^2}\mu(d\rho)\,d\xi \\ &= \int_0^{\infty}\mu(d\rho)\left\{(2\pi)^{-n}\int_{\mathbf{R}^n} e^{i\xi\cdot x}e^{-\rho|\xi|^2}\,d\xi\right\} \\ &= \int_0^{\infty}(4\pi\rho)^{-n/2}e^{-|x|^2/4\rho}\mu(d\rho) > 0. \end{aligned} \tag{3.4}$$

Therefore, by the expression (3.4) combined with (2.4) we obtain the following theorem.

THEOREM 3.2.

(i) *$W^{(\alpha)}(t, x) > 0$ for $t > 0$ and for $x \in \mathbf{R}^n$.*

(ii) *$W^{(\alpha)}(t, x)$ is monotone decreasing with respect to $|x|$, that is, $W^{(\alpha)}(t, x) > W^{(\alpha)}(t, y)$ if $|x| < |y|$.*

By Corollary 2.2 combined with the above theorem, we have

COROLLARY 3.1. *For each fixed $t > 0$, there exist positive constants $C_1$ and $C_2$ such that*

$$C_1(1+|x|)^{-n-2\alpha} \le W^{(\alpha)}(t,x) \le C_2(1+|x|)^{-n-2\alpha}, \quad \text{for } x \in \mathbf{R}^n. \tag{3.5}$$

REMARK 3.1. Nishio [N] also proves (3.5), using a potential theoretical method. However, the estimate (3.5) itself is not sufficient for our purpose. As we mentioned in the introduction, we need both Corollary 3.1 and the integrability of derivatives of $W^{(\alpha)}(t,x)$.

Next, we go into the comparison theorem.

THEOREM 3.3. *Let $u, v \in C((0,T); H_1^{2\alpha} \cap H_\infty^{2\alpha}) \cap C^1((0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)) \cap C([0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$ be nonnegative solutions to the following equations respectively.*

$$\partial_t u + (-\Delta)^\alpha u = f(u), \quad u(0,x) = u_0(x), \tag{3.6}$$

$$\partial_t v + (-\Delta)^\alpha v = g(v), \quad v(0,x) = v_0(x). \tag{3.7}$$

*We assume that $f$ and $g$ are continuous functions on $[0,\infty)$ and $u_0, v_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. In addition, we assume that $g$ is monotone increasing and satisfies the following condition. For any $M > 0$, there exists a constant $C_M$ such that*

$$\sup_{0 \le u, v \le M} \frac{g(u) - g(v)}{u - v} \le C_M \tag{3.8}$$

*Then we have*

(i) *If $u_0(x) \le v_0(x)$ for $x \in \mathbf{R}^n$ and $f(u) \le g(u)$ for $u \ge 0$, then $u(t,x) \le v(t,x)$ for $(t,x) \in [0,T) \times \mathbf{R}^n$.*

(ii) *Conversely, if $u_0(x) \ge v_0(x)$ for $x \in \mathbf{R}^n$ and $f(u) \ge g(u)$ for $u \ge 0$, then $u(t,x) \ge v(t,x)$ for $(t,x) \in [0,T) \times \mathbf{R}^n$.*

PROOF. We prove only (i) of the statement. By (3.6), (3.7), and the assumption that $f(u) \le g(u)$, we have

$$\begin{aligned} \partial_t(v-u) + (-\Delta)^\alpha(v-u) &= g(v) - f(u) \\ &= g(v) - g(u) + g(u) - f(u) \\ &\ge g(v) - g(u). \end{aligned} \tag{3.9}$$

Let $w = v - u$ and $w_0 = v_0 - u_0$. Moreover let

$$G(t,x) = \begin{cases} \frac{g(v(t,x)) - g(u(t,x))}{v(t,x) - u(t,x)}, & \text{if } v(t,x) \neq u(t,x) \\ 0, & \text{if } v(t,x) = u(t,x) \end{cases}. \tag{3.10}$$

Then by (3.9), Theorem 3.2 (i), and the assumption that $w_0 = v_0 - u_0 \geq 0$, we have

$$\begin{aligned} w(t,x) &\geq W^{(\alpha)}(t,\cdot) * w_0(x) \\ &\quad + \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s, x-y)\{g(v(s,y)) - g(u(s,y))\}\, dyds \\ &\geq \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s, x-y) G(s,y) w(s,y)\, dyds \end{aligned} \tag{3.11}$$

Let us fix any $T_0$ $(0 < T_0 < T)$ and let

$$M = \max\{sup_{0 \leq t \leq T_0} \|u(t,\cdot)\|_{L^\infty}, sup_{0 \leq t \leq T_0} \|v(t,\cdot)\|_{L^\infty}\}. \tag{3.12}$$

Then, by (3.8),

$$0 \leq G(t,x) \leq C_M, \quad \text{for } (t,x) \in [0,T_0] \times \mathbf{R}^n. \tag{3.13}$$

Now we introduce a linear operator $S : L^\infty([0,T_0] \times \mathbf{R}^n) \to L^\infty([0,T_0] \times \mathbf{R}^n)$ as follows.

$$(S\phi)(t,x) = \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s, x-y) G(s,y) \phi(s,y)\, dyds, \quad \text{for } \phi \in L^\infty([0,T_0] \times \mathbf{R}^n). \tag{3.14}$$

We see easily by induction that

$$|(S^N\phi)(t,x)| \leq \frac{C_M^N}{N!} t^N \|\phi\|_{L^\infty([0,T_0]\times\mathbf{R}^n)}, \quad N = 1,2,3,\ldots. \tag{3.15}$$

Hence

$$\|S^N\| \leq \frac{(C_M T_0)^N}{N!} \to 0, \quad \text{as } N \to \infty. \tag{3.16}$$

Since $W^{(\alpha)}(t,x) > 0$ (Theorem 3.2) and $G(t,x) \geq 0$, $S$ maps a nonnegative function to a nonnegative function. So if $\phi(t,x) \geq (S\phi)(t,x)$, then

$$\phi(t,x) \geq (S\phi)(t,x) \geq (S^2\phi)(t,x) \geq \cdots \geq (S^N\phi)(t,x). \tag{3.17}$$

Let $N \to \infty$. Then, by (3.16) we obtain $\phi(t,x) \geq 0$. Here note that inequality (3.11) means that $w(t,x) \geq (Sw)(t,x)$. Therefore, the above argument can be applied to $\phi = w$, which proves that $w(t,x) = v(t,x) - u(t,x) \geq 0$. □

Remark 3.2. Usually the comparison theorem for nonlinear parabolic differential equation is proved by the maximum principle of the corresponding linear parabolic differential equation. (See Protter and Weinberger [PW].) However, such a method can no longer be applied to this case, due to the fact that $(-\Delta)^\alpha$ is not a local operator.

## 4. Existence of Local Solutions

We begin with the integral equation arising from (1.1).

$$u(t,x) = \int_{\mathbf{R}^n} W^{(\alpha)}(t, x-y) u_0(y)\, dy + \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s, x-y) u(s,y)^p\, dyds, \tag{4.1}$$

where $W^{(\alpha)}(t,x)$ is the fundamental soluton to the linear equation $\partial_t u + (-\Delta)^\alpha u = 0$.

By contraction argument, we can prove the following.

Theorem 4.1. *Assume that* $u_0(x) \geq 0$ *and that* $u_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. *If* $T > 0$ *is sufficiently small, then the integral equation* (4.1) *has a unique nonnegative solution* $u \in C([0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$.

The problem is to prove that the above solution $u$ of the integral equation (4.1) satisfies the parabolic pseudo-differential equation (1.1).

The difficulty lies in the fact that $(-\Delta)^\alpha$ is no longer a local operator. In addition, the regularizing effect of $(-\Delta)^\alpha$ becomes weak if $\alpha$ is small.

Therfore, we need to consider the meaning of *a solution* to (1.1) rigorously. In this paper, we study the existence of a solution to (1.1) in the framework of $L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. So we define a solution of (1.1) by the following.

Definition 4.1. $u \in C([0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$ is called *a strong solution* of the semilinear parabolic pseudo-differential equation (1.1) in $[0,T) \times \mathbf{R}^n$ if $u$ satisfies the following conditions.

(i) $u \in C((0,T); H_1^{2\alpha} \cap H_\infty^{2\alpha})$.

(ii) $u \in C^1((0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$.

(iii) As an equality in $L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$, $u$ satisfies the equation

$$\partial_t u + (-\Delta)^\alpha u = u^p. \tag{4.2}$$

(iv) $\lim_{t \to +0} u(t, \cdot) = u_0$ in $L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$.

From now on, we assume that $\frac{1}{2} < \alpha < 1$.

First, let us consider the regularity of $u$ with respect to $x$.

LEMMA 4.1. (1) *We assume that* $u \in C([0, T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$. *Let*

$$\begin{aligned} \Phi u(t, x) &= \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s, x-y) u(s, y)^p \, dyds \\ &= \int_0^t \int_{\mathbf{R}^n} (t-s)^{-n/2\alpha} w^{(\alpha)}((t-s)^{-1/2\alpha}(x-y)) u(s, y)^p \, dyds, \end{aligned} \tag{4.3}$$

*where* $w^{(\alpha)}$ *is the function given by* (2.3). *Then, for each* $t \in [0, T)$, $\Phi u(t, \cdot) \in C^1(\mathbf{R}^n)$. *Moreover,* $\partial_{x_j}(\Phi u) \in C([0, T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$, $(1 \le j \le n)$.

(2) *In addition to the assumption in* (1), *we assume that* $u(t, \cdot) \in C^1(\mathbf{R}^n)$ *for each* $t \in (0, T)$ *and that* $\partial_{x_j} u \in C((0, T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$ $(1 \le j \le n)$. *Then,* $\Phi u(t, \cdot) \in C^2(\mathbf{R}^n)$ *for each* $t \in (0, T)$, *and* $\partial_{x_j}\partial_{x_k}(\Phi u) \in C((0, T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$, $(1 \le j, k \le n)$.

PROOF. By a straightforward computation, we have

$$\partial_{x_j}(\Phi u)(t, x) = \int_0^t \int_{\mathbf{R}^n} (t-s)^{-1/2\alpha} (\partial_{x_j} w^{(\alpha)})(z) u(s, x - (t-s)^{1/2\alpha} z)^p \, dzds. \tag{4.4}$$

So, by Corollary 2.2 and the assumption that $\frac{1}{2} < \alpha < 1$,

$$\|\partial_{x_j}(\Phi u)(t, \cdot)\|_{L^\infty(\mathbf{R}^n)} \le \|\partial_{x_j} w^{(\alpha)}\|_{L^1(\mathbf{R}^n)} \|u\|_{L^\infty(\mathbf{R}^n)} \times \int_0^t (t-s)^{-1/2\alpha} \, ds < +\infty. \tag{4.5}$$

Similarly we have

$$\|\partial_{x_j}(\Phi u)(t, \cdot)\|_{L^1(\mathbf{R}^n)} \le \|\partial_{x_j} w^{(\alpha)}\|_{L^1(\mathbf{R}^n)} \|u\|_{L^1(\mathbf{R}^n)} \times \int_0^t (t-s)^{-1/2\alpha} \, ds < +\infty. \tag{4.6}$$

The above two inequalities prove the assertion of (1).

If $u$ satisfies the assumptions in (2), so does $u^p$. Therefore, by integral by parts with respect to $x$, we have

(4.7) $$\partial_{x_j}\partial_{x_k}(\Phi u)(t,x) = \int_0^{t/2}\int_{\mathbf{R}^n}(\partial_{x_j}\partial_{x_k}W^{(\alpha)})(t-s,x-y)u(s,y)^p\,dyds + \int_{t/2}^{t}\int_{\mathbf{R}^n}(t-s)^{-1/2\alpha}(\partial_{x_j}w^{(\alpha)})(z)\partial_{x_k}\{u(s,x-((t-s)^{1/2\alpha}z))\}^p\,dzds.$$

In the integrand of the first term of L.H.S. of (4.7), $(\partial_{x_j}\partial_{x_k}W^{(\alpha)})(t-s,x-y)$ is integrable with respect to $(s,y)\in\left[0,\frac{t}{2}\right]\times\mathbf{R}^n$ due to Corollary 2.1 and Corollary 2.2. Moreover, in the integrand of the second term of L.H.S. of (4.7), $\partial_{x_k}\{u(s,x-((t-s)^{1/2\alpha}z))\}^p$ is bounded and integrable with respect to $x\in\mathbf{R}^n$ for each $s\in\left[\frac{t}{2},t\right]$ due to the assertion of (1). Therefore, it follows from (4.7) that $\partial_{x_j}\partial_{x_k}(\Phi u)\in C((0,T);L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$. □

PROPOSITION 4.1. *Let* $u\in C([0,T);L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$ *be the solution of the integral equation* (4.1) *in* $[0,T)\times\mathbf{R}^n$. *Then, for each* $t\in(0,T)$, $u(t,\cdot)\in C^2(\mathbf{R}^n)$. *Moreover,* $\partial_{x_j}u,\partial_{x_j}\partial_{x_k}u\in C((0,T);L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$ *for* $1\le j,k\le n$.

PROOF. Using (4.3) and (2.4), we rewrite integral equation (4.1) as

(4.8) $$u(t,x) = W^{(\alpha)}(t,\cdot)*u_0(x)+\Phi u(t,x).$$

Obviously, $W^{(\alpha)}(t,\cdot)*u_0\in C^\infty((0,T)\times\mathbf{R}^n)$ and $\partial_{x_j}W^{(\alpha)}(t,\cdot)*u_0,\partial_{x_j}\partial_{x_k}W^{(\alpha)}(t,\cdot)*u_0\in C((0,T);L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$. On the other hand, by Lemma 4.2 (1), $\Phi u(t,\cdot)\in C^1(\mathbf{R}^n)$, and $\partial_{x_j}(\Phi u)\in C([0,T);L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$, $(1\le j\le n)$. Therefore, by equation (4.8), $u(t,\cdot)\in C^1(\mathbf{R}^n)$ for $t\in(0,T)$, and $\partial_{x_j}u\in C((0,T);L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$. Then, we can apply Proposition 4.1 (2) to $u$. Thus, the assertion of the Proposition is proved. □

The above proposition yields the following.

PROPOSITION 4.2. *Let* $u\in C([0,T);L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$ *be the solution of the integral equation* (4.1) *in* $[0,T)\times\mathbf{R}^n$. *Then,* $(-\Delta)^\alpha u(t,\cdot)\in L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$, *for* $t\in(0,T)$.

PROOF. Proposition 4.1 shows that $u\in C((0,T);H_1^2(\mathbf{R}^n)\cap H_\infty^2(\mathbf{R}^n))$. Thus, $(1-\Delta)u\in C((0,T);L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n))$. As is well known, the operator $(-\Delta)^\alpha(1-\Delta)^{-1}$ is well defined as a bounded operator on $L^q(\mathbf{R}^n)$, $(1\le q\le\infty)$.

This fact is easily checked by the argument of Fourier multipliers. (See for example Bergh and Löfström [BL], Chapter 6, Theorem 6.2.3.)

Therefore, $(-\Delta)^{\alpha}u = (-\Delta)^{\alpha}(1-\Delta)^{-1}(1-\Delta)u \in C((0,T); L^1(\mathbf{R}^n) \cap L^{\infty}(\mathbf{R}^n))$. $\square$

Now we will show that $u$ satisfies the parabolic pseudo-differential equation (1.1). We start with the difference quotient of $\Phi u$.

$$\begin{aligned}(4.9) \qquad \frac{1}{h}\{\Phi u(t+h,x) - \Phi u(t,x)\} &= \frac{1}{h}\int_t^{t+h}\int_{\mathbf{R}^n} W^{(\alpha)}(t+h-s,x-y)u(s,y)^p\,dyds \\ &\quad + \frac{1}{h}\int_0^t\int_{\mathbf{R}^n}\{W^{(\alpha)}(t+h-s,x-y) \\ &\quad - W^{(\alpha)}(t-s,x-y)\}u(s,y)^p\,dyds \\ (\text{we put}) &= I(h) + J(h).\end{aligned}$$

Without loss of generality, we may assume that $h > 0$ when we let $h \to 0$ in the above equality. By the mean value theorem, there exists $\sigma \in (t, t+h)$ depending on $h$ and $z$ such that

$$(4.10) \qquad \int_t^{t+h} u(s, x+(t+h-s)^{1/2\alpha}z)^p\,ds = hu(\sigma, x+(t+h-\sigma)^{1/2\alpha}z)^p.$$

Then, we see easily that

$$\begin{aligned}(4.11) \qquad I(h) &= \frac{1}{h}\int_t^{t+h}\int_{\mathbf{R}^n} w^{(\alpha)}(z)u(s,x+(t+h-s)^{1/2\alpha}z)\,dzds \\ &= \frac{1}{h}\int_{\mathbf{R}^n} w^{(\alpha)}(z)\left\{\int_t^{t+h} u(s,x+(t+h-s)^{1/2\alpha}z)^p\,ds\right\}dz \\ &= \int_{\mathbf{R}^n} w^{(\alpha)}(z)u(\sigma,x+(t+h-\sigma)^{1/2\alpha}z)^p\,dz \\ &\to \int_{\mathbf{R}^n} w^{(\alpha)}(z)u(t,x)^p\,dz = u(t,x)^p, \\ &\quad \text{as } h \to 0, \text{ in } C((0,T), L^1(\mathbf{R}^n) \cap L^{\infty}(\mathbf{R}^n)).\end{aligned}$$

Next, for $\epsilon > 0$, let

$$\begin{aligned}(4.12) \qquad J^{(\epsilon)}(h) &= \frac{1}{h}\int_0^{t-\epsilon}\int_{\mathbf{R}^n}\{W^{(\alpha)}(t+h-s,x-y) \\ &\quad - W^{(\alpha)}(t-s,x-y)\}u(s,y)^p\,dyds.\end{aligned}$$

We note that $(\partial_t W^{(\alpha)})(t+h-s,x-y) = -((-\Delta)^\alpha W^{(\alpha)})(t+h-s,x-y)$ is bounded and integrable with respect to $(s,y) \in [0,t-\epsilon] \times \mathbf{R}^n$. So we have

$$
\begin{aligned}
(4.13) \qquad J^{(\epsilon)}(h) &= \int_0^{t-\epsilon}\int_{\mathbf{R}^n}\int_0^1 (\partial_t W^{(\alpha)})(t+h\tau-s,x-y)u(s,y)^p \, d\tau dy ds \\
&= -\int_0^{t-\epsilon}\int_{\mathbf{R}^n}\int_0^1 ((-\Delta)^\alpha W^{(\alpha)})(t+h\tau-s,x-y)u(s,y)^p \, d\tau dy ds \\
&= -(-\Delta)^\alpha \int_0^1\int_0^{t-\epsilon}\int_{\mathbf{R}^n} W^{(\alpha)}(t+h\tau-s,x-y)u(s,y)^p \, dy ds d\tau.
\end{aligned}
$$

We put

$$
(4.14) \qquad \Phi_{(h,\epsilon)}u(t,x) = \int_0^1\int_0^{t-\epsilon}\int_{\mathbf{R}^n} W^{(\alpha)}(t+h\tau-s,x-y)u(s,y)^p \, dy ds d\tau.
$$

Taking account of Lemma 4.1, we see that

$$
\begin{aligned}
(4.15) \qquad \Phi_{(h,\epsilon)}u(t,x) &\xrightarrow{\epsilon\to 0} \Phi_{(h,0)}u(t,x) \\
&= \int_0^1\int_0^t\int_{\mathbf{R}^n} W^{(\alpha)}(t+h\tau-s,x-y)u(s,y)^p \, dy ds d\tau, \\
&\text{in } C((0,T), H_1^2(\mathbf{R}^n) \cap H_\infty^2(\mathbf{R}^n)).
\end{aligned}
$$

As we explained bebore, $(-\Delta)^\alpha = (-\Delta)^\alpha(1-\Delta)^{-1}(1-\Delta)$ is a bounded operator from $H_1^2(\mathbf{R}^n) \cap H_\infty^2(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$. So we have

$$
\begin{aligned}
(4.16) \qquad J^{(\epsilon)}(h) = -(-\Delta)^\alpha \Phi_{(h,\epsilon)}u &\xrightarrow{\epsilon\to 0} -(-\Delta)^\alpha \Phi_{(h,0)}u \\
&\text{in } C((0,T), L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)).
\end{aligned}
$$

On the other hand, obviously $J^{(\epsilon)}(h) \to J(h)$ as $\epsilon \to 0$ in $C((0,T), L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$. Thus,

$$
\begin{aligned}
(4.17) \qquad J(h) &= -(-\Delta)^\alpha \Phi_{(h,0)}u(t,x) \\
&= -(-\Delta)^\alpha \int_0^1\int_0^t\int_{\mathbf{R}^n} W^{(\alpha)}(t+h\tau-s,x-y)u(s,y)^p \, dy ds d\tau.
\end{aligned}
$$

Moreover, we have

$$
(4.18) \qquad \lim_{h\to 0} \Phi_{(h,0)}u(t,x) = \Phi u(t,x), \quad \text{in } C((0,T), H_1^2(\mathbf{R}^n) \cap H_\infty^2(\mathbf{R}^n)).
$$

Therefore, we have

$$\lim_{h\to 0} J(h) = \lim_{h\to 0} -(-\Delta)^{\alpha}\Phi_{(h,0)}u(t,x) \tag{4.19}$$
$$= -(-\Delta)^{\alpha}\Phi u(t,x), \quad \text{in } C((0,T), L^1(\mathbf{R}^n)\cap L^{\infty}(\mathbf{R}^n)).$$

Therefore, by (4.9), (4.11) and (4.19),

$$\lim_{h\to 0}\frac{1}{h}\{\Phi u(t+h,x) - \Phi u(t,x)\} \tag{4.20}$$
$$= \lim_{h\to 0} I(h) + \lim_{h\to 0} J(h)$$
$$= u(t,x)^p - (-\Delta)^{\alpha}\Phi u(t,x), \quad \text{in } C((0,T), L^1(\mathbf{R}^n)\cap L^{\infty}(\mathbf{R}^n)).$$

Since $u$ is a solution to the integral equation (4.8), we have

$$\lim_{h\to 0}\frac{1}{h}\{u(t+h,x) - u(t,x)\} \tag{4.21}$$
$$= \partial_t W^{(\alpha)}(t,\cdot) * u_0(x) + \lim_{h\to 0}\frac{1}{h}\{\Phi u(t+h,x) - \Phi u(t,x)\}$$
$$= -(-\Delta)^{\alpha} W^{(\alpha)}(t,\cdot) * u_0(x) + u(t,x)^p - (-\Delta)^{\alpha}\Phi u(t,x)$$
$$(\text{by (4.8)}) = -(-\Delta)^{\alpha}\{W^{(\alpha)}(t,\cdot) * u_0(x) + \Phi u(t,x)\} + u(t,x)^p$$
$$= -(-\Delta)^{\alpha}u(t,x) + u(t,x)^p.$$

In the above equality, the limit exists in the topology of $C((0,T), L^1(\mathbf{R}^n)\cap L^{\infty}(\mathbf{R}^n))$. We note that in these limiting procedures $I(h), J(h) \in C((0,T-h)\times \mathbf{R}^n)$ and $I(h)\to u^p$, $J(h)\to -(-\Delta)^{\alpha}\Phi u$ respectively as $h\to 0$ in the topology of $C((0,T_0), L^1(\mathbf{R}^n)\cap L^{\infty}(\mathbf{R}^n))$ for any $T_0 < T$. Thus $\Phi u$ and $u$ are differentiable with respect to $t\in(0,T)$ and $\partial_t u, \partial_t \Phi u \in C((0,T)\times \mathbf{R}^n)$. Moreover, for the same reason, we see that $(-\Delta)^{\alpha}\Phi u$ and $(-\Delta)^{\alpha}u$ also belong to $C((0,T)\times\mathbf{R}^n)$.

Summarizing the above argument, we obtain the following.

Theorem 4.2. *We assume that* $\frac{1}{2} < \alpha < 1$. *Let* $u \in C([0,T); L^1(\mathbf{R}^n)\cap L^{\infty}(\mathbf{R}^n))$ *satisfy the integral equation* (4.1) *in* $[0,T)\times\mathbf{R}^n$. *Then, the following* (i), (ii), *and* (iii) *hold.*

(i) *$u$ is of class $C^1$ in $t\in(0,T)$ and of class $C^2$ in $x\in\mathbf{R}^n$. Moreover,* $u\in C((0,T), H_1^2(\mathbf{R}^n)\cap H_{\infty}^2(\mathbf{R}^n))\cap C^1((0,T); L^1(\mathbf{R}^n)\cap L^{\infty}(\mathbf{R}^n))$.

(ii) $(-\Delta)^{\alpha}u \in C((0,T)\times\mathbf{R}^n)$ *and* $(-\Delta)^{\alpha}u\in C((0,T); L^1(\mathbf{R}^n)\cap L^{\infty}(\mathbf{R}^n))$.

(iii) *$u$ is a unique strong solution of* (1.1) *in the sense of Definition* 4.1.

Theorem B in the introduction is a direct consequence of the above theorem.

Remark 4.1. In this paper, we do not discuss the case $0 < \alpha \le \frac{1}{2}$. For a technical reason, our method employed in this section cannot be applied in this case. Moreover, it seems that the solution of the integral equation (4.1) no longer becomes a strong solution in the sense of Definition 4.1.

## 5. Existence of Global Solutions and Lower Bounds of the Life Span

The purposes of this section are twofold, the first one is to prove the existence of global solutions to (1.1) for sufficiently small data in the case $p_*(\alpha) = 1 + \frac{2\alpha}{n} < p$, and the second one is to give lower bounds of the life span in the case $1 < p \le p_*(\alpha)$.

As we mentioned in the introduction, we assume the following condition on the initial data $u_0$.

$$\text{(A)} \qquad 0 \le u_0(x) \le \delta_0(1+|x|)^{-n-2\alpha}, \quad \text{where } \delta_0 > 0.$$

Due to Corollary 3.1, the condition (A) is equivalent to

$$\text{(A}'\text{)} \qquad 0 \le u_0(x) \le c_0 W^{(\alpha)}(\tau, x), \quad \text{where } c_0, \tau > 0.$$

For $0 < T \le \infty$, we define a Banach space $\mathscr{V}_T$ by the space of all measurable functions $v$ on $[0, T) \times \mathbf{R}^n$ satisfying

$$\|v\|_{\mathscr{V}_T} \stackrel{\text{def}}{=} \text{ess.sup}_{(t,x) \in [0,T) \times \mathbf{R}^n} \frac{|v(t,x)|}{\rho(t,x)} < +\infty, \tag{5.1}$$

where

$$\rho(t,x) = W^{(\alpha)}(1+t, x). \tag{5.2}$$

Moreover, we define a subset $\mathscr{V}_T^+$ of $\mathscr{V}_T$ by

$$\mathscr{V}_T^+ = \{v \in \mathscr{V}_T; v(t,x) \ge 0, \text{ for } (t,x) \in [0,T) \times \mathbf{R}^n\} \tag{5.3}$$

The following lemma is important for both global existence theorem and lower bounds of the life span.

Lemma 5.1. *Let $\Phi$ be the operator defined by* (4.3).

(i) *There exists a constant $C$ independent of $T$ such that*

$$0 \le (\Phi\rho)(t,x) \le C\rho(t,x) \int_0^T (1+s)^{-n(p-1)/2\alpha} \, ds \tag{5.4}$$

(ii) *Assume that* $1 < p \le p_*(\alpha)$. *Let* $\lambda > 0$. *If* $0 \le v(t,x) \le \lambda\rho(t,x)$ *for* $(t,x) \in [0,T) \times \mathbf{R}^n$, *then we have the estimate*

$$(5.5) \qquad 0 \le (\Phi v)(t,x) \le \begin{cases} C\lambda^p T^{1-n(p-1)/2\alpha}\rho(t,x), & \text{in the case } 1 < p < p_*(\alpha), \\ C\lambda^p \log(1+T)\rho(t,x), & \text{in the case } p = p_*(\alpha), \end{cases}$$

*for* $(t,x) \in [0,T) \times \mathbf{R}^n$. *Here the constant* $C$ *does not depend on* $T$.

(iii) *Assume that* $p_*(\alpha) < p$. *Let* $\lambda > 0$. *If* $0 \le v(t,x) \le \lambda\rho(t,x)$ *for* $(t,x) \in [0,\infty) \times \mathbf{R}^n$, *then we have the estimate*

$$(5.6) \qquad 0 \le (\Phi v)(t,x) \le C\lambda^p \rho(t,x).$$

*In particular*, $\Phi\rho \in \mathscr{V}_\infty^+$.

Proof.

$$\begin{aligned}(5.7) \qquad \rho(t,x)^p &= W^{(\alpha)}(1+t,x)^{p-1} W^{(\alpha)}(1+t,x) \\ &= \{(1+t)^{-n/2\alpha} w^{(\alpha)}((1+t)^{-1/2\alpha}x)\}^{p-1} W^{(\alpha)}(1+t,x) \\ &\le C(1+t)^{-n(p-1)/2\alpha} W^{(\alpha)}(1+t,x), \\ &\qquad \text{where } C = \left\{(2\pi)^{-n}\int_{\mathbf{R}^n} e^{-|\xi|^{2\alpha}}\, d\xi\right\}^{p-1}.\end{aligned}$$

Thus, we have

$$\begin{aligned}(5.8) \qquad 0 \le (\Phi\rho)(t,x) &= \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s,x-y)\rho(s,y)^p \, dyds \\ &\le C\int_0^t (1+s)^{-n(p-1)/2\alpha} \\ &\qquad \times \int_{\mathbf{R}^n} W^{(\alpha)}(t-s,x-y) W^{(\alpha)}(1+s,y)\, dyds \\ (\text{by semigroup property}) &= C\int_0^t (1+s)^{-n(p-1)/2\alpha}\, ds \times W^{(\alpha)}(1+t,x).\end{aligned}$$

The above inequality proves (i).

The assertions (ii) and (iii) follow easily from (i). □

*Existence of global solutions.*

Let us first consider the global existence of the solutions to (1.1) for sufficiently small data.

The method is similar to that of Fujita [Fu]. So we are brief.

Due to Lemma 5.1, we have

LEMMA 5.2. (i) *If* $v \in \mathscr{V}_\infty^+$, *then* $\Phi v \in \mathscr{V}_\infty^+$. *Moreover, we have*

$$\|\Phi v\|_{\mathscr{V}_\infty} \le C\|v\|_{\mathscr{V}_\infty}^p. \tag{5.9}$$

(ii) *Let* $v_1, v_2 \in \mathscr{V}_\infty^+$. *If* $\|v_1\|_{\mathscr{V}_\infty} \le M$ *and* $\|v_2\|_{\mathscr{V}_\infty} \le M$, *then we have*

$$\|\Phi v_1 - \Phi v_2\|_{\mathscr{V}_\infty} \le CpM^{p-1}\|v_1 - v_2\|_{\mathscr{V}_\infty}, \tag{5.10}$$

*In* (i) *and* (ii), *C is the constant given in* (iii) *of Lemma* 5.1.

Let us take $\delta_0$ such that

$$0 < \delta_0 \le C^{-1/(p-1)}2^{-p/(p-1)}, \quad Cp(2\delta_0)^{p-1} \le \frac{1}{2}. \tag{5.11}$$

For the above $\delta_0$, we take a closed convex set $\mathscr{B}^+(2\delta_0)$ in $\mathscr{V}_\infty$ as follows.

$$\mathscr{B}^+(2\delta_0) = \{v \in \mathscr{V}_\infty \mid \|v\|_{\mathscr{V}_\infty} \le 2\delta_0, v \ge 0\} \tag{5.12}$$

We define a mapping $\Psi : \mathscr{V}_\infty \to \mathscr{V}_\infty$ by

$$\begin{aligned}(\Psi v)(t,x) &= W^{(\alpha)}(t,\cdot) * u_0(x) + (\Phi v)(t,x) \\ &= \int_{\mathbf{R}^n} W^{(\alpha)}(t,x-y)u_0(y)\,dy \\ &\quad + \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s,x-y)v(s,y)^p\,dyds\end{aligned} \tag{5.13}$$

Due to the assumption (A) on $u_0$, $\Psi : \mathscr{V}_\infty \to \mathscr{V}_\infty$ is well-defined. Then, by Lemma 5.2 and the definition of $\delta_0$ we have

LEMMA 5.3. *If* $u_0$ *satisfies*

$$0 \le u_0(x) \le \delta_0 W^{(\alpha)}(1,x), \tag{5.14}$$

*then*

(i) $\Psi(\mathscr{B}^+(2\delta_0)) \subset \mathscr{B}^+(2\delta_0)$.

(ii) *For* $v_1, v_2 \in \mathscr{B}^+(2\delta_0))$,

$$\|\Psi v_1 - \Psi v_2\|_{\mathscr{V}_\infty} \le \frac{1}{2}\|v_1 - v_2\|_{\mathscr{V}_\infty}. \tag{5.15}$$

The above lemma means that $\Psi : \mathscr{B}^+(2\delta_0) \to \mathscr{B}^+(2\delta_0)$ is a contraction mapping. Therefore, by the fixed point theorem, there exists $u \in \mathscr{B}^+(2\delta_0)$ such that $\Psi u = u$. In other words, this $u$ satisfies the integral equation (4.1). So, by Theorem 4.2, this $u$ solves the equation (1.1) in $(0, \infty) \times \mathbf{R}^n$. Thus we obtain

Theorem 5.1. *Assume that* $p_*(\alpha) = 1 + \frac{2\alpha}{n} < p$ *and that* $\frac{1}{2} < \alpha < 1$. *Then, in the sense of Definition* 4.1, *there exists a global in time strong solution to the semilinear parabolic pseudo-differential equation* (1.1) *with the initial data* $u_0$ *satisfying the condition* (A) *for sufficiently small* $\delta_0 > 0$.

*Lower bounds of the life span.*

In this subsection, we will give lower bounds of the life span of the solution to the following Cauchy problem under the assumption that $\frac{1}{2} < \alpha < 1$ and $1 < p \leq p_*(\alpha) = 1 + \frac{2\alpha}{n}$.

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = u^p, & t \in (0, T),\ x \in \mathbf{R}^n, \\ u(0, x) = \lambda \psi(x), & x \in \mathbf{R}^n, \end{cases} \tag{5.16}$$

Here we assume that $\psi \not\equiv 0$ satisfies the condition (A).

Let $T(\lambda)$ be the life span of the solution to (5.16), that is,

$$T(\lambda) = \sup\{T > 0;\ \text{the strong solution of (5.16) exists in } (0, T) \times \mathbf{R}^n\}. \tag{5.17}$$

It is easily seen by Theorem 4.2 that $T(\lambda)$ coincides with the life span of the $L^1 \cap L^\infty$-solution to the integral equation arising from (5.16).

The purpose of this subsection is to prove the following proposition.

Proposition 5.1. *Assume that* $1 < p \leq p_*(\alpha) = 1 + \frac{2\alpha}{n}$. *Then there exists a costant* $C > 0$ *such that for sufficiently small* $\lambda > 0$

$$C\lambda^{-2\alpha(p-1)/n(p_*(\alpha)-p)} \leq T(\lambda), \quad \text{in the case } 1 < p < p_*(\alpha), \tag{5.18}$$

$$C\lambda^{-2\alpha/n} \leq \log T(\lambda), \quad \text{in the case } p = p_*(\alpha). \tag{5.19}$$

Proof. Similarly as in Lemma 5.2, we have

Lemma 5.4. *Assume that* $1 < p \leq p_*(\alpha)$ *and* $1 < T < \infty$.

(i) *Let* $v \in \mathscr{V}_T^+$. *Then* $\Phi v \in \mathscr{V}_T^+$. *Moreover, if* $\|v\|_{\mathscr{V}_T} \leq \lambda$, *then*

$$\|\Phi v\|_{\mathscr{V}_T} \leq \begin{cases} C\lambda^p T^{1-n(p-1)/2\alpha} & \text{in the case } 1 < p < p_*(\alpha) \\ C\lambda^p \log T & \text{in the case } p = p_*(\alpha) \end{cases} \tag{5.20}$$

(ii) *Let* $v_1, v_2 \in \mathscr{V}_T^+$. *If* $\|v_1\|_{\mathscr{V}_T} \le \lambda$ *and* $\|v_2\|_{\mathscr{V}_T} \le \lambda$, *then we have*

$$\|\Phi v_1 - \Phi v_2\|_{\mathscr{V}_T} \le Cp\lambda^{p-1}\|v_1 - v_2\|_{\mathscr{V}_T} \times \begin{cases} T^{1-n(p-1)/2\alpha}, & \text{in the case } 1 < p < p_*(\alpha), \\ \log T, & \text{in the case } p = p_*(\alpha). \end{cases} \tag{5.21}$$

*In* (i) *and* (ii), *the constant* $C$ *does not depend on either* $\lambda$ *or* $T$.

Since $\psi$ satisfies the assumption (A), we have

$$\begin{aligned} 0 \le W^{(\alpha)}(t,\cdot) * \psi(x) &\le c_0 W^{(\alpha)}(t,\cdot) * W^{(\alpha)}(1,\cdot)(x) \\ &= c_0 W^{(\alpha)}(t+1,x) = c_0\rho(t,x), \end{aligned} \tag{5.22}$$

for some positive costant $c_0$. Namely we have $\|W^{(\alpha)}(t,\cdot) * \psi\|_{\mathscr{V}_T} \le c_0$.

For the above constant $c_0$ and a parameter $\lambda > 0$, we define a convex subset $\mathscr{B}^+(2c_0\lambda; T)$ of $\mathscr{V}_T$ by

$$\mathscr{B}^+(2c_0\lambda; T) := \{v \in \mathscr{V}_T^+; \|v\|_{\mathscr{V}_T} \le 2c_0\lambda\}. \tag{5.23}$$

Next, for a positive constant $\mu_0$, let us take $T = T_*(\lambda)$ such that

$$\begin{cases} C\lambda^{p-1}T^{1-n(p-1)/2\alpha} = \mu_0, & \text{in the case } 1 < p < p_*(\alpha). \\ C\lambda^{p-1}\log T = \mu_0, & \text{in the case } p = p_*(\alpha). \end{cases} \tag{5.24}$$

Here in the above equalities $C$ is the same constant as in Lemma 5.4. The above contant $\mu_0$ will be determined later.

We now define a mapping $\Psi_\lambda : \mathscr{V}_T^+ \to \mathscr{V}_T^+$ by

$$\begin{aligned} (\Psi_\lambda v)(t,x) &= \lambda W^{(\alpha)}(t,\cdot) * \psi(x) + (\Phi v)(t,x) \\ &= \lambda \int_{\mathbf{R}^n} W^{(\alpha)}(t, x-y)\psi(y)\,dy \\ &\quad + \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s, x-y)v(s,y)^p\,dyds \end{aligned} \tag{5.25}$$

Then by Lemma 5.4, (i) and (5.24), if $v \in \mathscr{B}^+(2c_0\lambda; T)$

$$\|\Psi_\lambda v\|_{\mathscr{V}_T} \le c_0\lambda + (2c_0)^p\mu_0\lambda. \tag{5.26}$$

Moreover, if $v_1, v_2 \in \mathscr{B}^+(2c_0\lambda; T)$, then

$$\|\Psi_\lambda v_1 - \Psi_\lambda v_2\|_{\mathscr{V}_T} \le p(2c_0)^{p-1}\mu_0\|v_1 - v_2\|_{\mathscr{V}_T}. \tag{5.27}$$

So let us choose a positive constant $\mu_0$ such that

$$(5.28)\qquad (2c_0)^p\mu_0 \le c_0, \quad p(2c_0)^{p-1}\mu_0 \le \frac{1}{2}.$$

Taking (5.26), (5.27), and (5.28) into account we see that

$$(5.29)\qquad \begin{aligned} &v \in \mathscr{B}^+(2c_0\lambda; T) \Rightarrow \Psi v \in \mathscr{B}^+(2c_0\lambda; T), \\ &v_1, v_2 \in \mathscr{B}^+(2c_0\lambda; T) \Rightarrow \|\Psi_\lambda v_1 - \Psi_\lambda v_2\|_{\mathscr{V}_T} \le \frac{1}{2}\|v_1 - v_2\|_{\mathscr{V}_T}. \end{aligned}$$

(5.29) shows that $\Psi_\lambda : \mathscr{B}^+(2c_0\lambda; T) \to \mathscr{B}^+(2c_0\lambda; T)$ is a contraction mapping. Therefore, by the fixed point theorem, there exists an element $u \in \mathscr{B}^+(2c_0\lambda; T)$ such that $\Psi_\lambda u = u$. By Theorem 4.2, this $u$ solves the Cauchy problem (5.16).

Summarizing the above argument, we have the following.

Take a positive constant $\mu_0$ such that (5.28) holds. Next, take $T = T_*(\lambda)$ such that (5.24) holds for the above $\mu_0$. Then there exists the solution $u$ to the Cauchy problem (5.16) in $(0, T) \times \mathbf{R}^n$. It follows from (5.24) that the above $T = T_*(\lambda)$ satisfies

$$(5.30)\qquad \begin{aligned} &T_*(\lambda) = \mu_0^{\{1-n(p-1)/2\alpha\}^{-1}} C^{1-n(p-1)/2\alpha} \lambda^{-2\alpha(p-1)/n(p_*(\alpha)-p)}, \\ &\qquad \text{in the case } 1 < p < p_*(\alpha), \\ &\log T_*(\lambda) = \mu_0 C^{-1} \lambda^{-2\alpha/n}, \quad \text{in the case } p = p_*(\alpha), \end{aligned}$$

which proves the assertion of Proposition 5.1. □

## 6. Upper Bounds of the Life Span

In this section, we give upper bounds of the life span of the solution to (1.11) with nontrivial initial data.

We start with the integral equation arising from (1.11).

$$(6.1)\qquad u(t,x) = \lambda W^{(\alpha)}(t,\cdot) * \psi(x) + \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s, x-y) u(s,y)^p \, dyds.$$

For the solution $u_\lambda$ of (6.1), let

$$(6.2)\qquad F_\lambda(t) = \int_{\mathbf{R}^n} W^{(\alpha)}(t,x) u_\lambda(t,x) \, dx$$

Then, our first objective is to estimate the blowup time of $F_\lambda(t)$.

Here we note that without loss of generality we may assume that

$$\psi(x) \geq c_0 W^{(\alpha)}(1, x), \tag{6.3}$$

for some constant $c_0 > 0$. In fact, we have the following lemma.

LEMMA 6.1. *Assume that the intial data* $u_0$ *of* (1.1) *is nonzero, that is,* $u_0(x) \geq 0$, *and* $u_0 \not\equiv 0$. *For each* $t_0 > 0$, *the solution* $u$ *to* (1.1) *satisfies*

$$u(t_0, x) \geq c_0 W^{(\alpha)}(t_0, x), \tag{6.4}$$

*for some costant* $c_0 > 0$.

(This lemma is easily seen, so we omit the proof.)

As the first step, we will prove the following.

LEMMA 6.2.

$$F_\lambda(t) \geq \lambda c_0 W^{(\alpha)}(1, 0)(2t+1)^{-n/2\alpha} + (2t)^{-n/2\alpha} \int_0^t s^{n/2\alpha} F_\lambda(s)^p \, ds. \tag{6.5}$$

PROOF. Multiply both sides of (6.1) by $W^{(\alpha)}(t, x)$ and integrate with respect to $x$. Then we have

$$\begin{aligned} F_\lambda(t) = \lambda \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} W^{(\alpha)}(t, x) W^{(\alpha)}(t, x-y) \psi(y) \, dxdy \\ + \int_0^t \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} W^{(\alpha)}(t, x) W^{(\alpha)}(t-s, x-y) u_\lambda(s, y)^p \, dxdyds. \end{aligned} \tag{6.6}$$

Using the semigroup property

$$\int_{\mathbf{R}^n} W^{(\alpha)}(t, x) W^{(\alpha)}(t, x-y) \, dx = W^{(\alpha)}(2t, y), \tag{6.7}$$

we have

$$\begin{aligned} \text{The first term of RHS of (6.6)} &= \lambda \int_{\mathbf{R}^n} W^{(\alpha)}(2t, y) \psi(y) \, dy \\ \text{(by (6.3))} &\geq \lambda c_0 \int_{\mathbf{R}^n} W^{(\alpha)}(2t, y) W^{(\alpha)}(1, y) \, dy \\ \text{(by semigroup property)} &= \lambda c_0 W^{(\alpha)}(2t+1, 0) \\ &= \lambda c_0 W^{(\alpha)}(1, 0)(2t+1)^{-n/2\alpha}. \end{aligned} \tag{6.8}$$

Similarly we have

(6.9) The second term of RHS of (6.6)

$$= \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(2t-s, y) u_\lambda(s, y)^p \, dyds$$

$$\geq \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(2t, y) u_\lambda(s, y)^p \, dyds$$

$$\geq \int_0^t \int_{\mathbf{R}^n} \left(\frac{2t}{s}\right)^{-n/2\alpha} W^{(\alpha)}(s, y) u_\lambda(s, y)^p \, dyds$$

$$\geq (2t)^{-n/2\alpha} \int_0^t s^{n/2\alpha} F_\lambda(s)^p \, ds. \qquad \square$$

Let

$$G_\lambda(t) = t^{n/2\alpha} F_\lambda(t). \tag{6.10}$$

Then Lemma 6.2 yields the following.

LEMMA 6.3.

$$G_\lambda(t) \geq c_1 \lambda + 2^{-n/2\alpha} \int_1^t s^{(n/2\alpha)(1-p)} G_\lambda(s)^p \, ds, \quad \textit{for } t \geq 1, \tag{6.11}$$

*where* $c_1 = 3^{-n/2\alpha} c_0 W^{(\alpha)}(1, 0)$.

Let $H_\lambda(t)$ be the solution to the integral equation

$$H_\lambda(t) = c_1 \lambda + 2^{-n/2\alpha} \int_1^t s^{(n/2\alpha)(1-p)} H_\lambda(s)^p \, ds, \quad \text{for } t \geq 1. \tag{6.12}$$

By comparison theorem, we have

$$G_\lambda(t) \geq H_\lambda(t) \quad \text{for } t \geq 1. \tag{6.13}$$

Therefore,

$$F_\lambda(t) = t^{-n/2\alpha} G_\lambda(t) \geq t^{-n/2\alpha} H_\lambda(t) \quad \text{for } t \geq 1. \tag{6.14}$$

On the other hand, $H_\lambda(t)$ is given explicitely as follows.

$$H_\lambda(t) = \{c_1^{1-p}\lambda^{1-p} - (p-1)2^{-n/2\alpha}A_p(t)\}^{-1/(p-1)},$$

(6.15)

$$\text{Here}\quad A_p(t) = \begin{cases} \frac{1}{1-\frac{n}{2\alpha}(p-1)}\{t^{1-(n/2\alpha)(p-1)} - 1\}, & (1 < p < p_*(\alpha) = 1 + \frac{2\alpha}{n}) \\ \log t, & (p = p_*(\alpha) = 1 + \frac{2\alpha}{n}). \end{cases}$$

Obviously $H_\lambda(t)$ blows up in finite time. Thus by (6.14) $F_\lambda(t)$ also blows up in finite time. Let $T_F(\lambda)$ and $T_H(\lambda)$ be the blowup time of $F_\lambda$ and $H_\lambda$ respectively. Again by (6.14), $T_F(\lambda) \le T_H(\lambda)$. Here by (6.15)

(6.16)

$$T_H(\lambda) = (\lambda^{p-1} + c_2)\lambda^{-2\alpha(p-1)/n(p_*(\alpha)-p)}, \quad \text{in the case } 1 < p < p_*(\alpha),$$

$$\text{where } c_2 = \frac{\left(1 - \frac{n(p-1)}{2\alpha}\right)c_1^{1-p}}{(p-1)2^{-n/2\alpha}}$$

$$\log T_H(\lambda) = c_2\lambda^{-n/2\alpha}, \quad \text{in the case } p = p_*(\alpha),$$

$$\text{where } c_2 = \frac{c_1^{1-p}}{(p-1)2^{-n/2\alpha}}.$$

Therefore, we obtain

PROPOSITION 6.1. *$F_\lambda(t)$ blows up in finite time. In addition, the blowup time $T_F(\lambda)$ of $F_\lambda(t)$ is estimated as follows.*

$$T_F(\lambda) \le C\lambda^{-2\alpha(p-1)/n(p_*(\alpha)-p)}, \quad \textit{in the case } 1 < p < p_*(\alpha), \tag{6.17}$$

$$\log T_F(\lambda) \le C\lambda^{-n/2\alpha}, \quad \textit{in the case } p = p_*(\alpha), \tag{6.18}$$

*where $C$ is a constant depending only on $n$, $p$, and $\alpha$.*

The next objective is to estimate the blowup time of $u_\lambda$ using $T_F(\lambda)$.

Let $x = 0$ in the integral equation (6.1) and apply Jensen's inequality. Then

$$\begin{aligned} u_\lambda(t,0) &\ge \int_0^t \int_{\mathbf{R}^n} W^{(\alpha)}(t-s,y)u_\lambda(s,y)^p \, dyds \\ &\ge \int_0^t \left\{ \int_{\mathbf{R}^n} W^{(\alpha)}(t-s,y)u_\lambda(s,y)\, dy \right\}^p ds \\ &\ge \int_{(1/4)t}^{(1/2)t} \left\{ \int_{\mathbf{R}^n} W^{(\alpha)}(t-s,y)u_\lambda(s,y)\, dy \right\}^p ds. \end{aligned} \tag{6.19}$$

If $\frac{1}{4}t \le s \le \frac{1}{2}t$, then $\left|\left(\frac{s}{t-s}\right)^{1/2\alpha} y\right| \le |y|$ and $\frac{1}{3} \le \frac{s}{t-s}$. Thus by Theorem 3.2 (ii), we have

$$W^{(\alpha)}(t-s, y) = \left(\frac{s}{t-s}\right)^{n/2\alpha} W^{(\alpha)}\left(s, \left(\frac{s}{t-s}\right)^{1/2\alpha} y\right) \tag{6.20}$$

$$\ge 3^{-n/2\alpha} W^{(\alpha)}(s, y).$$

By making use of the above two inequalities, we obtain

$$u_\lambda(t, 0) \ge 3^{-np/2\alpha} \int_{(1/4)t}^{(1/2)t} \left\{ \int_{\mathbf{R}^n} W^{(\alpha)}(s, y) u_\lambda(s, y)\, dy \right\}^p ds \tag{6.21}$$

$$= 3^{-np/2\alpha} \int_{(1/4)t}^{(1/2)t} F_\lambda(s)^p\, ds.$$

If $\frac{1}{2}t \to T_F(\lambda) - 0$, then by Proposition 6.1 the RHS of inequality (6.21) (and thus the LHS of (6.21)) blow up. Here we note that the life span $T(\lambda)$ of $u_\lambda$ is nothing but the blowup time of $u_\lambda$. Therefore, we have

Proposition 6.2. *$u_\lambda$ blows up in finite time. Moreover, $T(\lambda) \le 2T_F(\lambda)$.*

Combining Proposition 5.1, Proposition 6.1, and Proposition 6.2, we obtain Theorem C in the introduction, namely,

Theorem 6.1. *There exist constants $C_1, C_2 > 0$ such that for sufficiently small $\lambda > 0$*

$$C_1 \lambda^{-2\alpha(p-1)/n(p_*(\alpha)-p)} \le T(\lambda) \le C_2 \lambda^{-2\alpha(p-1)/n(p_*(\alpha)-p)}, \quad \textit{in the case } 1 < p < p_*(\alpha), \tag{6.22}$$

$$C_1 \lambda^{-n/2\alpha} \le \log T(\lambda) \le C_2 \lambda^{-n/2\alpha}, \quad \textit{in the case } p = p_*(\alpha). \tag{6.23}$$

## 7. Some Generalization

In this section, we deal with a semilinear parablic pseudo-differential equation with a more generalized nonlinear term. Let us consider the following Cauchy problem.

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = g(u), & t \in (0, \infty),\ x \in \mathbf{R}^n, \\ u(0, x) = u_0(x) \ge 0, & x \in \mathbf{R}^n, \end{cases} \tag{7.1}$$

where $g$ is a real valued function.

Then, by the same argument as in Section 4, we have the following.

THEOREM 7.1. *If we assume the condition*
(H1): $g \in C^1([0,\infty))$.

*In addition, we assume that* $u_0 \in L^1 \cap L^\infty$. *Then*, (7.1) *has a unique strong solution* $u$ *in* $[0,T) \times \mathbf{R}^n$ *for some* $T > 0$ *in the following sense.*

(i) *$u$ is of class $C^1$ with respect to $t \in (0,T)$ and of class $C^2$ with respect to $x \in \mathbf{R}^n$. Moreover,* $u \in C((0,T), H_1^2(\mathbf{R}^n) \cap H_\infty^2(\mathbf{R}^n)) \cap C^1((0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$

(ii) $(-\Delta)^\alpha u \in C((0,T) \times \mathbf{R}^n)$ *and* $(-\Delta)^\alpha u \in C((0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$.

(iii) *$u$ satisfies the semilinear parabolic pseudo-differential equation*

$$\partial_t u + (-\Delta)^\alpha u = g(u),$$

*as an equality in* $C((0,T); L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$.

Next, let us consider the problem of life span. Let $T(\lambda; g, \psi)$ be the life span of the strong solution to the following Cauchy problem.

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u = g(u), & t \in (0,\infty),\ x \in \mathbf{R}^n, \\ u(0,x) = \lambda\psi(x), & x \in \mathbf{R}^n, \end{cases} \tag{7.2}$$

where $\psi(x)$ is a function which satisfies the condition (A) in Section 5.

Here in addition to (H1), we assume the following two conditions (H2) and (H3) on $g$.

(H2): $g'(u) \geq 0$ for $u \in [0,\infty)$.

(H3): There exist constants $C_1, C_2 > 0$ and $p > 1$ such that $C_1 u^p \leq g(u) \leq C_2 u^p$ for $u \in [0,\infty)$.

Then, we can apply the comparison theorem (Theorem 3.3) to (7.2). Thus we have

$$T(\lambda; C_2 u^p, \psi) \leq T(\lambda; g, \psi) \leq T(\lambda; C_1 u^p, \psi). \tag{7.3}$$

Therefore, by Theorem C, we obtain

THEOREM 7.2. (i) *If* $1 < p < p_*(\alpha) = 1 + \frac{2\alpha}{n}$, *then*

$$T(\lambda; g, \psi) \sim \lambda^{-2\alpha(p-1)/n(p_*(\alpha)-p)}, \quad \textit{as } \lambda \to 0. \tag{7.4}$$

(ii) *If* $p = p_*(\alpha) = 1 + \frac{2\alpha}{n}$, *then*

$$\log T(\lambda; g, \psi) \sim \lambda^{-2\alpha/n}, \quad \textit{as } \lambda \to 0. \tag{7.5}$$

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Department of Mathematics, Faculty of Science, Okayama University
E-mail address: kakehi@math.okayama-u.ac.jp

Fujitsu Terminal Systems
E-mail address: k-sakai@fjts.fujitsu.com