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ON THE CARTIER DUALITY OF CERTAIN FINITE GROUP SCHEMES OF TYPE $(p^n, \dots, p^n)$

By

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Abstract. In this paper, we determine the Cartier dual of certain finite group schemes of type $(p^n, \dots, p^n)$, restricting ourselves to positive characteristic p case. They are given by the kernel of certain endomormphisms of the fibre product $W_{l,A} \times_{\mathrm{Spec}\,A} \cdots \times_{\mathrm{Spec}\,A} W_{l,A}$ of the group scheme of Witt vectors of the length l. Moreover we can treat the kernel of the endomorphism of a type $F^n + \boldsymbol{a}_1 F^{n-1} + \cdots + \boldsymbol{a}_n : W_{l,A} \to W_{l,A}$ as our special class, where F is the Frobenius endomorphism and $\boldsymbol{a}_k$ $(k = 1, \dots, n)$ are suitable Witt vectors.

1. Introduction

Throughout this paper, we denote by p a prime number. Let A be a commutative unitary ring of characteristic p. For a group scheme G over A, we denote by $\hat{G}$ the formal completion of G along the zero section. Our argument is expanded on the group schemes introduced by T. Sekiguchi and N. Suwa [SS2, Theorem 3.2 and Theorem 3.3]

$$\mathscr{E}_n := \mathrm{Spec}\, A\left[X_0, X_1, \dots, X_{n-1}, \frac{1}{1+\lambda_1 X_0}, \frac{1}{D_1(X_0)+\lambda_2 X_1}, \frac{1}{D_{n-1}(X_0, \dots, X_{n-2}) + \lambda_n X_{n-1}}\right].$$

These group schemes are constructed inductively by the following extensions;

$$\mathscr{E}_1 = \mathscr{G}^{(\lambda_1)} = \mathrm{Spec}\, A\left[X_0, \frac{1}{1+\lambda_1 X_0}\right]$$

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$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{G}^{(\lambda_2)} & \longrightarrow & \mathscr{E}_2 = \operatorname{Spec} A\left[X_0, X_1, \dfrac{1}{1+\lambda_1 X_0}, \dfrac{1}{D_0(X_0)+\lambda_2 X_1}\right] & \longrightarrow & \mathscr{E}_1 & \longrightarrow & 0 \\
0 & \longrightarrow & \mathscr{G}^{(\lambda_3)} & \longrightarrow & \mathscr{E}_3 & \longrightarrow & \mathscr{E}_2 & \longrightarrow & 0 \\
 & & & & \vdots & & & & \\
0 & \longrightarrow & \mathscr{G}^{(\lambda_{n-1})} & \longrightarrow & \mathscr{E}_n & \longrightarrow & \mathscr{E}_{n-1} & \longrightarrow & 0.
\end{array}
$$

Here, $\lambda_1, \lambda_2, \dots, \lambda_n$ are elements of A, and $D_i(X_0, X_1, \dots, X_{n-1})$'s are given as elements of $\operatorname{Hom}_{A/\lambda_i}(\hat{\mathscr{E}}_{i-1}, \hat{\mathbf{G}}_{m,A/\lambda_i})$. For deciding D_i's more explicitly, they introduced the endomorphism $U^n : W_A^n \to W_A^n$ on the fibre product space of group schemes of Witt vectors, and showed the canonical isomorphism;

$$\operatorname{Ker}[U^n : W_A(A)^n \to W_A(A)^n] \simeq \operatorname{Hom}(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}).$$

Moreover they also showed the canonical isomorphism;

$$\operatorname{Coker}[U^n : W_A(A)^n \to W_A(A)^n] \simeq H_0^2(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}),$$

where $H_0^2(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A})$ means the Hochschild cohomology groups.

By these results, the homomorphism $D_i \in \operatorname{Hom}_{A/\lambda_i}(\hat{\mathscr{E}}_{i-1}, \hat{\mathbf{G}}_{m,A/\lambda_i})$ is given by an element in $\operatorname{Ker}[U^{i-1} : W_{A/\lambda_i}(A/\lambda_i)^{i-1} \to W_{A/\lambda_i}(A/\lambda_i)^{i-1}]$. The group scheme structure of $\mathscr{E}_n$ is given by the one which makes the morphism

$$\alpha^{(n)} : \mathscr{E}_n \to \mathbf{G}_{m,A}^n,$$

defined by

$$(x_0, x_1, \dots, x_{n-1}) \mapsto (1 + \lambda_1 x_0, D_1(x_0) + \lambda_2 x_1, \dots, D_{n-1}(x_0, \dots, x_{n-2}) + \lambda_n x_{n-1}),$$

a homomorphism of group schemes. Hereafter let l be a positive integer. For a given group scheme $\mathscr{E}_n$ such as above, if we take the p^l-th power of the data defining $\mathscr{E}_n$, then those data defines a group scheme $\mathscr{E}_n^{p^l}$;

$$\mathscr{E}_n^{p^l} := \operatorname{Spec} A\left[X_0, X_1, \dots, X_{n-1}, \frac{1}{1+\lambda_1^{p^l} X_0}, \frac{1}{D_1'(X_0) + \lambda_2^{p^l} X_1}, \frac{1}{D_{n-1}'(X_0, \dots, X_{n-2}) + \lambda_n^{p^l} X_{n-1}}\right].$$

The group scheme structure of $\mathscr{E}_n^{p^l}$ is given by the one which makes the morphism

$$\alpha^{(n)'} : \mathscr{E}_n^{p^l} \to \mathbf{G}_{m,A}^n,$$

defined by

$$(x_0, x_1, \dots, x_{n-1}) \mapsto (1 + \lambda_1^{p^l} x_0, D_1'(x_0) + \lambda_2^{p^l} x_1, \dots, D_{n-1}'(x_0, \dots, x_{n-2}) + \lambda_n^{p^l} x_{n-1}),$$

a homomorphism. And this satisfies the following commutative diagram;

$$\begin{array}{ccc} \mathscr{E}_n & \xrightarrow{\alpha^{(n)}} & \mathbf{G}_{m,A}^n \\ {\scriptstyle \psi^{(l)}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathscr{E}_n^{p^l} & \xrightarrow{\alpha^{(n)\prime}} & \mathbf{G}_{m,A}^n \end{array}$$

where φ is given by $\varphi(t_0, \dots, t_{n-1}) = (t_0^{p^l}, \dots, t_{n-1}^{p^l})$ and $\psi^{(l)}$ is isogeny defined by $\psi^{(l)}(x_0, \dots, x_{n-1}) = (x_0^{p^l}, \dots, x_{n-1}^{p^l})$. Then the kernel $N_l = \mathrm{Ker}\ \psi^{(l)}$ is given explicity by

$$N_l = \mathrm{Spec}\ A[X_0, \dots, X_{n-1}]/(X_0^{p^l}, \dots, X_{n-1}^{p^l}),$$

and we have the exact sequence;

$$0 \longrightarrow N_l \xrightarrow{\iota} \mathscr{E}_n \xrightarrow{\psi^{(l)}} \mathscr{E}_n^{p^l} \longrightarrow 0.$$

Note that the group scheme sturucture of N_l is the one induced from $\mathscr{E}_n$. In our argument, the important thing is that we can identify the finite group scheme N_l with the completion $\hat{N}_l$, because X_i's are nilpotents in the coordinate ring of N_l, and we can consider the exact sequence;

$$0 \longrightarrow N_l \xrightarrow{\iota} \hat{\mathscr{E}}_n \xrightarrow{\psi^{(l)}} \hat{\mathscr{E}}_n^{p^l} \longrightarrow 0.$$

By means of the definition of the endomorphism

$$U^n : W_A^n \to W_A^n,$$

it induces an endomorphism

$$U_l^n : W_{l,A}^n \to W_{l,A}^n$$

which makes the following commutative diagram;

$$\begin{array}{ccc} W_A^n & \xrightarrow{(R_l)^n} & W_{l,A}^n \\ {\scriptstyle U^n}\downarrow & & \downarrow{\scriptstyle U_l^n} \\ W_A^n & \xrightarrow{(R_l)^n} & W_{l,A}^n. \end{array}$$

Under these notations, our first main result is given as follows;

Theorem 1. *Assum that A is a commutative unitary ring of chracteritic p. Then the Cartier dual of N_l is canonically isomorphic to* $\mathrm{Ker}[U_l^n : W_{l,A}^n \to W_{l,A}^n]$.

Oort-Tate [OT] gave the result of Theorem 1 in the case of $l = n = 1$. Next M. Amano [A] proved Theorem 1 for any l and $n = 1$, and N. Aki and M. Amano [AA] proved Theorem 1 for any l and $n = 2$ by using the deformations of Artin-Hasse exponential series. We prove Theorem 1 in the general case by generalizing the argument in the previous paper [AA].

Let K be a perfect field of characteritic p. Then we have Dieudonné ring $\mathbf{D}_K$ and the isomorphism $\mathbf{D}_K/\mathbf{D}_K V^l \simeq \mathrm{Hom}(W_{l,K}, W_{l,K})$. ([DG, p. 550].) From this point of view, $F^n + \boldsymbol{a}_1 F^{n-1} + \cdots + \boldsymbol{a}_n$ is an element of $\mathbf{D}_K/\mathbf{D}_K V^l$ for Witt vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in W_{l,A}$. In Section 6, we give an isomorphism;

$$\mathrm{Ker}[U_l^n : W_{l,A}^n \to W_{l,A}^n] \simeq \mathrm{Ker}[F^n + \boldsymbol{a}_1 F^{n-1} + \cdots + \boldsymbol{a}_n : W_{l,A} \to W_{l,A}],$$

for some special type of Witt vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in W_{l,A}$, and we give the second asertion;

Theorem 2. *If we choose the base ring A of characteritic p and the group scheme $\mathscr{E}_n$ suitably, then the Cartier dual of N_l is canonically isomorphic to* $\mathrm{Ker}[F^n + \boldsymbol{a}_1 F^{n-1} + \cdots + \boldsymbol{a}_n : W_{l,A} \to W_{l,A}]$, *where for each $1 \leq k \leq n$, $\boldsymbol{a}_k$ is Witt vectors given by* $\boldsymbol{a}_k = \sum_{n \geq i_1 > i_2 \cdots > i_k \geq 1} (-1)^k [\prod_{j=1}^k \lambda_{i_k}^{(p-1)p^{n-i_j-(j-1)}}]$, *and $[\lambda_{i_k}]$ is the Teichmüller lifting $(\lambda_{i_k}, 0, \dots) \in W(A)$ of $\lambda_{i_k} \in A$.*

The contents of this paper is as follows. The next two sections are devoted to give the definitions and some reviews of properties of Witt vectors, the deformed Artin-Hasse exponential series and the group schemes $\mathscr{E}_n$ and $\mathscr{E}_n^{p^l}$. In Section 5 and Section 6 we give the proofs of Theorem 1 and Theorem 2.

Notations

$\mathbf{G}_{m,A}$: the multiplicative group scheme over A
$W_{n,A}$: the group scheme of Witt vectors of length n over A
W_A: the group scheme of Witt vectors over A
$\hat{\mathbf{G}}_{m,A}$: the multiplicative formal group scheme over A
$\hat{W}_{n,A}$: the formal group scheme of Witt vectors of length n over A
$\hat{W}_A$: the formal group scheme of Witt vectors over A
F: the Frobenius of endomorphism of W_A
V: the Verschiebung endomorphism of W_A

R_n: the restriction homomorphism of W_A to $W_{n,A}$
$[\lambda]$: the Teichmüller lifting $(\lambda, 0, \dots) \in W(A)$ of $\lambda \in A$
$\boldsymbol{a}^{(p)} := (a_0^p, a_1^p, \dots)(= F(\boldsymbol{a}))$ $(\boldsymbol{a} = (a_0, a_1, \dots) \in W(A))$
$F^{(\lambda)} := F - [\lambda^{p-1}]$
$\mathbf{X} := (X_0, X_1, \dots)$ (a sequence of variables)
$\mathbf{Y} := (Y_0, Y_1, \dots)$ (a sequence of variables)

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2. Witt Vectors

In this short section we recall necessary facts on Witt vectors for this paper. For details, see [DG, Chap. V] or [HZ, Chap. III].

2.1. Let $\mathbf{X} = (X_0, X_1, \dots)$ be a sequence of variables. For each $n \geq 0$, we denote by $\Phi_n(\mathbf{X}) = \Phi_n(X_0, X_1, \dots, X_n)$ the Witt polynomial

$$\Phi_n(\mathbf{X}) = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n$$

in $\mathbf{Z}[\mathbf{X}] = \mathbf{Z}[X_0, X_1, \dots]$. Let $W_{n,\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[X_0, X_1, \dots, X_{n-1}]$ be the n-dimentional affine space over $\mathbf{Z}$. We define a morphism (the so-called Phantom map) $\Phi^{(n)}$ by

$$\Phi^{(n)} : W_{n,\mathbf{Z}} \to \mathbf{A}_{\mathbf{Z}}^n; \quad \boldsymbol{x} \mapsto (\Phi_0(\boldsymbol{x}), \Phi_1(\boldsymbol{x}), \dots, \Phi_{n-1}(\boldsymbol{x})).$$

Note that $W_{n,\mathbf{Z}}$ has the ring so that $\Phi^{(n)}$ becomes a ring scheme homomorphism, when $\mathbf{A}_{\mathbf{Z}}^n$ is regarded as a ring scheme by coordinate-wise addition and multiplication.

2.2. The Verschiebung homomorphism V is defined by

$$V : W(A) \to W(A); \quad \boldsymbol{x} = (x_0, x_1, \dots) \mapsto V(\boldsymbol{x}) = (0, x_1, x_2, \dots).$$

The restriction homorphism R_n is defined by

$$R_n : W(A) \to W_n(A); \quad \boldsymbol{x} = (x_0, x_1, \dots) \mapsto \boldsymbol{x}_n = (x_0, x_1, \dots, x_{n-1}).$$

We define a morphism $F : W_n(A) \to W_{n-1}(A)$ by

$$\Phi_i(F\boldsymbol{x}) = \Phi_{i+1}(\boldsymbol{x})$$

for $\boldsymbol{x} \in W_n(A)$. If A is of characteristic p, F is noting but the usual Frobenius endomorphism. For $\lambda \in A$, $[\lambda]$ and $F^{(\lambda)}$ denote the Teichmüller lifting $[\lambda] = (\lambda, 0, \dots) \in W(A)$ and the endomorphism $F - [\lambda^{p-1}]$ of $W(A)$, respectively. For $\boldsymbol{a} = (a_0, a_1, \dots) \in W(A)$, we also define a morphism $T_{\boldsymbol{a}} : W(A) \to W(A)$ by

$$\Phi_n(T_{\boldsymbol{a}}\boldsymbol{x}) = a_0^{p^n}\Phi_n(\boldsymbol{x}) + pa_1^{p^{n-1}}\Phi_{n-1}(\boldsymbol{x}) + \cdots + p^n a_n \Phi_0(\boldsymbol{x})$$

for $\boldsymbol{x} \in W(A)$. Then it is known that this morphism satisfies the formula $T_{\boldsymbol{a}} = \sum_{k \geq 0} V^k \cdot [a_k]$. (cf. [SS2, Chap. 4, p. 20].)

3. Deformed Artin-Hasse Exponential Series

In this short section we recall nececssary facts on the deformed Artin-Hasse exponential series for this paper.

3.1. The Artin-Hasse exponential series $E_p(X)$ is given by

$$E_p(X) = \exp\left(\sum_{r \geq 0} \frac{X^{p^r}}{p^r}\right) \in \mathbf{Z}_{(p)}[[X]].$$

We define a formal power series $E_p(U, \lambda; X)$ in $\mathbf{Q}[U, \lambda][[X]]$ by

$$E_p(U, \Lambda; X) = (1 + \Lambda X)^{U/\Lambda} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} X^{p^k})^{(1/p^k)((U/\Lambda)^{p^k} - (U/\Lambda)^{p^{k-1}})}.$$

As in [SS1, Corollrary 2.5] or [SS2, Lemma 4.8], we see that this formal power series $E_p(U, \lambda; X)$ is integral over $\mathbf{Z}_{(p)}$.

Let A be a $\mathbf{Z}_{(p)}$-algebra. Let $\lambda \in A$ and $\boldsymbol{v} = (v_0, v_1, \dots) \in W(A)$. We define a formal power series $E_p(\boldsymbol{v}, \lambda; X)$ in $A[[X]]$ by

$$\begin{aligned} E_p(\boldsymbol{v}, \lambda; X) &= \prod_{k=0}^{\infty} E_p(v_k, \lambda^{p^k}; X^{p^k}) \\ &= (1 + \lambda X)^{v_0/\lambda} \prod_{k=1}^{\infty} (1 + \lambda^{p^k} X^{p^k})^{(1/p^k \lambda^{p^k})\Phi_{k-1}(F^{(\lambda)}\boldsymbol{v})}. \end{aligned}$$

Moreover we define a formal power series $F_p(\boldsymbol{v}, \lambda; X, Y)$ as follows;

$$F_p(\boldsymbol{v}, \lambda; X, Y) = \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k}(X + Y + \lambda XY)^{p^k}} \right)^{(1/p^k \lambda^{p^k})\Phi_{k-1}(\boldsymbol{v})}.$$

As in [SS1, Lemma 2.16] or [SS2, Lemma 4.9], we see that the formal power series $F_p(\boldsymbol{v}, \lambda; X, Y)$ satisfies the formula;

$$\frac{E_p(\boldsymbol{v}, \lambda; X) E_p(\boldsymbol{v}, \lambda; Y)}{E_p(\boldsymbol{v}, \lambda; X + Y + \lambda XY)} = F_p(F^{(\lambda)}\boldsymbol{v}, \lambda; X, Y).$$

4. Definitions of the Group Schemes $\mathscr{E}_n$ and $\mathscr{E}_n^{p^l}$

We review here the group schemes $\mathscr{E}_n$ briefly from T. Sekiguchi and N. Suwa [SS2, Theorem 3.3].

4.1. Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda, \lambda_1, \dots, \lambda_n$ be non-zero elements of A. For a vector $\bar{\boldsymbol{a}}$ of $W(A/\lambda)$, we denote by $\boldsymbol{a} \in W(A)$ a representative of $\bar{\boldsymbol{a}}$. Note that the formal completion $\hat{W}$ is characterized as a functor given by;

$$\hat{W}(A) = \{(a_0, a_1, \dots) \in W(A) \mid a_i \text{ are nilpotents and } a_i = 0 \text{ for almost all } i\}.$$

We choose Witt vectors

$$\bar{\boldsymbol{a}}^i = (\bar{\boldsymbol{a}}_j^i)_{1 \leqq j \leqq i} \in \operatorname{Ker}[U^i : \hat{W}(A/\lambda_{i+1})^i \to \hat{W}(A/\lambda_{i+1})^i]$$

inductively by the following recursive conditions;

$$U^1 := F^{(\lambda_1)}, \quad \bar{\boldsymbol{a}}^1 := \bar{\boldsymbol{a}}_1^1 \in \operatorname{Ker}[U^1 : \hat{W}(A/\lambda_2) \to \hat{W}(A/\lambda_2)]$$

$$\boldsymbol{b}_1^2 := \frac{1}{\lambda_2} F^{(\lambda_1)} \boldsymbol{a}_1^1, \quad U^2 := \begin{pmatrix} F^{(\lambda_1)} & -T_{\boldsymbol{b}_1^2} \\ 0 & F^{(\lambda_2)} \end{pmatrix},$$

and for $k \geqq 2$, we choose

$$\bar{\boldsymbol{a}}^k := (\bar{\boldsymbol{a}}_j^k)_{1 \leqq j \leqq k} \in \operatorname{Ker}[U^k : \hat{W}(A/\lambda_{k+1})^k \to \hat{W}(A/\lambda_{k+1})^k],$$

and we define

$$\boldsymbol{b}_j^{k+1} := \frac{1}{\lambda_{k+1}} \left(F^{(\lambda_j)} \boldsymbol{a}_j^k - \sum_{l=j+1}^{k} T_{\boldsymbol{b}_j^l} \boldsymbol{a}_l^k \right) \quad 1 \leqq j \leqq k-1$$

$$\boldsymbol{b}_k^{k+1} := \frac{1}{\lambda_{k+1}} F^{(\lambda_k)} \boldsymbol{a}_k^k$$

$$U^{k+1} := \begin{pmatrix} F^{(\lambda_1)} & -T_{\boldsymbol{b}_1^2} & \cdots & \cdots & -T_{\boldsymbol{b}_1^{k+1}} \\ 0 & F^{(\lambda_2)} & -T_{\boldsymbol{b}_2^3} & \cdots & -T_{\boldsymbol{b}_2^{k+1}} \\ 0 & \cdots & & & -T_{\boldsymbol{b}_k^{k+1}} \\ 0 & & \cdots & & F^{(\lambda_{k+1})} \end{pmatrix}.$$

Moreover we define formal power series $D_k(X_0, X_1, \dots, X_{k-1})$ $(k \geqq 1)$ by

$$D_0 = 1$$

$$D_1(X_0) = E_p(\bar{\boldsymbol{a}}_1^1, \lambda_1; X_0)$$

and for $k \geqq 1$

$$\begin{aligned} D_{k+1}(X_0, X_1, \dots, X_k) &= E_p(\bar{\boldsymbol{a}}^{k+1}, (\lambda_i)_{1 \leqq i \leqq k+1}; X_0, X_1, \dots, X_k) \\ &:= \prod_{i=1}^{k+1} E_p\left(\bar{\boldsymbol{a}}_i^{k+1}, \lambda_i; \frac{X_{i-1}}{D_{i-1}(X_0, X_1, \dots, X_{i-2})}\right). \end{aligned}$$

We put

$$\begin{aligned} \mathscr{E}_n := \operatorname{Spec} A\Bigg[X_0, X_1, \dots, X_{n-1}, \frac{1}{1 + \lambda_1 X_0}, \\ \frac{1}{D_1(X_0) + \lambda_2 X_1}, \dots, \frac{1}{D_{n-1}(X_0, \dots, X_{n-2}) + \lambda_n X_{n-1}}\Bigg]. \end{aligned}$$

Then by [SS2, Theorem 4.16 and Theorem 3.3], $\mathscr{E}_n$ becomes a group scheme and

$$D_i \in \operatorname{Hom}_{A/\lambda_{i+1}}(\mathscr{E}_n \otimes_A A/\lambda_{i+1}, \mathbf{G}_{A/\lambda_{i+1}}), \quad \text{for } i = 1, \dots, n-1.$$

4.2. In this subsection, let A be of characteristic p and $\lambda_1, \dots, \lambda_n \in A$. We will define a group scheme denoted by $\mathscr{E}_n^{p^l}$. Let l be positive interger and $(\lambda_i^{p^l})_{1 \leqq i \leqq k+1}$ be elements of A. We define Witt vectors inductively by the following recursive condisions.

For $\bar{\boldsymbol{a}}_1^1 \in \operatorname{Ker} U^1$, we have a relation

$$F^{(\lambda_1)}(\boldsymbol{a}_1^1) \equiv 0 \mod \lambda_2.$$

So we have also the following relation

$$F^{(\lambda_1^{p^l})}((\boldsymbol{a}_1^1)^{(p^l)}) \equiv 0 \mod \lambda_2^{p^l}.$$

We put $\overline{\mathbf{A}}_1^1 := \bar{\boldsymbol{a}}_1^{1(p^l)}$, then $\overline{\mathbf{A}}_1^1 \in \operatorname{Ker} F^{(\lambda_1^{p^l})}$. We define $(U^1)' := F^{(\lambda_1^{p^l})}$. Then we have the following congruences;

$$\overline{\mathbf{A}}^1 := \overline{\mathbf{A}}_1^1 \in \operatorname{Ker}[(U^1)' : \hat{W}(A/\lambda_2^{p^l}) \to \hat{W}(A/\lambda_2^{p^l})].$$

For $(\bar{\boldsymbol{a}}_i^k)_{1 \leqq i \leqq k} \in \operatorname{Ker} U^k$, we have following equations;

$$F^{(\lambda_1)}\boldsymbol{a}_1^k - T_{\boldsymbol{b}_1^2}\boldsymbol{a}_2^k - \cdots - T_{\boldsymbol{b}_1^2}\boldsymbol{a}_2^k \equiv 0 \mod \lambda_{k+1}$$
$$F^{(\lambda_2)}\boldsymbol{a}_2^k - \cdots - T_{\boldsymbol{b}_2^k}\boldsymbol{a}_k^k \equiv 0 \mod \lambda_{k+1}$$
$$\vdots$$
$$F^{(\lambda_k)}\boldsymbol{a}_k^k \equiv 0 \mod \lambda_{k+1}$$

and

$$(F^{(\lambda_1)}\boldsymbol{a}_1^k - T_{\boldsymbol{b}_1^2}\boldsymbol{a}_2^k - \cdots - T_{\boldsymbol{b}_1^2}\boldsymbol{a}_2^k)^{p^l} \equiv 0 \mod \lambda_{k+1}^{p^l}$$
$$(F^{(\lambda_2)}\boldsymbol{a}_2^k - \cdots - T_{\boldsymbol{b}_2^k}\boldsymbol{a}_k^k)^{p^l} \equiv 0 \mod \lambda_{k+1}^{p^l}$$
$$\vdots$$
$$(F^{(\lambda_k)}\boldsymbol{a}_k^k)^{p^l} \equiv 0 \mod \lambda_{k+1}^{p^l}.$$

Moreover we have the following equations;

$$F^{(\lambda_1^{p^l})}(\boldsymbol{a}_1^k)^{(p^l)} - \cdots - T_{\boldsymbol{b}_2^{k(p^l)}}(\boldsymbol{a}_k^k)^{(p^l)} \equiv 0 \mod \lambda_{k+1}^{p^l}$$
$$F^{(\lambda_2^{p^l})}(\boldsymbol{a}_2^k)^{(p^l)} - \cdots - T_{\boldsymbol{b}_2^{k(p^l)}}(\boldsymbol{a}_k^k)^{(p^l)} \equiv 0 \mod \lambda_{k+1}^{p^l}$$
$$\vdots$$
$$F^{(\lambda_k)^{p^l}}(\boldsymbol{a}_k^k)^{(p^l)} \equiv 0 \mod \lambda_{k+1}^{p^l}.$$

For $k \geqq 2$ we define

$$(U^2)' := \begin{pmatrix} F^{(\lambda_1^{p^l})} & -T_{\mathbf{B}_1^2} \\ 0 & F^{(\lambda_2^{p^l})} \end{pmatrix}$$

$$\mathbf{B}_1^2 := \frac{1}{\lambda_2^{p^l}} F^{(\lambda_1^{p^l})}\mathbf{A}_1^1 = \left(\frac{1}{\lambda_2}F^{(\lambda_1)}\boldsymbol{a}_1^1\right)^{p^l} = \bar{\boldsymbol{b}}_1^{2(p^l)}$$

$$\mathbf{B}_j^{k+1} := \frac{1}{\lambda_{k+1}}\left(F^{(\lambda_j^{p^l})}\mathbf{A}_j^k - \sum_{l=j+1}^{k} T_{\mathbf{B}_j^l}\mathbf{A}_l^k\right) \quad 1 \leqq j \leqq k-1$$

$$\mathbf{B}_k^{k+1} := \frac{1}{\lambda_{k+1}^{p^l}} F^{(\lambda_k^{p^l})}\mathbf{A}_k^k$$

$$(U^{k+1})' := \begin{pmatrix} F^{(\lambda_1^{p^l})} & -T_{\mathbf{B}_1^2} & \cdots & \cdots & -T_{\mathbf{B}_1^{k+1}} \\ 0 & F^{(\lambda_2^{p^l})} & -T_{\mathbf{B}_2^3} & \cdots & -T_{\mathbf{B}_2^{k+1}} \\ 0 & \cdots & & & -T_{\mathbf{B}_k^{k+1}} \\ 0 & & \cdots & & F^{(\lambda_{k+1}^{p^l})} \end{pmatrix}.$$

For $k \geqq 2$ we put

$$\overline{\mathbf{A}}^k := (\overline{\mathbf{A}}_j^k)_{1 \leqq j \leqq k} = (\bar{\boldsymbol{a}}_j^{k(p^l)})_{1 \leqq j \leqq k}.$$

And for $k \geqq 2$, we have relations

$$\overline{\mathbf{A}}^k = (\overline{\mathbf{A}}_j^k)_{1 \leqq j \leqq k} \in \mathrm{Ker}[(U^k)' : \hat{W}(A/\lambda_{k+1}^{p^l})^k \to \hat{W}(A/\lambda_{k+1}^{p^l})^k].$$

So we define formal power series $D_k'(X_0, X_1, \ldots, X_{k-1})$ $(k \geqq 1)$ by

$$D_0' = 1$$

$$D_1'(X_0) = E_p(\overline{\mathbf{A}}_1^1, \lambda_1^{p^l}; X_0)$$

and for $k \geqq 1$

$$\begin{aligned} D_{k+1}'(X_0, X_1, \ldots, X_k) &= E_p(\overline{\mathbf{A}}^{k+1}, (\lambda_i^{p^l})_{1 \leqq i \leqq k+1}; X_0, X_1, \ldots, X_k) \\ &:= \prod_{i=1}^{k+1} E_p\left(\overline{\mathbf{A}}_i^{k+1}, \lambda_i^{p^l}; \frac{X_{i-1}}{D_{i-1}'(X_0, X_1, \ldots, X_{i-2})}\right). \end{aligned}$$

Then we have a group scheme;

$$\mathrm{Spec}\, A\left[X_0, X_1, \ldots, X_{n-1}, \frac{1}{1 + \lambda_1^{p^l} X_0}, \frac{1}{D_1'(X_0) + \lambda_2^{p^l} X_1}, \ldots, \frac{1}{D_{n-1}'(X_0, \ldots, X_{n-2}) + \lambda_n^{p^l} X_{n-1}}\right]$$

satisfied the above conditions and in this case, we denote the group scheme by $\mathscr{E}_n^{p^l}$.

5. The Proof of Theorem 1

In this section we give our proof of Theorem 1. Suppose A is a commutative unitary ring of characteristic p. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be elements of A and $\mathscr{E}_n$ be a group scheme defined in section 4.1 and $\hat{\mathscr{E}}_n$ be the formal completion of $\mathscr{E}_n$ along the zero section. We can easily see that the map

$$\psi^{(l)} : \hat{\mathscr{E}}_n \to \hat{\mathscr{E}}_n^{p^l}; \quad (x_0, \dots, x_{n-1}) \mapsto (x_0^{p^l}, \dots, x_{n-1}^{p^l}),$$

is a homomorphism and the kernel of this isogeny is given by

$$\begin{aligned} N_l = \mathrm{Ker}\,\psi^{(l)} &= \mathrm{Spf}\ A[[X_0, X_1, \dots, X_{n-1}]]/(X_0^{p^l}, X_1^{p^l}, \dots, X_{n-1}^{p^l}) \\ &= \mathrm{Spec}\ A[X_0, X_1, \dots, X_{n-1}]/(X_0^{p^l}, X_1^{p^l}, \dots, X_{n-1}^{p^l}) \end{aligned}$$

since $X_0, X_1, \dots, X_{n-1}$ are nilpotents in the coordinate ring of N_l. The following exact sequence is induced by the homomorphism $\psi^{(l)}$;

$$0 \longrightarrow N_l \xrightarrow{\ \iota\ } \hat{\mathscr{E}}_n \xrightarrow{\psi^{(l)}} \hat{\mathscr{E}}_n^{p^l} \longrightarrow 0, \tag{1}$$

where ι is the canonical inclusion. This exact sequence (1) deduces the following long exact sequence;

$$\begin{array}{llllll} 0 \longrightarrow & \mathrm{Hom}(\hat{\mathscr{E}}_n^{p^l}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{\psi^{(l)^*}} & \mathrm{Hom}(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \mathrm{Hom}(N_l, \hat{\mathbf{G}}_{m,A}) \\ \xrightarrow{\ \partial\ } & \mathrm{Ext}^1(\hat{\mathscr{E}}_n^{p^l}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{\psi^{(l)^*}} & \mathrm{Ext}^1(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}) & \longrightarrow & \cdots. \end{array} \tag{2}$$

As a consequance of the argument in the proofs of Lemma 4 and Lemma 5, we will see that in the exact sequences, we can replace $\mathrm{Ext}^1(\hat{\mathscr{E}}_n^{p^l}, \hat{\mathbf{G}}_{m,A})$ and $\mathrm{Ext}^1(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A})$ with the Hochschild cohomology groups $H_0^2(\hat{\mathscr{E}}_n^{p^l}, \hat{\mathbf{G}}_{m,A})$ and $H_0^2(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A})$ respectively. Here $H_0^2(\hat{G}, \hat{H})$ denote the Hochschid cohomology group consisting of symmetric 2-cocycles of $\hat{G}$ with coefficients in $\hat{H}$ for formal group schemes G and H. (c.f. [DG, II.2 and Chap III.6].) Therfore we have the following exact sequence;

$$\begin{array}{llllll} 0 \longrightarrow & \mathrm{Hom}(\hat{\mathscr{E}}_n^{p^l}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{\psi^{(l)^*}} & \mathrm{Hom}(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \mathrm{Hom}(N_l, \hat{\mathbf{G}}_{m,A}) \\ \xrightarrow{\ \partial\ } & H_0^2(\hat{\mathscr{E}}_n^{p^l}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{\psi^{(l)^*}} & H_0^2(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}) & \longrightarrow & \cdots. \end{array} \tag{3}$$

On the other hand, as in the case n of [SS2, Theorem 5.1], the following morphisms are isomorphic;

$$\begin{aligned} \xi_0^n : \mathrm{Ker}[U^n : W(A)^n \to W(A)^n] &\to \mathrm{Hom}(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}); \\ \bar{\boldsymbol{v}}^n = (\bar{\boldsymbol{v}}_i^n) &\mapsto E_p(\bar{\boldsymbol{v}}^n, (\lambda_i); \mathbf{X}) \end{aligned} \tag{4}$$

$$\begin{aligned} \xi_1^n : \mathrm{Coker}[U^n : W(A)^n \to W(A)^n] &\to H_0^2(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}); \\ \overline{\boldsymbol{w}}^n = (\overline{\boldsymbol{w}}_i^n) &\mapsto F_p(\overline{\boldsymbol{w}}^n, (\lambda_i); \mathbf{X}, \mathbf{Y}). \end{aligned} \tag{5}$$

We put

$$U^n := \begin{pmatrix} F^{(\lambda_1)} & -T_{\boldsymbol{b}_1^2} & \cdots & \cdots & -T_{\boldsymbol{b}_1^n} \\ 0 & F^{(\lambda_2)} & -T_{\boldsymbol{b}_2^3} & \cdots & -T_{\boldsymbol{b}_2^n} \\ 0 & \cdots & & & -T_{\boldsymbol{b}_{n-1}^n} \\ 0 & & \cdots & & F^{(\lambda_n)} \end{pmatrix}$$

and

$$(U^n)' := \begin{pmatrix} F^{(\lambda_1^{p^l})} & -T_{\mathbf{B}_1^2} & \cdots & \cdots & -T_{\mathbf{B}_1^n} \\ 0 & F^{(\lambda_2^{p^l})} & -T_{\mathbf{B}_2^3} & \cdots & -T_{\mathbf{B}_2^n} \\ 0 & \cdots & & & -T_{\mathbf{B}_{n-1}^n} \\ 0 & & \cdots & & F^{(\lambda_n^{p^l})} \end{pmatrix}$$

where $\boldsymbol{b}_i^j$ and $\mathbf{B}_i^j$ (see section 4) are Witt vectors. We consider the following diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(A)^n & \xrightarrow{(V^l)^n} & W(A)^n & \xrightarrow{(R^l)^n} & W_l(A)^n & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle (U^n)'} & & \Big\downarrow{\scriptstyle U^n} & & \Big\downarrow{\scriptstyle U_l^n} & & \\ 0 & \longrightarrow & W(A)^n & \xrightarrow{(V^l)^n} & W(A)^n & \xrightarrow{(R^l)^n} & W_l(A)^n & \longrightarrow & 0, \end{array} \qquad (*)$$

where U_l^n is the restriction morphism of U^n to $W_l(A)^n$. Then we have the commutativity of this diagram.

Proposition 1. *The diagram* $(*)$ *is commutative.*

Lemma 1. $T_{\boldsymbol{b}_1^2} V^l = V^l T_{\mathbf{B}_1^2}$.

Proof. This follows from [AA, Sublemma 1].

By using this lemma, we get

$$\begin{aligned} \mathbf{B}_j^{k+1} &= \frac{1}{\lambda_{k+1}^{p^l}} \left(F^{(\lambda_j^{p^l})} \mathbf{A}_j^k - \sum_{l=j+1}^{k} T_{\mathbf{B}_j^l} \mathbf{A}_l^k \right) \\ &= \left(\frac{1}{\lambda_{k+1}} \left(F^{(\lambda_j)} \boldsymbol{a}_j^k - \sum_{l=j+1}^{k} T_{\boldsymbol{b}_j^l} \boldsymbol{a}_l^k \right) \right)^{p^l} \\ &= \boldsymbol{b}_i^{j\,(p^l)} \quad 1 \leqq j \leqq k-1. \end{aligned}$$

First we will check the equality $U^n \circ (V^l)^n = (V^l)^n \circ (U^n)'$.

In fact for $\boldsymbol{v}^n = (\boldsymbol{v}_i^n)_{1 \leqq i \leqq n} \in W(A)^n$, we have

$$U^n \circ (V^l)^n(\boldsymbol{v}^n) = \begin{pmatrix} F^{(\lambda_1)} V^l \boldsymbol{v}_1^n & -T_{\boldsymbol{b}_1^2} V^l \boldsymbol{v}_2^n & \cdots & \cdots & -T_{\boldsymbol{b}_1^n} V^l \boldsymbol{v}_n^n \\ 0 & F^{(\lambda_2)} V^l \boldsymbol{v}_2^n & -T_{\boldsymbol{b}_2^3} V^l \boldsymbol{v}_3^n & \cdots & -T_{\boldsymbol{b}_2^n} V^l \boldsymbol{v}_n^n \\ 0 & \cdots & & & -T_{\boldsymbol{b}_{n-1}^n} V^l \boldsymbol{v}_n^n \\ 0 & & \cdots & & F^{(\lambda_n)} V^l \boldsymbol{v}_n^n \end{pmatrix}$$

$$= \begin{pmatrix} V^l F^{(\lambda_1^{p^l})} \boldsymbol{v}_1^n & -V^l T_{\mathbf{B}_1^2} \boldsymbol{v}_2^n & \cdots & \cdots & -V^l T_{\mathbf{B}_1^n} \boldsymbol{v}_n^n \\ 0 & V^l F^{(\lambda_2^{p^l})} \boldsymbol{v}_2^n & -V^l T_{\mathbf{B}_2^3} \boldsymbol{v}_3^n & \cdots & -V^l T_{\mathbf{B}_2^n} \boldsymbol{v}_n^n \\ 0 & \cdots & & & -V^l T_{\mathbf{B}_{n-1}^n} \boldsymbol{v}_n^n \\ 0 & & \cdots & & V^l F^{(\lambda_n^{p^l})} \boldsymbol{v}_n^n \end{pmatrix}$$

$$= (V^l)^n \circ (U^n)'(\boldsymbol{v}^n). \qquad \square$$

The next equality $U_l^n \circ (R_l)^n = (R_l)^n \circ U^n$ is a direct conseqence of the definition of U_l^n. The exactness of the horizontal sequences are obvious. By applying the snake lemma to $(*)$, we have the following exact sequence;

$$\text{(6)} \quad \begin{aligned} 0 \longrightarrow \operatorname{Ker}(U^n)' &\xrightarrow{(V^l)^n} \operatorname{Ker} U^n \xrightarrow{(R_l)^n} \operatorname{Ker} U_l^n \\ \xrightarrow{\partial} \operatorname{Coker}(U^n)' &\xrightarrow{(V^l)^n} \operatorname{Coker} U^n \xrightarrow{(R_l)^n} \operatorname{Coker} U_l^n \longrightarrow 0. \end{aligned}$$

Then, we can combine the exact sequence (3), (6) and the isomorphisms (4), (5) we have the following diagram in which the two horizontal sequences are exact, and vertical morphisms except for ϕ are isomorphisms;

$$\text{(7)} \quad \begin{array}{ccccccc} \operatorname{Hom}(\hat{\mathscr{E}}_n^{p^l}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{\psi^{(l)*}} & \operatorname{Hom}(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \operatorname{Hom}(N_l, \hat{\mathbf{G}}_{m,A}) & & \\ \phi_1 \uparrow & & \phi_2 \uparrow & & \phi \uparrow & & \\ \operatorname{Ker}(U^n)' & \xrightarrow{(V^l)^n} & \operatorname{Ker} U^n & \xrightarrow{(R_l)^n} & \operatorname{Ker} U_l^n & & \\ & \xrightarrow{\partial} & H_0^2(\hat{\mathscr{E}}_n^{p^l}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{\psi^{(l)*}} & H_0^2(\hat{\mathscr{E}}_n, \hat{\mathbf{G}}_{m,A}) & & \\ & & \phi_3 \uparrow & & \phi_4 \uparrow & & \\ & \xrightarrow{\partial} & \operatorname{Coker}(U^n)' & \xrightarrow{(V^l)^n} & \operatorname{Coker} U^n & \xrightarrow{(R_l)^n} & \operatorname{Coker} U_l^n. \end{array}$$

Here, ϕ is the composite map $(\iota)^* \circ \xi_0^n$ of the morphism $(\iota)^*$ in (3) with the isomorphism ξ_0^n in (4). If the diagram (7) is proved to be true, we get the isomorphism $\phi : \operatorname{Ker}[U_l^n : W_l(A)^n \to W_l(A)^n] \simeq \operatorname{Hom}(N_l, \hat{\mathbf{G}}_{m,A})$ by the five lemma. So we obtain the Theorem 1. Next we will check the commutativity of (7).

Lemma 2. $(\psi^{(l)})^* \circ \phi_1 = \phi_2 \circ (V^l)^n$.

Proof. For $(\boldsymbol{v}^n) = (\boldsymbol{v}_i^n)_{1 \leqq i \leqq n}$, we have

$$\begin{aligned} E_p((\boldsymbol{v}^n), (\lambda_i^{p^l})_{1\leqq i\leqq n}; (x_{i-1})_{1\leqq i\leqq n}) &= \prod_{i=1}^n E_p\left(\boldsymbol{v}_i^n, (\lambda_i^{p^l}); \frac{x_{i-1}^{p^l}}{D_i'(x_0^{p^l}, x_1^{p^l}, \dots, x_{n-2}^{p^l})}\right) \\ &= \prod_{i=1}^n E_p\left(\boldsymbol{v}_i^n, (\lambda_i^{p^l}); \left(\frac{x_{i-1}}{D_i'(x_0, x_1, \dots, x_{n-2})}\right)^{p^l}\right) \\ &= \prod_{i=1}^n E_p\left(V^l\boldsymbol{v}_i^n, (\lambda_i); \frac{x_{i-1}}{D_i(x_0, x_1, \dots, x_{n-2})}\right) \\ &= E_p(V^l(\boldsymbol{v}^n), (\lambda_i)_{1\leqq i\leqq n}; (x_{i-1})_{1\leqq i\leqq n}). \end{aligned}$$

These equalities means our assertions. □

Lemma 3. $(\iota)^* \circ \phi_2 = \phi \circ (R_l)^n$.

Proof. This follows immediately from the definitions of ϕ and $(\iota)^*$. □

Lemma 4. $\partial \circ \phi = \phi_3 \circ \partial$.

Proof. For $(R_l)^n\boldsymbol{v}^n = (R_l\boldsymbol{v}_i^n)_{1\leqq i\leqq n} \in \operatorname{Ker} U_l^n$, we caculate $\partial E_p((R_l)^n\boldsymbol{v}^n, (\lambda_i)_{1\leqq i\leqq n}; (x_{i-1})_{1\leqq i\leqq n}) \in \operatorname{Ker} U_l^n$ on the fibre product $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n^{p^l}$ where the following diagram is commutative;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_l & \longrightarrow & \hat{\mathscr{E}}_n & \xrightarrow{\psi^{(l)}} & \hat{\mathscr{E}}_n^{p^l} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle E_p((R_l)^n\boldsymbol{v}^n, (\lambda_i); (x_{i-1}))} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\| & & \\ 0 & \longrightarrow & \hat{\mathbf{G}}_{m,A} & \longrightarrow & \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n^{p^l} & \longrightarrow & \hat{\mathscr{E}}_n^{p^l} & \longrightarrow & 0. \end{array} \tag{8}$$

By the above condition, we get Φ as the following map;

$$\Phi : \hat{\mathscr{E}}_n \to \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n^{p^l}; \quad (x_{i-1}) \mapsto (E_p((\boldsymbol{v}_i^n), (\lambda_i); (x_{i-1})), \psi^{(l)}((x_{i-1})))$$

so we must endow $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n^{p^l}$ with a group scheme structrures so that Φ is a homomorphism. This means the equality;

$$\Phi((x_{i-1}), (y_{i-1})) = \Phi((x_{i-1})) \cdot \Phi((y_{i-1})), \quad (x_{i-1}), (y_{i-1}) \in \hat{\mathscr{E}}_n,$$

where

$$\Phi((x_{i-1})\cdot(y_{i-1})) = (E_p((\boldsymbol{v}_i^n),(\lambda_i);(x_{i-1})\cdot(y_{i-1})), \psi^{(l)}((x_{i-1})\cdot\psi^{(l)}(y_{i-1}))),$$

$$\begin{aligned}\Phi((x_{i-1}))\cdot\Phi((y_{i-1})) &= (E_p((\boldsymbol{v}_i^n),(\lambda_i);(x_{i-1})), \psi^{(l)}((x_{i-1})))\\ &\quad\cdot(E_p((\boldsymbol{v}_i^n),(\lambda_i);(y_{i-1})), \psi^{(l)}((y_{i-1}))).\end{aligned}$$

For elements $(t_1,(z_{i-1})),(t_2,(w_{i-1}))$ of $\hat{\mathbf{G}}_{m,A}\times\hat{\mathscr{E}}_n^{p^l}$, we choose the inverse images (x_{i-1}) and (y_{i-1}) of (z_{i-1}) and (w_{i-1}) with respect to the $\psi^{(l)}$, respectively. Then the group sturucture of $\hat{\mathbf{G}}_{m,A}\times\hat{\mathscr{E}}_n^{p^l}$ should be given by

$$\begin{aligned}&(t_1,(z_{i-1}))\cdot(t_2,(w_{i-1}))\\ &\quad=\left(t_1t_2\cdot\frac{E_p((\boldsymbol{v}_i^n),(\lambda_i);(x_{i-1})\cdot(y_{i-1}))}{(E_p((\boldsymbol{v}_i^n),(\lambda_i);(x_{i-1})))\cdot(E_p((\boldsymbol{v}_i^n),(\lambda_i);(y_{i-1})))},(z_{i-1})\cdot(w_{i-1})\right)\\ &\quad=F_p(U^n(\boldsymbol{v}_i^n),(\lambda_i);(x_{i-1}),(y_{i-1})).\end{aligned}$$

Next we must show the following equation;

$$F_p(U^n(\boldsymbol{v}_i^n),(\lambda_i);(x_{i-1}),(y_{i-1})) = F_p((\boldsymbol{w}_i^n),(\lambda_i);(x_{i-1}),(y_{i-1}))$$

(note that $(V^l)^n(\boldsymbol{w}_i^n)=U^n(\boldsymbol{v}_i^n)$).

$$\begin{aligned}&F_p(U^n(\boldsymbol{v}_i^n),(\lambda_i);(x_{i-1}),(y_{i-1}))\\ &\quad=\prod_{i=1}^n F_p\left(V^l\boldsymbol{w}_i^n,\lambda_i;\frac{x_{i-1}}{D_{i-1}(x_0,\dots,x_{n-2})},\frac{y_{i-1}}{D_{i-1}(y_0,\dots,y_{n-2})}\right)\\ &\qquad\times\prod_{i=1}^n F_p\left(V^l\boldsymbol{w}_i^n,\lambda_i;H_{i-1},\frac{x_{i-1}}{D_{i-1}(x_0,\dots,x_{n-2})}\dot{+}\frac{y_{i-1}}{D_{i-1}(y_0,\dots,y_{n-2})}\right)\\ &\qquad\times\prod G_p(V^l\boldsymbol{w}_i^n,\lambda_i;F^{i-1})^{-1}\\ &\quad=\prod_{i=1}^n F_p\left(\boldsymbol{w}_i^n,\lambda_i^{p^l};\left(\frac{x_{i-1}}{D_{i-1}(x_0,\dots,x_{n-2})}\right)^{p^l},\left(\frac{y_{i-1}}{D_{i-1}(y_0,\dots,y_{n-2}}\right)^{p^l}\right)\\ &\qquad\times\prod_{i=1}^n F_p\left(V^l\boldsymbol{w}_i^n,\lambda_i^{p^l};H_{i-1}^{(p^l)},\left(\frac{x_{i-1}}{D_{i-1}(x_0,\dots,x_{n-2})}\dot{+}\frac{y_{i-1}}{D_{i-1}(y_0,\dots,y_{n-2})}\right)^{p^l}\right)\\ &\qquad\times\prod G_p(V^l\boldsymbol{w}_i^n,\lambda_i^{p^l};(F^{(i-1)})^{(p^l)})^{-1}\\ &\quad=F_p(U^n(\boldsymbol{w}_i^n),(\lambda_i^{p^l});(x_{i-1}^{p^l}),(y_{i-1}^{p^l})).\end{aligned}$$

This means our assertion. □

Lemma 5. $\psi^{(l)} \circ \phi_3 = \phi_4 \circ (V^l)^n$.

Proof. For $(\boldsymbol{v}^n) \in \mathrm{Coker}(U^n)'$, we can determine the direct image;

$$(\psi^{(l)})^* F_p((\boldsymbol{v}_i^n), (\lambda_i^{p^l}); (z_{i-1}), (w_{i-1}))$$

on the fibre product $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n$, so we look at the following commutative diagram;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{\mathbf{G}}_{m,A} & \longrightarrow & \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n & \longrightarrow & \hat{\mathscr{E}}_n & \longrightarrow & 0 \\ & & \| & & \Phi\downarrow & & \phi^{(l)}\downarrow & & \\ 0 & \longrightarrow & \hat{\mathbf{G}}_{m,A} & \longrightarrow & \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n^{p^l} & \longrightarrow & \hat{\mathscr{E}}_n^{p^l} & \longrightarrow & 0. \end{array} \tag{9}$$

By the condition of diagram (9), we have a map Φ given by

$$\Phi : \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n \to \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n^{p^l}; \quad (t, (x_{i-1})) \mapsto (t, \phi^{(l)}((x_{i-1}))).$$

We endow $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n$ with a group scheme structure so that Φ becomes a homomorphism. Let $(t_1, (x_{i-1}))$ and $(t_2, (y_{i-1}))$ be local sections in $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n$. If the product $(t_1, (x_{i-1})) \cdot (t_2, (y_{i-1}))$ is expressed as $(t_1, (x_{i-1})) \cdot (t_2, (y_{i-1})) = (t_1 t_2 G((x_{i-1}), (y_{i-1})), (x_{i-1}) \cdot (y_{i-1}))$ where $G((x_{i-1}), (y_{i-1}))$ is a cocycle on $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}_n$. Then we have the following equation;

$$\begin{aligned} \Phi((t_1, (x_{i-1})) \cdot (t_2, (y_{i-1}))) &= \Phi(t_1 t_2 G((x_{i-1}), (y_{i-1})), (x_{i-1}) \cdot (y_{i-1})) \\ &= (t_1 t_2 G((x_{i-1}), (y_{i-1})), \phi^{(l)}(x_{i-1}) \cdot \phi^{(l)}(y_{i-1})). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\Phi((t_1, (x_{i-1})) \cdot \Phi((t_2, (y_{i-1}))) \\ &\quad = (t_1, \phi^{(l)}(x_{i-1})) \cdot (t_2, \phi^{(l)}(y_{i-1})) \\ &\quad = (t_1 t_2 F_p((\boldsymbol{v}^n), (\lambda_i^{p^l}); \phi^{(l)}(x_{i-1}), \phi^{(l)}(y_{i-1})), \phi^{(l)}(x_{i-1}) \cdot \phi^{(l)}(y_{i-1})). \end{aligned}$$

Hence it is necessarly to have the following condition that Φ is a homomorphism;

$$G((x_{i-1}), (y_{i-1})) = F_p((\boldsymbol{v}^n), (\lambda_i^{p^l}); \phi^{(l)}(x_{i-1}), \phi^{(l)}(x_{i-1})).$$

We'll show the next equation to prove this;

$$F_p((\boldsymbol{v}^n), (\lambda_i^{p^l}); \phi^{(l)}(x_{i-1}), \phi^{(l)}(x_{i-1})) = F_p(V^l(\boldsymbol{v}^n), (\lambda_i); \phi^{(l)}(x_{i-1}), \phi^{(l)}(y_{i-1})).$$

(But it has already proved in [AA, lemma 3].)

6. The Kernel of the Type $F^n + \boldsymbol{a}_1 F^{n-1} + \cdots + \boldsymbol{a}_n$

In the previous paper [AA], we can construct the case where the endomorphism $T_{\boldsymbol{b}}$ becomes the identity map. Taking the similar method, we give such a description in the generalized situation when n is arbitrary, $T_{\boldsymbol{b}_1^2}, T_{\boldsymbol{b}_2^3}, \dots,$ and $T_{\boldsymbol{b}_{n-1}^n}$ are identity maps and the other $T_{\boldsymbol{b}_i^j}$'s are zero maps . So we get

$$U^n = \begin{pmatrix} F^{(\lambda_1)} & -1 & \cdots & \cdots & 0 \\ 0 & F^{(\lambda_2)} & -1 & \cdots & 0 \\ 0 & \cdots & & & -1 \\ 0 & & \cdots & & F^{(\lambda_n)} \end{pmatrix}.$$

Let $(\boldsymbol{v}_l^n)$ be the element of Ker U_l^n, we have following equations;

$$F^{(\lambda_1)}\boldsymbol{v}_l^1 - \boldsymbol{v}_l^2 = 0$$
$$F^{(\lambda_2)}\boldsymbol{v}_l^2 - \boldsymbol{v}_l^3 = 0$$
$$\vdots$$
$$F^{(\lambda_{n-1})}\boldsymbol{v}_l^{n-1} - \boldsymbol{v}_l^n = 0$$
$$F^{(\lambda_n)}\boldsymbol{v}_l^n = 0.$$

Hence we have the next equations;

$$\begin{aligned} \boldsymbol{v}_l^2 &= F^{(\lambda_1)}\boldsymbol{v}_l^1 \\ \boldsymbol{v}_l^3 &= F^{(\lambda_2)}\boldsymbol{v}_l^2 = F^{(\lambda_2)}F^{(\lambda_1)}\boldsymbol{v}_l^1 \\ &\vdots \\ \boldsymbol{v}_l^n &= F^{(\lambda_{n-1})}F^{(\lambda_{n-2})}\cdots F^{(\lambda_1)}\boldsymbol{v}_l^1 \\ 0 &= F^{(\lambda_n)}\boldsymbol{v}_l^n = F^{(\lambda_n)}F^{(\lambda_{n-1})}\cdots F^{(\lambda_1)}\boldsymbol{v}_l^1. \end{aligned}$$

Therefore in this case we have the canonical isomorphism;

$$\mathrm{Ker}[U_l^n : W_l^n \to W_l^n] \simeq \mathrm{Ker}[F^{(\lambda_n)}F^{(\lambda_{n-1})}\cdots F^{(\lambda_1)} : W_l \to W_l].$$

Moreover $F^{(\lambda_n)}F^{(\lambda_{n-1})}\cdots F^{(\lambda_1)}$ is given the following polynomial in F;

$$F^{(\lambda_n)}F^{(\lambda_{n-1})}\cdots F^{(\lambda_1)} = F^n + \boldsymbol{a}_1 F^{n-1} + \cdots + \boldsymbol{a}_n,$$

where the Witt vectors $\boldsymbol{a}_k$'s are given by;

$$\boldsymbol{a}_k = \sum_{n \geq i_1 > i_2 \cdots > i_k \geq 1} (-1)^k \left[\prod_{j=1}^{k} \lambda_{i_k}^{(p-1)p^{n-i_j-(j-1)}} \right] \quad k = 1, \ldots, n.$$

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