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HERMANN TYPE ACTIONS ON A PSEUDO-RIEMANNIAN SYMMETRIC SPACE

By

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Abstract. In this paper, we first investigate the shape operators of certain kind of orbits of the isotropy action of a semi-simple pseudo-Riemannian symmetric space. The investigation is performed by investigating the complexified action. Next, by using the fact obtained by the investigation, we show that certain kind of principal orbits of a Hermann type action on a semi-simple pseudo-Riemannian symmetric space are curvature-adapted proper complex equifocal submanifolds and that their shape operators are semi-simple. It follows from this fact that the principal orbits are isoparametric submanifolds with flat section. Also, we derive an interesting structure of a semi-simple pesudo-Riemannian symmetric space (in particular, the complexification of a Riemannian symmetric space) from two special Hermann type actions on the space.

1. Introduction

In Riemannian symmetric spaces, the notion of an equifocal submanifold was introduced by Terng-Thorbergsson in [36]. This notion is defined as a compact submanifold with flat section such that the normal holonomy group is trivial and that the focal radius functions for each parallel normal vector field are constant. However, the condition of the equifocality is rather weak in the case where the Riemannian symmetric spaces are of non-compact type and the submanifold is non-compact. So we [17, 18] have recently introduced the notion of a complex equifocal submanifold in a Riemannian symmetric space G/K of non-compact type. This notion is defined by imposing the constancy of the complex focal radius functions in more general. Here we note that the complex focal radii are

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the quantities indicating the positions of the focal points of the extrinsic complexification of the submanifold, where the submanifold needs to be assumed to be complete and of class C^{ω} (i.e., real analytic). On the other hand, Heintze-Liu-Olmos [13] has recently defined the notion of an isoparametric submanifold with flat section in a general Riemannian manifold as a submanifold such that the normal holonomy group is trivial, its sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction and that the image of the normal space at each point by the normal exponential map is flat and totally geodesic. We [18] showed that all isoparametric submanifolds with flat section in a Riemannian symmetric space G/K of non-compact type are complex equifocal and that conversely, all curvature-adapted and complex equifocal submanifolds are isoparametric ones with flat section. Here the curvature-adaptedness means that, for each normal vector v of the submanifold, the Jacobi operator $R(\cdot, v)v$ preserves the tangent space of the submanifold invariantly and the restriction of $R(\cdot, v)v$ to the tangent space commutes with the shape operator A_v, where R is the curvature tensor of G/K. As a subclass of the class of complex equifocal submanifolds, we [19] defined the notion of a proper complex equifocal submanifold in G/K as a complex equifocal submanifold such that its complex focal set at any point consists of infinitely many complex hyperplanes in the complexified normal space at the point and that the complex reflections of order two with respect to the complex hyperplanes generates a Coxeter group. Let G/K be a Riemannian symmetric space of non-compact type and H be a closed subgroup of G. If the H-action is proper and there exists a complete embedded flat submanifold meeting all H-orbits orthogonally, then it is called a *complex hyperpolar action*. Principal orbits of a complex hyperpolar action are complex equifocal (see [18]). If H is a symmetric subgroup of G (i.e., $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$ for some involution σ of G), then the H-action is called a *Hermann type action*, where $\mathrm{Fix}\,\sigma$ is the fixed point group of σ and $(\mathrm{Fix}\,\sigma)_0$ is the identity component of the group. Hermann type actions are complex hyperpolar. We ([18, 19]) showed the following facts.

Fact 1. *Let θ be the Cartan involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$, σ be an involution of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$ and $L := (\mathrm{Fix}(\sigma \circ \theta))_0$, where we may assume that $\theta \circ \sigma = \sigma \circ \theta$ by replacing H to its conjugate group. Then the orbit $H(eK)$ of the H-action on G/K is a reflective submanifold and it is homothetic to the Riemannian symmetric space $H/H \cap K$. For each $x \in H(eK)$, the section Σ_x of $H(eK)$ through x is homothetic to the Riemannian symmetric space $L/H \cap K$.*

FACT 2. *Principal orbits of a Hermann type action are curvature-adapted and proper complex equifocal. Hence it is an isoparametric submanifold with flat section.*

Similarly, we can define the notions of a complex equifocal submanifold and a proper complex equifocal one in a pseudo-Riemannian symmetric space, and the notion of an isoparametric submanifold with flat section in a general pseudo-Riemannian manifold. Also, we can define the notions of a complex hyperpolar action and a Hermann type action on a pseudo-Riemannian symmetric space. We [23] showed the following fact.

FACT 3. *All isoparametric submanifolds with flat section in a pseudo-Riemannian symmetric space G/K are complex equifocal. Conversely all curvature-adapted complex equifocal submanifolds such that A_v and $R(\cdot, v)v$ are semi-simple for any normal vector v are isoparametric ones with flat section, where A_v is the shape operator and R is the curvature tensor of G/K and the semi-simplenesses of A_v and $R(\cdot, v)v$ mean that their complexifications are diagonalizable.*

L. Geatti and C. Gorodski [9] has recently showed that a polar representation of a real reductive algebraic group on a pseudo-Euclidean space has the same closed orbits as the isotropy representation (i.e., the linear isotropy action) of a pseudo-Riemannian symmetric space (see Theorem 1 of [9]). Also, they showed that the principal orbits of the polar representation through a semi-simple element (i.e., the orbit through a regular element (in the sense of [9])) is an isoparametric submanifold by investigating the complexified representation (see Theorem 11 (also Example 12) of [9]), where an isoparametric submanifold means a pseudo-Riemannian submanifold (in a pseudo-Euclidean space) such that the (restricted) normal holonomy group is trivial and that the shape operator for each (local) parallel normal vector field is semi-simple and has constant complex principal curvature. All isoparametric submanifold (in a pseudo-Euclidean space) in this sense are isoparametric ones (with flat section) in the sense of [13]. Let G/H be a (semi-simple) pseudo-Riemannian symmetric space (equipped with the metric $\langle\,,\rangle$ induced from the Killing form of the Lie algebra $\mathfrak{g}$ of G). In this paper, we first investigate the complexified shape operators of the orbits of the isotropy action of G/H (i.e., the H-action on G/H) by investigating the orbits of the isotropy action of $G^{\mathbf{c}}/H^{\mathbf{c}}$ (see Section 3). Next, by using the investigation, we prove the following fact for the orbits of Hermann type action.

Theorem A. *Let* G/H *be a* (*semi-simple*) *pseudo-Riemannian symmetric space*, H' *be a symmetric subgroup of* G, σ (*resp.* σ') *be an involution of* G *with* $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$ (*resp.* $(\mathrm{Fix}\,\sigma')_0 \subset H' \subset \mathrm{Fix}\,\sigma'$), $L := (\mathrm{Fix}(\sigma \circ \sigma'))_0$ *and* $\mathfrak{l} := \mathrm{Lie}\,L$. *Assume that* G *is not compact and* $\sigma \circ \sigma' = \sigma' \circ \sigma$. *Then the following statements* (i) *and* (ii) *hold*:

(i) *The orbit* $H'(eH)$ *of the* H'*-action on* G/H *is a reflective pseudo-Riemannian submanifold and it is homothetic to the semi-simple pseudo-Riemannian symmetric space* $H'/H \cap H'$. *For each* $x \in H'(eH)$, *the section* Σ_x *of* $H'(eH)$ *through* x *is homothetic to the semi-simple pseudo-Riemannian symmetric space* $L/H \cap H'$.

(ii) *Let* M *be a principal orbit of the* H'*-action through a point* $\exp_G(w)H$ ($w \in \mathfrak{q} \cap \mathfrak{q}'$ *s.t.* $\mathrm{ad}(w)|_{\mathfrak{l}}$: *semi-simple*) *of* $\Sigma_{eH} \backslash F$, *where* $\mathfrak{q} := \mathrm{Ker}(\sigma + \mathrm{id})$ $(= T_{eH}(G/H))$, $\mathfrak{q}' := \mathrm{Ker}(\sigma' + \mathrm{id})$ *and* F *is a focal set of* $H'(eH)$. *Then* M *is curvature-adapted and proper complex equifocal*, *for any normal vector* v *of* M, $R(\cdot, v)v$ *and the shape operator* A_v *are semi-simple. Hence it is an isoparametric submanifold with flat section.*

Remark 1.1. (i) Since $\bigcup_{w \in \mathfrak{q}\cap\mathfrak{q}' \text{ s.t. } \mathrm{ad}(w)|_{\mathfrak{l}}:\text{semi-simple}} (H' \cap H)(\exp_G(w)H)$ is an open dense subset of $L(eH)$, it is shown that

$$\bigcup_{w \in \mathfrak{q}\cap\mathfrak{q}' \text{ s.t. } \mathrm{ad}(w)|_{\mathfrak{l}}:\text{semi-simple}} H'(\exp_G(w)H)$$

is an open dense subset of G/H.

(ii) In the case where G/H is a Riemannian symmetric space of non-compact type, $\mathrm{ad}(w)|_{\mathfrak{l}}$ is semi-simple for any $w \in \mathfrak{q} \cap \mathfrak{q}'$, $R(\cdot, v)v$ and A_v is semi-simple for any normal vector v of M, $F = \emptyset$ and $\bigcup_{x \in H'(eH)} \Sigma_x = G/H$. Hence the statement (ii) of Theorem A is a generalized result of the above Fact 2.

L. Geatti [8] has recently defined a pseudo-Kaehlerian structure on some G-invariant domain of the complexification $G^{\mathbf{c}}/H^{\mathbf{c}}$ of a semi-simple pseudo-Riemannian symmetric space G/H. On the other hand, we [23] have recently defined an anti-Kaehlerian structure on the whole of the complexification $G^{\mathbf{c}}/H^{\mathbf{c}}$. By applying Theorem A to the complexification $G^{\mathbf{c}}/H^{\mathbf{c}}$ (equipped with the natural anti-Kaehlerian structure) of a semi-simple pseudo-Riemannian symmetric space G/H and a symmetric subgroup G of $G^{\mathbf{c}}$, we recognize an interesting structure of $G^{\mathbf{c}}/H^{\mathbf{c}}$. Here we note that an involution σ of $G^{\mathbf{c}}$ with

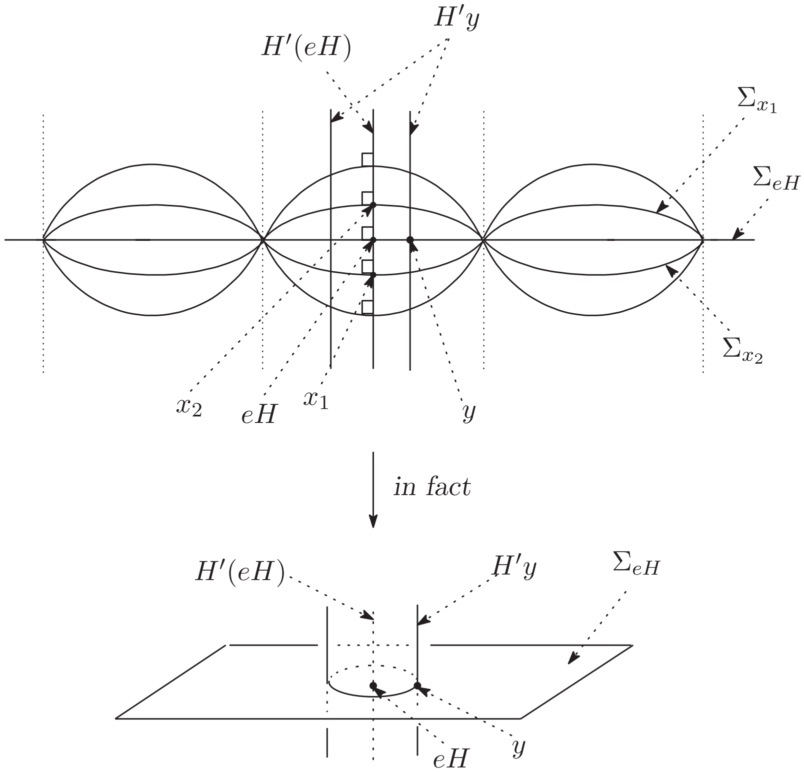

Figure 1

$(\mathrm{Fix}\,\sigma)_0 \subset H^{\mathbf{c}} \subset \mathrm{Fix}\,\sigma$ and the conjugation τ of $G^{\mathbf{c}}$ with respect to G are commutative. In this case, the group corresponding to L in the statement of Theorem A is the dual G^{*_H} of G with respect to H. Hence we have the following fact.

Corollary B. *Let $G^{\mathbf{c}}/H^{\mathbf{c}}$ and G^{*_H} be as above. Then the following statements* (i) *and* (ii) *hold*:

(i) *The orbit $G(eH^{\mathbf{c}})$ is a reflective pseudo-Riemannian submanifold and it is homothetic to the pseudo-Riemannian symmetric space G/H. For each $x \in G(eH^{\mathbf{c}})$, the section Σ_x of $G(eH^{\mathbf{c}})$ through x is homothetic to the pseudo-Riemannian symmetric space G^{*_H}/H.*

(ii) *For principal orbits of the G-action on $G^{\mathbf{c}}/H^{\mathbf{c}}$, the same fact as the statement* (ii) *of Theorem A holds.*

By considering two special Hermann type actions on a semi-simple pseudo-Riemannian symmetric space, we obtain the following interesting fact for the structure of the semi-simple pseudo-Riemannian symmetric space.

Theorem C. *Let G/H and σ be as in Theorem A, θ the Cartan involution of G with $\theta \circ \sigma = \sigma \circ \theta$, $K := (\mathrm{Fix}\,\theta)_0$ and $L := (\mathrm{Fix}(\sigma \circ \theta))_0$. Then the following statements* (i) *and* (ii) *hold*:

(i) *The orbits $K(eH)$ and $L(eH)$ are reflective submanifolds satisfying $T_{eH}(G/H) = T_{eH}(K(eH)) \oplus T_{eH}(L(eH))$ (orthogonal direct sum), $K(eH)$ is anti-homothetic to the Riemannian symmetric space $K/H \cap K$ of compact type and $L(eH)$ is homothetic to the Riemannian symmetric space $L/H \cap K$ of non-compact type. Also, the orbit $K(eH)$ has no focal point.*

(ii) *Let M_1 be a principal orbit of the K-action and M_2 be a principal orbit of the L-action through a point of $K(eH) \backslash F$, where F is the focal set of $L(eH)$. Then M_i $(i = 1, 2)$ are curvature-adapted and proper complex equifocal, for any normal vector v of M_i, $R(\cdot, v)v|_{T_x M_i}$ (x : the base point of v) and the shape operator A_v are diagonalizable. Hence they are isoparametric submanifolds with flat section.*

Remark 1.2. For any involution σ of G, the existence of a Cartan involution θ of G with $\theta \circ \sigma = \sigma \circ \theta$ is assured by Lemma 10.2 in [1].

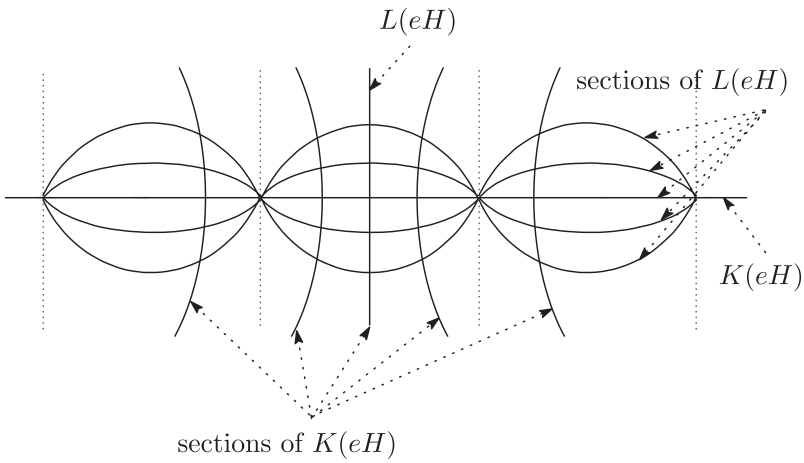

Figure 2

By applying Theorem C to the complexification $G^{\mathbf{c}}/K^{\mathbf{c}}$ (equipped with the natural anti-Kaehlerian structure) of a Riemannian symmetric space G/K of non-compact type, we can recognize the interesting structure of $G^{\mathbf{c}}/K^{\mathbf{c}}$. In this case, the groups corresponding to K, L and $H \cap K$ in the statement of Theroem C are the compact dual G^* of G, G and K, respectively. Hence we have the following fact.

Theorem D. *Let $G^{\mathbf{c}}/K^{\mathbf{c}}$ be the complexification* (*equipped with the natural anti-Kaehlerian structure*) *of a Riemannian symmetric space G/K of non-compact type and G^* be the compact dual of G. Then the following statements* (i) *and* (ii) *hold*:

(i) *The orbits $G^*(eK^{\mathbf{c}})$ and $G(eK^{\mathbf{c}})$ are reflective submanifolds of $G^{\mathbf{c}}/K^{\mathbf{c}}$ satisfying $T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}}) = T_{eK^{\mathbf{c}}}(G^*(eK^{\mathbf{c}})) \oplus T_{eK^{\mathbf{c}}}(G(eK^{\mathbf{c}}))$ (orthogonal direct sum), $G^*(eK^{\mathbf{c}})$ is anti-homothetic to the Riemannian symmetric space G^*/K of compact type and $G(eK^{\mathbf{c}})$ is homothetic to the Riemannian symmetric space G/K of non-compact type. Also, the orbit $G^*(eK^{\mathbf{c}})$ has no focal point.*

(ii) *For principal orbits of the G^*-action and G-action on $G^{\mathbf{c}}/K^{\mathbf{c}}$, the same fact as the statement* (ii) *of Theorem C holds.*

Homogeneous submanifolds with flat section in a pseudo-Riemannian symmetric space are complex equifocal. We obtain the following fact for a homogeneous submanifold with flat section in a semi-simple pseudo-Riemannian symmetric space which admits a reflective focal submanifold, where a reflective submanifold means a totally geodesic pseudo-Riemannian submanifold with section.

Theorem E. *Let M be a homogeneous submanifold with flat section in a semi-simple pseudo-Riemannian symmetric space G/H. Assume that M admits a reflective focal submanifold F such that $\mathfrak{n}_{\mathfrak{h}}(g_*^{-1}T_{gH}F)$ is a non-degenerate subspace of $\mathfrak{h}$, where gH is an arbitrary point of F and $\mathfrak{n}_{\mathfrak{h}}(g_*^{-1}T_{gH}F)$ is the normalizer of $g_*^{-1}T_{gH}F$ in $\mathfrak{h}$. Then M is a principal orbit of a Hermann type action.*

Remark 1.4. (i) For the H'-action in Theorem A, we have $\mathfrak{n}_{\mathfrak{h}}(T_{eH}(H'(eH))) = \mathfrak{n}_{\mathfrak{h}}(\mathfrak{q} \cap \mathfrak{h}') = \mathfrak{h} \cap \mathfrak{h}' + \mathfrak{z}_{\mathfrak{h}\cap\mathfrak{q}'}(\mathfrak{q} \cap \mathfrak{h}')$, where $\mathfrak{z}_{\mathfrak{h}\cap\mathfrak{q}'}(\mathfrak{q} \cap \mathfrak{h}')$ is the centralizer of $\mathfrak{q} \cap \mathfrak{h}'$ in $\mathfrak{h} \cap \mathfrak{q}'$. Hence, if $\mathfrak{z}_{\mathfrak{h}\cap\mathfrak{q}'}(\mathfrak{q} \cap \mathfrak{h}') = \{0\}$, then $\mathfrak{n}_{\mathfrak{h}}(T_{eH}(H'(eH)))$ is a non-degenerate subspace of $\mathfrak{h}$. Thus almost all principal orbits of the H'-action have $H'(eH)$ as a reflective focal submanifold as in the statement of Theorem E.

(ii) For the K-action in Theorem C, we have $\mathfrak{n}_{\mathfrak{h}}(T_{eH}(K(eH))) = \mathfrak{n}_{\mathfrak{h}}(\mathfrak{q} \cap \mathfrak{k}) = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{z}_{\mathfrak{h}\cap\mathfrak{p}}(\mathfrak{q} \cap \mathfrak{k})$. Hence, $\mathfrak{n}_{\mathfrak{h}}(T_{eH}(K(eH)))$ is a non-degenerate subspace of $\mathfrak{h}$. Similarly, for the L-action in Theorem C, it is shown that $\mathfrak{n}_{\mathfrak{h}}(T_{eH}(L(eH)))$ is a non-degenerate subspace of $\mathfrak{h}$. Thus almost all principal orbits of the K-action (resp. the L-action) have $K(eH)$ (resp. $L(eH)$) as a reflective focal submanifold as in the statement of Theorem E.

(iii) In the case where G/H is a Riemannian symmetric space of non-compact type, the statement of Theorem E has already been shown in [21], where we note that $\mathfrak{n}_{\mathfrak{h}}(g_*^{-1}T_{gH}F)$ is automatically a non-degenerate subspace of $\mathfrak{h}$ because H is compact.

2. New Notions in a Pseudo-Riemannian Symmetric Space

In this section, we shall define new notions in a (semi-simple) pseudo-Riemannian symmetric space, which are analogies of notions in a Riemannian symmetric space of non-compact type defined in [18]. Let M be an immersed pseudo-Riemannian submanifold with flat section (that, is, $g_*^{-1}T_x^{\perp}M$ is abelian for any $x = gH \in M$) in a (semi-simple) pseudo-Riemannian symmetric space $N = G/H$ (equipped with the metric induced from the Killing form of $\mathfrak{g} := \mathrm{Lie}\, G$), where $T_x^{\perp}M$ is the normal space of M at x. Denote by A the shape tensor of M. Let $v \in T_x^{\perp}M$ and $X \in T_xM$ $(x = gK)$, where T_xM is the tangent space of M at x. Denote by γ_v the geodesic in N with $\dot{\gamma}_v(0) = v$, where $\dot{\gamma}_v(0)$ is the velocity vector of γ_v at 0. The strongly M-Jacobi field Y along γ_v with $Y(0) = X$ (hence $Y'(0) = -A_vX$) is given by

$$Y(s) = (P_{\gamma_v|_{[0,s]}} \circ (D_{sv}^{co} - sD_{sv}^{si} \circ A_v))(X), \tag{2.1}$$

where $Y'(0) = \tilde{\nabla}_v Y$ ($\tilde{\nabla}$: the Levi-Civita connection of N), $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$ and D_{sv}^{co} (resp. D_{sv}^{si}) is given by

$$D_{sv}^{co} = g_* \circ \cos(\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v)) \circ g_*^{-1}$$

$$\left(\text{resp. } D_{sv}^{si} = g_* \circ \frac{\sin(\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v))}{\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v)} \circ g_*^{-1}\right).$$

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ and $\cos(\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v))$ (resp. $\frac{\sin(\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v))}{\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v)}$) is defined by

$$\cos(\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v)) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\,\mathrm{ad}(sg_*^{-1}v)^{2k}$$

$$\left(\text{resp. } \frac{\sin(\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v))}{\sqrt{-1}\,\mathrm{ad}(sg_*^{-1}v)} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\,\mathrm{ad}(sg_*^{-1}v)^{2k}\right).$$

All focal radii of M along γ_v are obtained as real numbers s_0 with $\mathrm{Ker}(D^{co}_{s_0v} - s_0 D^{si}_{s_0v} \circ A_v) \neq \{0\}$. So, we call a complex number z_0 with $\mathrm{Ker}(D^{co}_{z_0v} - z_0 D^{si}_{z_0v} \circ A^{\mathbf{c}}_v) \neq \{0\}$ a *complex focal radius of M along γ_v* and call $\dim \mathrm{Ker}(D^{co}_{z_0v} - z_0 D^{si}_{z_0v} \circ A^{\mathbf{c}}_v)$ the *multiplicity* of the complex focal radius z_0, where $A^{\mathbf{c}}_v$ is the complexification of A_v and $D^{co}_{z_0v}$ (resp. $D^{si}_{z_0v}$) is a $\mathbf{C}$-linear transformation of $(T_xN)^{\mathbf{c}}$ defined by

$$D^{co}_{z_0v} = g^{\mathbf{c}}_* \circ \cos(\sqrt{-1}\,\mathrm{ad}^{\mathbf{c}}(z_0 g_*^{-1}v)) \circ (g^{\mathbf{c}}_*)^{-1}$$

$$\left(\text{resp. } D^{si}_{sv} = g^{\mathbf{c}}_* \circ \frac{\sin(\sqrt{-1}\,\mathrm{ad}^{\mathbf{c}}(z_0 g_*^{-1}v))}{\sqrt{-1}\,\mathrm{ad}^{\mathbf{c}}(z_0 g_*^{-1}v)} \circ (g^{\mathbf{c}}_*)^{-1}\right),$$

where $g^{\mathbf{c}}_*$ (resp. $\mathrm{ad}^{\mathbf{c}}$) is the complexification of g_* (resp. ad). Here we note that, in the case where M is of class C^ω, complex focal radii along γ_v indicate the positions of focal points of the (extrinsic) complexification $M^{\mathbf{c}}(\hookrightarrow G^{\mathbf{c}}/H^{\mathbf{c}})$ of M along the complexified geodesic $\gamma^{\mathbf{c}}_{\iota_* v}$. Here $G^{\mathbf{c}}/H^{\mathbf{c}}$ is the pseudo-Riemannian symmetric space equipped with the metric induced from the Killing form of $\mathfrak{g}^{\mathbf{c}}$ regarded as a real Lie algebra (which is called the *anti-Kaehlerian symmetric space associated with G/H*) and ι is the natural embedding of G/H into $G^{\mathbf{c}}/H^{\mathbf{c}}$. See [23] ([18] also) about the definition of the (extrinsic) complexification $M^{\mathbf{c}}(\hookrightarrow G^{\mathbf{c}}/H^{\mathbf{c}})$. Furthermore, assume that the normal holonomy group of M is trivial. Let $\tilde{v}$ be a parallel unit normal vector field of M. Assume that the number (which may be 0 and ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Furthermore assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \dots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ & $\mathrm{Im}\, r_{i,x} = -\mathrm{Im}\, r_{i+1,x} < 0$".

Definition 2.1. Assume that M is a submanifold with flat section in N such that the normal holonomy group of M is trivial and that the number (which may be 0 and ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, where $\tilde{v}$ is as above. Define complex valued functions r_i $(i = 1, 2, \dots)$ on M by assigning $r_{i,x}$ to each $x \in M$. We call these functions r_i $(i = 1, 2, \dots)$ *complex focal radius functions for $\tilde{v}$* and $r_i\tilde{v}$ a *complex focal normal vector field for $\tilde{v}$*.

Remark 2.1. The complex focal radius functions r_i's $(i = 1, 2, \dots)$ are of class C^∞.

Definition 2.2. Let M be a submanifold with flat section in N such that the normal holonomy group of M is trivial. If, for any parallel normal vector field $\tilde{v}$, the number of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, if complex focal radius functions for any parallel normal vector field $\tilde{v}$ are constant over M and have constant multiplicities, then we call M a *complex equifocal submanifold.*

Let $N = G/H$ be a (semi-simple) pseudo-Riemannian symmetric space and π be the natural projection of G onto G/H. Let σ be an involution of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$ and denote by the same symbol σ the involution of $\mathfrak{g} := \mathrm{Lie}\, G$. Let $\mathfrak{h} := \{X \in \mathfrak{g} \mid \sigma(X) = X\}$ and $\mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$, which is identified with the tangent space $T_{eH}N$. Let $\langle\, , \rangle$ be the Killing form of $\mathfrak{g}$. Denote by the same symbol $\langle\, , \rangle$ both the bi-invariant pseudo-Riemannian metric of G induced from $\langle\, , \rangle$ and the pseudo-Riemannian metric of N induced from $\langle\, , \rangle$. Let θ be a Cartan involution of G with $\theta \circ \sigma = \sigma \circ \theta$. Denote by the same symbol θ the involution of $\mathfrak{g}$ induced from θ. Let $\mathfrak{f} := \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. From $\theta \circ \sigma = \sigma \circ \theta$, it follows that $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{f} + \mathfrak{h} \cap \mathfrak{p}$ and $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{f} + \mathfrak{q} \cap \mathfrak{p}$. Set $\mathfrak{g}_+ := \mathfrak{p}$, $\mathfrak{g}_- := \mathfrak{f}$ and $\langle\, , \rangle_{\mathfrak{g}_\pm} := -\pi_{\mathfrak{g}_-}^* \langle\, , \rangle + \pi_{\mathfrak{g}_+}^* \langle\, , \rangle$, where $\pi_{\mathfrak{g}_-}$ (resp. $\pi_{\mathfrak{g}_+}$) is the projection of $\mathfrak{g}$ onto $\mathfrak{g}_-$ (resp. $\mathfrak{g}_+$). Let $H^0([0,1], \mathfrak{g})$ be the space of all L^2-integrable paths $u : [0,1] \to \mathfrak{g}$ (with respect to $\langle\, , \rangle_{\mathfrak{g}_\pm}$). It is shown that $(H^0([0,1], \mathfrak{g}), \langle\, , \rangle_0)$ is a pseudo-Hilbert space, where $\langle\, , \rangle_0$ is defined by $\langle u, v\rangle_0 := \int_0^1 \langle u(t), v(t)\rangle\, dt$ $(u, v \in H^0([0,1], \mathfrak{g}))$. Let $H^1([0,1], G)$ be the Hilbert Lie group of all absolutely continuous paths $g : [0,1] \to G$ such that the weak derivative g' of g is squared integrable (with respect to $\langle\, , \rangle_{\mathfrak{g}_\pm}$), that is, $g_*^{-1} g' \in H^0([0,1], \mathfrak{g})$. Define a map $\phi : H^0([0,1], \mathfrak{g}) \to G$ by $\phi(u) = g_u(1)$ $(u \in H^0([0,1], \mathfrak{g}))$, where g_u is the element of $H^1([0,1], G)$ satisfying $g_u(0) = e$ and $g_{u*}^{-1} g_u' = u$. We call this map the *parallel transport map* (from 0 to 1). This submersion ϕ is a pseudo-Riemannian submersion of $(H^0([0,1], \mathfrak{g}), \langle\, , \rangle_0)$ onto $(G, \langle\, , \rangle)$. Denote by $\mathfrak{g}^{\mathbf{c}}$, $\mathfrak{h}^{\mathbf{c}}$, $\mathfrak{q}^{\mathbf{c}}$, $\mathfrak{f}^{\mathbf{c}}$, $\mathfrak{p}^{\mathbf{c}}$ and $\langle\, , \rangle^{\mathbf{c}}$ the complexifications of $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{q}$, $\mathfrak{f}$, $\mathfrak{p}$ and $\langle\, , \rangle$. Set $\mathfrak{g}_+^{\mathbf{c}} := \sqrt{-1}\mathfrak{f} + \mathfrak{p}$ and $\mathfrak{g}_-^{\mathbf{c}} := \mathfrak{f} + \sqrt{-1}\mathfrak{p}$. Set $\langle\, , \rangle' := 2\,\mathrm{Re}\langle\, , \rangle^{\mathbf{c}}$ and $\langle\, , \rangle'_{\mathfrak{g}_\pm^{\mathbf{c}}} := -\pi_{\mathfrak{g}_-^{\mathbf{c}}}^* \langle\, , \rangle' + \pi_{\mathfrak{g}_+^{\mathbf{c}}}^* \langle\, , \rangle'$, where $\pi_{\mathfrak{g}_-^{\mathbf{c}}}$ (resp. $\pi_{\mathfrak{g}_+^{\mathbf{c}}}$) is the projection of $\mathfrak{g}^{\mathbf{c}}$ onto $\mathfrak{g}_-^{\mathbf{c}}$ (resp. $\mathfrak{g}_+^{\mathbf{c}}$). Let $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ be the space of all L^2-integrable paths $u : [0,1] \to \mathfrak{g}^{\mathbf{c}}$ (with respect to $\langle\, , \rangle'_{\mathfrak{g}_\pm^{\mathbf{c}}}$). Define a non-degenerate symmetric bilinear form $\langle\, , \rangle'_0$ of $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ by $\langle u, v\rangle'_0 := \int_0^1 \langle u(t), v(t)\rangle'\, dt$. It is shown that $(H^0([0,1], \mathfrak{g}^{\mathbf{c}}), \langle\, , \rangle'_0)$ is an infinite dimensional anti-Kaehlerian space. See [18] about the definition of an infinite dimensional anti-Kaehlerian space. In similar to ϕ, the parallel transport map $\phi^{\mathbf{c}} : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ for $G^{\mathbf{c}}$ is defined. This submersion $\phi^{\mathbf{c}}$ is an anti-Kaehlerian submersion. Let $\pi : G \to G/H$ and $\pi^{\mathbf{c}} : G^{\mathbf{c}} \to$

$G^{\mathbf{c}}/H^{\mathbf{c}}$ be the natural projections. By imitating the proof of Theorem 1 of [18], we can show the following fact.

PROPOSITION 2.2. *For a* C^{ω}*-pseudo-Riemannian submanifold* M *in* G/H, *the following statements* (i)$\sim$(iii) *are equivalent*:

(i) M *is complex equifocal*,

(ii) *each component of* $(\pi \circ \phi)^{-1}(M)$ *is complex isoparametric*,

(iii) *each component of* $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ *is anti-Kaehlerian isoparametric*.

See [18] about the definitions of a complex isoparametric submanifold and an anti-Kaehlerian isoparametric submanifold.

DEFINITION 2.3. If each component of $(\pi \circ \phi)^{-1}(M)$ is proper complex isoparametric in the sense of [17] (i.e., it is complex isoparametric and, for its each normal vector v, there exists a pseudo-orthonormal base of the complexified tangent space consisting of the eigenvectors of the complexified shape operator for v), then we call M a *proper complex equifocal submanifold*.

REMARK 2.2. It is shown that the complex focal set of a proper complex isoparametric submanifold (in a pseudo-Hilbert space) at any point consists of infinitely many complex hyperplanes in the complexified normal space at the point and that the complex reflections of order two with respect to the complex hyperplanes generates a Coxeter group (see [18], [20]). From this fact, it follows that the same fact holds for a proper complex equifocal submanifold.

Now we shall define the notion of a Hermann type action on a semi-simple pseudo-Riemannin symmetric space G/H and that of a reflective submanifold in G/H.

DEFINITION 2.4. If H' is a symmetric subgroup of G (i.e., $(\mathrm{Fix}\,\sigma')_0 \subset H' \subset \mathrm{Fix}\,\sigma'$ for some involution σ' of G), then the H'-action on G/H is called a *Hermann type action*.

DEFINITION 2.5. Let M be a pseudo-Riemannian submanifold in a pseudo-Riemannian manifold N. If there exists an involutive isometry of a neighborhood of M having M as the fixed point set, then we call M a *reflective submanifold*.

REMARK 2.3. As in the case of a Reimannian submanifold, we should not define the reflectivity of a pseudo-Riemannian submanifold by the existence of a (global) involutive isometry of the ambient space having the submanifold as a component of the fixed point set (see Fig. 1).

As in the case of a Riemannian submanifold, we can show the following fact.

Proposition 2.3. *The following statements* (i) *and* (ii) *are equivalent*:

(i) M *is reflective*.

(ii) *The set* $\exp^\perp(T_x^\perp M)$ *is totally geodesic for each* x *of* M, *where* $\exp^\perp$ *is the normal exponential map of* M.

Next we shall recall the notions of a complex Jacobi field and the parallel translation along a holomorphic curve, which are introduced in [23], and we state some facts related to these notions. These notions and facts will be used in the next section. Let (M, J, g) be an anti-Kaehlerian manifold, ∇ (resp. R) be the Levi-Civita connection (resp. the curvature tensor) of g and $\nabla^{\mathbf{c}}$ (resp. $R^{\mathbf{c}}$) be the complexification of ∇ (resp. R). Let $(TM)^{(1,0)}$ be the holomorphic vector bundle consisting of complex vectors of M of type $(1,0)$. Note that the restriction of $\nabla^{\mathbf{c}}$ to $TM^{(1,0)}$ is a holomorphic connection of $TM^{(1,0)}$ (see Theorem 2.2 of [6]). For simplicity, assume that (M, J, g) is complete even if the discussion of this section is valid without the assumption of the completeness of (M, J, g). Let $\gamma : \mathbf{C} \to M$ be a complex geodesic, that is, $\gamma(z) = \exp_{\gamma(0)}\big((\mathrm{Re}\, z)\gamma_*\big(\big(\frac{\partial}{\partial s}\big)_0\big) + (\mathrm{Im}\, z)J_{\gamma(0)}\gamma_*\big(\big(\frac{\partial}{\partial s}\big)_0\big)\big)$, where (z) is the complex coordinate of $\mathbf{C}$ and $s := \mathrm{Re}\, z$. Let $Y : \mathbf{C} \to (TM)^{(1,0)}$ be a holomorphic vector field along γ. That is, Y assigns $Y_z \in (T_{\gamma(z)}M)^{(1,0)}$ to each $z \in \mathbf{C}$ and, for each holomorphic local coordinate $(U, (z_1, \dots, z_n))$ of M with $U \cap \gamma(\mathbf{C}) \neq \emptyset$, $Y_i : \gamma^{-1}(U) \to \mathbf{C}$ $(i = 1, \dots, n)$ defined by $Y_z = \sum_{i=1}^{n} Y_i(z)\big(\frac{\partial}{\partial z_i}\big)_{\gamma(z)}$ are holomorphic.

Definition 2.6. If Y satisfies $\nabla^{\mathbf{c}}_{\gamma_*(d/dz)}\nabla^{\mathbf{c}}_{\gamma_*(d/dz)}Y + R^{\mathbf{c}}\big(Y, \gamma_*\big(\frac{d}{dz}\big)\big)\gamma_*\big(\frac{d}{dz}\big) = 0$, then we call Y a *complex Jacobi field along* γ.

Let $\delta : \mathbf{C} \times D(\varepsilon) \to M$ be a holomorphic two-parameter map, where $D(\varepsilon)$ is the ε-disk centered at 0 in $\mathbf{C}$. Denote by z (resp. u) the first (resp. second) parameter of δ.

Definition 2.7. If $\delta(\cdot, u_0) : \mathbf{C} \to M$ is a complex geodesic for each $u_0 \in D(\varepsilon)$, then we call δ a *complex geodesic variation*.

It is shown that, for a complex geodesic variation δ, the complex variational vector field $Y := \delta_*(\frac{\partial}{\partial u}|_{u=0})$ is a complex Jacobi field along $\gamma := \delta(\cdot, 0)$. A vector field X on M is said to be *real holomorphic* if the Lie derivation $L_X J$ of J with respect to X vanishes. It is known that X is a real holomorphic vector field if and only if the complex vector field $X - \sqrt{-1}JX$ is holomorphic. Let $\gamma : \mathbf{C} \to M$ be a complex geodesic and Y be a holomorphic vector field along γ. Denote by $Y_{\mathbf{R}}$ the real part of Y. Then the following fact holds.

Proposition 2.4 ([23]). *Y is a complex Jacobi field along γ if and only if, for any $z_0 \in \mathbf{C}$, $s \mapsto (Y_{\mathbf{R}})_{sz_0}$ is a Jacobi field along the geodesic $\gamma_{z_0}(\underset{\text{def}}{\Leftrightarrow} \gamma_{z_0}(s) := \gamma(sz_0))$.*

Next we shall recall the notion of the parallel translation along a holomorphic curve. Let $\alpha : D \to (M, J, g)$ be a holomorphic curve, where D is an open set of $\mathbf{C}$. Let Y be a holomorphic vector field along α. If $\nabla^{\mathbf{c}}_{\alpha_*(d/dz)} Y = 0$, then we say that Y is *parallel*. For $z_0 \in D$ and $v \in (T_{\alpha(z_0)}M)^{(1,0)}$, there uniquely exists a parallel holomorphic vector field Y along α with $Y_{z_0} = v$.

Definition 2.8. For each $z_1 \in D$, we define a $\mathbf{C}$-linear isomorphism $(P_\alpha)_{z_0, z_1}(v)$ of $(T_{\alpha(z_0)}M)^{(1,0)}$ onto $(T_{\alpha(z_1)}M)^{(1,0)}$ by $(P_\alpha)_{z_0, z_1}(v) := Y_{z_1}$ $(v \in (T_{\alpha(z_0)}M)^{(1,0)})$, where Y is the parallel holomorphic vector field along α with $Y_{z_0} = v$. We call $(P_\alpha)_{z_0, z_1}$ the *parallel translation along α from z_0 to z_1*.

We consider the case where (M, J, g) is an anti-Kaehlerian symmetric space $G^{\mathbf{c}}/H^{\mathbf{c}}$. For $v \in (T_{g_0 H^{\mathbf{c}}}(G^{\mathbf{c}}/H^{\mathbf{c}}))^{\mathbf{c}}$, we define $\mathbf{C}$-linear transformations $\hat{D}^{co}_v$ and $\hat{D}^{si}_v$ of $(T_{g_0 H^{\mathbf{c}}}(G^{\mathbf{c}}/H^{\mathbf{c}}))^{\mathbf{c}}$ by $\hat{D}^{co}_v := g^{\mathbf{c}}_{0*} \circ \cos(\sqrt{-1}\, \mathrm{ad}^{\mathbf{c}}_{\mathfrak{g}^{\mathbf{c}}}((g^{\mathbf{c}}_{0*})^{-1} v)) \circ (g^{\mathbf{c}}_{0*})^{-1}$ and $\hat{D}^{si}_v := g^{\mathbf{c}}_{0*} \circ \frac{\sin(\sqrt{-1}\, \mathrm{ad}^{\mathbf{c}}_{\mathfrak{g}^{\mathbf{c}}}((g^{\mathbf{c}}_{0*})^{-1} v))}{\sqrt{-1}\, \mathrm{ad}^{\mathbf{c}}_{\mathfrak{g}^{\mathbf{c}}}((g^{\mathbf{c}}_{0*})^{-1} v)} \circ (g^{\mathbf{c}}_{0*})^{-1}$, respectively, where $\mathrm{ad}^{\mathbf{c}}_{\mathfrak{g}^{\mathbf{c}}}$ is the complexification of the adjoint representation $\mathrm{ad}_{\mathfrak{g}^{\mathbf{c}}}$ of $\mathfrak{g}^{\mathbf{c}}$. Let Y be a holomorphic vector field along $\gamma^{\mathbf{c}}_v$. Define $\hat{Y} : D \to (T_{g_0 K^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}}))^{(1,0)}$ by $\hat{Y}_z := (P_{\gamma^{\mathbf{c}}_v})_{z,0}(Y_z)$ $(z \in D)$, where D is the domain of $\gamma^{\mathbf{c}}_v$. Then we have

Proposition 2.5 ([23]). *The following relation holds*:

$$Y_z = (P_{\gamma^{\mathbf{c}}_v})_{0,z}\left(\hat{D}^{co}_{zv_{(1,0)}}(Y_0) + z\hat{D}^{si}_{zv_{(1,0)}}\left(\frac{d\hat{Y}}{dz}\bigg|_{z=0}\right)\right). \tag{2.2}$$

3. The Isotropy Action of a Pseudo-Riemannian Symmetric Space

In this section, we investigate the complexified shape operators of the orbits of the isotropy action of a semi-simple pseudo-Riemannian symmetric space by investigating the complexified action. Let G/H be a (semi-simple) pseudo-Riemannian symmetric space (equipped with the metric $\langle\,,\,\rangle$ induced from the Killing form B of $\mathfrak{g}$) and σ be an involution of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$. Denote by the same symbol σ the differential of σ at e. Let $\mathfrak{h} := \mathrm{Lie}\,H$ and $\mathfrak{q} := \mathrm{Ker}(\sigma + \mathrm{id})$, which is identified with $T_{eH}(G/H)$. Let θ be a Cartan involution of G with $\theta \circ \sigma = \sigma \circ \theta$, $\mathfrak{k} := \mathrm{Ker}(\theta - \mathrm{id})$ and $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id})$. Let $\mathfrak{g}^{\mathbf{c}}$, $\mathfrak{h}^{\mathbf{c}}$, $\mathfrak{q}^{\mathbf{c}}$, $\mathfrak{k}^{\mathbf{c}}$, $\mathfrak{p}^{\mathbf{c}}$ and $\langle\,,\,\rangle^{\mathbf{c}}$ be the complexifications of $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{q}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\langle\,,\,\rangle$, respectively. The complexification $\mathfrak{q}^{\mathbf{c}}$ is identified with $T_{eH^{\mathbf{c}}}(G^{\mathbf{c}}/H^{\mathbf{c}})$. Under this identification, $\sqrt{-1}X \in \mathfrak{q}^{\mathbf{c}}$ corresponds to $J_{eH^{\mathbf{c}}}X \in T_{eH^{\mathbf{c}}}(G^{\mathbf{c}}/H^{\mathbf{c}})$, where J is the complex structure of $G^{\mathbf{c}}/H^{\mathbf{c}}$. Give $G^{\mathbf{c}}/H^{\mathbf{c}}$ the metric (which also is denoted by $\langle\,,\,\rangle$) induced from the Killing form B_A of $\mathfrak{g}^{\mathbf{c}}$ regarded as a real Lie algebra. Note that B_A coincides with $2\,\mathrm{Re}\,B^{\mathbf{c}}$ and $(J, \langle\,,\,\rangle)$ is an anti-Kaehlerian structure of $G^{\mathbf{c}}/H^{\mathbf{c}}$, where $B^{\mathbf{c}}$ is the complexification of B. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$ (that is, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ and each element of $\mathfrak{a}$ is semi-simple). The dimension of $\mathfrak{a}$ is called the *rank* of G/H. Without loss of generality, we may assume that $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{k} + \mathfrak{a} \cap \mathfrak{p}$. Let $\mathfrak{q}^{\mathbf{c}}_\alpha := \{X \in \mathfrak{q}^{\mathbf{c}} \mid \mathrm{ad}(a)^2 X = \alpha(a)^2 X \text{ for all } a \in \mathfrak{a}^{\mathbf{c}}\}$ and $\mathfrak{h}^{\mathbf{c}}_\alpha := \{X \in \mathfrak{h}^{\mathbf{c}} \mid \mathrm{ad}(a)^2 X = \alpha(a)^2 X \text{ for all } a \in \mathfrak{a}^{\mathbf{c}}\}$ for each $\alpha \in (\mathfrak{a}^{\mathbf{c}})^*$ ($(\mathfrak{a}^{\mathbf{c}})^*$: the ($\mathbf{C}$-)dual space of $\mathfrak{a}^{\mathbf{c}}$) and $\triangle := \{\alpha \in (\mathfrak{a}^{\mathbf{c}})^* \setminus \{0\} \mid \mathfrak{q}^{\mathbf{c}}_\alpha \neq \{0\}\}$. Then we have

$$\mathfrak{q}^{\mathbf{c}} = \mathfrak{a}^{\mathbf{c}} + \sum_{\alpha \in \triangle_+} \mathfrak{q}^{\mathbf{c}}_\alpha \quad \text{and} \quad \mathfrak{h}^{\mathbf{c}} = \mathfrak{z}_{\mathfrak{h}^{\mathbf{c}}}(\mathfrak{a}^{\mathbf{c}}) + \sum_{\alpha \in \triangle_+} \mathfrak{h}^{\mathbf{c}}_\alpha, \tag{3.1}$$

where $\triangle_+ (\subset \triangle)$ is the positive root system under some lexicographical ordering and $\mathfrak{z}_{\mathfrak{h}^{\mathbf{c}}}(\mathfrak{a}^{\mathbf{c}})$ is the centralizer of $\mathfrak{a}^{\mathbf{c}}$ in $\mathfrak{h}^{\mathbf{c}}$. Let $\tilde{\mathfrak{a}}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ and $\mathfrak{g}^{\mathbf{c}}_{\tilde\alpha} := \{X \in \mathfrak{g}^{\mathbf{c}} \mid \mathrm{ad}(a)X = \tilde\alpha(a)X \text{ for all } a \in \tilde{\mathfrak{a}}^{\mathbf{c}}\}$ for each $\tilde\alpha \in (\tilde{\mathfrak{a}}^{\mathbf{c}})^*$ and $\tilde\triangle := \{\tilde\alpha \in (\tilde{\mathfrak{a}}^{\mathbf{c}})^* \setminus \{0\} \mid \mathfrak{g}^{\mathbf{c}}_{\tilde\alpha} \neq \{0\}\}$. Then we have $\mathfrak{g}^{\mathbf{c}} = \tilde{\mathfrak{a}}^{\mathbf{c}} + \sum_{\tilde\alpha \in \tilde\triangle} \mathfrak{g}^{\mathbf{c}}_{\tilde\alpha}$ and $\dim_{\mathbf{c}} \mathfrak{g}^{\mathbf{c}}_{\tilde\alpha} = 1$ for each $\tilde\alpha \in \tilde\triangle$. Also, we have $\triangle = \{\tilde\alpha|_{\mathfrak{a}^{\mathbf{c}}} \mid \tilde\alpha \in \tilde\triangle\} \setminus \{0\}$, $\mathfrak{q}^{\mathbf{c}}_\alpha = \left(\sum_{\tilde\alpha \in \tilde\triangle \text{ s.t. } \tilde\alpha|_{\mathfrak{a}^{\mathbf{c}}} = \pm\alpha} \mathfrak{g}^{\mathbf{c}}_{\tilde\alpha}\right) \cap \mathfrak{q}^{\mathbf{c}}$ ($\alpha \in \triangle$) and $\mathfrak{h}^{\mathbf{c}}_\alpha = \left(\sum_{\tilde\alpha \in \tilde\triangle \text{ s.t. } \tilde\alpha|_{\mathfrak{a}^{\mathbf{c}}} = \pm\alpha} \mathfrak{g}^{\mathbf{c}}_{\tilde\alpha}\right) \cap \mathfrak{h}^{\mathbf{c}}$ ($\alpha \in \triangle$). The following fact is well-known.

Lemma 3.1. *For each $\alpha \in \triangle$, $\alpha(\mathfrak{a} \cap \mathfrak{p}) \subset \mathbf{R}$ and $\alpha(\mathfrak{a} \cap \mathfrak{k}) \subset \sqrt{-1}\mathbf{R}$.*

Remark 3.1. Each element of $\mathfrak{a} \cap \mathfrak{p}$ (resp. $\mathfrak{a} \cap \mathfrak{k}$) is called a *hyperbolic* (resp. *elliptic*) *element*.

For each $\alpha \in \triangle$, define $a_\alpha \in \mathfrak{a}^{\mathbf{c}}$ by $\alpha(a) = B^{\mathbf{c}}(a_\alpha, a)$ ($a \in \mathfrak{a}^{\mathbf{c}}$). Take $E_{\tilde\alpha} (\neq 0) \in \mathfrak{g}^{\mathbf{c}}_{\tilde\alpha}$ for each $\tilde\alpha \in \tilde\triangle$ and set $Z_{\tilde\alpha} := c_{\tilde\alpha}(E_{\tilde\alpha} + \sigma E_{\tilde\alpha})$ and $Y_{\tilde\alpha} := c_{\tilde\alpha}(E_{\tilde\alpha} - \sigma E_{\tilde\alpha})$, where $c_{\tilde\alpha}$ is one of two solutions of the complex equation

$$z^2 = \frac{\alpha(a_\alpha)}{B^{\mathbf{c}}(E_{\tilde{\alpha}} - \sigma E_{\tilde{\alpha}}, E_{\tilde{\alpha}} - \sigma E_{\tilde{\alpha}})}.$$

Then we have $\mathrm{ad}(a)Z_{\tilde{\alpha}} = \tilde{\alpha}(a)Y_{\tilde{\alpha}}$ and $\mathrm{ad}(a)Y_{\tilde{\alpha}} = \tilde{\alpha}(a)Z_{\tilde{\alpha}}$ for any $a \in \mathfrak{a}^{\mathbf{c}}$. Hence we have $Z_{\tilde{\alpha}} \in \mathfrak{h}^{\mathbf{c}}_{\tilde{\alpha}|_{\mathfrak{a}^{\mathbf{c}}}}$ and $Y_{\tilde{\alpha}} \in \mathfrak{q}^{\mathbf{c}}_{\tilde{\alpha}|_{\mathfrak{a}^{\mathbf{c}}}}$. Furthermore, for $\alpha \in \mathfrak{a}^{\mathbf{c}}$, it is shown that $\mathfrak{h}^{\mathbf{c}}_\alpha$ (resp. $\mathfrak{q}^{\mathbf{c}}_\alpha$) is spanned by $\{Z_{\tilde{\alpha}} \mid \tilde{\alpha} \in \tilde{\triangle} \text{ s.t. } \tilde{\alpha}|_{\mathfrak{a}^{\mathbf{c}}} = \alpha\}$ (resp. $\{Y_{\tilde{\alpha}} \mid \tilde{\alpha} \in \tilde{\triangle} \text{ s.t. } \tilde{\alpha}|_{\mathfrak{a}^{\mathbf{c}}} = \alpha\}$). Then $[Z_{\tilde{\alpha}}, Y_{\tilde{\alpha}}] = \alpha(a_\alpha)a_\alpha$ is shown. L. Verhoczki [38] investigated the shape operators of orbits of the isotropy action of a Riemannian symmetric space of compact type. By applying his method of investigation to the isotropy action of the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/H^{\mathbf{c}}$, we prove the following fact for orbits of the isotropy action of G/H.

Proposition 3.2. *Let M be an orbit of the isotropy action* (*i.e.*, *the H-action*) *on G/H through $x := \exp_G(w)H$* ($w \in \mathfrak{q}$ *s.t.* $\mathrm{ad}(w)$: *semi-simple*) *and A be the shape tensor of M. For simplicity, set $g := \exp_G(w)$. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$ containing w and $\mathfrak{q}^{\mathbf{c}} = \mathfrak{a}^{\mathbf{c}} + \sum_{\alpha \in \triangle_+} \mathfrak{q}^{\mathbf{c}}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}^{\mathbf{c}}$. Then the following statements* (i) *and* (ii) *hold*:

(i) *We have*

$$g_*^{-1}(T_xM)^{\mathbf{c}} = \sum_{\substack{\alpha \in \triangle_+ \\ \text{s.t. } \alpha(w) \notin \sqrt{-1}\pi\mathbf{Z}}} \mathfrak{q}^{\mathbf{c}}_\alpha$$

and

$$g_*^{-1}(T_x^\perp M)^{\mathbf{c}} = \mathfrak{a}^{\mathbf{c}} + \sum_{\substack{\alpha \in \triangle_+ \\ \text{s.t. } \alpha(w) \in \sqrt{-1}\pi\mathbf{Z}}} \mathfrak{q}^{\mathbf{c}}_\alpha$$

hold. In particular, if M is a principal orbit, then we have $g_*^{-1}(T_xM)^{\mathbf{c}} = \sum_{\alpha \in \triangle_+} \mathfrak{q}^{\mathbf{c}}_\alpha$ and $g_*^{-1}(T_x^\perp M)^{\mathbf{c}} = \mathfrak{a}^{\mathbf{c}}$.

(ii) *Let H_x be the isotropy group of H at x and set $H_x(g_*\mathfrak{a}) := \{h_{*x}g_*a \mid a \in \mathfrak{a}, h \in H_x\}$. Then $H_x(g_*\mathfrak{a})$ is open in $T_x^\perp M$ and, for any $v := h_{*x}g_*a \in H_x(g_*\mathfrak{a})$* ($a \in \mathfrak{a}, h \in H_x$), *we have $A^{\mathbf{c}}_v|_{h_{*x}g_*\mathfrak{q}^{\mathbf{c}}_\alpha} = -\frac{\sqrt{-1}\alpha(a)}{\tan(\sqrt{-1}\alpha(w))}\,\mathrm{id}$* ($\alpha \in \triangle_+$ *s.t.* $\alpha(w) \notin \sqrt{-1}\pi\mathbf{Z}$), *where $A^{\mathbf{c}}$ is the complexification of A.*

Proof. First we shall show the statement (i) by imitating the proof of Proposition 3 in [38]. Let $M^{\mathbf{c}}$ be the extrinsic complexification of M, that is, $M^{\mathbf{c}} := H^{\mathbf{c}} \cdot x (\subset G^{\mathbf{c}}/H^{\mathbf{c}})$, where G/H is identified with $G(eH^{\mathbf{c}})$. We shall investigate $T_x(M^{\mathbf{c}})$ instead of $(T_xM)^{\mathbf{c}}$ because $(T_xM)^{\mathbf{c}}$ is identified with $T_x(M^{\mathbf{c}})$. Let a_α $(\alpha \in \triangle)$, $\tilde{\triangle}$, $Z_{\tilde{\alpha}}$ and $Y_{\tilde{\alpha}}$ $(\tilde{\alpha} \in \tilde{\triangle})$ be the above quantities defined for $\mathfrak{a}$ and a Cartan subalgebra $\tilde{\mathfrak{a}}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. Let $\tilde{\alpha} \in \tilde{\triangle}$ and $\alpha := \tilde{\alpha}|_{\mathfrak{a}^{\mathbf{c}}}$. Since $[Z_{\tilde{\alpha}}, w] = -\alpha(w) Y_{\tilde{\alpha}}$ and $[Z_{\tilde{\alpha}}, Y_{\tilde{\alpha}}] = \alpha(a_\alpha) a_\alpha$, we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{G^{\mathbf{c}}}(\exp tZ_{\tilde{\alpha}}) w = -\alpha(w) Y_{\tilde{\alpha}},$$

where $\mathrm{Ad}_{G^{\mathbf{c}}}$ is the adjoint representation of $G^{\mathbf{c}}$. Hence we have

$$T_w \,\mathrm{Ad}_{G^{\mathbf{c}}}(H^{\mathbf{c}}) w = \sum_{\alpha \in \triangle_+ \text{ s.t. } \alpha(w) \neq 0} \mathfrak{q}_\alpha^{\mathbf{c}}.$$

Denote by Exp the exponential map of the anti-Kaehlerian symmetric space $(G^{\mathbf{c}}/H^{\mathbf{c}}, J, \langle\,,\rangle)$. Assume that $\alpha(w) \neq 0$. Define a complex geodesic variation $\delta : \mathbf{C}^2 \to G^{\mathbf{c}}/H^{\mathbf{c}}$ of the complex geodesic $\gamma_w^{\mathbf{c}}(z) = \mathrm{Exp}(zw)$ by

$$\delta(z,u) := \mathrm{Exp}\left(z\left(\cos u \cdot w + \sin u \sqrt{\frac{\langle w, w\rangle}{\langle Y_{\tilde{\alpha}}, Y_{\tilde{\alpha}}\rangle}} Y_{\tilde{\alpha}}\right)\right)$$

$((z,u) \in \mathbf{C}^2)$. Set $W := \left.\frac{\partial \delta}{\partial u}\right|_{u=0}$, which is a complex Jacobi field along $\gamma_w^{\mathbf{c}}$. Hence it follows from (2.2) that

$$W_1 = \frac{\sin(\sqrt{-1}\alpha(w))}{\sqrt{-1}\alpha(w)} \sqrt{\frac{\langle w, w\rangle}{\langle Y_{\tilde{\alpha}}, Y_{\tilde{\alpha}}\rangle}} g_* Y_{\tilde{\alpha}},$$

where $W_1 := W|_{z=1}$. On the other hand, we have $W_1 = (d\,\mathrm{Exp})_w\left(\sqrt{\frac{\langle w,w\rangle}{\langle Y_{\tilde{\alpha}}, Y_{\tilde{\alpha}}\rangle}} Y_{\tilde{\alpha}}\right)$. Hence we have

$$(d\,\mathrm{Exp})_w(Y_{\tilde{\alpha}}) = \frac{\sin(\sqrt{-1}\alpha(w))}{\sqrt{-1}\alpha(w)} g_* Y_{\tilde{\alpha}}. \tag{3.1}$$

Since $M^{\mathbf{c}} = \mathrm{Exp}(\mathrm{Ad}_{G^{\mathbf{c}}}(H^{\mathbf{c}})w)$, we have $T_x(M^{\mathbf{c}}) = (d\,\mathrm{Exp})_w(T_w(\mathrm{Ad}_{G^{\mathbf{c}}}(H^{\mathbf{c}})w))$. Hence the relations in the statement (i) follow from (3.1).

Next we shall show the statement (ii). The H_x-action on $T_x(G/H)$ preserves T_xM and $T_x^\perp M$ invariantly, respectively. The H_x-action on $T_x^\perp M$ is so-called slice representation and it is equivalent to an s-representation (the isotropy representation of a pseudo-Riemannian symmetric space) (see Page 359–360 of [39]). Therefore $H_x(g_*\mathfrak{a})$ is open in $T_x^\perp M$ (see [12]). In the sequel, we shall show the remaining part of the statement (ii) by imitating the proof of Theorem 1 in [38] for the isotropy action of a Riemannian symmetric space of compact type.

Denote by $\hat{A}$ the shape tensor of $M^{\mathbf{c}}$. Under the identification of $(T_xM)^{\mathbf{c}}$ with $T_x(M^{\mathbf{c}})$, the complexified shape operator $A_w^{\mathbf{c}}$ is identified with $\hat{A}_w$. Hence we suffice to investigate $\hat{A}_w$ instead of $A_w^{\mathbf{c}}$. Let α be an element of $\triangle_+$ with $\alpha(w) \notin \sqrt{-1}\pi\mathbf{Z}$. Take $\tilde{\alpha}_1 \in \tilde{\triangle}$ with $\tilde{\alpha}_1|_{\mathfrak{a}^{\mathbf{c}}} = \alpha$. Also, in case of $2\alpha \in \triangle$, $\tilde{\alpha}_2 \in \tilde{\triangle}$ with $\tilde{\alpha}_2|_{\mathfrak{a}^{\mathbf{c}}} = 2\alpha$. Set $\hat{\mathfrak{h}}_\alpha^{\mathbf{c}} := \mathfrak{z}_{\mathfrak{h}^{\mathbf{c}}}(\mathfrak{a}^{\mathbf{c}}) + \mathfrak{h}_\alpha^{\mathbf{c}} + \mathfrak{h}_{2\alpha}^{\mathbf{c}}$ ($\mathfrak{h}_{2\alpha}^{\mathbf{c}} = \{0\}$ in case of $2\alpha \notin \triangle$) and $\hat{H}_\alpha^{\mathbf{c}} := \exp_{G^{\mathbf{c}}}(\hat{\mathfrak{h}}_\alpha^{\mathbf{c}})$. Easily we can show

$$\mathrm{Ad}_{G^{\mathbf{c}}}(\exp zZ_{\tilde{\alpha}_k})a_\alpha = \cos(k^2 z\alpha(a_\alpha))a_\alpha - \frac{1}{k}\sin(k^2 z\alpha(a_\alpha))\,Y_{\tilde{\alpha}_k} \quad (k = 1, 2).$$

From this relation, it follows that $\mathrm{Ad}(\hat{H}_\alpha^{\mathbf{c}})(a_\alpha)$ is a complex hypersurface in $\hat{\mathfrak{q}}_\alpha^{\mathbf{c}} := \mathbf{C}a_\alpha + \mathfrak{q}_\alpha^{\mathbf{c}} + \mathfrak{q}_{2\alpha}^{\mathbf{c}}$ ($\mathfrak{q}_{2\alpha}^{\mathbf{c}} = \{0\}$ in case of $2\alpha \notin \triangle$). On the other hand, it is clear that $\mathrm{Ad}(\hat{H}_\alpha^{\mathbf{c}})(a_\alpha)$ is contained in the complex hypersphere $(B^{\mathbf{c}}|_{\mathfrak{q}_\alpha^{\mathbf{c}} \times \mathfrak{q}_\alpha^{\mathbf{c}}})(\mathbf{z}, \mathbf{z}) = B^{\mathbf{c}}(a_\alpha, a_\alpha)$ of $\hat{\mathfrak{q}}_\alpha^{\mathbf{c}}$. Hence $\mathrm{Ad}(\hat{H}_\alpha^{\mathbf{c}})(a_\alpha)$ coincides with this complex hypersphere. The vector w is expressed as $w = \frac{\alpha(w)}{\alpha(a_\alpha)}a_\alpha + b$ for some $b \in \alpha^{-1}(0)$. Then we have

$$\mathrm{Ad}_{G^{\mathbf{c}}}(\exp zZ_{\tilde{\alpha}_k})w = b + \frac{\alpha(w)}{\alpha(a_\alpha)}\left(\cos(k^2 z\alpha(a_\alpha))a_\alpha - \frac{1}{k}\sin(k^2 z\alpha(a_\alpha))\,Y_{\tilde{\alpha}_k}\right)$$

$(k = 1, 2)$. From this relation, it follows that $\mathrm{Ad}(\hat{H}_\alpha^{\mathbf{c}})(w)$ coincides with the complex hypersphere $(B^{\mathbf{c}}|_{\hat{\mathfrak{q}}_\alpha^{\mathbf{c}} \times \hat{\mathfrak{q}}_\alpha^{\mathbf{c}}})(\mathbf{z} - b, \mathbf{z} - b) = \frac{\alpha(w)^2}{\alpha(a_\alpha)}$ of $b + \hat{\mathfrak{q}}_\alpha^{\mathbf{c}}$. Set $\hat{Q}_\alpha^{\mathbf{c}} := \mathrm{Exp}(\hat{\mathfrak{q}}_\alpha^{\mathbf{c}})$ and $\hat{Q}_\alpha^{\mathbf{c}}(b) := \mathrm{Exp}(b + \hat{\mathfrak{q}}_\alpha^{\mathbf{c}})$. It is easy to show that $\hat{Q}_\alpha^{\mathbf{c}}$ is a totally geodesic complex rank one anti-Kaehlerian symmetric space in $G^{\mathbf{c}}/H^{\mathbf{c}}$. Furthermore, by imitating the proof of Proposition 4 in [38], it is shown that $\hat{Q}_\alpha^{\mathbf{c}}(b)$ is a totally geodesic complex rank one anti-Kaehlerian symmetric space and it is isometric to $\hat{Q}_\alpha^{\mathbf{c}}$. In fact, a map $\phi : \hat{Q}_\alpha^{\mathbf{c}} \to \hat{Q}_\alpha^{\mathbf{c}}(b)$ defined by $\phi(\mathrm{Exp}\,\mathbf{z}) = \mathrm{Exp}(\mathbf{z} + b)$ ($\mathbf{z} \in \hat{\mathfrak{q}}_\alpha^{\mathbf{c}}$) is an isometry. Since $\mathrm{Ad}(\hat{H}_\alpha^{\mathbf{c}})(w)$ is equal to the complex hypersphere of complex radius $\sqrt{\frac{\alpha(w)^2}{\alpha(a_\alpha)}}$ of $b + \hat{\mathfrak{q}}_\alpha^{\mathbf{c}}$, $\hat{H}_\alpha^{\mathbf{c}} \cdot x$ is a complex geodesic hypersphere of complex radius $\sqrt{\alpha(w)}$ in $\hat{Q}_\alpha^{\mathbf{c}}(b)$. Set $\hat{Q}_\alpha^{\mathbf{c}\prime} := \mathrm{Exp}(\mathfrak{a}^{\mathbf{c}} + \mathfrak{q}_\alpha^{\mathbf{c}} + \mathfrak{q}_{2\alpha}^{\mathbf{c}})$, which is isometric to the anti-Kaehlerian product $\hat{Q}_\alpha^{\mathbf{c}}(b) \times \mathbf{C}^{r-1}$ ($r := \mathrm{rank}(G/H)$).

We have $\hat{H}_\alpha^{\mathbf{c}} \cdot x \subset M^{\mathbf{c}} \cap \hat{Q}_\alpha^{\mathbf{c}}(b) \subset M^{\mathbf{c}} \cap \hat{Q}_\alpha^{\mathbf{c}\prime}$. Also, since $T_x(M^{\mathbf{c}}) = g_*\left(\sum_{\alpha \in \triangle_+ \text{ s.t. } \alpha(w) \notin \sqrt{-1}\pi\mathbf{Z}} \mathfrak{q}_\alpha^{\mathbf{c}}\right)$ and $T_x\hat{Q}_\alpha^{\mathbf{c}\prime} = g_*(\mathfrak{a}^{\mathbf{c}} + \mathfrak{q}_\alpha^{\mathbf{c}} + \mathfrak{q}_{2\alpha}^{\mathbf{c}})$, we have $T_x(M^{\mathbf{c}} \cap \hat{Q}_\alpha^{\mathbf{c}\prime}) = \mathfrak{q}_\alpha^{\mathbf{c}} + \mathfrak{q}_{2\alpha}^{\mathbf{c}}$ and hence $\dim T_x(M^{\mathbf{c}} \cap \hat{Q}_\alpha^{\mathbf{c}\prime}) = \dim(\hat{H}_\alpha^{\mathbf{c}} \cdot x)$. Therefore $\hat{H}_\alpha^{\mathbf{c}} \cdot x$ is a component of $M^{\mathbf{c}} \cap \hat{Q}_\alpha^{\mathbf{c}\prime}$. Denote by $\bar{A}$ the shape tensor of $\hat{H}_\alpha^{\mathbf{c}} \cdot x \hookrightarrow \hat{Q}_\alpha^{\mathbf{c}\prime}$. Since $\hat{Q}_\alpha^{\mathbf{c}\prime}$ is totally geodesic in $G^{\mathbf{c}}/H^{\mathbf{c}}$ and $T_x^\perp(M^{\mathbf{c}})$ contains the normal space of $\hat{H}_\alpha^{\mathbf{c}} \cdot x$ in $\hat{Q}_\alpha^{\mathbf{c}\prime}$, it follows from pseudo-Riemannian version of Lemma 6 of [38] that $\hat{A}_{g_*a_\alpha}$ preserves $T_x(\hat{H}_\alpha^{\mathbf{c}} \cdot x)$ invariantly and that $\hat{A}_{g_*a_\alpha} = \bar{A}_{g_*a_\alpha}$ on $T_x(\hat{H}_\alpha^{\mathbf{c}} \cdot x)$. Let ϕ be the above isometry of $\hat{Q}_\alpha^{\mathbf{c}}$ onto $\hat{Q}_\alpha^{\mathbf{c}}(b)$. Set $r_0 := \frac{\alpha(w)}{\alpha(a_\alpha)}$ and denote by $\bar{A}'$ the shape tensor of $\hat{H}_\alpha^{\mathbf{c}} \cdot (r_0 a_\alpha) \hookrightarrow \hat{Q}_\alpha^{\mathbf{c}\prime}$. Clearly we have $\phi(\hat{H}_\alpha^{\mathbf{c}} \cdot (r_0 a_\alpha)) = \hat{H}_\alpha^{\mathbf{c}} \cdot x$ and

$\phi_*((\exp_{G^{\mathbf{c}}}(r_0a_\alpha))_*(a_\alpha)) = g_*a_\alpha$. Hence we have $\bar{A}_{g_*a_\alpha} = \phi_* \circ \bar{A}'_{(\exp_{G^{\mathbf{c}}}(r_0a_\alpha))_*(a_\alpha)} \circ \phi_*^{-1}$. For simplicity, set $\bar{g} := \exp_{G^{\mathbf{c}}}(r_0a_\alpha)$. Now we shall investigate $\bar{A}'_{\bar{g}_*a_\alpha}$. Define a complex geodesic variation $\delta : \mathbf{C}^2 \to G^{\mathbf{c}}/H^{\mathbf{c}}$ by

$$\delta(z,u) := \mathrm{Exp}\left(z\left(r_0 \cos u \cdot a_\alpha + \sqrt{\frac{r_0^2\langle a_\alpha, a_\alpha\rangle}{\langle Y_{\tilde{\alpha}_1}, Y_{\tilde{\alpha}_1}\rangle}} \sin u \cdot Y_{\tilde{\alpha}_1}\right)\right) \quad ((z,u) \in \mathbf{C}^2).$$

Set $W := \frac{\partial\delta}{\partial u}\big|_{u=0}$. Since W is a complex Jacobi field along $\gamma^{\mathbf{c}}_{r_0a_\alpha}$, it follows from (2.2) that

$$W_z = \frac{\sin(\sqrt{-1}z\alpha(r_0a_\alpha))}{\sqrt{-1}\alpha(r_0a_\alpha)}\sqrt{\frac{r_0^2\langle a_\alpha, a_\alpha\rangle}{\langle Y_{\tilde{\alpha}_1}, Y_{\tilde{\alpha}_1}\rangle}}(P_{\gamma^{\mathbf{c}}_{r_0a_\alpha}})_{0,z}(Y_{\tilde{\alpha}_1}). \tag{3.2}$$

We have

$$\begin{aligned}\tilde{\nabla}_{(\partial\delta/\partial u)|_{z=1,u=0}}\frac{\partial\delta}{\partial z} &= \tilde{\nabla}_{(\partial\delta/\partial z)|_{z=1,u=0}}\frac{\partial\delta}{\partial u} = W'_1\\ &= \cos(\sqrt{-1}\alpha(r_0a_\alpha))\sqrt{\frac{r_0^2\langle a_\alpha, a_\alpha\rangle}{\langle Y_{\tilde{\alpha}_1}, Y_{\tilde{\alpha}_1}\rangle}}\bar{g}_*Y_{\tilde{\alpha}_1} \in T_{\mathrm{Exp}(r_0a_\alpha)}\hat{H}^{\mathbf{c}}_\alpha \cdot (r_0a_\alpha)\end{aligned}$$

and hence

$$\bar{A}'_{\bar{g}_*(r_0a_\alpha)}W_1 = -\cos(\sqrt{-1}\alpha(r_0a_\alpha))\sqrt{\frac{r_0^2\langle a_\alpha, a_\alpha\rangle}{\langle Y_{\tilde{\alpha}_1}, Y_{\tilde{\alpha}_1}\rangle}}\bar{g}_*Y_{\tilde{\alpha}_1},$$

which together with (3.2) and $\alpha(b) = 0$ deduces

$$\bar{A}'_{\bar{g}_*a_\alpha}\bar{g}_*Y_{\tilde{\alpha}_1} = -\frac{\sqrt{-1}\alpha(a_\alpha)}{\tan(\sqrt{-1}\alpha(w))}\bar{g}_*Y_{\tilde{\alpha}_1}.$$

Therefore we have

$$\hat{A}_{g_*a_\alpha}g_*Y_{\tilde{\alpha}_1} = -\frac{\sqrt{-1}\alpha(a_\alpha)}{\tan(\sqrt{-1}\alpha(w))}g_*Y_{\tilde{\alpha}_1}.$$

Similarly we have

$$\hat{A}_{g_*a_\alpha}g_*Y_{\tilde{\alpha}_2} = -\frac{2\sqrt{-1}\alpha(a_\alpha)}{\tan(2\sqrt{-1}\alpha(w))}g_*Y_{\tilde{\alpha}_2}.$$

Take $\bar{b} \in \alpha^{-1}(0)$. Since $\hat{Q}^{\mathbf{c}}_\alpha(b)$ is totally geodesic and $T^\perp\hat{Q}^{\mathbf{c}}_\alpha(b)|_{\hat{H}^{\mathbf{c}}_\alpha\cdot x} \cap T^\perp M|_{\hat{H}^{\mathbf{c}}_\alpha\cdot x}$ is parallel along $\hat{H}^{\mathbf{c}}_\alpha \cdot x$ with respect to the normal connection of $\hat{Q}^{\mathbf{c}}_\alpha(b) \hookrightarrow G^{\mathbf{c}}/H^{\mathbf{c}}$, we have

$$\hat{A}_{g_*\bar{b}}g_*Y_{\tilde{\alpha}_1} = \hat{A}_{g_*\bar{b}}g_*Y_{\tilde{\alpha}_2} = 0.$$

Take an arbitrary $a \in \mathfrak{a}$. We can express as $a = \frac{\alpha(a)}{\alpha(a_\alpha)} a_\alpha + \hat{b}$ for some $\hat{b} \in \alpha^{-1}(0)$. Thus, for each $a \in \mathfrak{a}$, we have

$$\hat{A}_{g_*a}|_{g_*\mathfrak{q}^{\mathbf{c}}_\beta} = -\frac{\sqrt{-1}\beta(a)}{\tan(\sqrt{-1}\beta(w))} \text{ id} \quad (\beta \in \triangle_+ \text{ s.t. } \beta(w) \notin \sqrt{-1}\pi\mathbf{Z}).$$

Take an arbitrary $h_{*x}g_*a \in H_x(g_*\mathfrak{a})$ $(a \in \mathfrak{a},\ h \in H_x)$. Since h is an isometry of $G^{\mathbf{c}}/H^{\mathbf{c}}$, we have $\hat{A}_{h_{*x}g_*a} = h_{*x} \circ \hat{A}_{g_*a} \circ h_{*x}^{-1}$. Hence we have

$$\hat{A}_{h_{*x}g_*a}|_{h_{*x}g_*\mathfrak{q}^{\mathbf{c}}_\beta} = -\frac{\sqrt{-1}\beta(a)}{\tan(\sqrt{-1}\beta(w))} \text{ id} \quad (\beta \in \triangle_+ \text{ s.t. } \beta(w) \notin \sqrt{-1}\pi\mathbf{Z}).$$

Therefore, we obtain the relation in the statement (ii). q.e.d.

4. Shape Operators of Partial Tubes

In this section, we investigate the shape operators of partial tubes over a pseudo-Riemannian submanifold with section in a (semi-simple) pseudo-Riemannian symmetric space G/H equipped with the metric induced from the Killing form of $\mathfrak{g} := \mathrm{Lie}\, G$. Let M be a pseudo-Riemannian submanifold with section in G/H, that is, for each $x = gH$ of M, $g_*^{-1}T_x^\perp M$ is a Lie triple system. Let $t(M)$ be a connected submanifold in the normal bundle $T^\perp M$ of M such that, for any curve $c : [0,1] \to M$, $P_c^\perp(t(M) \cap T_{c(0)}^\perp M) = t(M) \cap T_{c(1)}^\perp M$ holds, where $P_c^\perp$ is the parallel transport along c with respect to the normal connection. Denote by F the set of all critical points of the normal exponential map $\exp^\perp$ of M. Assume that $t(M) \cap F = \emptyset$. Then the restriction $\exp^\perp|_{t(M)}$ of $\exp^\perp$ to $t(M)$ is an immersion of $t(M)$ into G/H. Assume that $\exp^\perp|_{t(M)} : t(M) \hookrightarrow G/H$ is a pseudo-Riemannian submanifold. Then we call $t(M)$ a *partial tube over* M. Define a distribution D^V on $t(M)$ by $D_v^V = T_v(t(M) \cap T_{\pi(v)}^\perp M)$ $(v \in t(M))$, where π is the bundle projection of $T^\perp M$. We call this distribution a *vertical distribution* on $t(M)$. Let $X \in T_{\pi(v)}M$. Take a curve c in M with $\dot{c}(0) = X$. Let $\tilde{v}$ be a parallel normal vector field along c with $\tilde{v}(0) = v$. Denote by $\tilde{X}_v$ the velocity vector $\dot{\tilde{v}}(0)$ of the curve $\tilde{v}$ in $T^\perp M$ at 0. We call $\tilde{X}_v$ the *horizontal lift of* X *to* v. Define a distribution D^H on $t(M)$ by $D_v^H = \{\tilde{X}_v \mid X \in T_{\pi(v)}M\}$ $(v \in t(M))$. We call this distribution a *horizontal distribution* on $t(M)$. From (2.1), we have

$$\exp_*^\perp(\tilde{X}_v) = P_{\gamma_v}\big(D_v^{co}X - D_v^{si}(A_vX)\big). \tag{4.1}$$

Assume that $t(M)$ is contained in the ε-tube $t_\varepsilon(M) := \left\{v \in T^\perp M \,\middle|\, \frac{\langle v,v\rangle}{\sqrt{|\langle v,v\rangle|}} = \varepsilon\right\}$ $(\varepsilon \neq 0)$. Define a subbundle $D^\perp$ of the normal bundle $T^\perp t(M)$ of $t(M)$ by $D_v^\perp := T_v^\perp t(M) \cap T_v(t_\varepsilon(M))$ $(v \in t(M))$. Clearly we have $T_vt(M) = D_v^H \oplus D_v^V$ (orthogonal direct sum) and $T_v^{\perp}t(M) = D_v^{\perp} \oplus \mathrm{Span}\{\dot{\bar{\gamma}}_v(1)\}$ (orthogonal direct sum), where $\bar{\gamma}_v$ is defined by $\bar{\gamma}_v(t) := tv$. Denote by A (resp. A^t) the shape tensor of M (resp. $t(M)$). Also, denote by A^x that of a submanifold $t(M) \cap T_x^{\perp}M$ in $\exp^{\perp}(T_x^{\perp}M)$ immersed by $\exp^{\perp}|_{t(M)\cap T_x^{\perp}M}$. In the sequel, we omit $\exp_*^{\perp}$. For a real analytic function F and $v \in T_{gH}(G/H)$, we denote the operator $g_* \circ F(\mathrm{ad}(g_*^{-1}v)) \circ g_*^{-1}$ by $F(\mathrm{ad}(v))$ for simplicity. Then, by imitating the proof of Proposition 3.1 in [19], we can show the the following relations.

Proposition 4.1. *Let $v \in t(M)$ and $w \in D_v^{\perp}$. Also, let $\pi(v) = g_1H$, $g_2 := \exp_G(g_{1*}^{-1}v)$ and $g := g_1g_2g_1^{-1}$, where $\exp_G$ is the exponential map of the Lie group G.*

(i) *For $Y \in D_v^V$, we have*

$$A_{g_*v}^t Y = A_{g_*v}^{\pi(v)} Y, \quad A_w^t Y = A_w^{\pi(v)} Y. \tag{4.2}$$

(ii) *Assume that $\mathrm{Span}\{g_{1*}^{-1}v, (g_1g_2)_*^{-1}w\}$ is abelian. Then, for $X \in T_{\pi(v)}M$, we have*

$$\begin{aligned} A_w^t \tilde{X}_v = {} & \sqrt{-1}\,\mathrm{ad}(g_*^{-1}w)\sin(\sqrt{-1}\,\mathrm{ad}(v))(X) \\ & - \frac{\sqrt{-1}\sin(\sqrt{-1}\,\mathrm{ad}(v))}{\mathrm{ad}(v)}(A_{g_*^{-1}w}X) \\ & + \left(\frac{\cos(\sqrt{-1}\,\mathrm{ad}(v)) - \mathrm{id}}{\mathrm{ad}(v)} + \frac{\sqrt{-1}\sin(\sqrt{-1}\,\mathrm{ad}(v)) + \mathrm{ad}(v)}{\mathrm{ad}(v)^2}\right) \\ & \times \mathrm{ad}(g_*^{-1}w)(A_vX). \end{aligned} \tag{4.3}$$

Remark 4.1. The parallel translation P_{γ_v} along γ_v is equal to g_*.

5. Proper Complex Equifocality

In this section, we investigate the proper complex equifocality of a complex equifocal submanifold in a pseudo-Riemannian symmetric space. Let G/H be a (semi-simple) pseudo-Riemannian symmetric space and R be the curvature tensor of G/H. First we prepare the following lemma for a curvature-adapted submanifold with flat section such that the normal holonomy group is trivial.

Lemma 5.1. *Let M be a curvature-adapted submanifold in G/H with flat section such that the normal holonomy group is trivial. Assume that, for any normal* *vector v of M, A_v and $\mathrm{ad}(g_*^{-1}v)$ are semi-simple, where A is the shape tensor of M and g is an element of G such that gH is the base point of v. Then, for any $x \in M$, $\{A_v \mid v \in T_x^\perp M\} \cup \{R(\cdot, v)v|_{T_xM} \mid v \in T_x^\perp M\}$ is a commuting family of linear transformations of T_xM.*

Proof. Let $v_i \in T_x^\perp M$ $(i = 1, 2)$. Since M has flat section, $R(\cdot, v_1)v_1|_{T_xM}$ and $R(\cdot, v_2)v_2|_{T_xM}$ commute with each other. Since M has flat section and the normal holonomy group is trivial, A_{v_1} and A_{v_2} commute with each other. In the sequel, we shall show that $R(\cdot, v_1)v_1|_{T_xM}$ and A_{v_2} commute with each other. Let $x = gH$. Since $g_*^{-1}T_x^\perp M$ is abelian and, for any $v \in T_x^\perp M$, $\mathrm{ad}(g_*^{-1}v)$ is semi-simple, there exists a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}(= T_{eH}(G/H))$ containing $\mathfrak{b} := g_*^{-1}(T_x^\perp M)$. Let $\triangle$ be the root system with respect to $\mathfrak{a}^{\mathbf{c}}$ and set $\overline{\triangle} := \{\alpha|_{\mathfrak{b}^{\mathbf{c}}} \mid \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}^{\mathbf{c}}} \neq 0\}$. For each $\beta \in \overline{\triangle}$, we set $\mathfrak{q}_\beta^{\mathbf{c}} := \{X \in \mathfrak{q}^{\mathbf{c}} \mid \mathrm{ad}(b)^2(X) = \beta(b)^2 X \ \ (\forall b \in \mathfrak{b}^{\mathbf{c}})\}$. Then we have $\mathfrak{q}^{\mathbf{c}} = \mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) + \sum_{\beta \in \overline{\triangle}_+} \mathfrak{q}_\beta^{\mathbf{c}}$, where $\overline{\triangle}_+$ is the positive root system under some lexicographical ordering and $\mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}})$ is the centralizer of $\mathfrak{b}^{\mathbf{c}}$ in $\mathfrak{q}^{\mathbf{c}}$. Consider

$$D := \{v \in (T_x^\perp M)^{\mathbf{c}} \mid \beta(g_*^{-1}v)\text{'s } \ (\beta \in \overline{\triangle}_+) \ \text{are mutually distinct}\}.$$

It is clear that D is open and dense in $(T_x^\perp M)^{\mathbf{c}}$. Take $v \in D$. Since $\beta(g_*^{-1}v)$'s $(\beta \in \overline{\triangle}_+)$ are mutually distinct, the decomposition $(T_xM)^{\mathbf{c}} = g_*(\mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) \ominus \mathfrak{b}^{\mathbf{c}}) + \sum_{\beta \in \overline{\triangle}_+} g_*\mathfrak{q}_\beta^{\mathbf{c}}$ is the eigenspace decomposition of $R^{\mathbf{c}}(\cdot, v)v|_{(T_xM)^{\mathbf{c}}}$, where we note that $R^{\mathbf{c}}(\cdot, v)v = -g_* \circ \mathrm{ad}^{\mathbf{c}}(g_*^{-1}v)^2 \circ g_*^{-1}$. Since M is curvature-adapted and hence $[R^{\mathbf{c}}(\cdot, v)v|_{(T_xM)^{\mathbf{c}}}, A_v^{\mathbf{c}}] = 0$, we have

$$(5.1) \qquad (T_xM)^{\mathbf{c}} = \sum_{\lambda \in \mathrm{Spec}\, A_v^{\mathbf{c}}} (g_*(\mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) \ominus \mathfrak{b}^{\mathbf{c}}) \cap \mathrm{Ker}(A_v^{\mathbf{c}} - \lambda\, \mathrm{id})) + \sum_{\lambda \in \mathrm{Spec}\, A_v^{\mathbf{c}}} \sum_{\beta \in \overline{\triangle}_+} (g_*\mathfrak{q}_\beta^{\mathbf{c}} \cap \mathrm{Ker}(A_v^{\mathbf{c}} - \lambda\, \mathrm{id})).$$

Suppose that (5.1) does not hold for some $v_0 \in (T_x^\perp M)^{\mathbf{c}} \setminus D$. Then it is easy to show that there exists a neighborhood U of v_0 in $(T_x^\perp M)^{\mathbf{c}}$ such that (5.1) does not hold for any $v \in U$. Clearly we have $U \cap D = \emptyset$. This contradicts the fact that D is dense in $(T_x^\perp M)^{\mathbf{c}}$. Hence (5.1) holds for any $v \in (T_x^\perp M)^{\mathbf{c}} \setminus D$. Therefore, (5.1) holds for any $v \in (T_x^\perp M)^{\mathbf{c}}$. In particular, (5.1) holds for v_2. On the other hand, the decomposition $(T_xM)^{\mathbf{c}} = g_*(\mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) \ominus \mathfrak{b}^{\mathbf{c}}) + \sum_{\beta \in \overline{\triangle}_+} g_*\mathfrak{q}_\beta^{\mathbf{c}}$ is the common eigenspace decomposition of $R^{\mathbf{c}}(\cdot, v)v|_{(T_xM)^{\mathbf{c}}}$'s $(v \in (T_x^\perp M)^{\mathbf{c}})$. From these facts, it follows that $R^{\mathbf{c}}(\cdot, v_1)v_1|_{(T_xM)^{\mathbf{c}}}$ and $A_{v_2}^{\mathbf{c}}$ commute with each other. This completes the proof. q.e.d.

By this lemma, Lemma 5.3, Propositions 5.6 and 5.7 of [17] (these lemmas are valid even if the ambient space is a pseudo-Riemannian symmetric space), we can show the following fact.

Proposition 5.2. *Let M be a curvature-adapted complex equifocal submanifold in G/H. Assume that, for any normal vector v of M, A_v and* $\mathrm{ad}(g_*^{-1}v)$ *are semi-simple and that* $\pm\beta(g_*^{-1}v) \notin \mathrm{Spec}\, A_v^{\mathbf{c}}|_{g_*\mathfrak{q}_\beta^{\mathbf{c}}}$ $(\beta \in \overline{\triangle}_+)$, *where g is an element of G such that gH is the base point of v. Then M is proper complex equifocal.*

Proof. Let $\tilde{M} := (\pi \circ \phi)^{-1}(M)$ and denote by $\tilde{A}$ the shape tensor of $\tilde{M}$. Fix $u \in \tilde{M}$ and $\tilde{v} \in T_u^{\perp}\tilde{M}$. For simplicity, set $x(= gH) = (\pi \circ \phi)(u)$ and $v := (\pi \circ \phi)_*(\tilde{v})$. According to Lemma 5.1, it follows from the assumptions that $A_v^{\mathbf{c}}$ commutes with $R^{\mathbf{c}}(\cdot, w)w|_{(T_xM)^{\mathbf{c}}}$'s $(w \in (T_x^{\perp}M)^{\mathbf{c}})$. Also, it follows from the assumptions that $A_v^{\mathbf{c}}$ and $R^{\mathbf{c}}(\cdot, w)w|_{T_xM}$'s $(w \in (T_x^{\perp}M)^{\mathbf{c}})$ are diagonalizable. Hence they are simultaneously diagonalizable, that is, we have the relation (5.1). On the other hand, by the assumption, we have $\pm\beta(g_*^{-1}v) \notin \mathrm{Spec}(A_v^{\mathbf{c}}|_{g_*\mathfrak{q}_\beta^{\mathbf{c}}})$ for each $\beta \in \overline{\triangle}_+$. Therefore, it follows from Lemma 5.3, Propositions 5.6 and 5.7 of [17] that there exists a pseudo-orthonormal base of $(T_u\tilde{M})^{\mathbf{c}}$ consisting of eigenvectors of $\tilde{A}_{\tilde{v}}^{\mathbf{c}}$. Therefore $\tilde{M}$ is proper complex isoparametric, that is, M is proper complex equifocal. q.e.d.

6. Proof of Theorems A, C and E

In this section, we shall prove Theorems A, C and E. First we prove Theorem A in terms of Propositions 3.2, 4.1 and 5.2.

Proof of Theorem A. Since $T_{eH}(H'(eH)) = \mathfrak{q} \cap \mathfrak{h}'$ and $\mathfrak{q} \cap \mathfrak{h}'$ is a non-degenerate subsapce of $\mathfrak{q}$, we see that $H'(eH)$ is a pseudo-Riemannian submanifold. Since $\sigma \circ \sigma' = \sigma' \circ \sigma$, we can show that $H'(eH)$ is a reflective submanifold by imitating the first-half part of the proof of Lemma 4.2 in [19]. Thus the first-half part of the statement (i) is shown. Furthermore, by imitating the second-half part of the proof of Lemma 4.2 in [19], we can show the second-half part of the statement (i). In the sequel, we shall show the statement (ii). Let M be a principal orbit of the H'-action as in the statement (ii). For simplicity, set $x := \exp_G(w)H$ and $g := \exp_G(w)$, where w is as in the statement (ii). By imitating the second-half part of the proof of Lemma 4.2 in [17], it is shown that M is a partial tube over $H'(eH)$ and $M \cap \Sigma_{eH}$ is an orbit of the isotropy action of the symmetric space $\Sigma_{eH}(\cong L/H \cap H')$. Since M is a principal orbit, $M \cap \Sigma_{eH}$ is a principal orbit of the isotropy action. Hence, since $\mathrm{ad}(w)|_{\mathfrak{l}}$ is semi-simple,

$\mathfrak{b} := g_*^{-1} T_x^{\perp} M$ is a Cartan subspace of $\mathfrak{q} \cap \mathfrak{q}'$ by (i) of Proposition 3.2. Take a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$ containing $\mathfrak{b}$. Let $\mathfrak{q}^{\mathbf{c}} = \mathfrak{a}^{\mathbf{c}} + \sum_{\alpha \in \triangle_+} \mathfrak{q}_{\alpha}^{\mathbf{c}}$ be the root space decomposition with respect to $\mathfrak{a}^{\mathbf{c}}$. Set $\triangle_{\mathfrak{b}^{\mathbf{c}}} := \{\alpha|_{\mathfrak{b}^{\mathbf{c}}} \,|\, \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}^{\mathbf{c}}} \neq 0\}$ and $\mathfrak{q}_{\beta}^{\mathbf{c}} := \sum_{\alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}^{\mathbf{c}}} = \beta} \mathfrak{q}_{\alpha}^{\mathbf{c}}$ $(\beta \in \triangle_{\mathfrak{b}^{\mathbf{c}}})$. Then we have $\mathfrak{q}^{\mathbf{c}} = \mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) + \sum_{\beta \in (\triangle_{\mathfrak{b}^{\mathbf{c}}})_+} \mathfrak{q}_{\beta}^{\mathbf{c}}$, where $(\triangle_{\mathfrak{b}^{\mathbf{c}}})_+$ is the positive root system under some lexicographical ordering. Also, since $\mathfrak{q}^{\mathbf{c}} \cap \mathfrak{h}'^{\mathbf{c}}$ and $\mathfrak{q}^{\mathbf{c}} \cap \mathfrak{q}'^{\mathbf{c}}$ are $\mathrm{ad}(b)^2$-invariant for any $b \in \mathfrak{b}^{\mathbf{c}}$, we have $\mathfrak{q}^{\mathbf{c}} \cap \mathfrak{h}'^{\mathbf{c}} = \mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) \cap \mathfrak{h}'^{\mathbf{c}} + \sum_{\beta \in (\triangle_{\mathfrak{b}^{\mathbf{c}}})_+} (\mathfrak{q}_{\beta}^{\mathbf{c}} \cap \mathfrak{h}'^{\mathbf{c}})$ and $\mathfrak{q}^{\mathbf{c}} \cap \mathfrak{q}'^{\mathbf{c}} = \mathfrak{b}^{\mathbf{c}} + \sum_{\beta \in (\triangle_{\mathfrak{b}^{\mathbf{c}}})_+} (\mathfrak{q}_{\beta}^{\mathbf{c}} \cap \mathfrak{q}'^{\mathbf{c}})$. Hence we have

$$(T_x M)^{\mathbf{c}} = g_*^{\mathbf{c}}(\mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) \cap \mathfrak{h}'^{\mathbf{c}}) + \sum_{\beta \in (\triangle_{\mathfrak{b}^{\mathbf{c}}})_+} (g_*^{\mathbf{c}}(\mathfrak{q}_{\beta}^{\mathbf{c}} \cap \mathfrak{h}'^{\mathbf{c}}) + g_*^{\mathbf{c}}(\mathfrak{q}_{\beta}^{\mathbf{c}} \cap \mathfrak{q}'^{\mathbf{c}})),$$

$$(T_{eH}(H'(eH)))^{\mathbf{c}} = \mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) \cap \mathfrak{h}'^{\mathbf{c}} + \sum_{\beta \in (\triangle_{\mathfrak{b}^{\mathbf{c}}})_+} (\mathfrak{q}_{\beta}^{\mathbf{c}} \cap \mathfrak{h}'^{\mathbf{c}})$$

and

$$(T_x(M \cap \Sigma_{eH}))^{\mathbf{c}} = \sum_{\beta \in (\triangle_{\mathfrak{b}^{\mathbf{c}}})_+} g_*^{\mathbf{c}}(\mathfrak{q}_{\beta}^{\mathbf{c}} \cap \mathfrak{q}'^{\mathbf{c}}).$$

Also we have $T_x^{\perp} M = g_* \mathfrak{b}$. Take $v \in T_x^{\perp} M = g_* \mathfrak{b}$. It is clear that $R(\cdot, v)v$ is semi-simple. Since $H'(eH)$ is totally geodesic, it follows from (ii) of Proposition 4.1 and (4.1) that $A_v^{\mathbf{c}} \tilde{X}_w = 0$ $(X \in \mathfrak{z}_{\mathfrak{q}^{\mathbf{c}}}(\mathfrak{b}^{\mathbf{c}}) \cap \mathfrak{h}'^{\mathbf{c}})$ and

$$A_v^{\mathbf{c}} \tilde{X}_w = \sqrt{-1}\beta(g_*^{-1} v) \tan(\sqrt{-1}\beta(w)) \tilde{X}_w \quad (X \in \mathfrak{q}_{\beta}^{\mathbf{c}} \cap \mathfrak{h}'^{\mathbf{c}} \ (\beta \in (\triangle_{\mathfrak{b}^{\mathbf{c}}})_+)). \tag{6.1}$$

Also, since $M \cap \Sigma_{eH}$ is a principal orbit of the isotropy action of $\Sigma_{eH} (\cong L/H \cap K)$, it follows from Proposition 3.2 and (i) of Proposition 4.1 that

$$A_v^{\mathbf{c}} Y = -\frac{\sqrt{-1}\beta(g_*^{-1} v)}{\tan(\sqrt{-1}\beta(w))} Y \quad (Y \in g_*(\mathfrak{q}_{\beta}^{\mathbf{c}} \cap \mathfrak{q}'^{\mathbf{c}})) \tag{6.2}$$

up to constant-multiple, where we note that the induced metric on $\Sigma_{eH} (= L/H \cap K)$ is homothetic to the metric induced from the Killing form of $\mathfrak{l}$. Thus $A_v^{\mathbf{c}}$ is diagonalizable, that is, A_v is semi-simple. Also we have $[A_v^{\mathbf{c}}, R^{\mathbf{c}}(\cdot, v)v|_{(T_x M)^{\mathbf{c}}}] = 0$ and hence $[A_v, R(\cdot, v)v|_{T_x M}] = 0$. Therefore M is curvature-adapted. Next we shall show that M is proper complex equifocal. Since $g_*^{-1} T_x^{\perp} M$ is a Cartan subspace of $\mathfrak{q} \cap \mathfrak{q}'$ for each $x (= gH) \in M$, M has flat section. Since M is a principal orbit of the H'-action, each normal vector of M extend to an H'-equivariant normal vector field, which is parallel with respect to the normal connection of M because M has flat section. From this fact, it follows that the normal holonomy group of M is trivial. Furthermore, it follows from the homogeneity of M that M is complex equifocal, where we use Fact 3 stated in Introduction. From (6.1) and (6.2), we have $\mathrm{Spec}(A_v^{\mathbf{c}}|_{g_*^{\mathbf{c}}\mathfrak{q}_\beta^{\mathbf{c}}}) \subset \left\{\sqrt{-1}\beta(g_*^{-1}v)\tan(\sqrt{-1}\beta(w)), -\frac{\sqrt{-1}\beta(g_*^{-1}v)}{\tan(\sqrt{-1}\beta(w))}\right\}$ $(\beta \in (\triangle_{\mathfrak{b}^{\mathbf{c}}})_+)$, that is, $\pm\beta(g_*^{-1}v) \notin \mathrm{Spec}\, A_v^{\mathbf{c}}|_{g_*^{\mathbf{c}}\mathfrak{q}_\beta^{\mathbf{c}}}$. Therefore, it follows from Proposition 5.2 that M is proper complex equifocal. Furthermore it follows from the result of [23] stated in Introduction that M is an isoparametric submanifold with flat section. This completes the proof. q.e.d.

Next we prove Theorem C.

Proof of Theorem C. According to Theorem A, we have only to show that $K(eH)$ has no focal point and that, for any normal vector v of M_i, $R(\cdot, v)v|_{T_xM_i}$ and A_v are diagonalizable. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ associated with θ. Take an arbitrary normal vector v of $K(eH)$ at eH. Take a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{q} \cap \mathfrak{p}$ containing v and a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$ containing $\mathfrak{b}$. Let $\mathfrak{q}^{\mathbf{c}} = \mathfrak{a}^{\mathbf{c}} + \sum_{\alpha\in\triangle_+}\mathfrak{q}_\alpha^{\mathbf{c}}$ be the root space decomposition of $\mathfrak{q}^{\mathbf{c}}$ with respect to $\mathfrak{a}^{\mathbf{c}}$. Let $\triangle_{\mathfrak{b}} := \{\alpha|_{\mathfrak{b}} \mid \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$ and $\mathfrak{q}_\beta := (\sum_{\alpha\in\triangle \text{ s.t. } \alpha|_{\mathfrak{b}}=\beta}\mathfrak{q}_\alpha^{\mathbf{c}}) \cap \mathfrak{q}$ $(\beta \in \triangle_{\mathfrak{b}})$. Since $\mathfrak{b} \subset \mathfrak{p}$, we have $\beta(\mathfrak{b}) \subset \mathbf{R}$ $(\beta \in \triangle_{\mathfrak{b}})$ (see Lemma 3.1) and hence $\mathfrak{q} = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{b}) + \sum_{\beta\in(\triangle_{\mathfrak{b}})_+}\mathfrak{q}_\beta$. Furthermore, since $\mathrm{ad}(b)^2(\mathfrak{q}\cap\mathfrak{k}) \subset \mathfrak{q}\cap\mathfrak{k}$ for any $b \in \mathfrak{b}$, we have $\mathfrak{q}\cap\mathfrak{k} = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{b})\cap\mathfrak{k} + \sum_{\beta\in(\triangle_{\mathfrak{b}})_+}(\mathfrak{q}_\beta\cap\mathfrak{k})$. Let $X \in \mathfrak{q}_\beta\cap\mathfrak{k}$ $(\beta \in (\triangle_{\mathfrak{b}})_+)$, Y be the strongly $K(eH)$-Jacobi field along γ_v with $Y(0) = X$. Since $K(eH)$ is totally geodesic, we have $Y(s) = \cosh(s\beta(v))P_{\gamma_v|_{[0,s]}}(X)$. Since $\beta(v)$ is a real number, Y has no zero point. Also any strongly $K(eH)$-Jacobi field $\hat{Y}$ along γ_v with $\hat{Y}(0) \in \mathfrak{z}_{\mathfrak{q}}(\mathfrak{b})\cap\mathfrak{k}$ is expressed as $\hat{Y}(s) = P_{\gamma_v|_{[0,s]}}(\hat{Y}(0))$ and hence it has no zero point. On the other hand, since $K(eH)$ is reflective and hence it has section, any non-strongly $K(eH)$-Jacobi field along γ_v has no zero point. After all there exists no focal point of $K(eH)$ along γ_v. From the arbitrariness of v, it follows that $K(eH)$ has no focal point. For convenience, set $H_1 := K$, $H_2 := L$, $\mathfrak{h}_1 := \mathfrak{k}$, $\mathfrak{h}_2 := \mathfrak{l}$, $\mathfrak{q}_1 := \mathfrak{p}$ and $\mathfrak{q}_2 := \mathfrak{k}\cap\mathfrak{q} + \mathfrak{p}\cap\mathfrak{h}$. Let M_1 (resp. M_2) be a principal orbit of the H_1-action (resp. the H_2-action) through $x_1 = \exp_G(w_1)H \in H_2(eH)$ $(w_1 \in \mathfrak{q}\cap\mathfrak{q}_1)$ (resp. $x_2 = \exp_G(w_2)H \in H_1(eH)\setminus F$ $(w_2 \in \mathfrak{q}\cap\mathfrak{q}_2)$). Set $g_i := \exp_G(w_i)$ $(i = 1, 2)$. Since $\mathfrak{b}_1 := g_{1*}^{-1}(T_{x_1}^\perp M_1)$ and $\mathfrak{b}_2 := g_{2*}^{-1}(T_{x_2}^\perp M_2)$ are maximal abelian subspaces of $\mathfrak{q}\cap\mathfrak{p}$ and $\mathfrak{q}\cap\mathfrak{k}$, respectively, they are maximal split abelian subspaces of $\mathfrak{q}$. Hence we have the root space decomposition $\mathfrak{q} = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{b}_i) + \sum_{\beta\in\triangle_+^i}\mathfrak{q}_\beta$ of $\mathfrak{q}$ with respect to $\mathfrak{b}_i$ $(i = 1, 2)$, where $\mathfrak{q}_\beta := \{X \in \mathfrak{q} \mid \mathrm{ad}(b)^2(X) = (-1)^{\varepsilon_i}\beta(b)^2X \ (\forall b \in \mathfrak{b}_i)\}$ $(\beta \in \mathfrak{b}_i^*)$ $(\varepsilon_1 = 0$ and $\varepsilon_2 = 1$ by Lemma 3.1) and $\triangle_+^i$ is the positive root system of $\triangle^i := \{\beta \in \mathfrak{b}_i^* \mid \mathfrak{q}_\beta \neq \{0\}\}$ with respect to a lexicographical ordering of $\mathfrak{b}_i^*$.

Also, it is easy to show that $\mathfrak{q} \cap \mathfrak{h}_i = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{b}_i) \cap \mathfrak{h}_i + \sum_{\beta \in \triangle_+^i} (\mathfrak{q}_\beta \cap \mathfrak{h}_i)$ and $\mathfrak{q} \cap \mathfrak{q}_i = \mathfrak{b}_i + \sum_{\beta \in \triangle_+^i} (\mathfrak{q}_\beta \cap \mathfrak{q}_i)$, where $i = 1, 2$. Hence we have

$$T_{x_i} M_i = g_{i*}(\mathfrak{z}_{\mathfrak{q}}(\mathfrak{b}_i) \cap \mathfrak{h}_i) + \sum_{\beta \in \triangle_+^i} (g_{i*}(\mathfrak{q}_\beta \cap \mathfrak{h}_i) + g_{i*}(\mathfrak{q}_\beta \cap \mathfrak{q}_i)),$$

$$T_{eH}(H_i(eH)) = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{b}_i) \cap \mathfrak{h}_i + \sum_{\beta \in \triangle_+^i} (\mathfrak{q}_\beta \cap \mathfrak{h}_i)$$

and

$$T_{x_i}(M_i \cap \Sigma_{eH}^i) = \sum_{\beta \in \triangle_+^i} g_{i*}(\mathfrak{q}_\beta \cap \mathfrak{q}_i),$$

where Σ_{eH}^i is the section of $H_i(eH)$ through eH. Take $v_i \in T_{x_i}^\perp M_i = g_{i*}\mathfrak{b}_i$. It is clear that $R(\cdot, v_i)v_i$ is diagonalizable. Denote by A^i the shape tensor of M_i. By using Propositions 3.2, 4.1 and (4.1), we can show $A_{v_i}^i \tilde{X}_{w_i} = 0$ $(X \in \mathfrak{z}_{\mathfrak{q}}(\mathfrak{b}_i) \cap \mathfrak{h}_i)$,

$$A_{v_i}^i \tilde{X}_{w_i} = \sqrt{-1}^i \beta(g_{i*}^{-1} v_i) \tan(\sqrt{-1}^i \beta(w_i)) \tilde{X}_{w_i} \quad (X \in \mathfrak{q}_\beta \cap \mathfrak{h}_i \ \ (\beta \in \triangle_+^i))$$

and

$$A_{v_i}^i Y = -\frac{\sqrt{-1}^i \beta(g_{i*}^{-1} v_i)}{\tan(\sqrt{-1}^i \beta(w_i))} Y \quad (Y \in g_{i*}(\mathfrak{q}_\beta \cap \mathfrak{q}_i) \ \ (\beta \in \triangle_+^i)).$$

Thus $A_{v_i}^i$ is diagonalizable. This completes the proof. q.e.d.

Next we shall prove Theorem E. By imitating the proof of Lemma 2.1 of [21], we can show the following fact.

Lemma 6.1. *Let $G(= (G \times G)/\triangle G)$ be a semi-simple Lie group equipped with the bi-invariant pseudo-Riemannian metric induced from the Killing form of $\mathfrak{g} + \mathfrak{g}$, H' be a closed subgroup of $G \times G$ and $\mathfrak{a}$ be an abelian subspace of the normal space $T_e^\perp(H' \cdot e)$ of $H' \cdot e$. Set $\Sigma := \exp_G(\mathfrak{a})$. Then all H'-orbits through Σ meet Σ orthogonally.*

By using this lemma and imitating the proof of Lemma 2.2 of [21], we can show tyhe following fact.

Lemma 6.2. *Let G/H be a semi-simple pseudo-Riemannian symmetric space, H' be a closed subgroup of G and $\mathfrak{a}$ be an abelian subspace of the normal space*

$T^{\perp}_{eH}H(eH)$ *of* $H'(eH)$. *Set* $\Sigma := \mathrm{Exp}(\mathfrak{a})$. *Then all* H'*-orbits through* Σ *meet* Σ *orthogonally.*

By using this lemma, we prove Theorem E.

Proof of Theorem E. Let M, F and G/H be as in the statement of Theorem E. Without loss of generality, we may assume that G is simply connected. Since M is homogeneous, there exists a closed subgroup H_1 of G having M as an orbit. Without loss of generality, we may assume that $H_1(eH) = M$. Set $\Sigma := \mathrm{Exp}(T^{\perp}_{eH}M)$. Since M has flat section, that is, $T^{\perp}_{eH}M$ is abelian, it follows from Lemma 6.2 that all H_1-orbits through Σ meet Σ orthogonally. Hence their dimensions are lower than $\dim M + 1$. This fact implies that all H_1-orbits through W are of the same dimension as $\dim M$ for some neighborhood W of eH in Σ. Hence they are principal orbits or exceptional orbits of the H_1-action. By imitating the proof of the fact that a hyperpolar action has no exceptional orbit (see [28]), we can show that there exists no exceptonal orbit among the H_1-orbits through W. Hence the H_1-orbits through W are principal. Set $U := H_1 \cdot W$, which is an open set of G/H. Fix $g_0H \in F$. Set $H_2 := g_0^{-1}H_1g_0$, $\mathfrak{t} := T_{eH}g_0^{-1}F$ and $\mathfrak{t}^{\perp} := T^{\perp}_{eH}g_0^{-1}F$. Furthermore set $\mathfrak{h}' := \mathfrak{n}_{\mathfrak{h}}(\mathfrak{t}) + \mathfrak{t}$ and $\mathfrak{q}' := (\mathfrak{h} \ominus \mathfrak{n}_{\mathfrak{h}}(\mathfrak{t})) + \mathfrak{t}^{\perp}$. Since $\mathfrak{n}_{\mathfrak{h}}(\mathfrak{t})$ is a non-degenerate subspace of $\mathfrak{h}$ by the assumption, we have $\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{q}'$ (orthogonal direct sum). Since F is a reflective by the assumption, $\mathfrak{t}$ and $\mathfrak{t}^{\perp}$ are Lie triple systems. By using this fact, we can show $[\mathfrak{h}', \mathfrak{h}'] \subset \mathfrak{h}'$, $[\mathfrak{h}', \mathfrak{q}'] \subset \mathfrak{q}'$ and $[\mathfrak{q}', \mathfrak{q}'] \subset \mathfrak{h}'$. Thus the connected subgroup H' of G having $\mathfrak{h}'$ as its Lie algebra is symmetric, where we use the simply connectedness of G. That is, the H'-action on G/H is a Hermann type action. Easily we can show $T_e((H_2 \times H) \cdot e) = \mathrm{pr}_{\mathfrak{q}}(\mathfrak{h}_2) + \mathfrak{h}$ and $T_e((H' \times H) \cdot e) = \mathrm{pr}_{\mathfrak{q}}(\mathfrak{h}') + \mathfrak{h} = \mathfrak{t} + \mathfrak{h}$, where $\mathrm{pr}_{\mathfrak{q}}$ is the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{q}$ and $\mathfrak{h}_2 := \mathrm{Lie}\, H_2$. Since $\pi^{-1}(H_2(eH)) = (H_2 \times H) \cdot e$, we have $T_{eH}(H_2(eH)) = \mathrm{pr}_{\mathfrak{q}}(T_e((H_2 \times H) \cdot e)) = \mathrm{pr}_{\mathfrak{q}}(\mathfrak{h}_2)$, that is, $\mathrm{pr}_{\mathfrak{q}}(\mathfrak{h}_2) = \mathfrak{t}$. Hence we have $T_e((H' \times H) \cdot e) = T_e((H_2 \times H) \cdot e))$, which implies $(H' \times H) \cdot e = (H_2 \times H) \cdot e$. Therefore we have $H'(eH) = H_2(eH)$. Set $\Sigma' := \mathrm{Exp}(T^{\perp}_{g_0^{-1}H}(g_0^{-1}M))$, which passes through eH. Set $\mathfrak{a}' := T_{eH}\Sigma'$, which is abelian. Since $T^{\perp}_{eH}(H'(eH)) = T^{\perp}_{eH}(H_2(eH))$ includes $\mathfrak{a}'$, it follows from Lemma 6.2 that all H'-orbits and all H_2-orbits through Σ' meet Σ' orthogonally. Since all H_2-orbits through $g_0^{-1}W(\subset \Sigma')$ are principal and hence $T^{\perp}_{gH}(H_2(gH)) = T_{gH}\Sigma'$ for all $gH \in g_0^{-1}W$, we have $T_{gH}(H'(gH)) \subset T_{gH}(H_2(gH))$ for all $gH \in g_0^{-1}W$. On the other hand, we have $[\mathrm{pr}_{\mathfrak{h}}(\mathfrak{h}_2), \mathfrak{t}] = \mathrm{pr}_{\mathfrak{q}}([\mathfrak{h}_2, \mathfrak{t}]) \subset \mathrm{pr}_{\mathfrak{q}}(T_e((H_2 \times H) \cdot e)) = T_{eH}((H_2(eH)) = \mathfrak{t}$, that is, $\mathrm{pr}_{\mathfrak{h}}(\mathfrak{h}_2) \subset \mathfrak{n}_{\mathfrak{h}}(\mathfrak{t})$, where $\mathrm{pr}_{\mathfrak{h}}$ is the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{h}$. Hence we have $\mathfrak{h}_2 \subset \mathrm{pr}_{\mathfrak{h}}(\mathfrak{h}_2) + \mathrm{pr}_{\mathfrak{q}}(\mathfrak{h}_2) \subset \mathfrak{h}'$, that is, $H_2 \subset H'$.

Therefore we see that $H'(gH) = H_2(gH)$ for all $gH \in g_0^{-1}W$. In particular, $g_0^{-1}M$ is a principal orbit of the H'-action. Hence M is a principal orbit of the Hermann type action $g_0H'g_0^{-1}$. This completes the proof. q.e.d.

7. Cohomogeneities of Special Hermann Type Actions

In this section, we shall list up the cohomogeneities of the K-action and the L-action as in Theorem C on irreducible (semi-simple) pseudo-Riemannian symmetric spaces G/H in terms of the fact that the cohomogeneity of the K-action (resp. L-action) is equal to the rank of $L/H \cap K$ (resp. $K/H \cap K$). In Tables 1$\sim$5, $A \cdot B$ denotes $A \times B/\Pi$, where Π is the discrete center of $A \times B$. The symbol $\widetilde{SO_0(1,8)}$ in Table 6 denotes the universal covering of $SO_0(1,8)$ and the symbol α in Table 6 denotes an outer automorphism of G_2^2.

Table 1.

G/H	K	L
	cohom_K	cohom_L
$SL(n,\mathbf{R})/SO_0(p,n-p)$ $\left(p \leq \frac{n}{2}\right)$	$SO(n)$	$(SL(p,\mathbf{R}) \times SL(n-p,\mathbf{R})) \cdot \mathbf{R}_*$
	$n-1$	p
$SL(n,\mathbf{R})/(SL(p,\mathbf{R}) \times SL(n-p,\mathbf{R})) \cdot \mathbf{R}_*$ $\left(p \leq \frac{n}{2}\right)$	$SO(n)$	$SO_0(p,n-p)$
	p	p
$SL(2n,\mathbf{R})/Sp(n,\mathbf{R})$	$SO(2n)$	$SL(n,\mathbf{C}) \cdot U(1)$
	$n-1$	$\left[\frac{n}{2}\right]$
$SL(2n,\mathbf{R})/SL(n,\mathbf{C}) \cdot U(1)$	$SO(2n)$	$Sp(n,\mathbf{R})$
	n	$\left[\frac{n}{2}\right]$
$SU^*(2n)/SO^*(2n)$	$Sp(n)$	$SL(n,\mathbf{C}) \cdot U(1)$
	$n-1$	n
$SU^*(2n)/SL(n,\mathbf{C}) \cdot U(1)$	$Sp(n)$	$SO^*(2n)$
	$\left[\frac{n}{2}\right]$	n
$SU^*(2n)/Sp(p,n-p)$ $\left(p \leq \frac{n}{2}\right)$	$Sp(n)$	$SU^*(2p) \times SU^*(2n-2p) \times U(1)$
	$n-1$	p
$SU^*(2n)/(SU^*(2p) \times SU^*(2n-2p) \times U(1))$ $\left(p \leq \frac{n}{2}\right)$	$Sp(n)$	$Sp(p,n-p)$
	p	p

Table 1 (continued)

G/H	K	L
	cohom_K	cohom_L
$SU(p,q)/SO_0(p,q)$ $(p \le q)$	$S(U(p) \times U(q))$	$SO_0(p,q)$
	p	$n-1$
$SU(p,p)/SO^*(2p)$	$S(U(p) \times U(p))$	$Sp(p,\mathbf{R})$
	p	$p-1$
$SU(p,p)/Sp(p,\mathbf{R})$	$S(U(p) \times U(p))$	$SO^*(2p)$
	$\left[\frac{p}{2}\right]$	$p-1$
$SU(p,p)/SL(p,\mathbf{C}) \cdot U(1)$	$S(U(p) \times U(p))$	$SL(p,\mathbf{C}) \cdot U(1)$
	p	$p-1$
$SU(2p,2q)/Sp(p,q)$ $(p \le q)$	$S(U(2p) \times U(2q))$	$Sp(p,q)$
	p	$n-1$
$SU(p,q)/S(U(i,j) \times U(p-i,q-j))$	$S(U(p) \times U(q))$	$S(U(p-i,j) \times U(i,q-j))$
	$\min\{p-i,j\}$ $+\min\{i,q-j\}$	$\min\{i,p-i\}$ $+\min\{j,q-j\}$

Table 2.

G/H	K	L
	cohom_K	cohom_L
$SL(n,\mathbf{C})/SO(n,\mathbf{C})$	$SU(n)$	$SL(n,\mathbf{R})$
	$n-1$	$n-1$
$SL(n,\mathbf{C})/SL(n,\mathbf{R})$	$SU(n)$	$SO(n,\mathbf{C})$
	$\left[\frac{n}{2}\right]$	$n-1$
$SL(n,\mathbf{C})/(SL(p,\mathbf{C}) \times SL(n-p,\mathbf{C}) \times U(1))$ $\left(p \le \frac{n}{2}\right)$	$SU(n)$	$SU(p,n-p)$
	p	p
$SL(n,\mathbf{C})/SU(p,n-p)$ $\left(p \le \frac{n}{2}\right)$	$SU(n)$	$SL(p,\mathbf{C}) \times SL(n-p,\mathbf{C})$ $\times U(1)$
	$n-2$	p
$SL(2n,\mathbf{C})/Sp(n,\mathbf{C})$	$SU(2n)$	$SU^*(2n)$
	$n-1$	$n-1$

Table 2 (continued)

G/H	K	L
	cohom_K	cohom_L
$SL(2n,\mathbf{C})/SU^*(2n)$	$SU(2n)$	$Sp(n,\mathbf{C})$
	n	$n-1$
$SO_0(p,q)/SO_0(i,j)\times SO_0(p-i,q-j)$	$SO(p)\times SO(q)$	$SO_0(p-i,j)\times SO_0(i,q-j)$
	$\min\{p-i,j\}$ $+\min\{i,q-j\}$	$\min\{i,p-i\}+\min\{j,q-j\}$
$SO_0(p,p)/SO(p,\mathbf{C})$	$SO(p)\times SO(p)$	$SL(p,\mathbf{R})\cdot U(1)$
	p	$\left[\frac{p}{2}\right]$
$SO_0(p,p)/SL(p,\mathbf{R})\cdot U(1)$	$SO(p)\times SO(p)$	$SO(p,\mathbf{C})$
	$\left[\frac{p}{2}\right]$	$\left[\frac{p}{2}\right]$
$SO_0(2p,2q)/SU(p,q)\cdot U(1)\ (p\le q)$	$SO(2p)\times SO(2q)$	$SU(p,q)\cdot U(1)$
	p	$\left[\frac{p}{2}\right]+\left[\frac{q}{2}\right]$
$SO^*(2n)/SO^*(2p)\times SO^*(2n-2p)\ \left(p\le\frac{n}{2}\right)$	$U(n)$	$SU(p,n-p)\cdot U(1)$
	p	p
$SO^*(2n)/SU(p,n-p)\cdot U(1)\ \left(p\le\frac{n}{2}\right)$	$U(n)$	$SO^*(2p)\times SO^*(2n-2p)$
	$\left[\frac{p}{2}\right]+\left[\frac{n-p}{2}\right]$	p
$SO^*(2n)/SO(n,\mathbf{C})$	$U(n)$	$SO(n,\mathbf{C})$
	$\left[\frac{n}{2}\right]$	n
$SO^*(4n)/SU^*(2n)\cdot U(1)$	$U(2n)$	$SU^*(2n)\cdot U(1)$
	$n-1$	$n-1$
$SO(n,\mathbf{C})/SO(p,\mathbf{C})\times SO(n-p,\mathbf{C})\ \left(p\le\frac{n}{2}\right)$	$SO(n)$	$SO_0(p,n-p)$
	p	p
$SO(n,\mathbf{C})/SO_0(p,n-p)\ \left(p\le\frac{n}{2}\right)$	$SO(n)$	$SO(p,\mathbf{C})\times SO(n-p,\mathbf{C})$
	$\left[\frac{p}{2}\right]+\left[\frac{n-p}{2}\right]$	p
$SO(2n,\mathbf{C})/SL(n,\mathbf{C})\cdot SO(2,\mathbf{C})$	$SO(2n)$	$SO^*(2n)$
	$\left[\frac{n}{2}\right]$	$\left[\frac{n}{2}\right]$
$SO(2n,\mathbf{C})/SO^*(2n)$	$SO(2n)$	$SL(n,\mathbf{C})\cdot SO(2,\mathbf{C})$
	n	$\left[\frac{n}{2}\right]$

Table 3.

G/H	K / cohom_K	L / cohom_L
$Sp(n,\mathbf{R})/SU(p,n-p)\cdot U(1)$ $(p \le \frac{n}{2})$	$U(n)$	$Sp(p,\mathbf{R})\times Sp(n-p,\mathbf{R})$
	n	p
$Sp(n,\mathbf{R})/Sp(p,\mathbf{R})\times Sp(n-p,\mathbf{R})$ $(p \le \frac{n}{2})$	$U(n)$	$SU(p,n-p)\cdot U(1)$
	p	p
$Sp(n,\mathbf{R})/SL(n,\mathbf{R})\cdot U(1)$	$U(n)$	$SL(n,\mathbf{R})\cdot U(1)$
	$n-1$	$n-1$
$Sp(2n,\mathbf{R})/Sp(n,\mathbf{C})$	$U(2n)$	$Sp(n,\mathbf{C})$
	n	n
$Sp(p,q)/SU(p,q)\cdot U(1)$	$Sp(p)\times Sp(q)$	$SU(p,q)\cdot U(1)$
	p	$p+q$
$Sp(p,p)/SU^*(2p)\cdot U(1)$	$Sp(p)\times Sp(p)$	$Sp(p,\mathbf{C})$
	p	p
$Sp(p,p)/Sp(p,\mathbf{C})$	$Sp(p)\times Sp(p)$	$SU^*(2p)\cdot U(1)$
	$p-1$	p
$Sp(p,q)/Sp(i,j)\times Sp(p-i,q-j)$	$Sp(p)\times Sp(q)$	$Sp(p-i,j)\times Sp(i,q-j)$
	$\min\{p-i,j\}$ $+\min\{i,q-j\}$	$\min\{i,p-i\}+\min\{j,q-j\}$
$Sp(n,\mathbf{C})/SL(n,\mathbf{C})\cdot SO(2,\mathbf{C})$	$Sp(n)$	$Sp(n,\mathbf{R})$
	n	n
$Sp(n,\mathbf{C})/Sp(n,\mathbf{R})$	$Sp(n)$	$SL(n,\mathbf{C})\cdot SO(2,\mathbf{C})$
	n	n
$Sp(n,\mathbf{C})/Sp(p,\mathbf{C})\times Sp(n-p,\mathbf{C})$ $(p \le \frac{n}{2})$	$Sp(n)$	$Sp(p,n-p)$
	p	p
$Sp(n,\mathbf{C})/Sp(p,n-p)$ $(p \le \frac{n}{2})$	$Sp(n)$	$Sp(p,\mathbf{C})\times Sp(n-p,\mathbf{C})$
	n	p

Table 4.

G/H	K	L	cohom_K	cohom_L
$E_6^6/Sp(4,\mathbf{R})$	$Sp(4)/\{\pm 1\}$	$SL(6,\mathbf{R})\times SL(2,\mathbf{R})$	6	4
$E_6^6/SL(6,\mathbf{R})\times SL(2,\mathbf{R})$	$Sp(4)/\{\pm 1\}$	$Sp(4,\mathbf{R})$	4	4
$E_6^6/Sp(2,2)$	$Sp(4)/\{\pm 1\}$	$SO_0(5,5)\cdot\mathbf{R}$	6	2
$E_6^6/SO_0(5,5)\cdot\mathbf{R}$	$Sp(4)/\{\pm 1\}$	$Sp(2,2)$	2	2
$E_6^6/SU^*(6)\cdot SU(2)$	$Sp(4)/\{\pm 1\}$	F_4^4	4	1
E_6^6/F_4^4	$Sp(4)/\{\pm 1\}$	$SU^*(6)\cdot SU(2)$	2	1
$E_6^2/Sp(1,3)$	$SU(6)\cdot SU(2)$	F_4^4	4	2
E_6^2/F_4^4	$SU(6)\cdot SU(2)$	$Sp(1,3)$	1	2
$E_6^2/Sp(4,\mathbf{R})$	$SU(6)\cdot SU(2)$	$Sp(4,\mathbf{R})$	4	2
$E_6^2/SU(2,4)\cdot SU(2)$	$SU(6)\cdot SU(2)$	$SO_0(4,6)\cdot U(1)$	4	2
$E_6^2/SO_0(4,6)\cdot U(1)$	$SU(6)\cdot SU(2)$	$SU(2,4)\cdot SU(2)$	2	2
$E_6^2/SU(3,3)\cdot SL(2,\mathbf{R})$	$SU(6)\cdot SU(2)$	$SU(3,3)\cdot SL(2,\mathbf{R})$	4	4
$E_6^2/SO^*(10)\cdot U(1)$	$SU(6)\cdot SU(2)$	$SO^*(10)\cdot U(1)$	2	2
$E_6^{-14}/Sp(2,2)$	$Spin(10)\cdot U(1)$	$Sp(2,2)$	2	6
$E_6^{-14}/SU(2,4)\cdot SU(2)$	$Spin(10)\cdot U(1)$	$SU(2,4)\cdot SU(2)$	2	4
$E_6^{-14}/SU(1,5)\cdot SL(2,\mathbf{R})$	$Spin(10)\cdot U(1)$	$SO^*(10)\cdot U(1)$	2	2
$E_6^{-14}/SO^*(10)\cdot U(1)$	$Spin(10)\cdot U(1)$	$SU(1,5)\cdot SL(2,\mathbf{R})$	2	2
$E_6^{-14}/SO_0(2,8)\cdot U(1)$	$Spin(10)\cdot U(1)$	$SO_0(2,8)\cdot U(1)$	2	2
E_6^{-14}/F_4^{-20}	$Spin(10)\cdot U(1)$	F_4^{-20}	1	2
$E_6^{-26}/Sp(1,3)$	F_4	$SU^*(6)\cdot SU(2)$	2	4
$E_6^{-26}/SU^*(6)\cdot SU(2)$	F_4	$Sp(1,3)$	1	4
$E_6^{-26}/SO_0(1,9)\cdot U(1)$	F_4	F_4^{-20}	1	1
E_6^{-26}/F_4^{-20}	F_4	$SO_0(1,9)\cdot U(1)$	2	1
$E_6^{\mathbf{c}}/E_6^6$	E_6	$Sp(4,\mathbf{C})$	4	6
$E_6^{\mathbf{c}}/Sp(4,\mathbf{C})$	E_6	E_6^6	6	6
$E_6^{\mathbf{c}}/E_6^2$	E_6	$SL(6,\mathbf{C})\cdot SL(2,\mathbf{C})$	6	4
$E_6^{\mathbf{c}}/SL(6,\mathbf{C})\cdot SL(2,\mathbf{C})$	E_6	E_6^2	4	4

Table 4 (continued)

G/H	K	L	cohom_K	cohom_L
$E_6^{\mathbf{c}}/E_6^{-14}$	E_6	$SO(10,\mathbf{C})\cdot Sp(1)$	6	2
$E_6^{\mathbf{c}}/SO(10,\mathbf{C})\cdot Sp(1)$	E_6	E_6^{-14}	2	2
$E_6^{\mathbf{c}}/F_4^{\mathbf{C}}$	E_6	E_6^{-26}	2	2
$E_6^{\mathbf{c}}/E_6^{-26}$	E_6	$F_4^{\mathbf{C}}$	4	2

Table 5.

G/H	K	L	cohom_K	cohom_L
$E_7^7/SL(8,\mathbf{R})$	$SU(8)/\{\pm 1\}$	$SL(8,\mathbf{R})$	7	7
$E_7^7/SU^*(8)$	$SU(8)/\{\pm 1\}$	$E_6^6\cdot U(1)$	7	3
$E_7^7/E_6^6\cdot U(1)$	$SU(8)/\{\pm 1\}$	$SU^*(8)$	3	3
$E_7^7/SU(4,4)$	$SU(8)/\{\pm 1\}$	$SO_0(6,6)\cdot SL(2,\mathbf{R})$	7	4
$E_7^7/SO_0(6,6)\cdot SL(2,\mathbf{R})$	$SU(8)/\{\pm 1\}$	$SU(4,4)$	4	4
$E_7^7/SO^*(12)\cdot SU(2)$	$SU(8)/\{\pm 1\}$	$E_6^2\cdot U(1)$	4	2
$E_7^7/E_6^2\cdot U(1)$	$SU(8)/\{\pm 1\}$	$SO^*(12)\cdot SU(2)$	3	2
$E_7^{-5}/SU(4,4)$	$SO'(12)\cdot SU(2)$	$SU(4,4)$	4	7
$E_7^{-5}/SU(2,6)$	$SO'(12)\cdot SU(2)$	$E_6^2\cdot U(1)$	4	3
$E_7^{-5}/E_6^2\cdot U(1)$	$SO'(12)\cdot SU(2)$	$SU(2,6)$	2	3
$E_7^{-5}/SO^*(12)\cdot SL(2,\mathbf{R})$	$SO'(12)\cdot SU(2)$	$SO^*(12)\cdot SL(2,\mathbf{R})$	4	4
$E_7^{-5}/SO_0(4,8)\cdot SU(2)$	$SO'(12)\cdot SU(2)$	$SO_0(4,8)\cdot SU(2)$	4	4
$E_7^{-5}/E_6^{-14}\cdot U(1)$	$SO'(12)\cdot SU(2)$	$E_6^{-14}\cdot U(1)$	2	3
$E_7^{-25}/SU^*(8)$	$E_6\cdot U(1)$	$SU^*(8)$	3	7
$E_7^{-25}/SU(2,6)$	$E_6\cdot U(1)$	$SO^*(12)\cdot SU(2)$	3	5
$E_7^{-25}/SO^*(12)\cdot SU(2)$	$E_6\cdot U(1)$	$SU(2,6)$	2	5
$E_7^{-25}/SO_0(2,10)\cdot SL(2,\mathbf{R})$	$E_6\cdot U(1)$	$E_6^{-14}\cdot U(1)$	2	2
$E_7^{-25}/E_6^{-14}\cdot U(1)$	$E_6\cdot U(1)$	$SO_0(2,10)\cdot SL(2,\mathbf{R})$	3	2
$E_7^{-25}/E_6^{-26}\cdot U(1)$	$E_6\cdot U(1)$	$E_6^{-26}\cdot U(1)$	2	3

Table 5 (continued)

G/H	K	L	cohom_K	cohom_L
$E_7^{\mathbf{c}}/E_7^7$	E_7	$SL(8,\mathbf{C})$	7	7
$E_7^{\mathbf{c}}/SL(8,\mathbf{C})$	E_7	E_7^7	7	7
$E_7^{\mathbf{c}}/E_7^{-5}$	E_7	$SO(12,\mathbf{C})\cdot SL(2,\mathbf{C})$	7	4
$E_7^{\mathbf{c}}/SO(12,\mathbf{C})\cdot SL(2,\mathbf{C})$	E_7	E_7^{-5}	4	4
$E_7^{\mathbf{c}}/E_7^{-25}$	E_7	$E_6^{\mathbf{c}}\cdot\mathbf{C}^*$	7	3
$E_7^{\mathbf{c}}/E_6^{\mathbf{c}}\cdot\mathbf{C}^*$	E_7	E_7^{-25}	3	3
$E_8^8/SO^*(16)$	$SO'(16)$	$E_7^7\cdot SL(2,\mathbf{R})$	4	4
$E_8^8/E_7^7\cdot SL(2,\mathbf{R})$	$SO'(16)$	$SO^*(16)$	4	4
$E_8^8/SO_0(8,8)$	$SO'(16)$	$SO_0(8,8)$	8	8
$E_8^8/E_7^{-5}\cdot Sp(1)$	$SO'(16)$	$E_7^{-5}\cdot Sp(1)$	4	4
$E_8^{-24}/SO^*(16)$	$E_7\cdot Sp(1)$	$SO^*(16)$	4	8
$E_8^{-24}/SO_0(4,12)$	$E_7\cdot Sp(1)$	$E_7^{-5}\cdot Sp(1)$	4	4
$E_8^{-24}/E_7^{-5}\cdot Sp(1)$	$E_7\cdot Sp(1)$	$SO_0(4,12)$	4	4
$E_8^{-24}/E_7^{-25}\cdot SL(2,\mathbf{R})$	$E_7\cdot Sp(1)$	$E_7^{-25}\cdot SL(2,\mathbf{R})$	4	4
$E_8^{\mathbf{c}}/E_8^8$	E_8	$SO(16,\mathbf{C})$	8	8
$E_8^{\mathbf{c}}/SO(16,\mathbf{C})$	E_8	E_8^8	8	8
$E_8^{\mathbf{c}}/E_8^{-24}$	E_8	$E_7^{\mathbf{c}}\times SL(2,\mathbf{C})$	8	4
$E_8^{\mathbf{c}}/E_7^{\mathbf{c}}\times SL(2,\mathbf{C})$	E_8	E_8^{-24}	4	4

Table 6.

G/H	K	L	cohom_K	cohom_L
$F_4^4/Sp(1,2)\cdot Sp(1)$	$Sp(3)\cdot Sp(1)$	$SO_0(4,5)$	4	1
$F_4^4/SO_0(4,5)$	$Sp(3)\cdot Sp(1)$	$Sp(1,2)\cdot Sp(1)$	1	1
$F_4^4/Sp(3,\mathbf{R})\cdot SL(2,\mathbf{R})$	$Sp(3)\cdot Sp(1)$	$Sp(3,\mathbf{R})\cdot SL(2,\mathbf{R})$	4	4
$F_4^{-20}/Sp(1,2)\cdot Sp(1)$	$Spin(9)$	$\widetilde{SO_0(1,8)}$	1	1
$F_4^{-20}/\widetilde{SO_0(1,8)}$	$Spin(9)$	$Sp(1,2)\cdot Sp(1)$	1	1
$F_4^{\mathbf{C}}/F_4^4$	F_4	$Sp(3,\mathbf{C})\cdot SL(2,\mathbf{C})$	4	4
$F_4^{\mathbf{C}}/Sp(3,\mathbf{C})\cdot SL(2,\mathbf{C})$	F_4	F_4^4	4	4
$F_4^{\mathbf{C}}/F_4^{-20}$	F_4	$SO(9,\mathbf{C})$	4	1
$F_4^{\mathbf{C}}/SO(9,\mathbf{C})$	F_4	F_4^{-20}	1	1
$G_2^2/SL(2,\mathbf{R})\times SL(2,\mathbf{R})$	$SO(4)$	$\alpha(SO(4))$	2	2
$G_2^2/\alpha(SO(4))$	$SO(4)$	$SL(2,\mathbf{R})\times SL(2,\mathbf{R})$	2	2
$G_2^{\mathbf{c}}/G_2^2$	G_2	$SL(2,\mathbf{C})\times SL(2,\mathbf{C})$	2	2
$G_2^{\mathbf{c}}/SL(2,\mathbf{C})\times SL(2,\mathbf{C})$	G_2	G_2^2	2	2

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