TSUKUBA J. MATH.
Vol. 34 No. 1 (2010), 117–128

ODD DIMENSIONAL RIEMANNIAN SUBMANIFOLDS ADMITTING THE ALMOST CONTACT METRIC STRUCTURE IN A EUCLIDEAN SPHERE

By

Kazuhiro Okumura

Abstract. We investigate some odd dimensional Rimannian submanifolds admitting the almost contact metric structure $(\phi, \xi, \eta, \langle\, , \rangle)$ of a certain Euclidean sphere from the viewpoint of the weakly ϕ-invariance of the second fundamental form. The family of such submanifolds contains some homogeneous submanifolds of the ambient sphere. In the latter half of this paper, we caluculate the mean curvature and the length of the derivative of the mean curvature vector of these homogeneous submanifolds.

1. Introduction

We denote by (M^{2n-1}, ι_M) a real hypersurface M^{2n-1} of an n-dimensional complex projective space $\mathbf{C}P^n(c)$ of constant holomorphic sectional curvature $c(>0)$ through an isometric immersion $\iota_M : M \to \mathbf{C}P^n(c)$. In [3], Maeda and Udagawa considered a Riemannian submanifold $(M^{2n-1}, f_1 \circ \iota_M)$ of a certain Euclidean sphere by using the isometric parallel minimal embedding $f_1 : \mathbf{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$, where $S^{n(n+2)-1}((n+1)c/(2n))$ is an $(n(n+2)-1)$-dimensional sphere of constant sectional curvature $(n+1)c/(2n)$. Note that this submanifold $(M^{2n-1}, f_1 \circ \iota_M)$ has an almost contact metric structure $(\phi, \xi, \eta, \langle\, , \rangle)$ induced from the Kähler structure J and the standard metric $\langle\, , \rangle$ of $\mathbf{C}P^n(c)$.

In this paper, we pay particular attention on the structure tensor ϕ of $(M^{2n-1}, f_1 \circ \iota_M)$. Motivated by Maeda and Naitoh's work [2], we define the

---

2000 *Mathematics Subject Classification*. Primary 53B25, Secondary 53C40.
Key words and phrases. real hypersurfaces, complex projective spaces, real hypersurfaces of type (A), Hopf hypersurfaces, ruled real hypersurfaces, homogeneous submanifold, strongly ϕ-invariant, weakly ϕ-invariant, the first standard minimal embedding, Euclidean spheres, mean curvature vector, length of the mean curvature vector.
Received December 4, 2009.
Revised March 15, 2010.

notion of weakly ϕ-invariance of the second fundamental form of the immersion $f_1 \circ \iota_M$ (for details, see Section 4). By virtue of this point of view we can characterize two interesting examples of Riemannian submanifolds $(M^{2n-1}, f_1 \circ \iota_M)$ of the sphere $S^{n(n+2)-1}((n+1)c/(2n))$ (Theorem 1). One is a homogeneous submanifold and the other is a non-homogeneous submanifold of this sphere.

In this context, we are interested in a homogeneous submanifold $(M^{2n-1}, f_1 \circ \iota_M)$ with weakly ϕ-invariant shape operator in $S^{n(n+2)-1}((n+1)c/(2n))$. We remark that this manifold M^{2n-1} is congruent to a homogeneous real hypersurface of type (A) in $\mathbf{C}P^n(c)$, namely M^{2n-1} is congruent to a tube of radius r $(0 < r < \pi/\sqrt{c})$ around a totally geodesic Kähler submanifold $\mathbf{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-1)$ of $\mathbf{C}P^n(c)$, so that M^{2n-1} is an orbit of some subgroup of the isometry group $SU(n+1)$ of $\mathbf{C}P^n(c)$. We compute the mean curvature and the length of the derivative of the mean curvature vector of this homogeneous submanifold $(M^{2n-1}, f_1 \circ \iota_M)$ in the sphere $S^{n(n+2)-1}((n+1)c/(2n))$ (Theorem 2).

The author would like to thank Professor S. Maeda for his valuable suggestions and encouragement during the preparation of this paper. The author would also like to thank the referee for valuable comments.

2. Real Hypersurface of Type (A) in $\mathbf{C}P^n(c)$

Let M^{2n-1} be a real hypersurface with a unit local vector field $\mathcal{N}$ of $\mathbf{C}P^n(c)$ through an isometric immersion ι_M. The Riemannian connections $\nabla^{(1)}$ of $\mathbf{C}P^n(c)$ and ∇ of M are related by

$$\nabla^{(1)}_X Y = \nabla_X Y + \langle AX, Y \rangle \mathcal{N}, \tag{2.1}$$

$$\nabla^{(1)}_X \mathcal{N} = -AX \tag{2.2}$$

for vector fields X and Y tangent to M, where $\langle\ ,\ \rangle$ denotes the induced metric from the standard Riemannian metric of $\mathbf{C}P^n(c)$ and A is the shape operator of M in $\mathbf{C}P^n(c)$. (2.1) is called *Gauss's formula*, and (2.2) is called *Weingarten's formula*. It is known that M admits an *almost contact metric structure* $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the Kähler structure J of $\mathbf{C}P^n(c)$. The characteristic vector field ξ of M is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1 \quad \text{and} \quad \langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \tag{2.3}$$

where I denotes the identity map of the tangent bundle TM of M. It follows from the fact that $\nabla^{(1)}J = 0$ and Equations (2.1) and (2.2) that

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal vectors* of M in $\mathbf{C}P^n(c)$, respectively. In the following, we denote by V_λ the eigenspace associated to the principal curvature λ, namely we set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$.

We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of $\mathbf{C}P^n(c)$ is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurface is natural in the theory of real hypersurfaces in $\mathbf{C}P^n(c)$ (see [4]).

The following lemma clarifies a fundamental property on Hopf hypersurfaces in $\mathbf{C}P^n(c)$, $n \geqq 2$.

Lemma 1. *For a Hopf hypersurface M^{2n-1} $(n \geqq 2)$ with principal curvature α corresponding to the characteristic vector field ξ in $\mathbf{C}P^n(c)$, we have the following*:

(1) *α is locally constant on M*;

(2) *If X is a tangent vector of M perpendicular to ξ with $AX = \lambda X$, then*

$$A\phi X = \frac{\alpha\lambda + (c/2)}{2\lambda - \alpha}\phi X.$$

The following real hypersurfaces are so-called real hypersurfaces of type (A_1) and (A_2), respectively.

(A_1) A geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbf{C}P^n(c)$.

(A_2) A tube of radius r $(0 < r < \pi/\sqrt{c})$ around a totally geodesic Kähler submanifold $\mathbf{C}P^\ell(c)$ in $\mathbf{C}P^n(c)$ with $1 \leqq \ell \leqq n-2$.

In this paper, summing up the real hypersurfaces of type (A_1) and type (A_2), we call them *the real hypersurfaces of type* (A). The real hypersurfaces of type (A) are known as typical examples of Hopf hypersurfaces. The tangent bundle TM of real hypersurfaces M of type (A_1) with radius r $(0 < r < \pi/\sqrt{c})$ is decomposed as $TM = \{\xi\}_{\mathbf{R}} \oplus V_\lambda$ with $\alpha = \sqrt{c}\cot(\sqrt{c}r)$, $\lambda = (\sqrt{c}/2)\cot(\sqrt{c}r/2)$, $\dim_{\mathbf{R}} V_\lambda = 2n-2$ and $\phi V_\lambda = V_\lambda$. The tangent bundle TM of real hypersurfaces M of type (A_2) with radius r $(0 < r < \pi/\sqrt{c})$ is decomposed as $TM = \{\xi\}_{\mathbf{R}} \oplus V_{\lambda_1} \oplus V_{\lambda_2}$ with $\alpha = \sqrt{c}\cot(\sqrt{c}r)$, $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}r/2)$, $\lambda_2 = -(\sqrt{c}/2)\tan(\sqrt{c}r/2)$, $\dim_{\mathbf{R}} V_{\lambda_1} = 2n - 2\ell - 2$, $\dim_{\mathbf{R}} V_{\lambda_2} = 2\ell$ and $\phi V_i = V_i$ $(i = 1, 2)$.

Remark 1. A geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbf{C}P^n(c)$ is congruent to a tube of radius $(\pi/\sqrt{c}) - r$ around totally geodesic $\mathbf{C}P^{n-1}(c)$ of $\mathbf{C}P^n(c)$. In fact, $\lim_{r\to\pi/\sqrt{c}} G(r) = \mathbf{C}P^{n-1}(c)$.

For the later use we here prepare the following which is a characterization of the real hypersurfaces of type (A) (see [4]).

Lemma 2. *Let M be a real hypersurface in $\mathbf{C}P^n(c)$ $(n \geqq 2)$. Then the following conditions are mutually equivalent*:

(1) *M is locally congruent to a real hypersurface of type* (A);
(2) $\phi A = A\phi$;
(3) $\langle(\nabla_X A)Y, Z\rangle = (c/4)(-\eta(Y)\langle\phi X, Z\rangle - \eta(Z)\langle\phi X, Y\rangle)$ *for arbitrary vectors X, Y and Z on M.*

3. Ruled Real Hypersurfaces in $\mathbf{C}P^n(c)$

We recall ruled real hypersurfaces in $\mathbf{C}P^n(c)$, which are typical examples of non-Hopf hypersurfaces. A real hypersurface M is called a *ruled real hypersurface* of $\mathbf{C}P^n(c)$ $(n \geqq 2)$ if the holomorphic distribution T^0M defined by $T^0M = \{X \in TM \mid X \perp \xi\}$ is integrable and each of its maximal integral manifolds is a totally geodesic complex hyperplane $\mathbf{C}P^{n-1}(c)$ of $\mathbf{C}P^n(c)$. A ruled real hypersurface is constructed in the following manner. Given an arbitrary regular real curve γ in $\mathbf{C}P^n(c)$ which is defined on an interval I we have at each point $\gamma(t)$ $(t \in \mathrm{I})$ a totally geodesic complex hyperplane $\mathbf{C}P_t^{n-1}(c)$ orthogonal to the plane spanned by $\{\dot\gamma(t), J\dot\gamma(t)\}$. Then we see that $M = \bigcup_{t\in I} \mathbf{C}P_t^{n-1}(c)$ is a ruled real hypersurface in $\mathbf{C}P^n(c)$. The following gives a characterization of ruled real hypersurfaces in terms of the shape operator A of M (see [4]).

Lemma 3. *For a real hypersurface M in $\mathbf{C}P^n(c)$ $(n \geqq 2)$, the following conditions* (1), (2) *and* (3) *are mutually equivalent*:

(1) *M is a ruled real hypersurface*;
(2) *The shape operator A of M satisfies the following equalities on the open dense subset $M_1 = \{x \in M \mid \nu(x) \neq 0\}$ with a unit vector field U orthogonal to ξ : $A\xi = \mu\xi + \nu U$, $AU = \nu\xi$, $AX = 0$ for an arbitrary tangent vector X orthogonal to ξ and U, where μ, ν are differentiable functions on M_1 by $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$*;
(3) *The shape operator A of M satisfies $\langle Av, w\rangle = 0$ for arbitrary tangent vectors $v, w \in T_xM$ orthogonal to ξ_x at each point $x \in M$.*

We treat a ruled real hypersurface locally, because generally this hypersurface has self-intersections and sigularities. When we study ruled real hypersurfaces, we usually omit points where ξ is principal and suppose that v does not vanish everywhere, namely a ruled hypersurface M is usually supposed $M_1 = M$.

4. Preliminaries and Statements of Results

In order to obtain Theorems 1 and 2, we first define ϕ-invariances of the second fundamental form of a submanifold $(M^{2n-1}, f_1 \circ \imath_M)$ in the sphere $S^{n(n+2)-1}((n+1)c/(2n))$. Since $(M^{2n-1}, \imath_M)$ is a real hypersurface of $\mathbf{C}P^n(c)$, the manifold M^{2n-1} has an almost contact metric structure $(\phi, \xi, \eta, \langle\,,\rangle)$ from the Kähler structure J and the standard metric $\langle\,,\rangle$ of $\mathbf{C}P^n(c)$ (see Section 2). Then the second fundamental form σ of the immersion $f_1 \circ \imath_M$ is called *strongly ϕ-invariant* if σ satisfies $\sigma(\phi X, \phi Y) = \sigma(X, Y)$ for all vectors X and Y on M. Also, it is called *weakly ϕ-invariant* if σ satisfies $\sigma(\phi X, \phi Y) = \sigma(X, Y)$ for all vectors X and Y on M orthogonal to the characteristic vector ξ on M.

We next recall fundamental geometric properties of the isometric embedding $f_1 : \mathbf{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$. The inner product of the first normal space of f_1 is given by

$$(4.1)\quad \langle\sigma_1(X, Y), \sigma_1(Z, W)\rangle = -(c/(2n))\langle X, Y\rangle\langle Z, W\rangle + (c/4)(\langle X, W\rangle\langle Y, Z\rangle + \langle X, Z\rangle\langle Y, W\rangle + \langle JX, W\rangle\langle JY, Z\rangle + \langle JX, Z\rangle\langle JY, W\rangle).$$

Here, σ_1 denotes the second fundamental form of f_1. Equation (4.1) shows the following properties of f_1:

(i) f_1 is minimal;

(ii) $\sigma_1(JX, JY) = \sigma_1(X, Y)$ for $\forall X, Y \in T\mathbf{C}P^n(c)$ (namely, σ_1 is J-invariant);

(iii) $\|\sigma_1(X, X)\| = \sqrt{(n-1)c/(2n)}$ for each unit vector X on $\mathbf{C}P^n(c)$ (that is, f_1 is $\sqrt{(n-1)c/(2n)}$-isotropic in the sense of O'Neill [5]).

Note that σ_1 is J-invariant is equivalent to saying that the second fundamental form σ_1 of our embedding f_1 is parallel (for example see [2], Proposition 3). The embedding f_1 is usually called *the first standard minimal embedding*.

Theorem 1. (1) *There exists no Riemannian submanifold $(M^{2n-1}, f_1 \circ \imath_M)$ in $S^{n(n+2)-1}((n+1)c/(2n))$ such that the second fundamental form σ of the immersion $f_1 \circ \imath_M$ is strongly ϕ-invariant.*

(2) *Let* $(M^{2n-1}, f_1 \circ \iota_M)$ *be a Riemannian submanifold in* $S^{n(n+2)-1}((n+1)c/(2n))$ *such that the second fundamental form* σ *of the immersion* $f_1 \circ \iota_M$ *is weakly* ϕ*-invariant. Then the following hold*:

2i) *If* (M^{2n-1}, ι_M) *is a Hopf hypersurface in* $\mathbf{C}P^n(c)$, *then* (M^{2n-1}, ι_M) *is locally a real hypersurface of type* (A) *in* $\mathbf{C}P^n(c)$;

2ii) *If the holomorphic distribution* T^0M *of* (M^{2n-1}, ι_M) *is integrable, then* (M^{2n-1}, ι_M) *is a ruled real hypersurface in* $\mathbf{C}P^n(c)$.

Proof. Let A be a shape operator of M in $\mathbf{C}P^n(c)$. Then, second fundamental form σ of the immersion $f_1 \circ \iota_M$ is given by

$$\sigma(X, Y) = \langle AX, Y\rangle\mathcal{N} + \sigma_1(X, Y) \tag{4.2}$$

for all vectors X and Y on M.

(1) Suppose that σ is strongly ϕ-invariant. Then, from (4.2) we have

$$\begin{cases} \langle A\phi X, \phi Y\rangle = \langle AX, Y\rangle, \\ \sigma_1(\phi X, \phi Y) = \sigma_1(X, Y) \end{cases}$$

for all vectors X and Y on M. However, the equation $\sigma_1(\phi X, \phi Y) = \sigma_1(X, Y)$ for $\forall X, Y \in TM$ does not hold. Indeed, setting $X = Y = \xi$, we get $\sigma_1(\xi, \xi) = 0$, which cotradicts the property (iii) of the immersion f_1. Therefore σ is *not* strongly ϕ-invariant.

(2) Since σ is weakly ϕ-invariant, from (4.2) we have

$$\begin{cases} \langle A\phi X, \phi Y\rangle = \langle AX, Y\rangle, \\ \sigma_1(\phi X, \phi Y) = \sigma_1(X, Y) \end{cases}$$

for all vectors $X, Y \in T^0M$. On the other hand, using the property (ii) of f_1, we have

$$\begin{aligned} \sigma_1(\phi X, \phi Y) &= \sigma_1(JX - \langle JX, J\xi\rangle J\xi, JY - \langle JY, J\xi\rangle J\xi) \\ &= \sigma_1(JX, JY) - \langle JY, J\xi\rangle\sigma_1(JX, J\xi) \\ &\quad - \langle JX, J\xi\rangle\sigma_1(JY, J\xi) + \langle JX, J\xi\rangle\langle JY, J\xi\rangle\sigma_1(J\xi, J\xi) \\ &= \sigma_1(X, Y) - \langle Y, \xi\rangle\sigma_1(X, \xi) - \langle X, \xi\rangle\sigma_1(Y, \xi) \\ &\quad + \langle X, \xi\rangle\langle Y, \xi\rangle\sigma_1(\xi, \xi). \\ &= \sigma_1(X, Y) \quad \text{for all vectors } X, Y \in T^0M. \end{aligned} \tag{4.3}$$

The above equation implies that in the case 2i) we have only to determine Hopf hypersurfaces M of $\mathbf{C}P^n(c)$ having $\langle A\phi X, \phi Y\rangle = \langle AX, Y\rangle$ for all $X, Y \in T^0M$. We take an unit vector $X \in T^0M$ with $AX = \lambda X$. Then, by weakly ϕ-invariance, we see that

$$\langle A\phi X, \phi X\rangle = \langle AX, X\rangle = \lambda.$$

Moreover, using the assumption that M is a Hopf hypersurface, we obtain $\phi A\xi = 0 = A\phi\xi$. Hence, we can see that $\phi A = A\phi$ holds on TM, so that by Lemma 2, M is a real hypersurface of type (A). Thus we have proved Statement of 2i).

Next, we investigate Statement 2ii). So, we suppose that the holomorphic distribution T^0M of M is integrable. This assumpution is equivalent to saying that

$$\langle(\phi A + A\phi)X, Y\rangle = 0 \quad \text{for } \forall X, Y \in T^0M. \tag{4.4}$$

(See Proposion 5 of Kimura and Maeda [1]).

We next show that our real hypersurface M satsfies the following:

$$\langle AX, Y\rangle = 0 \quad \text{for arbitrary } X, Y \in T^0M.$$

Indeed,

$$\begin{aligned}\langle AX, Y\rangle &= \langle A\phi X, \phi Y\rangle \\ &= -\langle \phi AX, \phi Y\rangle \quad \text{(from (4.4))} \\ &= \langle AX, \phi^2 Y\rangle \quad \text{(from the skew-symmetry of } \phi) \\ &= -\langle AX, Y\rangle = 0,\end{aligned}$$

so that by (3) of Lemma 3, M is a ruled real hypersurface. Therefore we get Statement 2ii). □

Theorem 2. *For a real hypersurface (M, ι_M) of type* (A) *in $\mathbf{C}P^n(c)$, namely a tube (M, ι_M) of radius r $(0 < r < \pi/\sqrt{c})$ around totally geodesic $\mathbf{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-1)$ in $\mathbf{C}P^n(c)$, we denote by $\mathfrak{h}_\ell(r)$ the mean curvature vector of the immersion $f_1 \circ \iota_M : M \to S^{n(n+2)-1}((n+1)c/(2n))$. Then we obtain the following statements* (1) *and* (2).

(1) *The mean curvature $H_\ell(r) := \|\mathfrak{h}_\ell(r)\|$ is given by*

$$H_\ell(r)^2 = \frac{1}{(2n-1)^2}\left[(\text{Trace } A)^2 + \frac{(n-1)c}{2n}\right]. \tag{4.5}$$

Here, A is the shape operator of our real hypersurface (M, ι_M) in $\mathbf{C}P^n(c)$ and Trace A *is given by*

$$\text{(4.6)} \qquad \operatorname{Trace} A = (2n-2\ell-1)\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+1)\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right).$$

In this case, when Trace $A = 0$, *i.e.* $\tan^2(\sqrt{c}r/2) = (2n-2\ell-1)/(2\ell+1)$, *$H_\ell(r)$ has the minimal value $\sqrt{((n-1)c)/2n(2n-1)^2}$ which is independent of ℓ $(1 \leqq \ell \leqq n-1)$.*

(2) *The length of the derivative of the mean curvature vector $\|D\mathfrak{h}_\ell(r)\|$ is given by*

$$\text{(4.7)} \qquad \|D\mathfrak{h}_\ell(r)\|^2 = \frac{c^2}{8(2n-1)^2}\left[(n-\ell-1)\left\{(2n-2\ell+1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+1)\tan\left(\frac{\sqrt{c}}{2}r\right)\right\}^2 + \ell\left\{(2n-2\ell-1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+3)\tan\left(\frac{\sqrt{c}}{2}r\right)\right\}^2\right],$$

where D is the normal connection of the immersion $f_1 \circ \iota_M$. In particular, $D\mathfrak{h}_\ell(r) = 0$, namely the immersion $f_1 \circ \iota_M : M \to S^{n(n+2)-1}((n+1)c/(2n))$ has parallel mean curvature vector with respect to the normal connection D if and only if $\ell = n-1$ and $\tan^2(\sqrt{c}r/2) = (2n-2\ell-1)/(2\ell+3)$. This means that $D\mathfrak{h}_\ell(r) = 0$ if and only if M is a tube of radius $r = (2/\sqrt{c})\tan^{-1}(1/\sqrt{2n+1})$ around totally geodesic $\mathbf{C}P^{n-1}(c)$ in $\mathbf{C}P^n(c)$, that is M is a geodesic sphere $G(r)$ of radius $r = (2/\sqrt{c})\tan^{-1}(\sqrt{2n+1})$ of $\mathbf{C}P^n(c)$.

Proof. (1) We take a local field of orthonormal frame $\{e_1, \dots, e_{n-1}, \phi e_1 (= Je_1), \dots \phi e_{n-1} (= Je_{n-1}), \xi\}$ on M. Then, as a matter of course $\{e_1, \dots, e_{n-1}, Je_1, \dots Je_{n-1}, \xi, J\xi (= \mathcal{N})\}$ is a local field of orthonormal frames of $\mathbf{C}P^n(c)$ along M. It follows from the definition of the mean curvature vector that $\mathfrak{h}_\ell(r)$ is expressed as

$$\mathfrak{h}_\ell(r) = \frac{1}{2n-1}\left[(\operatorname{Trace} A)\mathcal{N} + \sum_{i=1}^{n-1}(\sigma_1(e_i, e_i) + \sigma_1(Je_i, Je_i)) + \sigma_1(\xi, \xi)\right],$$

so that

$$\text{(4.8)} \qquad \mathfrak{h}_\ell(r) = \frac{1}{2n-1}[(\operatorname{Trace} A)\mathcal{N} - \sigma_1(\xi, \xi)].$$

Here, we have used the property (i) of the immersion f_1. Hence, it follows from the property (iii) of the immersion f_1 that we obtain (4.5). We find easily that

$$\operatorname{Trace} A = (2n - 2\ell - 2)\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right) - 2\ell\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right) + \sqrt{c}\cot(\sqrt{c}r).$$

This, together with $\sqrt{c}\cot(\sqrt{c}r) = (\sqrt{c}/2)\cot(\sqrt{c}r/2) - (\sqrt{c}/2)\tan(\sqrt{c}r/2)$, yields (4.6).

(2) We take again a local field of orthonormal frames $\{e_1, \ldots, e_{2n-2l-2}, f_1, \ldots, f_{2l}, \xi\}$ on M such that $e_i \in V_{(\sqrt{c}/2)\cot(\sqrt{c}r/2)}$ $(1 \leqq i \leqq 2n - 2\ell - 2)$ and $f_j \in V_{-(\sqrt{c}/2)\tan(\sqrt{c}r/2)}$ $(1 \leqq j \leqq 2\ell)$. Then $\|D\mathfrak{h}_\ell(r)\|$ is expressed as

$$\|D\mathfrak{h}_\ell(r)\|^2 = \|D_\xi\mathfrak{h}_\ell(r)\|^2 + \sum_{i=1}^{2n-2\ell-2}\|D_{e_i}\mathfrak{h}_\ell(r)\|^2 + \sum_{j=1}^{2\ell}\|D_{f_j}\mathfrak{h}_\ell(r)\|^2. \tag{4.9}$$

We shall calculate $\|D_\xi\mathfrak{h}_\ell(r)\|$, $\|D_{e_i}\mathfrak{h}_\ell(r)\|$ and $\|D_{f_j}\mathfrak{h}_\ell(r)\|$ one by one. In the following computation, we shall use fundamental equations and terminologies in submanifold theory without explanation. Now we denote by $(*)^\perp$ the normal component of the vector $(*)$ along M in $TS^{n(n+2)-1}((n+1)c/(2n))$, $\nabla^{(2)}$ the Riemannian connection of $S^{n(n+2)-1}((n+1)c/(2n))$, $A^{(1)}$ the shape oparetor of $\mathbf{C}P^n(c)$ in $S^{n(n+2)-1}((n+1)c/(2n))$ and $D^{(1)}$ the normal connection of $\mathbf{C}P^n(c)$ in $S^{n(n+2)-1}((n+1)c/(2n))$.

We have

$$\begin{aligned}
D_\xi\mathcal{N} &= (\nabla^{(2)}_\xi\mathcal{N})^\perp = (\nabla^{(1)}_\xi\mathcal{N} + \sigma_1(\xi,\mathcal{N}))^\perp \quad \text{(from (2.1))}\\
&= (-A\xi + \sigma_1(\xi, J\xi))^\perp \quad \text{(from (2.2))}\\
&= \sigma_1(\xi, J\xi) = -\sigma_1(J\xi,\xi) = 0 \quad \text{(from the property (ii) of } f_1)
\end{aligned}$$

and

$$\begin{aligned}
&D_\xi(\sigma_1(\xi,\xi))\\
&\quad = (\nabla^{(2)}_\xi(\sigma_1(\xi,\xi)))^\perp\\
&\quad = (-A^{(1)}_{\sigma_1(\xi,\xi)}\xi + D^{(1)}_\xi(\sigma_1(\xi,\xi)))^\perp \quad \text{(from Weingarten's formula)}\\
&\quad = -\langle A^{(1)}_{\sigma_1(\xi,\xi)}\xi, J\xi\rangle J\xi + 2\sigma_1(\nabla^{(1)}_\xi\xi,\xi) \quad \text{(from the parallelism of } \sigma_1)\\
&\quad = -\langle\sigma_1(\xi,\xi), \sigma_1(\xi, J\xi)\rangle J\xi + 2\sigma_1(\nabla^{(1)}_\xi\xi,\xi)
\end{aligned}$$

$$= 2\sigma_1(\nabla_\xi \xi + \langle A\xi, \xi\rangle J\xi, \xi) \quad \text{(from the property (ii) of } f_1 \text{ and (2.1))}$$

$$= 2\sigma_1(\phi A\xi, \xi) = 0 \quad \text{(from (2.4) and the property (ii) of } f_1).$$

Hence, we obtain

$$D_\xi \mathfrak{h}_\ell(r) = 0. \tag{4.10}$$

Next, for e_i $(1 \leqq i \leqq 2n - 2\ell - 2)$ we see that

$$D_{e_i}\mathcal{N} = (\nabla^{(2)}_{e_i}\mathcal{N})^\perp = (\nabla^{(1)}_{e_i}\mathcal{N} + \sigma_1(e_i, \mathcal{N}))^\perp$$
$$= (-Ae_i + \sigma_1(e_i, J\xi))^\perp = \sigma_1(e_i, J\xi)$$

and

$$D_{e_i}(\sigma_1(\xi, \xi)) = (\nabla^{(2)}_{e_i}(\sigma_1(\xi, \xi)))^\perp$$
$$= (-A^{(1)}_{\sigma_1(\xi,\xi)}e_i + D^{(2)}_{e_i}(\sigma_1(\xi, \xi)))^\perp \quad \text{(from Weingarten's formula)}$$
$$= -\langle A^{(1)}_{\sigma_1(\xi,\xi)}e_i, J\xi\rangle J\xi + 2\sigma_1(\nabla^{(1)}_{e_i}\xi, \xi) \quad \text{(from the parallelism of } \sigma_1)$$
$$= -\langle \sigma_1(\xi, \xi), \sigma_1(e_i, J\xi)\rangle J\xi + 2\sigma_1(\nabla_{e_i}\xi + \langle Ae_i, \xi\rangle J\xi, \xi) \quad \text{(from (2.1))}$$
$$= 2\sigma_1(\nabla_{e_i}\xi, \xi) \quad \text{(from (4.1) and the property (ii) of } f_1)$$
$$= 2\sigma_1(\phi Ae_i, \xi) \quad \text{(from (2.4))}$$
$$= \sqrt{c}\cot\left(\frac{\sqrt{c}}{2}r\right)\sigma_1(\phi e_i, \xi) = \sqrt{c}\cot\left(\frac{\sqrt{c}}{2}r\right)\sigma_1(Je_i, \xi)$$
$$= -\sqrt{c}\cot\left(\frac{\sqrt{c}}{2}r\right)\sigma_1(e_i, J\xi).$$

Thus we get

$$D_{e_i}\mathfrak{h}_\ell(r) = \frac{\sqrt{c}}{2(2n-1)}\left[(2n - 2\ell + 1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell + 1)\tan\left(\frac{\sqrt{c}}{2}r\right)\right]\sigma_1(e_i, J\xi). \tag{4.11}$$

Similarly,

$$D_{f_j}\mathfrak{h}_\ell(r) = \frac{\sqrt{c}}{2(2n-1)}\left[(2n - 2\ell - 1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell + 3)\tan\left(\frac{\sqrt{c}}{2}r\right)\right]\sigma_1(f_j, J\xi). \tag{4.12}$$

By (4.1) we have

$$\langle \sigma_1(e_i, J\xi), \sigma_1(e_i, J\xi) \rangle = \frac{c}{4}. \tag{4.13}$$

Hence, from (4.11), (4.12) and (4.13) we can get

$$\|D_{e_i}\mathfrak{h}_\ell(r)\|^2 = \frac{c^2}{16(2n-1)^2}\left[(2n-2\ell+1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+1)\tan\left(\frac{\sqrt{c}}{2}r\right)\right]^2, \tag{4.14}$$

$$\|D_{f_j}\mathfrak{h}_\ell(r)\|^2 = \frac{c^2}{16(2n-1)^2}\left[(2n-2\ell-1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+3)\tan\left(\frac{\sqrt{c}}{2}r\right)\right]^2. \tag{4.15}$$

Thus from (4.9), (4.10), (4.14) and (4.15) we obtain (4.7).

Finally, we shall show that $D\mathfrak{h}_\ell(r) = 0$ if and only if $\ell = n-1$ and $\tan^2(\sqrt{c}r/2) = 1/(2n+1)$. It is obvious the "if" part.

So, we suppose that $D\mathfrak{h}_\ell(r) = 0$. Since $\ell > 0$ and $n - \ell - 1 \geqq 0$, from (4.7) and $D\mathfrak{h}_\ell(r) = 0$ we find the following equations:

$$(n-\ell-1)\left\{(2n-2\ell+1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+1)\tan\left(\frac{\sqrt{c}}{2}r\right)\right\} = 0 \tag{4.16}$$

and

$$(2n-2\ell-1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+3)\tan\left(\frac{\sqrt{c}}{2}r\right) = 0. \tag{4.17}$$

So, from (4.17) we have

$$\tan^2\left(\frac{\sqrt{c}}{2}r\right) = \frac{2n-2\ell-1}{2\ell+3}. \tag{4.18}$$

We here remark that in (4.16)

$$(2n-2\ell+1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+1)\tan\left(\frac{\sqrt{c}}{2}r\right) \neq 0.$$

In fact we suppose that

$$(2n-2\ell+1)\cot\left(\frac{\sqrt{c}}{2}r\right) - (2\ell+1)\tan\left(\frac{\sqrt{c}}{2}r\right) = 0,$$

which implies that

$$\tan^2\left(\frac{\sqrt{c}}{2}r\right) = \frac{2n - 2\ell + 1}{2\ell + 1}. \tag{4.19}$$

Solving equations (4.18) and (4.19), we find that $n = -1$, which contradicts $n \geqq 2$. Therefore, we get $\ell = n - 1$. □

Remark 2. It is known that an isometric immersion f of a Kähler manifold M with Kähler structure J into a sphere has parallel second fundamental form σ if and only if σ is J-invariant, that is $\sigma(JX, JY) = \sigma(X, Y)$ holds for each vector X, Y on M. On the other hand, there exist no submanifolds $(M^{2n-1}, f_1 \circ \iota_M)$ with parallel second fundamental form in the sphere $S^{n(n+2)-1}((n+1)c/(2n))$. However our homogeneous submanifolds $(M^{2n-1}, f_1 \circ \iota_M)$ stated in Theorem 2 have the weakly ϕ-invariant second fundamental form.

References

[1] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. **202** (1989), 299–311.

[2] S. Maeda and H. Naitoh, Real Hypersurfaces with ϕ-invariant shape operator in a complex projective space, preprint.

[3] S. Maeda and S. Udagawa, Construction of a certain homogeneous submanifold with nonzero parallel mean curvature vector in Euclidean sphere, preprint.

[4] R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space forms, in: Tight and Taut submanifolds, T. E. Cecil and S. S. Chern (eds.), Cambridge Univ. Press, 1998, 233–305.

[5] B. O'Neill, Isotropic immersions and Kaehler immersions, Canad. J. Math. **17** (1965), 905–915.

Department of Mathematics
Saga University
1 Honzyo, Saga, 840-8502, Japan
E-mail address: 09632302@edu.cc.saga-u.ac.jp