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BERNOULLI-TYPE RELATIONS IN SOME NONCOMMUTATIVE POLYNOMIAL RING

By

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Abstract. We find particular relations which we call "Bernoulli-type" in some noncommutative polynomial ring with a single non-trivial relation. More precisely, our ring is isomorphic to the universal enveloping algebra of a two-dimensional non-abelian Lie algebra. From these Bernoulli-type relations in our ring, we can obtain a representation on a certain left ideal with the Bernoulli numbers as structure constants.

1. Introduction

Bernoulli numbers are rational numbers with connections to many branches of mathematics. Especially, they are closely related to the values of the Riemann zeta function at negative integers [1], [2]. In this paper, we show a certain connection between some noncommutative polynomial ring and Bernoulli numbers. We let $K[x, y]$ be a noncommutative polynomial ring in two indeterminates x, y over a field K of characteristic zero. Now, we define $I = \langle xy - yx - x \rangle$ to be the ideal of $K[x, y]$ generated by $xy - yx - x$, and let A be $K[x, y]/I$, the quotient of $K[x, y]$ by I. Again we use x, y as $\bar{x} = x + I$, $\bar{y} = y + I$ respectively (if there is no confusion). We note that A is isomorphic to the universal enveloping algebra of a two-dimensional non-abelian Lie algebra (cf. Remark 4.1). Then, our main result is the following:

THEOREM (Bernoulli-type relations). *Let A be as above. We put*

$$w_{k,\ell} = (xy^k - y^k x)x^\ell \in A \quad (k \geq 1, \ell \geq 0).$$

Then, the following relations hold.

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$$xw_{k,\ell} = \sum_{i=1}^{k} \binom{k}{i} w_{i,\ell+1}$$

$$yw_{k,\ell} = \frac{k}{k+1} w_{k+1,\ell} - \sum_{i=1}^{k} \frac{1}{k+1} \binom{k+1}{i} B_{k+1-i} w_{i,\ell} \qquad \square$$

Here, B_{k+1-i} are Bernoulli numbers defined in Definition 2.1. Hence, we call the above relations "Bernoulli-type relations". Put $W = \bigoplus_{m\geq 1, n\geq 0} Kx^m y^n \subseteq A$, which is a direct sum by PBW theorem. Then W becomes a two-sided ideal of A. Using the Bernoulli-type relations, we see that W is generated by $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$. We can also see that $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ is a basis of W.

To start with our motivation, we will explain Bernoulli-type relations in terms of Lie algebras. We began this study from [3] on some factorizations in universal enveloping algebras. In [3], they deal with universal enveloping algebras of three-dimensional Lie algebras. Then they obtained certain general relations. Let $\mathfrak{L}$ be a three-dimensional Lie algebra over K and denote by $U(\mathfrak{L})$ the universal enveloping algebra of $\mathfrak{L}$. Assume that $\mathfrak{L}$ is generated by two elements x, y. Then, the general relations in $U(\mathfrak{L})$ are given as follows:

$$(\mathrm{A}_k) \quad yxy^k \equiv \frac{k}{k+1} xy^{k+1} + \frac{1}{k+1} y^{k+1}x \pmod{U_k},$$

$$(\mathrm{B}_k) \quad y^k xy \equiv \frac{1}{k+1} xy^{k+1} + \frac{k}{k+1} y^{k+1}x \pmod{U_k},$$

$$(\mathrm{C}_k) \quad yU_k \subseteq U_{k+1}, \quad U_k y \subseteq U_{k+1}, \quad \text{where}$$

$$U_k = \sum_{0\leq m\leq k} (Kxy^m + Ky^m x + Ky^m) \quad (k \geq 0). \qquad \square$$

The remainder terms, $u = \sum_{1\leq p,q,r\leq k} a_p xy^p + b_q y^q x + c_r y^r + dx \in U_k$ with $a_p, b_q, c_r, d \in K$, of (A_k), (B_k) are determined by the generators x, y and types of $\mathfrak{L}$. In the paper [3], they determine some exact forms of u along with a classification of $\mathfrak{L}$ in Jacobson's book [6].

Here we introduce a rough classification. We put $\mathfrak{L} = Ke \oplus Kf \oplus Kg$ with its basis (e, f, g). Let $\mathfrak{L}'$ be the derived ideal of $\mathfrak{L}$ and $\mathfrak{C}$ be the center of $\mathfrak{L}$. Then our classification is given as follows:

(a) If $\mathfrak{L}' = 0$, $\mathfrak{L}$ is abelian.
(b) If $\dim \mathfrak{L}' = 1$ and $\mathfrak{L}' \subseteq \mathfrak{C}$, the multiplication table of the basis is

$$[e, f] = g, \quad [e, g] = [f, g] = 0.$$

(c) If dim $\mathfrak{L}' = 1$ and $\mathfrak{L}' \not\subseteq \mathfrak{C}$, the multiplication table of the basis is

$$[e, f] = e, \quad [e, g] = [f, g] = 0.$$

(d) If dim $\mathfrak{L}' = 2$, the multiplication tables of the basis are

$$\text{(d)-}(\alpha) \quad [e, f] = 0, \quad [e, g] = e, \quad [f, g] = \alpha f,$$

$$\text{(d)-}(+) \quad [e, f] = 0, \quad [e, g] = e + f, \quad [f, g] = f,$$

where α in $K^{\times}$. Different choices of α give different algebras unless $\alpha\alpha' = 1$.

(e) dim $\mathfrak{L}' = 3$, the multiplication table of the basis is

$$[e, f] = g, \quad [g, e] = 2e, \quad [g, f] = -2f.$$

In the type (d) or (e), we suppose that K is algebraically closed (just for our rough explanation). As it is well-known, the type (b) gives a Heisenberg Lie algebra $\mathfrak{H}_K$ and the type (e) gives a special linear Lie algebra $\mathfrak{sl}_2(K)$. In the paper [3], they determined the exact forms of u for $\mathfrak{H}_F$ or $\mathfrak{sl}_2(F)$ with the above generators e, f including the case if F is a field of characteristic zero. They also showed that $\mathfrak{L}$ can not be two generated if $\mathfrak{L}$ is the type (a) or the type (d)-($\alpha = 1$). Hence, we were interested in determining the forms in U_k for the remaining type of $\mathfrak{L}$. For our purpose, we explain some results in the author's master thesis [10] written in Japanese. In [10], we obtained some properties between u and the types of $\mathfrak{L}$, and determined the exact forms of u if $\mathfrak{L}$ is the type (d)-(+). The properties between u and the types of $\mathfrak{L}$ are given as follows:

- We always have $u = 0$ regardless of the choice of generators if $\mathfrak{L}$ is the type (b).
- We always have $u \neq 0$ regardless of the choice of generators if $\mathfrak{L}$ is the type (e).
- We can get $u = 0$ by some special generators if $\mathfrak{L}$ is the type (c) or (d). (We also get $u \neq 0$ by another generators.)

The exact forms of u are determined if $\mathfrak{L}$ is the type (d)-(+) with the generators e and g. The formulas in $U(\mathfrak{L})$ are given as follows:

$$(\mathrm{P}_k) \quad geg^k = \frac{k}{k+1} eg^{k+1} + \frac{1}{k+1} g^{k+1} e - eg^k + \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} g^i e,$$

$$(\mathrm{Q}_k) \quad g^k eg = \frac{1}{k+1} eg^{k+1} + \frac{k}{k+1} g^{k+1} e + \frac{1}{k+1} \sum_{i=0}^{k} (-1)^{k+1-i} \binom{k+1}{i} eg^i + g^k e.$$

$\square$

These are a brief summary of [10]. After we obtained these results, we could establish the formulas if $\mathfrak{L}$ is the type (c) with the generators $e+g$ and $f+g$ in the above classification. Then we noticed that our formulas can be reduced to the two-dimensional case. That is, we put $L = Kx \oplus Ky$ as a two-dimensional Lie algebra satisfying $[x, y] = x$ and denote by $U(L)$ the universal enveloping algebra of L. Then, the formulas in $U(L)$ are given as follows:

$$
\begin{aligned}
(\mathrm{P}_k) \quad yxy^k = {} & \frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}y^{k+1}x \\
& - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}xy^i + \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}y^ix,
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{Q}_k) \quad y^kxy = {} & \frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x \\
& + \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}xy^i \\
& - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}y^ix. \qquad \square
\end{aligned}
$$

At first these formulas were shown without the Bernoulli-type relations. But using the Bernoulli-type relations, we could simplify their proofs as in Proposition 4.2.

We will review the Bernoulli numbers B_n with $B_1 = 1/2$ in Section 2. In Section 3, we will show the Bernoulli-type relations and study W introduced before. In Section 4, we will show the above formulas and explain a connection to Lie algebras. We also mention that $U(L)$ is isomorphic to A, and that $A = \bigoplus_{m,n\geq 0} Kx^my^n$ by PBW theorem.

2. Preliminaries

In this paper, K is a field of characteristic zero. We denote a left hand side (resp: right hand side) by (LHS) (resp: (RHS)). We also denote by B_n the Bernoulli numbers.

In this section, we review the Bernoulli numbers with $B_1 = 1/2$. We aim a self-contained explanation in this paper. Thus we confirm our setting here.

Definition 2.1 (The Bernoulli numbers). *We define the Bernoulli numbers B_n recursively as follows*:

$$\sum_{i=0}^{n} \binom{n+1}{i} B_i = n+1. \qquad \square$$

Remark 2.2. In general, the Bernoulli numbers are also given by a generating function. The generating function in our condition is given as follows:

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \qquad \square$$

Here we describe the Bernoulli numbers up to $n = 10$.

Figure 1 The Bernoulli numbers

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

As well known, there are the other definition of the Bernoulli numbers. If we denote by $\hat{B}_n$ the Bernoulli numbers with $\hat{B}_1 = -1/2$, then $\hat{B}_n$ are given by $(-1)^n B_n$ for $n \geq 0$.

Remark 2.3. In the first half of eighteenth century, the Bernoulli numbers were discovered around the same time by Jacob Bernoulli and Kowa Seki independently. At first, both Bernoulli and Seki took $B_1 = 1/2$. Hence, historically, our definition is an original version. $\square$

3. Bernoulli-Type Relations and the Ideal W

In this section, we show the main theorem and some corollaries. Now, we set $A = K[x, y]/I$, where $K[x, y]$ is a noncommutative polynomial ring in two indeterminates x, y and $I = \langle xy - yx - x \rangle$ is the two-sided ideal of $K[x, y]$ generated by $xy - yx - x$. At first, we confirm several elementary formulas for proving the main theorem.

Proposition 3.1. (i) *For integers $k \geq i \geq j \geq 0$, we have*

$$\binom{k}{i}\binom{i}{j} = \binom{k}{j}\binom{k-j}{i-j}.$$

(ii) *Let A be as above. Then the following formula holds.*

$$xy^k = \sum_{i=0}^{k} \binom{k}{i} y^i x$$

(iii) *Let A be as above. Then the following formula holds.*

$$y^k x = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} xy^i$$

PROOF. (i) We can calculate

$$\begin{aligned} \binom{k}{i}\binom{i}{j} &= \frac{k!}{(k-i)!i!} \frac{i!}{(i-j)!j!} \\ &= \frac{k!}{(k-j)!j!} \frac{(k-j)!}{\{(k-j)-(i-j)\}!(i-j)!} \\ &= \binom{k}{j}\binom{k-j}{i-j}. \end{aligned}$$

(ii) Since $xy = yx + x = y(x+1)$,

$$\begin{aligned} xy^k &= (yx + x)y^{k-1} = (y+1)xy^{k-1} \\ &= \cdots \\ &= (y+1)^k x = \sum_{i=0}^{k} \binom{k}{i} y^i x. \end{aligned}$$

(iii) Since $yx = xy - x = x(y-1)$,

$$\begin{aligned} y^k x &= y^{k-1}(xy - x) = y^{k-1}x(y-1) \\ &= \cdots \\ &= x(y-1)^k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} xy^i. \end{aligned}$$

Therefore, we obtain the desired results. □

Now, we prove the main theorem.

THEOREM 3.2. *Let A be as above. We take*

$$w_{k,\ell} = (xy^k - y^k x)x^\ell \in A \quad (k \geq 1, \ell \geq 0).$$

Then, the following relations hold.

$$(BR1) \quad xw_{k,\ell} = \sum_{i=1}^{k} \binom{k}{i} w_{i,\ell+1}$$

$$(BR2) \quad yw_{k,\ell} = \frac{k}{k+1} w_{k+1,\ell} - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i} w_{i,\ell}$$

Proof. At first, we show (BR1). Using Proposition 3.1 (ii), we can compute

$$\begin{aligned}
xw_{k,\ell} &= x(xy^k - y^k x)x^\ell \\
&= \{x(xy^k) - (xy^k)x\}x^\ell \\
&= \left\{ x\left(\sum_{i=0}^{k} \binom{k}{i} y^i x\right) - \left(\sum_{i=0}^{k} \binom{k}{i} y^i x\right) x \right\} x^\ell \\
&= \left\{ \sum_{i=0}^{k} \binom{k}{i} xy^i - \sum_{i=0}^{k} \binom{k}{i} y^i x \right\} x^{\ell+1} \\
&= \sum_{i=0}^{k} \binom{k}{i} (xy^i - y^i x) x^{\ell+1} \\
&= \sum_{i=0}^{k} \binom{k}{i} w_{i,\ell+1}.
\end{aligned}$$

Next, we show (BR2) by computing from (RHS) to (LHS). Using Proposition 3.1 (ii), we can compute

$$\begin{aligned}
(RHS) &= \frac{k}{k+1} w_{k+1,\ell} - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i} w_{i,\ell} \\
&= \frac{k}{k+1} (xy^{k+1} - y^{k+1}x)x^\ell - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i} (xy^i - y^i x)x^\ell \\
&= \frac{k}{k+1} \left\{ \sum_{i=0}^{k+1} \binom{k+1}{i} y^i x - y^{k+1} x \right\} x^\ell \\
&\quad - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i} \left\{ \sum_{j=0}^{i} \binom{i}{j} y^j x - y^i x \right\} x^\ell \\
&= \frac{k}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} y^i x^{\ell+1} - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i} \sum_{j=0}^{i-1} \binom{i}{j} y^j x^{\ell+1}.
\end{aligned}$$

We divide (RHS) into three terms such as $x^{\ell+1}$ and $y^k x^{\ell+1}$ and otherwise. Then we have

$$(RHS) = \frac{k}{k+1}\binom{k+1}{k} y^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$+ \frac{k}{k+1}\binom{k+1}{0} y^0 x^{\ell+1} - \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i} B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j} y^j x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i} B_{k+1-i}\binom{i}{0} y^0 x^{\ell+1}.$$

Since we can replace B_{k+1-i} with B_i in the last term, we have

$$(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} + \frac{k}{k+1} x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i} B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j} y^j x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i} B_i x^{\ell+1}$$

$$= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i} B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j} y^j x^{\ell+1} + \frac{k}{k+1} x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i} B_i x^{\ell+1} + \frac{1}{k+1}\binom{k+1}{0} x^{\ell+1}.$$

In the fifth term, using Definition 2.1, we get

$$(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i} B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j} y^j x^{\ell+1}$$

$$+ \frac{k}{k+1} x^{\ell+1} - \frac{1}{k+1}(k+1) x^{\ell+1} + \frac{1}{k+1} x^{\ell+1}.$$

$$= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i} B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j} y^j x^{\ell+1}.$$

Replacing the index i with $i+1$ in the third term, we obtain

$$(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i+1} B_{k-i}\sum_{j=1}^{i}\binom{i+1}{j} y^j x^{\ell+1}.$$

Then, changing addition method in the third term, we obtain

$$(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=j}^{k-1}\binom{k+1}{i+1}\binom{i+1}{j} B_{k-i}\right\} y^j x^{\ell+1}.$$

In the third term, using Proposition 3.1 (i), we get

$$(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=j}^{k-1}\binom{k+1}{j}\binom{k+1-j}{i+1-j} B_{k-i}\right\} y^j x^{\ell+1}.$$

Then, replacing the index $i+1-j$ with i, we get

$$(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=1}^{k-j}\binom{k+1}{j}\binom{k+1-j}{i} B_{k-(i+j-1)}\right\} y^j x^{\ell+1}$$

$$= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1}$$

$$- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=1}^{k-j}\binom{k+1}{j}\binom{k-j+1}{i} B_{k-j+1-i}\right\} y^j x^{\ell+1}.$$

Since we have $\binom{k-j+1}{i} = \binom{k-j+1}{k-j+1-i}$, we can replace $B_{k-j+1-i}$ with B_i. Hence we have

$$\begin{aligned}(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} \\ &\quad - \frac{1}{k+1}\sum_{j=1}^{k-1}\binom{k+1}{j}\left\{\sum_{i=1}^{k-j}\binom{k-j+1}{i}B_i\right\} y^j x^{\ell+1} \\ &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} \\ &\quad - \frac{1}{k+1}\sum_{j=1}^{k-1}\binom{k+1}{j}\left\{\sum_{i=0}^{k-j}\binom{k-j+1}{i}B_i - \binom{k-j+1}{0}B_0\right\} y^j x^{\ell+1}.\end{aligned}$$

In the third term, using Definition 2.1, we get

$$\begin{aligned}(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} \\ &\quad - \frac{1}{k+1}\sum_{j=1}^{k-1}\binom{k+1}{j}\{(k-j+1)-1\} y^j x^{\ell+1}.\end{aligned}$$

Then, replacing the index j with i in the third term, we have

$$\begin{aligned}(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}(k-i) y^i x^{\ell+1} \\ &= ky^k x^{\ell+1} + \frac{1}{k+1}\sum_{i=1}^{k-1}k\binom{k+1}{i} y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}(k-i) y^i x^{\ell+1} \\ &= ky^k x^{\ell+1} + \frac{1}{k+1}\sum_{i=1}^{k-1}i\binom{k+1}{i} y^i x^{\ell+1} \\ &= \binom{k}{k-1} y^k x^{\ell+1} + \sum_{i=1}^{k-1}\binom{k}{i-1} y^i x^{\ell+1} \\ &= \sum_{i=1}^{k}\binom{k}{i-1} y^i x^{\ell+1}.\end{aligned}$$

Replacing the index i with $i+1$, we get

$$(RHS) = \sum_{i=0}^{k-1} \binom{k}{i} y^{i+1} x^{\ell+1}.$$

Regarding $y^{i+1}x^{\ell+1}$ as $y(y^i x^{\ell+1})$, we have

$$\begin{aligned}
(RHS) &= y \sum_{i=0}^{k-1} \binom{k}{i} y^i x^{\ell+1} \\
&= y\left(\sum_{i=0}^{k} \binom{k}{i} y^i x^{\ell+1} - y^k x^{\ell+1}\right) \\
&= y\left(\sum_{i=0}^{k} \binom{k}{i} y^i x - y^i x\right) x^{\ell} \\
&= y(xy^k - y^k x)x^{\ell} \\
&= y w_{k,\ell} = (LHS).
\end{aligned}$$

Therefore, we obtain desired results. □

From the theorem, we can get some corollaries. As has been mentioned in the introduction, A is isomorphic to the universal enveloping algebra of a two-dimensional non-abelian Lie algebra. Thus, using PBW theorem, we can put

$$W = \bigoplus_{m \geq 1, n \geq 0} K x^m y^n.$$

Here we put

$$W' = \left\{ \sum_{k,\ell} c_{k,\ell} w_{k,\ell} \,\middle|\, \begin{array}{l} k \geq 1,\ \ell \geq 0,\ c_{k,\ell} \in K, \\ c_{k,\ell} = 0 \text{ for all but finitely many pairs } (k,\ell) \end{array} \right\}.$$

Then, the following statements hold.

Corollary 3.3. *Notation is as above. Then, W' is a two-sided ideal of A. In particular, $W = W'$.*

Proof. From Theorem 3.2, it is clear that W' becomes a left ideal of A. Again using Theorem 3.2, we can see

$$W' = A w_{1,0} = Ax.$$

Then, we have

$$W'x = (Ax)x \subseteq W'$$

and

$$W'y = (Ax)y = A(xy) = A(yx + x) = A(y+1)x \subseteq W'.$$

Hence, W' is a two-sided ideal of A. Using Proposition 3.1, we can obtain

$$x^m y^n = x^{m-1}(xy^n) = x^{m-1}\left(\sum_{i=0}^{n}\binom{n}{i}y^i\right)x,$$

which implies $W = Ax$ and $W = W'$. Therefore, we obtain the desired result. □

Next, we see that $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ is a basis of W'.

COROLLARY 3.4. *Notation is as above. Then, $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ is a basis of W, that is, $W = \bigoplus_{k\geq 1,\ell\geq 0} Kw_{k,\ell}$.*

PROOF. We show $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ are linearly independent. We assume

$$\sum_{\ell=1}^{n}\sum_{k=1}^{m} c_{k,\ell}(xy^k - y^k x)x^\ell = 0 \quad (m, n < \infty)$$

with $c_{k,\ell} \in K$. From Proposition 3.1 (ii), we obtain

$$\begin{aligned}(LHS) &= \sum_{\ell=1}^{n}\sum_{k=1}^{m} c_{k,\ell}(xy^k - y^k x)x^\ell \\ &= \sum_{\ell=1}^{n}\sum_{k=1}^{m} c_{k,\ell}\sum_{i=0}^{k-1}\binom{k}{i}y^i x^{\ell+1}.\end{aligned}$$

Hence, we have

$$\begin{aligned}(LHS) &= \sum_{\ell=1}^{n} c_{m,\ell}\sum_{i=0}^{m-1}\binom{m}{i}y^i x^{\ell+1} + \sum_{\ell=1}^{n}\sum_{k=1}^{m-1} c_{k,\ell}\sum_{i=0}^{k-1}\binom{k}{i}y^i x^{\ell+1} \\ &= \sum_{\ell=1}^{n} c_{m,\ell}\binom{m}{m-1}y^{m-1}x^{\ell+1} + \sum_{\ell=1}^{n} c_{m,\ell}\sum_{i=0}^{m-2}\binom{m}{i}y^i x^{\ell+1} \\ &\quad + \sum_{\ell=1}^{n}\sum_{k=1}^{m-1} c_{k,\ell}\sum_{i=0}^{k-1}\binom{k}{i}y^i x^{\ell+1}.\end{aligned}$$

Then, the term $y^{m-1}x^{\ell+1}$ appears only in the first term. Using PBW theorem, we get $c_{m,\ell} = 0$ for all ℓ. Hence, the second term vanishes. That is, we have

$$(LHS) = \sum_{\ell=1}^{n} \sum_{k=1}^{m-1} c_{k,\ell} \sum_{i=0}^{k-1} \binom{k}{i} y^i x^{\ell+1}.$$

Continuing this operation, we get $c_{k,\ell} = 0$ for all k. Namely, we get $c_{k,\ell} = 0$ for all k and ℓ. Hence, $\{w_{k,\ell}\}_{k \geq 1, \ell \geq 0}$ is a basis of W. □

Next, we show a variation of the Bernoulli-type relations.

Corollary 3.5. *Let A be as above. We take*

$$w_k = xy^k - y^k x \in A \quad (k \geq 1).$$

Then, the following relations hold.

$$(SBR1) \quad yw_k = \frac{k}{k+1} w_{k+1} - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i} w_i$$

$$(SBR2) \quad w_k y = \frac{k}{k+1} w_{k+1} - \frac{1}{k+1} \sum_{i=1}^{k} (-1)^{k+1-i} \binom{k+1}{i} B_{k+1-i} w_i$$

Proof. In Theorem 3.2, if we take $\ell = 0$, then (SBR1) holds.

Next, we show (BR2) by computing from (RHS) to (LHS). Using Proposition 3.1 (iii), we can compute

$$\begin{aligned}
(RHS) &= \frac{k}{k+1} w_{k+1} - \frac{1}{k+1} \sum_{i=1}^{k} (-1)^{k+1-i} \binom{k+1}{i} B_{k+1-i} w_i \\
&= \frac{k}{k+1} (xy^{k+1} - y^{k+1}x) - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i} (xy^i - y^i x). \\
&= \frac{k}{k+1} \left(xy^{k+1} - \sum_{i=0}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} xy^i \right) \\
&\quad - \frac{1}{k+1} \sum_{i=1}^{k} (-1)^{k+1-i} \binom{k+1}{i} B_{k+1-i} \left(xy^i - \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} xy^j \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{k}{k+1}\sum_{i=0}^{k}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=0}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j.
\end{aligned}
$$

We divide (RHS) into three terms such as x and $y^k x$ and otherwise. Then we have

$$
\begin{aligned}
(RHS) &= \frac{(-1)^{k-k}k}{k+1}\binom{k+1}{k}xy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad + \frac{(-1)^{k-0}}{k+1}\binom{k+1}{0}xy^0 \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}(-1)^{i-1-0}\binom{i}{0}xy^0.
\end{aligned}
$$

Since we can replace B_{k+1-i} with B_i in the last term, we have

$$
\begin{aligned}
(RHS) &= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i + \frac{(-1)^k}{k+1}x \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j \\
&\quad - \frac{(-1)^k}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_i x \\
&= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j \\
&\quad + \frac{(-1)^k}{k+1}x - \frac{(-1)^k}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i x - \frac{(-1)^k}{k+1}\binom{k+1}{0}B_0 x.
\end{aligned}
$$

In the fifth term, using Definition 2.1, we get

$$\begin{aligned}(RHS) = {} & kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ & - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j \\ & + \frac{(-1)^k}{k+1}x - \frac{(-1)^k}{k+1}(k+1)x - \frac{(-1)^k}{k+1}x \\ = {} & kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ & - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j.\end{aligned}$$

Replacing the index i with $i+1$ in the third term, we obtain

$$\begin{aligned}(RHS) = {} & kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ & - \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i+1}B_{k-i}\sum_{j=1}^{i}(-1)^{i-j}\binom{i+1}{j}xy^j.\end{aligned}$$

Then, changing addition method in the third term, we obtain

$$\begin{aligned}(RHS) = {} & kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ & - \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=j}^{k-1}(-1)^{k-j}\binom{k+1}{i+1}\binom{i+1}{j}B_{k-i}\right\}xy^j.\end{aligned}$$

In the third term, using Proposition 3.1 (i), we get

$$\begin{aligned}(RHS) = {} & kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ & - \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=j}^{k-1}(-1)^{k-j}\binom{k+1}{j}\binom{k+1-j}{i+1-j}B_{k-i}\right\}xy^j.\end{aligned}$$

Then, replacing the index $i+1-j$ with i, we get

$$\begin{aligned}(RHS) = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=1}^{k-j}(-1)^{k-j}\binom{k+1}{j}\binom{k+1-j}{i}B_{k-(i+j-1)}\right\}xy^j \\ = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=1}^{k-j}(-1)^{k-j}\binom{k+1}{j}\binom{k-j+1}{i}B_{k-j+1-i}\right\}xy^j.\end{aligned}$$

Since we have $\binom{k-j+1}{i} = \binom{k-j+1}{k-j+1-i}$, we can replace $B_{k-j+1-i}$ with B_i. Hence we have

$$\begin{aligned}(RHS) = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k+1}{j}\left\{\sum_{i=1}^{k-j}\binom{k-j+1}{i}B_i\right\}xy^j \\ = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k+1}{j}\left\{\sum_{i=0}^{k-j}\binom{k-j+1}{i}B_i - \binom{k-j+1}{0}B_0\right\}xy^j.\end{aligned}$$

In the third term, using Definition 2.1, we get

$$\begin{aligned}(RHS) = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k+1}{j}\{(k-j+1)-1\}xy^j.\end{aligned}$$

Then, replacing the index j with i in the third term, we have

$$
\begin{aligned}
(RHS) &= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}(k-i)xy^j \\
&= kxy^k + \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}k\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}(k-i)\binom{k+1}{i}xy^j \\
&= kxy^k + \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}i\binom{k+1}{i}xy^i \\
&= \binom{k}{k-1}xy^k + \sum_{i=1}^{k-1}(-1)^{k-i}\binom{k}{i-1}xy^i \\
&= \sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i-1}xy^i.
\end{aligned}
$$

Replacing the index i with $i+1$, we get

$$
(RHS) = \sum_{i=0}^{k-1}(-1)^{k+1-i}\binom{k}{i}xy^{i+1}.
$$

Regarding $y^{i+1}x^{\ell+1}$ as $y(y^i x^{\ell+1}$, we have

$$
\begin{aligned}
(RHS) &= \left(\sum_{i=0}^{k-1}(-1)^{k+1-i}\binom{k}{i}xy^i\right)y \\
&= \left(xy^k - \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}xy^i\right)y \\
&= (xy^k - y^k x)y \\
&= w_k y = (LHS).
\end{aligned}
$$

Therefore, we obtain desired results. □

We can easily see that w_k is $w_{k,0}$ in Theorem 3.2. We will investigate connections between Bernoulli-type relations and Lie algebras in the next section. Using Corollary 3.5, we will show the formulas with respect to Lie algebras.

4. A Connection between Bernoulli-Type Relations and Lie Algebras

In this section, we consider a connection between the Bernoulli-type relations and Lie algebras. In the introduction, we roughly reviewed the classification of three-dimensional Lie algebras. We let $\mathfrak{L}$ be a three-dimensional Lie algebra over a field K of characteristic zero and denote by $U(\mathfrak{L})$ the universal enveloping algebra of $\mathfrak{L}$. Then we also explain that if $\mathfrak{L}$ is the type (c), we have a two-dimensional Lie subalgebra L of $\mathfrak{L}$. Then, L is a non-abelian two-dimensional Lie algebra. That is, we can write $L = Kx \oplus Ky$ with $[x, y] = x$.

Now, we recall our settings in Section 3. We let $K[x, y]$ be a noncommutative polynomial ring generated by x, y and define $I = \langle xy - yx - x \rangle$ to be the ideal of $K[x, y]$ generated by $xy - yx - x$. We let A be $K[x, y]/I$. Then, if we denote by $U(L)$ the universal enveloping algebra of L, then we can see the following:

REMARK 4.1. Notation is as above. Then we have $A \cong U(L)$. □

From Remark 4.1, we can use the Bernoulli-type relations for $U(L)$. Conversely, it is the reason that we can use PBW theorem in A. Using the relations in Section 3, we will show the next formulas in $U(L)$.

PROPOSITION 4.2. *Let L be as above. Then in $U(L)$, we have*

$$(\mathrm{P}_k)\quad yxy^k = \frac{k}{k+1}xy^{k+1} + \frac{1}{k+1}x^{k+1}x - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}xy^i + \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}y^i x,$$

$$(\mathrm{Q}_k)\quad y^k xy = \frac{1}{k+1}xy^{k+1} + \frac{k}{k+1}y^{k+1}x + \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}xy^i - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}y^i x.$$

Proof. Using the Corollary 3.5, we see that (SBR1) implies (P_k) and (SBR2) implies (Q_k). □

Remark 4.3. The above formulas (P_k) and (Q_k) completely give the remaining terms of (A_k) and (B_k) in case of the type (c) if we replace x, y by $e+g$, $f+g$ respectively.

Remark 4.4. Using the theory of linear algebras, we can establish two-dimensional Lie algebras as follows:

Let V be a vector space over K, and $\mathrm{End}(V)$ be its endmorphism ring. Put $\mathfrak{g} = \mathrm{End}(V) \oplus V$, and we define

$$[f_1 + v_1, f_2 + v_2] = (f_1 f_2 - f_2 f_1) + (f_1(v_2) - f_2(v_1))$$

for all $f_1, f_2 \in \mathrm{End}(V)$ and $v_1, v_2 \in V$. Then $\mathfrak{g}$ becomes a Lie algebra. Suppose that $f \in \mathrm{End}(V)$ and $v \in V$ satisfy $f(v) = cv$ for some $c \in K$. Put $\mathfrak{a} = Kf \oplus Kv$ as a Lie subalgebra of $\mathfrak{g}$. Then, we have

$$\begin{cases} \mathfrak{a} \text{ is abelian} & (\text{if } c = 0), \\ \mathfrak{a} \cong L & (\text{if } c \neq 0). \end{cases}$$

Remark 4.5. If K is algebraically closed, then three-dimensional Lie algebras of type (d) corresponding to $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ in Jacobson's book [6], on page 12, are not according to β. Hence, in this paper, we introduce the exact one type as (d)-(+) at the introduction.

Remark 4.6. The following equation is pointed out by Mitsuhiro Takeuchi:

$$\begin{aligned} y\{(y+1)^k - y^k\} = {} & \frac{k}{k+1}\{(y+1)^{k+1} - y^{k+1}\} \\ & - \sum_{i=1}^{k} \frac{1}{k+1} \binom{k+1}{i} B_{k+1-i}\{(y+1)^i - y^i\}, \end{aligned}$$

which gives another proof of (BR2) in Theorem 3.2.

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