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A PRODUCT FORMULA DEFINED BY THE BETA FUNCTION AND GAUSS'S HYPERGEOMETRIC FUNCTION

By

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Abstract. Let c be a constant in $\mathbf{R}^+$. For a plane algebraic curve $r^{2m-n} = 2c^n \cos n\theta$, which depends on m and n in $\mathbf{N}$, we show that the whole length of the curve are given by a value of a product formula defined by the Beta function and Gauss's hypergeometric function depending m and n in $\mathbf{N}$. Besides, we point out the fact to be a similar model and an expansion for the complete elliptic integral of the second kind. Last, we give a background for the fact explaining the special case $m = n$.

1. Introduction and a Main Theorem

Gauss's hypergeometric function is defined by the power series

$$F(\alpha, \beta, \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n,$$

where the *Pochhammer symbol* $(x)_n$ is defined by $(x)_0 := 1$ and $(x)_n := x(x+1)\cdots(x+n-1)$. α, β and γ are any complex constants and γ is not any negative integer or 0. Let $y = F(\alpha, \beta, \gamma; z)$, then the hypergeometric function satisfies the following differential equation

$$z(1-z)y'' + (\gamma - (\alpha + \beta + 1)z)y' - \alpha\beta y = 0. \tag{1.1}$$

The hypergeometric function reduces to many elementary functions, for example,

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$$(t+u)^n = t^n F\left(-n, c, c; -\frac{u}{t}\right),$$

$$\log(1+z) = zF(1,1,2;-z),$$

$$\sin nz = n \sin zF\left(\frac{1+n}{2}, \frac{1-n}{2}, \frac{3}{2} : \sin^2 z\right).$$

We can see a value of argument 1 of the hypergeometric function in [1], [3], and [8].

PROPOSITION 1.1 (Gauss, [3, Proposition 3.21], [8] cf. [1]). *Suppose that* $a, b, c \in \mathbf{R}$, $c \notin \mathbf{Z}_{\leq 0}$ *and* $c > a+b$. *Then*

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

We denote by $\Gamma(s)$ the Gamma function. This can be proven by evaluation of Euler's integral using the Euler Beta function. These facts mentioned above can see some articles [1], [2], [3], [8], [9] and [10].

We have a concern for some values of the hypergeometric function. M. Kontsevich and D. Zagier introduce *periods* in [4].

DEFINITION 1.2 (see [4, p772]). A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbf{R}^n$ given by polynomial inequalities with rational coefficients.

They explain a relation between *periods* and linear differential equations as follows:

> By definition, periods are the values of integrals of algebraically defined differential forms over certain chains in algebraic varieties. If these forms and chains depend on parameters, then the integrals, considered as functions of the parameters, typically satisfy linear differential equations with algebraic coefficients. The periods then appear as special values of the solutions of these differential equations at algebraic arguments. This leads to a fascinating and very productive interplay between the study of periods and theory of linear differential equations (see [4, p778]).

The hypergeometric function satisfies a differential equation (1.1). After giving the explanation above, they introduce some values of the hypergeometric function. The following are contents of [5]. Article [5] has been treated in [4] again. We can see some values of the hypergeometric function in this here.

PROPOSITION 1.3 (F. Beukers and J. Wolfart, [5, Theorem 3]).

$$F\left(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{1323}{1331}\right)=\frac{3}{4}\sqrt[4]{11},\quad F\left(\frac{1}{12},\frac{7}{12},\frac{2}{3};\frac{64000}{64009}\right)=\frac{2}{3}\sqrt[6]{253}.$$

PROPOSITION 1.4 (F. Beukers and J. Wolfart, [5, Theorem 4]).

$$F\left(1-3a,3a,a;\frac{1}{2}\right)=2^{2-3a}\cos \pi a,\quad F\left(2a,1-4a,1-a;\frac{1}{2}\right)=4^{a}\cos \pi a.$$

$$F\left(\frac{7}{48},\frac{31}{48},\frac{29}{24};-\frac{1}{3}\right)=2^{5/24}3^{-11/12}5\sqrt{\frac{\sin\frac{\pi}{24}}{\sin\frac{5\pi}{24}}},$$

$$F\left(-\frac{7}{6},-\frac{2}{3},\frac{1}{18};\frac{1}{9}\right)=2^{29/2}3^{-7/6}\sin\frac{4\pi}{9}.$$

In this paper, we focus on some values of the Beta function and the hypergeometric function. We have two equalities in the following.

PROPOSITION 1.5 (complete elliptic integral, see [8] or [9]). *We denote by $K(k)$ the complete elliptic integral of the first kind. We denote by $E(k)$ the complete elliptic integral of the second kind. By using the hypergeometric function and the Euler integral, $K(k)$ and $E(k)$ are given by*

$$K(k):=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2},1;k^2\right),$$

$$E(k):=\int_0^1\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2},1;k^2\right).$$

Especially for the complete elliptic integral of the second kind $E(k)$, for an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\quad (a>b>0,a,b\in\mathbf{Q}^+), \tag{1.2}$$

we denote by $L_{a,b}$ the whole length. We can get $L_{a,b} = 4aE(k)$, $(k^2 = 1 - (b/a)^2)$. We treat $L_{a,b}$ as follows. We denote by $B(p,q)$ the Beta function.

$$(1.3) \quad L_{a,b} = 4aE(k) = 4a\int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = 2\pi aF\left(-\frac{1}{2},\frac{1}{2},1;1-\left(\frac{b}{a}\right)^2\right)$$

$$= 2aB\left(\frac{1}{2},\frac{1}{2}\right)F\left(-\frac{1}{2},\frac{1}{2},1;1-\left(\frac{b}{a}\right)^2\right).$$

If $a = b$, then equality (1.2) changes the form from the ellipse to a circle $x^2 + y^2 = a^2$ and we get the whole length

$$(1.4) \qquad L_{a,a} = 2\pi a = 2aB\left(\frac{1}{2},\frac{1}{2}\right).$$

Equality (1.2) includes a circle $x^2 + y^2 = a^2$ for *a special case*. Equality (1.3) includes equality (1.4) for *a special case*. Besides, from three equalities (1.2) (1.3) and (1.4), we can see a matching relation between the figure and the length of the ellipse (1.2). This is a viewpoint of the complete elliptic integral of the second kind. For the viewpoint, we obtain a similar model and an expansion for the complete elliptic integral of the second kind as follows. This is an our result.

THEOREM 1.6 (main theorem, [16, Theorem 4.2]). *Let $c > 0$ be a constant in* **R**. *For the following plane algebraic curve, which depends on m and n in* **N** *such that*

$$(1.5) \qquad (x^2 + y^2)^m = 2c^n g_n(x,y),$$

the polar form is given by

$$(1.6) \qquad r^{2m-n} = 2c^n \cos n\theta,$$

where $g_n(x,y)$ be functions satisfying the following equality

$$g_n(r\cos\theta, r\sin\theta) = r^n \cos n\theta.$$

For m and n in **N**, *if*

$$l := 2m - n > 0 \quad \text{and} \quad \frac{n}{2} < m < \frac{n}{2}(1+\sqrt{2}),$$

then the whole length $L_{m,n}$ of the plane algebraic curves (1.5) *are given by*

$$L_{m,n} = \sqrt[l]{2c^n}\frac{n}{l}B\left(\frac{1}{2l},\frac{1}{2}\right)F\left(-\frac{1}{2},\frac{1}{2l},\frac{1}{2l}+\frac{1}{2};1-\left(\frac{l}{n}\right)^2\right), \tag{1.7}$$

$$\textit{where } l := 2m - n, \quad (m, n \in \mathbf{N}).$$

Especially if $m = n$, then we can easily see $l = m = n$ and the whole length $L_{n,n}$ are given by

$$L_{n,n} = \sqrt[n]{2}cB\left(\frac{1}{2n},\frac{1}{2}\right). \tag{1.8}$$

We denote by $B(p,q)$ the Beta function. $F(\alpha,\beta,\gamma;x)$ denotes Gauss's hypergeometric function.

Remark 1.7. For Theorem 1.6 above, comparing with three equalities (1.2) (1.3) and (1.4), we obtain a similar model and an expansion for the complete elliptic integral of the second kind. If $l = 1$, then $L_{m,n}$ is the complete elliptic integral of the second kind. Besides, from Theorem 1.6, we obtain a function, which a product formula, defined by the beta function and the hypergeometric function, give the length of the plane algebraic curve (1.5). Considering the product formula (1.7), for the length of the plane algebraic curve (1.5), the value

$$B\left(\frac{1}{2l},\frac{1}{2}\right)$$

determine a fundamental measure of the length and the value

$$F\left(-\frac{1}{2},\frac{1}{2l},\frac{1}{2l}+\frac{1}{2};1-\left(\frac{l}{n}\right)^2\right)$$

determine a rate of the transition of the length.

Remark 1.8. By Schneider [6, 7], we have two following facts.

(1) For all rational number, non-integer a, b, the beta function

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

is a transcendental number.

(2) For an ellipse (1.2), we have the whole length $L_{a,b} = 4aE(k)$, $(k^2 = 1-(b/a)^2)$, where $E(k)$ denotes the complete elliptic integral of the

second kind. If a and b be positive algebraic numbers, then $L_{a,b}$ is a transcendental number.

From these mentioned facts, Theorem 1.6 includes some transcendental numbers.

From these mentioned facts, we think that the present paper is giving a contribution for a article [4, p778–p781].

Here is a short description of what to expect in this paper. First, we give a direct proof of Theorem 1.6 (main theorem). Second, explaining the special case $m = n$, we show *the background* of the main theorem. The plane algebraic curve (1.5) includes the circle and the lemniscate for *a special case*. Last, we would like to exhibit some figures and examples of the plane algebraic curve (1.5). From these figures, we can again recognize a similar model and an expansion for the complete elliptic integral of the second kind.

2. A Proof of Theorem

Since $x = r\cos\theta$ and $y = r\sin\theta$, therefore the line element for the polar form is given by

$$dL = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2\,d\theta^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

It is easy to see that the polar form of the plane algebraic curve (1.5) are given by (1.6) such that

$$r^{2m-n} = 2c^n \cos n\theta. \tag{1.6}$$

From the equality (1.6) above, we have

$$\frac{dr}{d\theta} = -\frac{2c^n n \sin n\theta}{lr^{l-1}}, \quad \text{where } l := 2m - n.$$

From the facts mentioned above

$$\begin{aligned} dL &= \sqrt{r^2 + \left(\frac{2c^n n \sin n\theta}{lr^{l-1}}\right)^2}\,d\theta = \sqrt{r^2 + \frac{4c^{2n}n^2\sin^2 n\theta}{l^2 r^{2l-2}}}\,d\theta \\ &= r\sqrt{1 + \frac{4c^{2n}n^2\sin^2 n\theta}{l^2 4c^{2n}\cos^2 n\theta}}\,d\theta = r\sqrt{1 + \frac{n^2}{l^2}\tan^2 n\theta}\,d\theta \\ &= \sqrt[l]{2c^n}\cos^{1/l} n\theta\sqrt{1 + \frac{n^2}{l^2}\tan^2 n\theta}\,d\theta, \end{aligned}$$

therefore we obtain

$$dL = \sqrt[l]{2c^n} \cos^{1/l} n\theta \sqrt{1 + \frac{n^2}{l^2} \tan^2 n\theta} \, d\theta.$$

The interval of θ is $-\pi \leq \theta < \pi$. However, from equality (1.6), $r \geq 0$ and $c > 0$, hence it is enough to consider in a interval

$$-\frac{\pi}{2n} \leq \theta \leq \frac{\pi}{2n}.$$

Therefore (pay attention to times n), the whole length $L_{m,n}$ of the plane algebraic curve (1.6) are given by

$$L_{m,n} = \sqrt[l]{2c^n} n \int_{-\pi/2n}^{\pi/2n} \cos^{1/l} n\theta \sqrt{1 + \frac{n^2}{l^2} \tan^2 n\theta} \, d\theta.$$

From the equality above, change of variables $n\theta = X$, and after some computing we get

$$\begin{aligned} L_{m,n} &= \sqrt[l]{2c^n} \int_{-\pi/2}^{\pi/2} \cos^{1/l} X \sqrt{1 + \frac{n^2}{l^2} \tan^2 X} \, dX \\ &= \sqrt[l]{2c^n} \frac{n}{l} 2 \int_0^{\pi/2} \cos^{1/l} X \sqrt{\tan^2 X + \left(\frac{l}{n}\right)^2} \, dX. \end{aligned}$$

We use the following computations:

For the integral

$$\int_0^{\pi/2} \cos^{1/l} X \sqrt{\tan^2 X + \left(\frac{l}{n}\right)^2} \, dX,$$

change of variables $x = 1/\cos^2 X (= 1 + \tan^2 X)$,

$$\begin{aligned} dx &= \frac{2 \cos X \sin X}{\cos^4 X} \, dX = \frac{2 \sin X}{\cos^3 X} dX = \frac{2\sqrt{1 - \cos^2 X}}{\cos^3 X} \, dX \\ &= 2x\sqrt{x}\sqrt{1 - \frac{1}{x}} \, dX = 2x\sqrt{x - 1} \, dX, \end{aligned}$$

hence

$$dX = \frac{dx}{2x\sqrt{x - 1}};$$

The other parts are

$$\cos^2 X = \frac{1}{x}, \quad \cos X = \frac{1}{\sqrt{x}}, \quad \cos^{1/l} X = (\cos X)^{1/l} = x^{-1/2l},$$

$$\tan^2 X = x - 1;$$

By using these calculations above, we get

$$\int_0^{\pi/2} \cos^{1/l} X \sqrt{\tan^2 X + \left(\frac{l}{n}\right)^2}\, dX = \int_1^{\infty} \left(\frac{1}{\sqrt{x}}\right)^{1/l} \sqrt{x - 1 + \left(\frac{l}{n}\right)^2}\, \frac{1}{2x\sqrt{x-1}}\, dx$$

$$= \frac{1}{2}\int_1^{\infty} x^{-1/2l-1}(x-1)^{-1/2}(x-a)^{1/2}\, dx, \quad \text{where } a := 1 - \left(\frac{l}{n}\right)^2.$$

Therefore, we obtain

$$L_{m,n} = \sqrt[l]{2c^n}\,\frac{n}{l}\int_1^{\infty} x^{-1/2l-1}(x-1)^{-1/2}(x-a)^{1/2}\, dx, \quad \text{where } a := 1 - \left(\frac{l}{n}\right)^2.$$

For the equality above, change of variables $t = 1/x$, then

$$dt = \frac{-1}{x^2}\, dx = -t^2\, dx,$$

$$dx = -\frac{1}{t^2}\, dt.$$

Compute the length $L_{m,n}$, then

$$L_{m,n} = \sqrt[l]{2c^n}\,\frac{n}{l}\int_1^{0} t^{1/2l+1}\left(\frac{1}{t} - 1\right)^{-1/2}\left(\frac{1}{t} - a\right)^{1/2}\left(-\frac{1}{t^2}\right) dt$$

$$= \sqrt[l]{2c^n}\,\frac{n}{l}\int_0^{1} t^{1/2l+1}\left(\frac{1-t}{t}\right)^{-1/2}\left(\frac{1-at}{t}\right)^{1/2}\left(\frac{1}{t^2}\right) dt$$

$$= \sqrt[l]{2c^n}\,\frac{n}{l}\int_0^{1} t^{1/2l-1}(1-t)^{-1/2}(1-at)^{1/2}\, dt,$$

therefore we get

$$L_{m,n} = \sqrt[l]{2c^n}\,\frac{n}{l}\int_0^{1} t^{1/2l-1}(1-t)^{-1/2}(1-at)^{1/2}\, dt, \quad \text{where } a := 1 - \left(\frac{l}{n}\right)^2. \tag{2.1}$$

We have the following well-known fact, which is the Euler's integral representation of the Gauss's hypergeometric function (see [10, p51])

$$F(\alpha,\beta,\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}\,dt, \tag{2.2}$$

where for its convergence,

$$\mathrm{Re}(\alpha)>0,\quad \mathrm{Re}(\gamma-\alpha)>0,\quad \text{and}\quad |x|<1. \tag{2.3}$$

Comparing equality (2.1) with equality (2.2), let

$$\alpha:=\frac{1}{2l},\quad \gamma:=\frac{1}{2l}+\frac{1}{2},\quad \beta:=-\frac{1}{2},\quad \text{and}\quad x:=1-\left(\frac{l}{n}\right)^2,$$

we obtain the whole length $L_{m,n}$ of the plane algebraic curve (1.6) as the following form.

$$L_{m,n}=\sqrt[l]{2c^n}\,\frac{n}{l}\,\frac{\Gamma\left(\frac{1}{2l}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2l}+\frac{1}{2}\right)}F\left(\frac{1}{2l},-\frac{1}{2},\frac{1}{2l}+\frac{1}{2};1-\left(\frac{l}{n}\right)^2\right).$$

By using the following well-known two equalities

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\quad F(\alpha,\beta,\gamma;x)=F(\beta,\alpha,\gamma;x),$$

we obtain a equality (1.7). The assumption of this theorem

$$l:=2m-n>0\quad\text{and}\quad \frac{n}{2}<m<\frac{n}{2}(1+\sqrt{2})$$

depends on a convergence of the equality (2.2), in short the assumption depends on the relations (2.3). □

3. Background

From the polar form (1.6), we give a proof of Theorem 1.6. *The plane algebraic curve* (1.5) *includes the circle and the lemniscate for a special case.* In fact, the plane algebraic curve (1.5) is an expansion of the polar form for the plane algebraic curve (3.7). We can see some similarities of the circular function and the lemniscate function [11], [12], [13], and [14]. Considering based on the similarities, we get a plane algebraic curve (3.7), which depends on n in $\mathbf{N}$. The plane algebraic curve (3.7) is a special case for Theorem 1.6 and its includes the circle and the lemniscate.

The following is the contents of [14, 15, 16].

First, we recall the construction method of the plane algebraic curve the unit circle $x^2 + y^2 = 1$ and the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$. After that, we exhibit a construction method of the plane algebraic curve (3.7), which depends on n in $\mathbf{N}$.

3.1. A construction method of the plane algebraic curves. The construction of the unit circle $x^2 + y^2 = 1$ is the following.

Put $P := (x, y)$ in $\mathbf{R}^2$. Let $F := (a, b)$ be a fixed point in $\mathbf{R}^2$, and $c > 0$ be a constant in $\mathbf{R}$. For P, F, and c, consider the plane algebraic curve which satisfy the following condition such that

$$|PF| = c, \quad \left(|PF| := \sqrt{(x-a)^2 + (y-b)^2}\right). \tag{3.1}$$

The unit circle $x^2 + y^2 = 1$ is a special case of the condition (3.1). For condition (3.1), if $F = (0, 0)$ and $c = 1$, then the plane algebraic curve is the unit circle $x^2 + y^2 = 1$.

On the other hand, the construction of the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$ is the following.

Put $P := (x, y)$ in $\mathbf{R}^2$. Let $F_1 := (a_1, b_1)$ and $F_2 := (a_2, b_2)$ $(F_1 \neq F_2)$ be two fixed points in $\mathbf{R}^2$, and $c > 0$ be a constant in $\mathbf{R}$. For P, F_1, F_2, and c, consider the plane algebraic curve which satisfy the following condition such that

$$|PF_1|\,|PF_2| = c, \quad \left(|PF_i| := \sqrt{(x-a_i)^2 + (y-b_i)^2} \quad (i = 1, 2)\right). \tag{3.2}$$

The lemniscate $(x^2 + y^2)^2 = x^2 - y^2$ is a special case of the condition (3.2). If $F_1 = (-1/\sqrt{2}, 0)$, $F_2 = (1/\sqrt{2}, 0)$ and $c = 1/2$, then the plane algebraic curve is the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$.

Based on the condition (3.1) and (3.2) of the plane algebraic curve mentioned above, we give a generalization of the condition.

Put $P := (x, y)$ in $\mathbf{R}^2$. Let $F_i := (a_i, b_i)$ $(i = 1, 2, \ldots, n.\ i \neq j, F_i \neq F_j)$ be some fixed points in $\mathbf{R}^2$, and $c > 0$ be a constant in $\mathbf{R}$. For P, F_i, and c, consider the plane algebraic curve which satisfy the following condition such that

$$\begin{aligned} &|PF_1|\,|PF_2| \cdots |PF_n| = c, \\ &\left(|PF_i| := \sqrt{(x-a_i)^2 + (y-b_i)^2}, \quad (i = 1, 2, \ldots, n)\right). \end{aligned} \tag{3.3}$$

Furthermore, in addition to assume the fact which F_n depends on n in $\mathbf{N}$, we consider the following plane algebraic curve which depends on n in $\mathbf{N}$. We denote by $f_n(x, y) = 0$ the plane algebraic curve.

$$(3.4) \qquad f_n(x, y) = 0 \begin{cases} f_1(x, y) = 0 \Leftrightarrow |PF_1| = c, \\ f_2(x, y) = 0 \Leftrightarrow |PF_1|\,|PF_2| = c, \\ \cdots, \\ f_k(x, y) = 0 \Leftrightarrow |PF_1|\,|PF_2| \cdots |PF_k| = c, \\ \cdots. \end{cases}$$

This is a fundamental construction method of the plane algebraic curve which depends on n in $\mathbf{N}$.

Based on the construction method above (3.4), we consider the plane algebraic curve which for all fixed points depends on n in $\mathbf{N}$ as follows.

Put $P := (x, y)$ in $\mathbf{R}^2$. Let $F_{ij} := (a_{ij}, b_{ij})$ be some fixed points in $\mathbf{R}^2$ which depend on i, j in $\mathbf{N}$ $(1 \le i \le n, 1 \le j \le n)$. Prepare a function $\phi : \mathbf{N} \to \mathbf{R}$. We denote by $f_n(x, y) = 0$ the plane algebraic curve. The plane algebraic curve $f_n(x, y) = 0$ satisfies the following condition.

$$(3.5) \qquad f_n(x, y) = 0 \begin{cases} f_1(x, y) = 0 \Leftrightarrow |PF_{11}| = \phi(1), \\ f_2(x, y) = 0 \Leftrightarrow |PF_{21}|\,|PF_{22}| = \phi(2), \\ f_3(x, y) = 0 \Leftrightarrow |PF_{31}|\,|PF_{32}|\,|PF_{33}| = \phi(3), \\ \cdots, \\ f_k(x, y) = 0 \Leftrightarrow |PF_{k1}|\,|PF_{k2}|\,|PF_{k3}| \cdots |PF_{kk}| = \phi(k), \\ \cdots. \end{cases}$$

For the construction method (3.5) of the plane algebraic curve $f_n(x, y) = 0$, we give a sample which can be considered important for number theory as follows.

Let $c > 0$ be a constant in $\mathbf{R}$. By using c, we define F_{nk} and a function $\phi : \mathbf{N} \to \mathbf{R}$ as follows:

$$(3.6) \qquad \begin{cases} F_{nk} := \left(c \cos \dfrac{2k\pi}{n}, c \sin \dfrac{2k\pi}{n}\right), \quad (k = 1, 2, \ldots, n, n \in \mathbf{N}); \\ \phi(n) := c^n. \end{cases}$$

This is the plane algebraic curve (3.7) of Theorem 3.1, which is a special case for Theorem 1.6. After some computing, we get the plane algebraic curve (3.7).

THEOREM 3.1 (see [14, Theorem 4.1], [16, Theorem 4.1]). *Let $c > 0$ be a constant in* **R**. *For the following plane algebraic curve $f_n(x, y) = 0$ which depends on n in* **N** *such that*

$$(3.7)\qquad \prod_{k=1}^{n}\left(x^2 + y^2 - 2c\left(x\cos\frac{2k\pi}{n} + y\sin\frac{2k\pi}{n}\right) + c^2\right) - c^{2n} = 0,$$

the whole length L_n of the plane algebraic curve $f_n(x, y) = 0$ are given by

$$L_n = \sqrt[n]{2}cB\left(\frac{1}{2n}, \frac{1}{2}\right).$$

We denote by $B(p, q)$ the Beta function such that

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad (p, q > 0).$$

3.2. Proof of Theorem 3.1 [14, 16]. There are two steps in this proof. The first step includes some important meanings.

The first step. Considering the plane algebraic curve (3.7) over **C**, in other words, replacing $P := (x, y)$ in $\mathbf{R}^2$ with $z := x + iy$ in **C**, and rewriting the construction method of the plane algebraic curve (3.5), (3.6), we obtain a polar form and a polynomial form for a complex variable z of the plane algebraic curve (3.7).

F_{nk} has changed

$$\alpha_{nk} := c\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right), \quad (k = 1, 2, \ldots, n, n \in \mathbf{N}).$$

For a complex variable $z := x + iy$, we have

$$|z| := \sqrt{x^2 + y^2}, \quad |z_1 z_2| = |z_1|\,|z_2|.$$

By using three equalities above, the condition (3.5) and (3.6) have just changed

$$|z - \alpha_{n1}|\,|z - \alpha_{n2}|\,|z - \alpha_{n3}| \cdots |z - \alpha_{nn}| = c^n.$$

Hence

$$|(z - \alpha_{n1})(z - \alpha_{n2})(z - \alpha_{n3}) \cdots (z - \alpha_{nn})| = c^n.$$

In fact α_{nk} is a root of the algebraic equation $z^n - c^n = 0$. Therefore, we obtain

$$|z^n - c^n| = c^n. \tag{3.8}$$

This is a polynomial form for a complex variable z of the plane algebraic curve (3.7).

Let r and θ be polar coordinates of $z := x + iy$. Since $x = r \cos \theta$, and $y = r \sin \theta$, z can be written in polar form and Euler's formula as

$$z = r(\cos \theta + i \sin \theta) = re^{i\theta}.$$

By using equality above and de Moivre's formula $(e^{i\theta})^n = e^{in\theta}$, equality (3.8) can be written

$$r^n = 2c^n \cos n\theta. \tag{3.9}$$

This is a polar form of the plane algebraic curve (3.7) of Theorem 3.1 and a special case for equality (1.6) of Theorem 1.6.

The second step. Since $x = r \cos \theta$ and $y = r \sin \theta$, therefore the line element for the polar form is given by

$$dL = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2\, d\theta^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

From equality (3.9), we have

$$\frac{dr}{d\theta} = -\frac{2c^n \sin n\theta}{r^{n-1}}.$$

By using two equalities above,

$$\begin{aligned} dL &= \sqrt{r^2 + \left(\frac{2c^n \sin n\theta}{r^{n-1}}\right)^2}\, d\theta = \sqrt{r^2 + \frac{4c^{2n} \sin^2 n\theta}{r^{2n-2}}}\, d\theta \\ &= \sqrt{\frac{r^{2n} + 4c^{2n} \sin^2 n\theta}{r^{2n-2}}}\, d\theta = \sqrt{\frac{4c^{2n} \cos^2 n\theta + 4c^{2n} \sin^2 n\theta}{r^{2n-2}}}\, d\theta \\ &= \sqrt{\frac{4c^{2n} r^2}{r^{2n}}}\, d\theta = \sqrt{\frac{4c^{2n} r^2}{4c^{2n} \cos^2 n\theta}}\, d\theta = \frac{r}{\cos n\theta}\, d\theta = \sqrt[n]{2}c \cos^{1/n-1} n\theta\, d\theta. \end{aligned}$$

Therefore, we get

$$dL = \sqrt[n]{2}c \cos^{1/n-1} n\theta\, d\theta. \tag{3.10}$$

The interval of θ is $-\pi \le \theta < \pi$. However, from equality (3.9), $r \ge 0$ and $c > 0$, it is enough to consider in a interval

$$-\frac{\pi}{2n} \le \theta \le \frac{\pi}{2n}.$$

Therefore (pay attention to times n), the whole length L_n of the plane algebraic curve (3.7) are given by

$$L_n = \sqrt[n]{2}cn \int_{-\pi/2n}^{\pi/2n} \cos^{1/n-1} n\theta \, d\theta.$$

From the equalities above, change of variables $n\theta = X$, and after some computing we get

$$L_n = \sqrt[n]{2}c2 \int_{0}^{\pi/2} \cos^{1/n-1} X \, dX. \tag{3.11}$$

For the equality (3.11), applying the properties of the Beta-function:

(definition) $\quad B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}\, dx \quad (p, q > 0);$

(property 1) $\quad B(p,q) = B(q,p);$

(property 2) $\quad \int_0^{\pi/2} \sin^a \theta \cos^b \theta \, d\theta = \frac{1}{2} B\left(\frac{a+1}{2}, \frac{b+1}{2}\right),$

we obtain the whole length L_n as the following form.

$$L_n = \sqrt[n]{2}cB\left(\frac{1}{2}, \frac{1}{2n}\right) = \sqrt[n]{2}cB\left(\frac{1}{2n}, \frac{1}{2}\right). \qquad \square$$

3.3. Setting a problem, the answer, and some related articles. From the proof of Theorem 3.1, especially for the step 1, we can easily see three forms of the plane algebraic curve (3.7). From the form (3.8), which is an expression of the absolute value of the complex variable polynomial, we think Theorem 3.1 to be a special case for two articles [17], [18].

For the plane algebraic curve (3.7), from some calculations, figures, and the polar form (3.9), we obtain a problem. The following are the calculations for the plane algebraic curve (3.7).

n	$f_n(x,y)=0$
1	$x^2+y^2-2cx=0$
2	$(x^2+y^2)^2-2c^2(x^2-y^2)=0$
3	$(x^2+y^2)^3-2c^3(x^3-3xy^2)=0$
4	$(x^2+y^2)^4-2c^4(x^4-6x^2y^2+y^4)=0$
5	$(x^2+y^2)^5-2c^5(x^5-10x^3y^2+5xy^4)=0$
6	$(x^2+y^2)^6-2c^6(x^6-15x^4y^2+15x^2y^4-y^6)=0$

For n in $\mathbf{N}$, Let $g_n(x,y)$ be functions satisfying the following equality

$$g_n(r\cos\theta, r\sin\theta) = r^n \cos n\theta.$$

n	$g_n(x,y)$
1	x
2	x^2-y^2
3	x^3-3xy^2
4	$x^4-6x^2y^2+y^4$
5	$x^5-10x^3y^2+5xy^4$
6	$x^6-15x^4y^2+15x^2y^4-y^6$

Let $c>0$ be a constant in $\mathbf{R}$. Based on the calculations above, we consider the length of the following plane algebraic curve, which depends on m and n in $\mathbf{N}$ such that

$$(x^2+y^2)^m = 2c^n g_n(x,y). \tag{3.12}$$

Can we give the length of the plane algebraic curve (3.12)?

Theorem 1.6 is the answer. Comparing two theorems Theorem 1.6 and Theorem 3.1, we can also see an expansion from a view point of the length. In other words, Theorem 1.6 includes Theorem 3.1 for a special case. We exhibit a connection between some plane algebraic curves and the length. *Through Theorem* 1.6 *and Theorem* 3.1, *we obtain a rule, which the Beta function and Gauss's hypergeometric function gives the length for the plane algebraic curve* (1.6) *and* (3.7). *Besides, from the figures and a view point some equalities* (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), *and* (1.8), *we obtain a similar model and an expansion for the complete elliptic integral of the second kind.*

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Figures

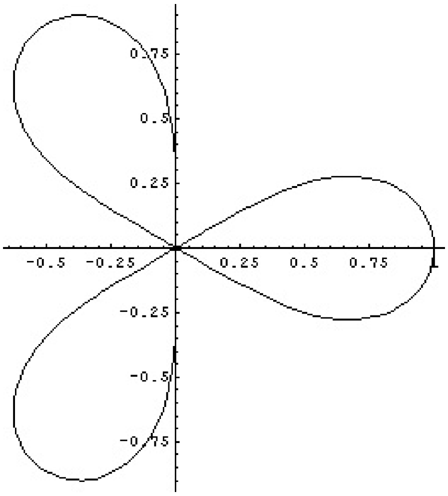

Figure 1: $(x^2+y^2)^3-(x^3-3xy^2)=0$.

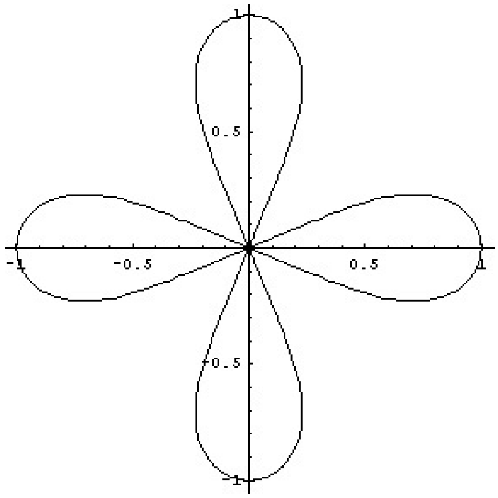

Figure 2: $(x^2+y^2)^4-(x^4-6x^2y^2+y^4)=0$.

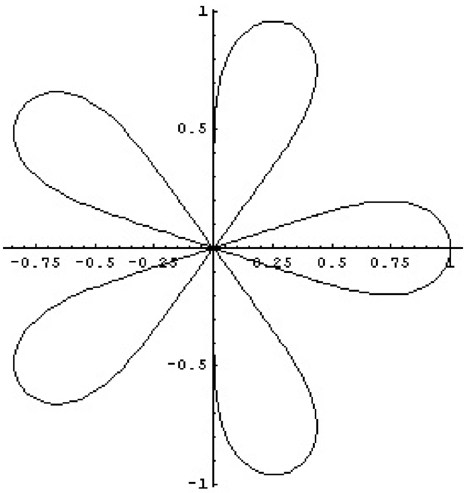

Figure 3: $(x^2+y^2)^5-(x^5-10x^3y^2+5xy^4)=0$.

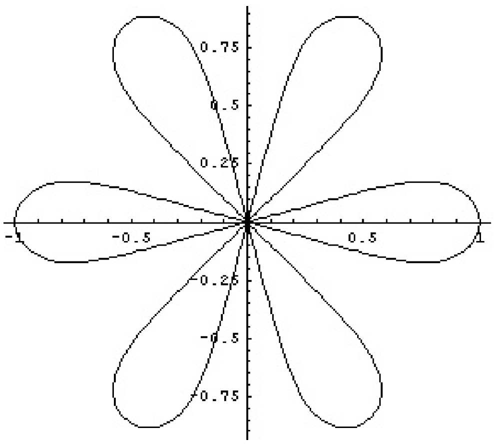

Figure 4: $(x^2+y^2)^6-(x^6-15x^4y^2+15x^2y^4-y^6)=0$.

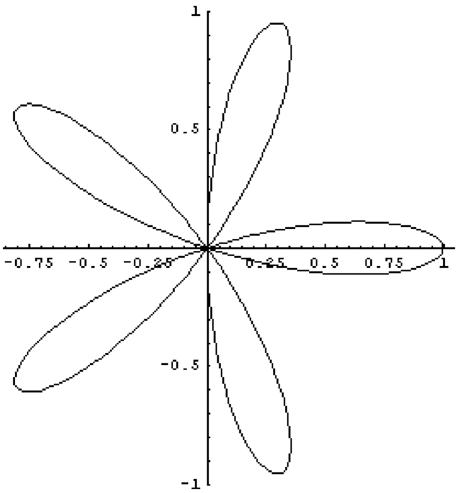

Figure 5: $(x^2 + y^2)^3 - (x^5 - 10x^3y^2 + 5xy^4) = 0$.

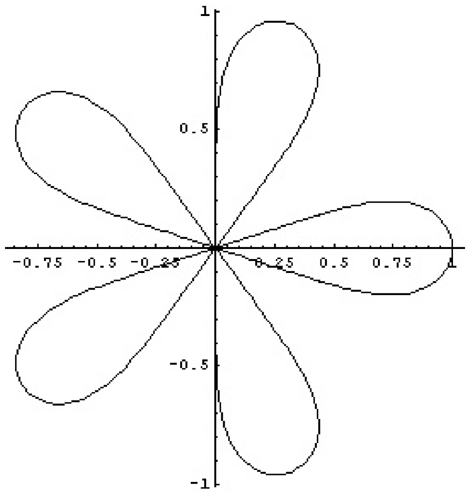

Figure 6: $(x^2 + y^2)^6 - (x^5 - 10x^3y^2 + 5xy^4) = 0$.

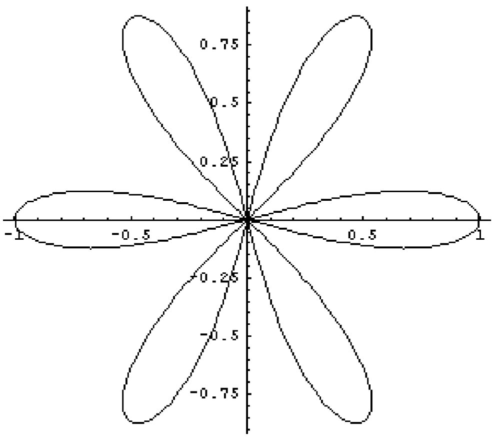

Figure 7: $(x^2 + y^2)^4 - (x^6 - 15x^4y^2 + 15x^2y^4 - y^6) = 0$.

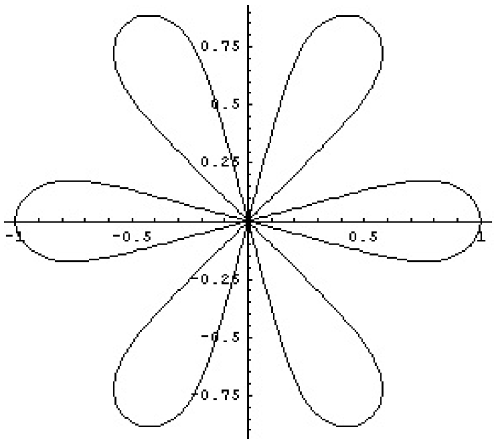

Figure 8: $(x^2 + y^2)^7 - (x^6 - 15x^4y^2 + 15x^2y^4 - y^6) = 0$.

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$$1 + \frac{\alpha\beta}{1 \cdot \gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)}x^3 + etc.,$$

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