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Influence of the density pole on the performances of its gamma-kernel estimator

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Abstract. In this paper, we aim at highlighting the influence of the density pole on the performances of its gamma-kernel estimator. To do this, we performed a comparative study for the performances of the gamma-kernel estimators with those provided by other bias effect correction techniques at the bounds, using the simulation technique. In conclusion, the results obtained confirm those provided in the literature and show that in some cases the normalization of the gamma estimators can considerably improve local and global performances of the gamma-kernel estimators.

Résumé. Dans ce papier, afin de mettre en évidence l'influence du pôle d'une densité sur les performances de son estimateur à noyau gamma, nous avons réalisé une étude comparative des performances des estimateurs à noyau gamma avec ceux fournis par d'autres techniques de correction du biais aux bornes, en utilisant la technique de simulation. Les résultats obtenus montrent que, dans certains cas, la normalisation des noyaux gamma peut améliorer considérablement les performances locales et globales des estimateurs à noyau gamma.

Key words: Shoulder condition; Pole density; Normalization; Performances comparison.
AMS 2010 Mathematics Subject Classification: 62G07, 62G20.

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1. Introduction

Let $X_1,\ X_2,\ldots,\ X_n$ be a random sample from a distribution with an unknown probability density function f defined on $[0,\ \infty[$ and it is twice continuously differentiable ($f \in \mathcal{C}^2([0,\infty[)$). It is well known that the using of the standard (symmetric) kernel in the nonparametric estimation of the density f, may lead to non-consistency estimators at the boundary. One of the old techniques introduced to remedy the problem of non-consistency is called the kernel method "cut-and-normalized" (see Gasser and Müller (1979)). Nevertheless, the bias of the cut-and-normalized kernel density estimator still converges to zero at the slower rate $O(h)$ compared to the usual rate $O(h^2)$ and there is still more bias at the boundary compared to the interior of the support. For this reason, numerous other techniques are proposed in the literature, with the aim at obtaining the same bias order over the whole support. We can cite, for instance, the data reflection methods Karunamuni and Zhang (2008); Schuster (1985), the generalized Jackknife Kyung-Joon and Schucany (1998); Jones (1993); Jones and Foster (1996); Jones *et al.* (1999), the data transformation Hall and Park (2002) and the using of the asymmetric kernels (constructed from gamma distribution) Chen (2000).

Moreover, Chen (2000) proposed to replace the standard kernel estimator given by Parzen-Rosenblatt (see Parzen (1962) and Rosenblatt (1956)) by:

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n} K_{(\rho_h(x),h)}(X_i), \tag{1}$$

where, $h = h(n)$ is the smoothing parameter satisfying $h \to 0$ and $nh \to \infty$ as $n \to \infty$ and $K_{(\rho_h(x),h)}$ is the density function of the gamma distribution with parameters $(\rho_h(x), h)$, given by the following formula:

$$K_{(\rho_h(x),h)}(t) = \frac{t^{\rho_h(x)-1}e^{-t/h}}{h^{\rho_h(x)}\Gamma(\rho_h(x))}, \tag{2}$$

with,

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx, \quad p > 0.$$

The first version of the gamma kernel density estimator, denoted $\hat{f}_1(x)$, is obtained by replacing $\rho_h(x)$ by $x/h + 1$, in formula (1). Because of the undesired involvement of f' (the first derivative of f) in the bias of $\hat{f}_1(x)$, a second version of $\hat{f}_1(x)$, called a modified gamma kernel estimator (denoted $\hat{f}_2(x)$), is proposed by Chen (2000). Indeed, this version is obtained by replacing $\rho_h(x)$, in formula (1), by the following quantity:

$$\begin{aligned} \rho_h(x) &= (x/h)\mathbb{I}_{\{x\geq 2h\}} + \left(\frac{1}{4}(x/h)^2 + 1\right)\mathbb{I}_{\{x\in[0,2h[\}} \\ &= \begin{cases} x/h, & \text{si } x \geq 2h, \\ \frac{1}{4}(x/h)^2 + 1, & \text{si } x \in [0, 2h[. \end{cases} \end{aligned} \tag{3}$$

The crucial difference between the standard kernel and the gamma kernel is that the form and the amount of smoothing, of the latter kernel, vary according to the position where the density is estimated (an adaptive density estimator). The gamma kernel density estimator is easy to implement, free of boundary bias, always non-negative and achieves the optimal rate of convergence for the mean integrated squared error. Furthermore, it reduces the variance when the position (in smoothing case) moves away from the boundary. Numerous other proprieties of the gamma kernel estimators are well documented in the literature. Bouezmarni and Scaillet (2003) state the uniform weak consistency for the gamma kernel estimator on $[0, +\infty[$ as well as the weak convergence in $MIAE$ sense (Mean Integer Absolute Error). For the unbounded densities at the origin (in the neighborhood of zero), the same authors examined the performance of this estimator by simulation studies and proved the convergence in probability. Fernandez and Monteiro (2005) established the central limit theorem for the functional gamma kernel estimator.

However, a drawback is that the gamma kernels may be inefficient in some situations. For example, in the work of Zhang (2010), the author has studied the performances of the gamma kernel estimators at the boundary and have shown that, in the boundary region, the gamma kernel estimator even outperforms some widely used boundary corrected density estimators such as the boundary kernel estimator when the estimated density has a shoulder. For densities not satisfying the shoulder condition, the author has shown that the gamma kernel estimator has a severe boundary problem and its performance is inferior to that of the boundary kernel estimator. Recently, Cherfaoui *et al.* (2015) were interested in the properties of gamma kernel estimators for finite samples. By simulation studies, the authors analyzed some proprieties of the gamma kernel density estimators for the different variants of the gamma kernel. In particular, they showed the nonunit mass of the gamma (gamma and modified gamma) kernel estimator for the finite samples (i,e. $\int \hat{f}_i(x)dx \neq 1, i = 1, 2.$). To correct the problem the authors propose two ways to normalize the estimators $\hat{f}_1(x)$ and $\hat{f}_2(x)$, that is: a Macro-normalization, which consist to divide the estimators on its integral, and a Micro-Normalization which consist to divide, for any x, the estimators on its integral:

- Macro-normalization

$$\hat{f}_{M1}(x) = \frac{\hat{f}_1(x)}{\int_0^\infty \hat{f}_1(x)dx}, \tag{4}$$

  and

$$\hat{f}_{M2}(x) = \frac{\hat{f}_2(x)}{\int_0^\infty \hat{f}_2(x)dx}. \tag{5}$$

- Micro-Normalization

$$\hat{f}_{m1}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{(x/b+1,b)}(X_i)}{\int_0^\infty K_{(x/b+1,b)}(X_i)dx}, \tag{6}$$

  and

$$\hat{f}_{m2}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{K_{(\rho_h(x),h)}(X_i)}{\int_0^\infty K_{(\rho_h(x),h)}(X_i)dx}. \tag{7}$$

Moreover, in order to predict the behavior of the two proposed normalization methods, Cherfaoui *et al.* (2015) carried out a comparative study on finite artificial samples having different size and they illustrated the necessity of the normalization through an application example using Markov chains.

In the present work, we propose to study the effect and the performances of the normalization techniques, given in Cherfaoui *et al.* (2015), on the quality of the gamma kernel estimators at the boundary (at the point $x = 0$), according to the type of the density to estimate (with or without pole), compared to other existing boundary corrected kernel estimators. Indeed, we give, in first time, some local proprieties of the gamma kernel estimators at the point $x = 0$. In a second time a numerical comparative study, which is similar to that presented in Zhang (2010), is carried out where we compare the local and the global performances of the considered kernel estimators. Moreover, the obtained results (numerical and graphical) show that the normalized gamma kernel estimators works significantly better than the others estimators (the standard gamma kernel estimators, the boundary kernel estimators, the cut-and-normalized kernel estimator and Jone-Foster estimators) and this, independently of the type of the target density.
The rest of the document is organized as follows: in the second section we present the boundary characteristics of the gamma kernel estimators. A detailed analysis of these characteristics is presented in the third section. In the fourth section, we focalize on the presentation of the simulation study realized with the aim at comparing the boundary performances of the normalized gamma kernel estimators with those obtained via others boundary corrected kernel techniques. Discussions and concluding remarks on the obtained results will be presented in the last Section.

2. The boundary proprieties of the gamma kernel estimators

Let $X_1,\ X_2, ...,\ X_n$ be a random sample from a distribution with an unknown probability density function f defined on the positive support $[0,\ \infty[$ and twice twice-continuously differentiable function ($f \in \mathcal{C}^2([0, \infty[)$). By definition, the gamma kernel estimation of the f at the boundary ($x = 0$) is given by:

$$\hat{f}_1(0) = \hat{f}_2(0) = \hat{f}_G(0) = \frac{1}{nh}\sum_{i=1}^{n} e^{-X_i/h}, \tag{8}$$

where h is the smoothing parameter.

The aim of this section, is to provide the locals properties of the estimators $\hat{f}_G(x)$ at the point $x = 0$ for some target densities. To do this, we take the same examples treated in Zhang (2010), which are summarized in the choice of the following densities:

- PDF_1: $f(x) = e^{-x}$ (the exponential density),
- PDF_2: $f(x) = \frac{2}{\sqrt{(2\pi)}}e^{-\left(\frac{x^2}{2}\right)}$ (the half normal density),
- PDF_3: $f(x) = \left(\frac{1}{10} + \frac{9}{10}x\right)e^{-x}$ (the mixture exponential density).

The choice of the previous density is motivated by the fact that it represents the three possible situations of the pole (see Figure 1).

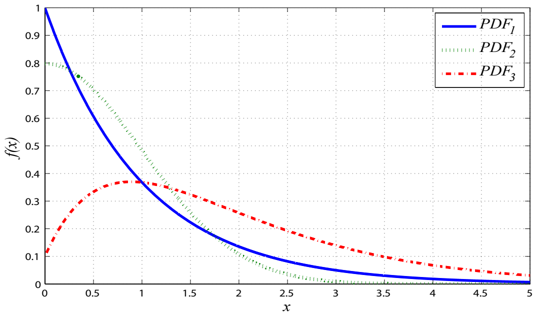

Fig. 1. Curves of the three target distributions PDF_i, $i = \overline{1,3}$

Now, we give the locals proprieties (mean and variance) of the estimator $\hat{f}_G(0)$ for an arbitrarily density.

$$
\begin{aligned}
E(\hat{f}_G(0)) &= E\left(\frac{1}{nh}\sum_{i=1}^{n} e^{-x_i/h}\right) = \frac{1}{nh}\sum_{i=1}^{n} E\left(e^{-X_i/h}\right) \\
&= \frac{1}{nh}\sum_{i=1}^{n} E\left(e^{-X/h}\right) = \frac{1}{h} E\left(e^{-X/h}\right). \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
Var(\hat{f}_G(0)) &= Var\left(\frac{1}{nh}\sum_{i=1}^{n} e^{-X_i/h}\right) = \frac{1}{(nh)^2}\sum_{i=1}^{n} Var\left(e^{-X_i/h}\right) \\
&= \frac{1}{(nh)^2}\sum_{i=1}^{n} Var\left(e^{-X/h}\right) = \frac{1}{nh^2} Var\left(e^{-X/h}\right) \\
&= \frac{1}{nh^2}\left[E\left(e^{-2X/h}\right) - E\left(e^{-X/h}\right)^2\right]. \qquad (10)
\end{aligned}
$$

In the following, we are interested in the local proprieties of the estimator $\hat{f}_G(0)$ for the chosen densities (PDF_i, $i = \overline{1,3}$). To do this, it will be enough to compute the two quantities $E\left(e^{-X/h}\right)$ and $E\left(e^{-2X/h}\right)$ for the considered densities and to replace them them in the formulas (9) and (10) by the found expressions.

2.1. The exponential distribution

Let f be an exponential density with parameter λ, defined by:

$$
f(x) = \frac{1}{\lambda} e^{-x/\lambda}, \; x \geq 0, \quad \lambda > 0.
$$

We gave:

$$\begin{aligned} E(e^{-X/h}) &= \int_0^{+\infty} e^{-x/h} f(x)dx \\ &= \frac{1}{\lambda}\int_0^{+\infty} e^{-((1/h)+(1/\lambda))x}dx \\ &= \frac{h}{\lambda+h}. \end{aligned}$$

By the same way as the above, we obtain:

$$E(e^{-2X/h}) = \frac{h}{2\lambda+h}.$$

From the formulas (9) and (10), we deduce the following:

$$E(\hat{f}_G(0)) = \frac{1}{\lambda+h}, \tag{11}$$

$$Var(\hat{f}_G(0)) = \frac{\lambda^2}{nh(h+2\lambda)(h+\lambda)^2}, \tag{12}$$

$$Bias(\hat{f}_G(0)) = E(\hat{f}_G(0)) - f(0) = \frac{-h}{\lambda(\lambda+h)}, \tag{13}$$

$$MSE(\hat{f}(0), f(0)) = Biais(\hat{f}_G(0))^2 + Var(\hat{f}_G(0)) = \frac{nh^4 + 2nh^3\lambda + \lambda^4}{nh\lambda^2(h+2\lambda)(h+\lambda)^2}. \tag{14}$$

2.2. The half normal distribution

Suppose that f is a half normal distribution, defined as follows:

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)},\ x \geq 0,$$

so, we can show that:

$$\begin{aligned} E(e^{-(X/h)}) &= e^{\frac{1}{(2h^2)}}\left(1 - \mathbf{erf}\left(\frac{\sqrt{2}}{2h}\right)\right), \\ E(e^{-2X/h}) &= e^{\frac{2}{h^2}}\left(1 - \mathbf{erf}\left(\frac{\sqrt{2}}{h}\right)\right), \end{aligned}$$

which leads to

$$
\begin{aligned}
E(\hat{f}_G(0)) &= \frac{e^{\frac{1}{(2h^2)}}\left(1-\mathbf{erf}\left(\frac{\sqrt{2}}{2h}\right)\right)}{h}, \qquad (15)\\
Var(\hat{f}_G(0)) &= \frac{1}{h^2 n}\left(e^{\frac{2}{h^2}}\left(1-\mathbf{erf}\left(\frac{\sqrt{2}}{h}\right)\right)-e^{\frac{1}{h^2}}\left(1-\mathbf{erf}\left(\frac{\sqrt{2}}{2h}\right)\right)^2\right), \qquad (16)\\
Bias(\hat{f}_G(0)) &= \frac{e^{\frac{1}{(2h^2)}}\left(1-\mathbf{erf}\left(\frac{\sqrt{2}}{2h}\right)\right)}{h}-\sqrt{\frac{2}{\pi}}, \qquad (17)\\
MSE(\hat{f}_G(0), f(0)) &= Bias(\hat{f}_G(0))^2 + Var(\hat{f}_G(0)), \qquad (18)
\end{aligned}
$$

where, $\mathbf{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = 2\Phi(\sqrt{2}x) - 1$, with $\Phi(.)$ is the standard normal probability distribution.

2.3. The mixture exponential density

Suppose now that f is a density function obtained by the mixing of two exponential densities, defined by:

$$f(x) = \left(\frac{1}{10} + \frac{9}{10}x\right)e^{-x},\ x \geq 0.$$

We have

$$
\begin{aligned}
E(e^{-(X/h)}) &= \frac{h(10h+1)}{10(h+1)^2},\\
E(e^{-2X/h}) &= \frac{h(10h+2)}{10(h+2)^2},
\end{aligned}
$$

and deduce the following characteristics:

$$
\begin{aligned}
E(\hat{f}_G(0)) &= \frac{(10h+1)}{10(h+1)^2}, \qquad (19)\\
Var(\hat{f}_G(0)) &= (199h^3 + 436h^2 + 176h + 20)/(100nh(h+1)^4(h+2)^2), \qquad (20)\\
Bias(\hat{f}_G(0)) &= \frac{-h(h-8)}{10(h+1)^2}, \qquad (21)
\end{aligned}
$$

$$MSE = \frac{20 + 176h + 436h^2 + (199 + 256n)h^3 + 192nh^4 + 4nh^5 - 12nh^6 + nh^7}{100nh(h+2)^2(h+1)^4}. \qquad (22)$$

3. Impact of the density pole on the local proprieties

In this section, we are interested in the analysis of the pole effect on the performances of gamma kernel estimators. We begin firstly to give an idea on the exponential density estimator characteristic, at the point $x = 0$ versus the parameters λ and h. Note that, the parameter λ indicates the type of the density pole in question (see Figure 2). After that, we present a comparative discussion of the local characteristics of the estimators associate to the above densities.

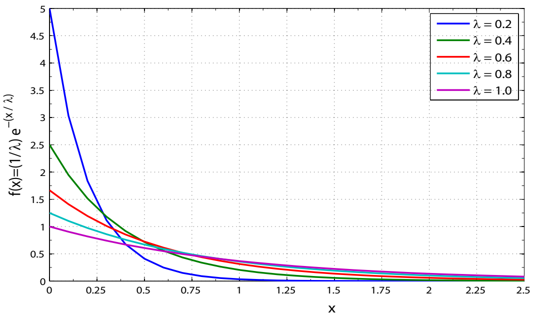

Fig. 2. Variation of the exponential pole-density versus the parameter λ

From Figure 3 and the formulas (12)–(14), we see that:

(a) The bias of f is strongly related to the density pole. Indeed, as the gradient of the pole $(\frac{-1}{\lambda} < 0)$ decreases, its bias increases and vice versa.

(b) For a finite smoothing parameter (finite sample), one can see that the variance tends to zero when λ tends to zero. In the contrary case, the bias tends to $-\infty$ considerably compared with the diminution of the variance. This remark can be explained by the fact that the MSE increases when the value of λ decreases.

(c) Finally, if λ tends to $+\infty$ (without the pole), then the three quantities (Bias, variance and MSE) simultaneously tend to zero.

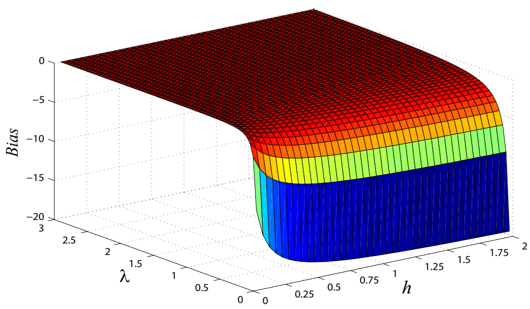

Fig. 3. Variation of the Bias of the exponential density estimator versus the parameters (λ, h)

From the obtained results (theoretical and graphical) on the boundary characteristics of the gamma kernel estimators for the three targets densities, according the smoothing parameter h and the sample size n, we can underline the following facts

1. From the formulas (12)–(14), (16)–(18) and (20)–(22), it easy to verify that the three quantities Bias, variance and MSE, converge under the same usual conditions on the smoothing parameter h. Indeed, the bias tends to zero for $h \to 0$, the variance converges to zero for $nh \to +\infty$ and the MSE converges to zero for $h \to 0$ and $nh \to +\infty$ (see Figures 4–6).
2. The worst estimator is obtained in the case of the exponential density and this for the three criteria considered (the case of a distribution having a negative pole equals to -1). The best one is obtained in the case of the PDF_3 where the gradient of its pole equals to $+\frac{4}{5}$ (the density presents a shoulder). Whereas, the medium situation is obtained in the case of the truncated and normalized Gaussian distribution i.e. the case where the density satisfies the shoulder condition $f'(0) = 0$ (see Figures 4–6).
3. The degradation of the estimator quality generated by the pole can be compensated by a large number of observations (which is not obvious in practice). On the one hand, this remark proves the convergence of the estimator towards the true value sought $(f(0))$, and on the other hand, it confirms the convergence conditions cited in the first point.

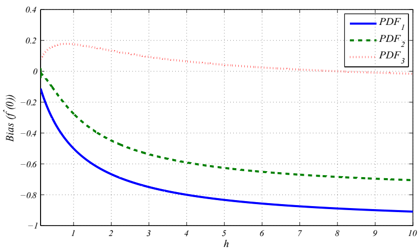

Fig. 4. Variation of the Bias versus the smoothing parameter h

4. Comparative study between the boundary correction methods

The aim of this part is to carry out a comparison study of the performances of the normalized gamma kernel estimators, proposed in the paper of Cherfaoui *et al.* (2015), with those obtained with other boundary correction approaches. To do this, we consider the same examples, the same approaches and the same steps simulation treated in Zhang (2010), namely:

– Compare the performances of the following estimators: gamma kernels ($\hat{f}_1$ and $\hat{f}_2$), normalized gamma kernels ($\hat{f}_{M1}$, $\hat{f}_{m1}$, $\hat{f}_{M2}$ and $\hat{f}_{m2}$), cut-and-normalized ($\hat{f}_{cn}$), boundary kernel ($\hat{f}_b$) and Jone-Foster ($\hat{f}_{JF}$).
– For each density (PDF_1, PDF_2, PDF_3), we generate 1000 samples of size $n = 100$.
– The bandwidths will be chosen by minimizing the average of the integrated squared errors (ISE).

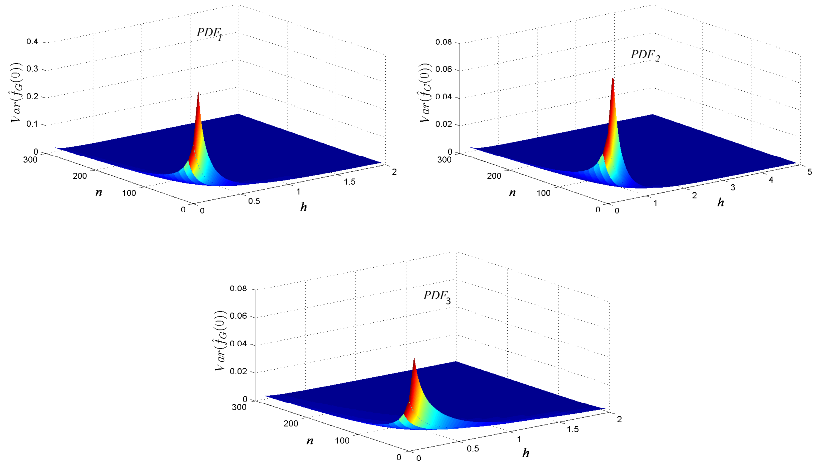

Fig. 5. Variation of the variance versus the parameters h and n

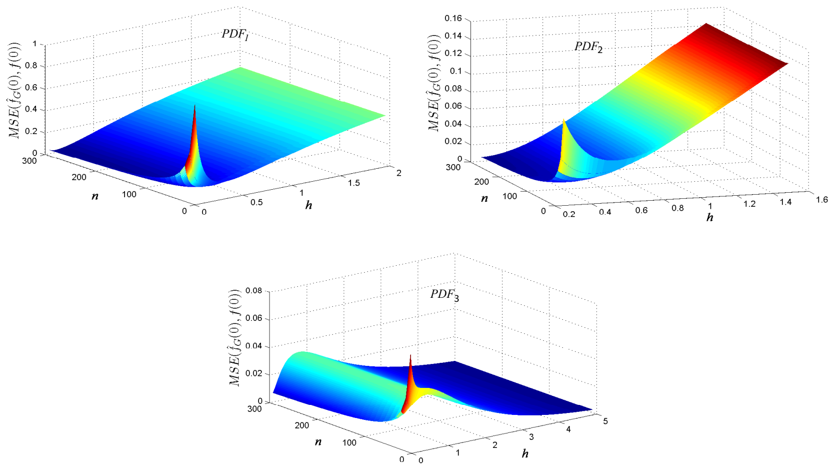

Fig. 6. Variation of the MSE versus the parameters h and n

We first, designed a simulator under the Matlab environment. The latter allows us to have two types of results: numerical results (local and global performance measures) and graphical results.

The execution of our program provides numerical and graphical results associated with the local and global proprieties of each estimator for the different target densities. Tables 1 and 2 summarize the local proprieties at the point $x = 0$. For other values of x, these proprieties are presented in Figures 7–15. However, Table 3 reports the global performances of each estimator.

Proprieties	Density	$\hat{f}_1$	$\hat{f}_{M1}$	$\hat{f}_{m1}$	$\hat{f}_2$	$\hat{f}_{M2}$	$\hat{f}_{m2}$
Bias	PDF_1	-0.1580	-0.0673	-0.4361	-0.1039	-0.1575	-0.0010
	PDF_2	-0.0386	0.0093	0.2098	-0.0566	-0.0923	-0.1226
	PDF_3	0.0840	0.0888	0.0131	0.0988	0.0909	0.0579
Variance	PDF_1	0.0161	0.0245	0.0022	0.0238	0.0194	0.0458
	PDF_2	0.0247	0.0170	0.0126	0.0132	0.0144	0.0271
	PDF_3	0.0038	0.0039	0.0026	0.0038	0.0039	0.0030
MSE	PDF_1	0.0410	0.0290	0.1924	0.0346	0.0442	0.0458
	PDF_2	0.0262	0.0171	0.0565	0.0164	0.0229	0.0419
	PDF_3	0.0109	0.0118	0.0028	0.0136	0.0122	0.0064

Table 1: Bias, variance and MSE of the different gamma kernel estimators at the boundary $x = 0$ for $n = 100$

Proprieties	Density	$\hat{f}_{cn}$	$\hat{f}_b$	$\hat{f}_{JF}$
Bias	PDF_1	-0.1671	-0.0903	-0.1217
	PDF_2	-0.0454	0.0121	-0.1149
	PDF_3	0.1495	0.1401	0.1492
Variance	PDF_1	0.0176	0.0259	0.0126
	PDF_2	0.0346	0.0171	0.1124
	PDF_3	0.0205	0.0124	0.0105
MSE	PDF_1	0.0455	0.0260	0.0274
	PDF_2	0.0367	0.0172	0.1256
	PDF_3	0.0429	0.0320	0.0328

Table 2: Bias, variance and MSE of the others estimators at the point $x = 0$ for $n = 100$

	PDF_1		PDF_2		PDF_3	
Estimators	h^*	ISE^*	h^*	ISE^*	h^*	ISE^*
$\hat{f}_1$	0.1200	0.0110	0.1188	0.0090	0.1559	0.0059
$\hat{f}_{M1}$	0.2208	0.0079	0.7896	0.0190	0.1113	0.0077
$\hat{f}_{m1}$	0.7408	0.0048	0.1658	0.0089	0.1736	0.0056
$\hat{f}_2$	0.1660	0.0111	0.1962	0.0065	0.1623	0.0058
$\hat{f}_{M2}$	0.1483	0.0107	0.1379	0.0097	0.1648	0.0056
$\hat{f}_{m2}$	0.0985	0.0142	0.1165	0.0132	0.1463	0.0057
$\hat{f}_{cn}$	0.6229	0.0054	0.4774	0.0134	0.7926	0.0090
$\hat{f}_b$	1.0628	0.0024	0.8252	0.0076	0.6937	0.0092
$\hat{f}_{JF}$	0.6648	0.0050	0.7198	0.0111	0.7842	0.0091

Table 3: The smoothing parameters and the optimal mean ISE of the three target densities

5. Discussion of the results and concluding remarks

According of the ranked results in Tables 1–3, we conclude that:

- The macro-normalization of the estimator $\hat{f}_1$ reduces the bias and increases the variance. Although, the reduction of the bias is more considerable than the growth of the variance, this can be justified by the reduction of MSE in this situation.
- The micro-normalization of the estimator $\hat{f}_1$ reduces the variance, but it increases the bias. Moreover, the increasing of latter it more considerable than the reduction of the variance this can be justified by the increasing of the MSE.
- The micro-normalization of the estimator $\hat{f}_2$ behaves in the same way as the macro-normalization of the estimator $\hat{f}_1$. However, unlike the macro-normalization of the estimator $\hat{f}_1$, the macro-normalization of the estimator $\hat{f}_2$ reduces the variance and increases the bias. As a result, the increase in bias is more significant than the reduction in variance (see the MSE of this situation).

From the results reported in Figures 7–15, we see, for the three target densities, that:

A. **The case PDF_1:**
  - Contrary to the estimators $\hat{f}_{M1}$, the estimator $\hat{f}_{m1}$ has local bias with a non-regular variation (see Figure 7 (a)).
  - The estimators $\hat{f}_1$, $\hat{f}_{M1}$ and $\hat{f}_{m1}$ have a local variance with a regular variation (see Figure 10 (a)).
  - By comparing the MSE of the estimators $\hat{f}_1$, $\hat{f}_{M1}$ and $\hat{f}_{m1}$ at the boundary (see Figure 13 (a)), it can be clearly seen that $\hat{f}_{m1}$ is the most efficient estimator while $\hat{f}_1$ is the least efficient one.
  - The estimators $\hat{f}_1$, $\hat{f}_{M1}$ and $\hat{f}_{m1}$ have a local proprieties (bias, variances and MSE) of non-regular variation (see Figures 7 (b), 10 (b) and 13 (b)). By comparing their MSE in the neighborhood of zero, we see that $\hat{f}_1$ is the most efficient estimator, but beyond $x = 2h$ it becomes the least efficient one.
  - Looking at the MSE of the three estimators $\hat{f}_1$, $\hat{f}_{M1}$ and $\hat{f}_{m1}$ in the neighborhood of zero (see Figure 13 (b)), we deduce that $\hat{f}_{m1}$ is the best one one while $\hat{f}_1$ is the least efficient one.

B. **The case PDF_2:**
  - The estimators $\hat{f}_1$, $\hat{f}_{M1}$, $\hat{f}_{m1}$, $\hat{f}_2$, $\hat{f}_{M2}$ and $\hat{f}_{m2}$, have local variances that vary in regular way (see Figure 11). But, their bias vary according to x in a non-regular way (see Figure 8). These bias lead to the non-regularity of the variation of their MSE (see Figure 14).
  - By comparing the MSE of the estimators $\hat{f}_1$, $\hat{f}_{M1}$ and $\hat{f}_{m1}$ (see Figure 14 (a)), we conclude that $\hat{f}_{M1}$ is the best estimator. Besides, comparing the MSE of the estimators $\hat{f}_2$, $\hat{f}_{M2}$ and $\hat{f}_{m2}$ (see Figure 14 (b)), we see that $\hat{f}_2$ is the best one in the neighborhood of zero but moving away from the point $x = 2h$ it becomes the least efficient of these estimators.

C. **The case PDF_3:**

- The estimators $\hat{f}_1$, $\hat{f}_{M1}$ and $\hat{f}_{m1}$, have local properties (bias, variances and MSE), which vary in regular way (see Figures 9 (a), 12 (a) and 15 (a)).
- By comparing the MSE of the estimators $\hat{f}_1$, $\hat{f}_{M1}$ and $\hat{f}_{m1}$ (see Figure 15 (a)), we see that $\hat{f}_{M1}$ and $\hat{f}_1$ are equivalent and the best one is the estimator $\hat{f}_{m1}$.
- The estimators $\hat{f}_2$, $\hat{f}_{M2}$ and $\hat{f}_{m2}$, have local properties (bias, variances and MSE), which vary in non-regular way, according to the position x (see Figures 9 (b), 12 (b) and 15 (b)).
- By looking at MSE's of the estimators $\hat{f}_2$, $\hat{f}_{M2}$ and $\hat{f}_{m2}$ in the neighborhood of zero (see Figure 15 (b)), we conclude that $\hat{f}_2$ is the best but beyond the position $x = h$ it quickly loses this quality.

Finally, this work permits us to conclude that the macro-normalization of the gamma kernel estimator provides us, generally, a new efficient estimator in the neighborhood of zero (in the neighborhood of the boundary) regardless of the target density choice (with or without pole). However, in the worst-case scenario, its performances are equivalent to those of the standard gamma kernel. Whereas, for the modified gamma kernel estimator, it is preferable to avoid normalization, since generally the standard modified gamma kernel estimator works significantly better than the normalized one.

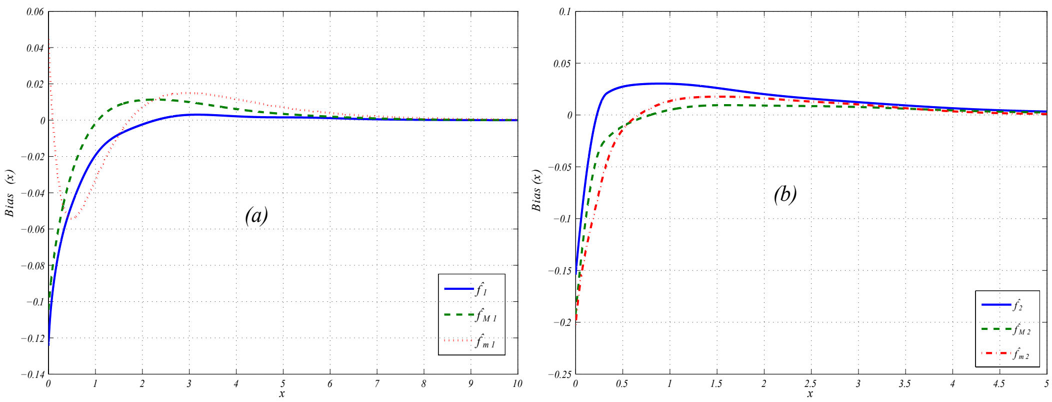

Fig. 7. Bias variation of gamma kernel estimators, case PDF_1

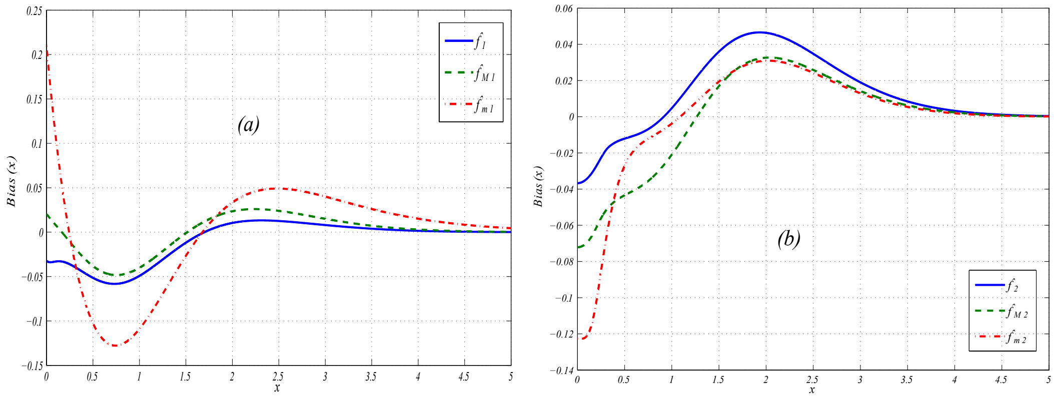

Fig. 8. Bias variation of gamma kernel estimators, case PDF_2

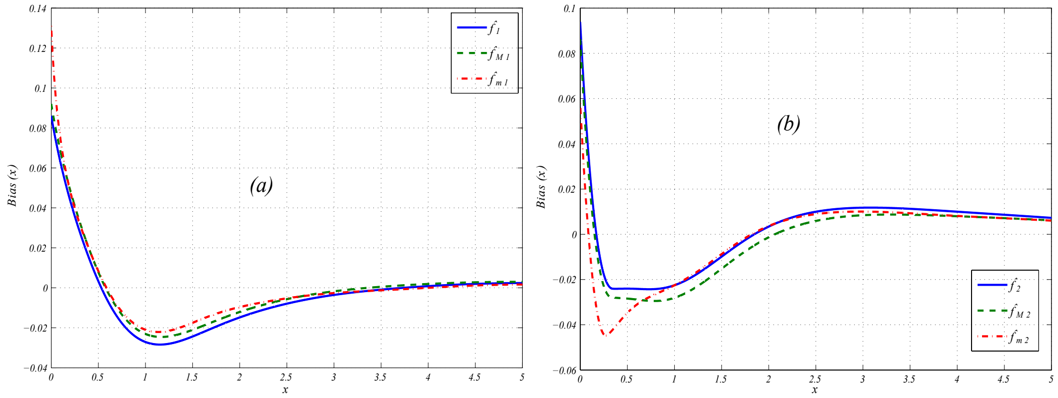

Fig. 9. Bias variation of gamma kernel estimators, case PDF_3

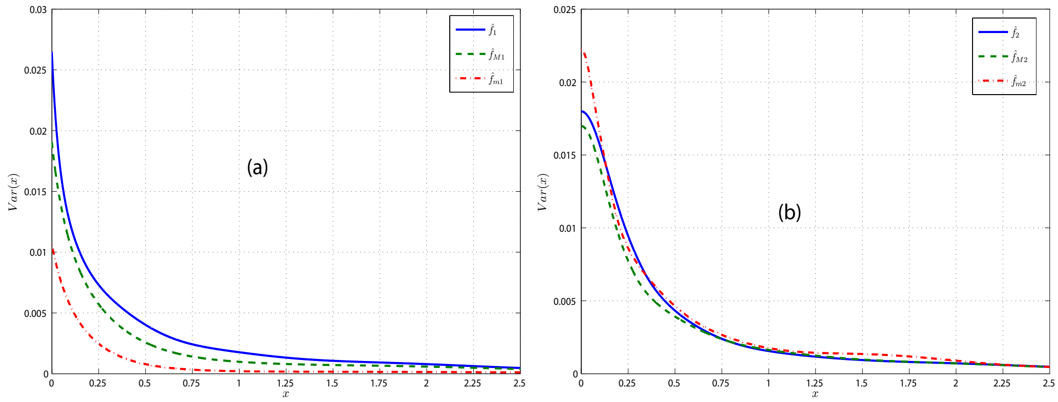

Fig. 10. Variance variation of gamma kernel estimators, case PDF_1

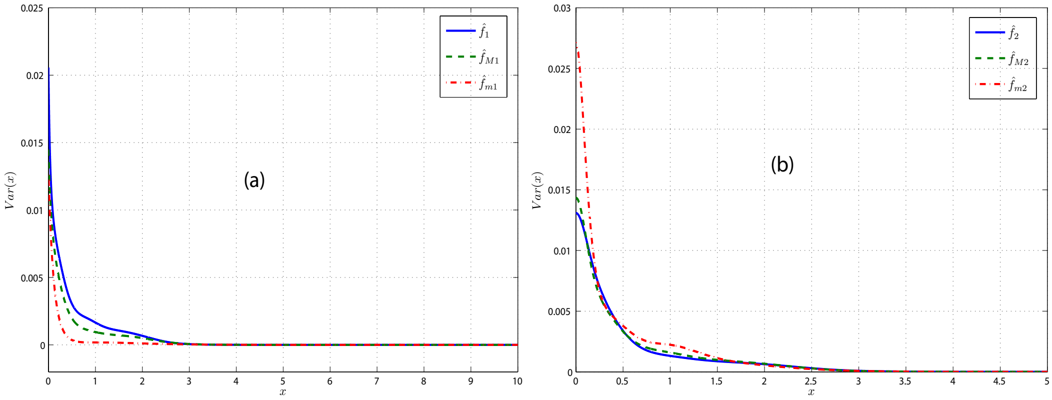

Fig. 11. Variance variation of gamma kernel estimators, case PDF_2

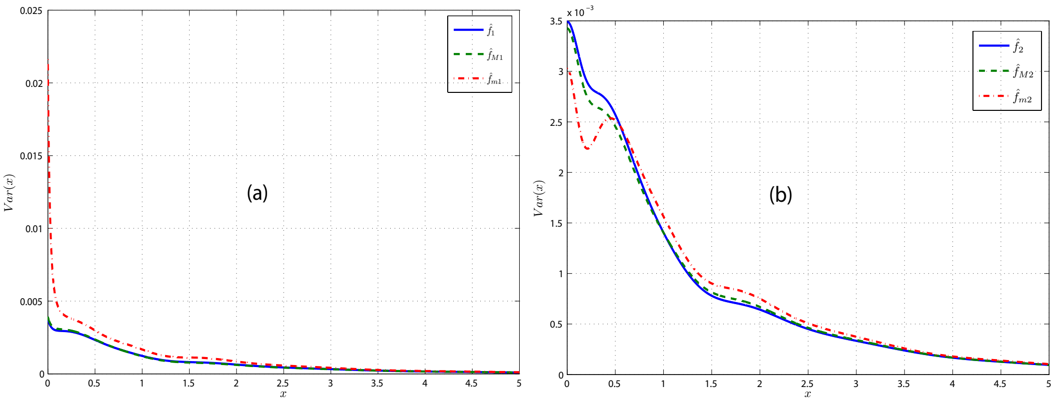

Fig. 12. Variance variation of gamma kernel estimators, case PDF_3

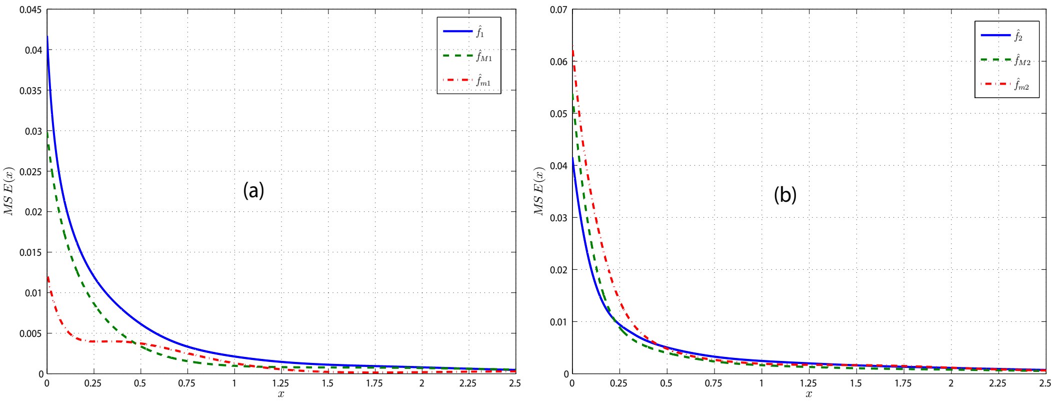

Fig. 13. MSE variation of gamma kernel estimators, case PDF_1

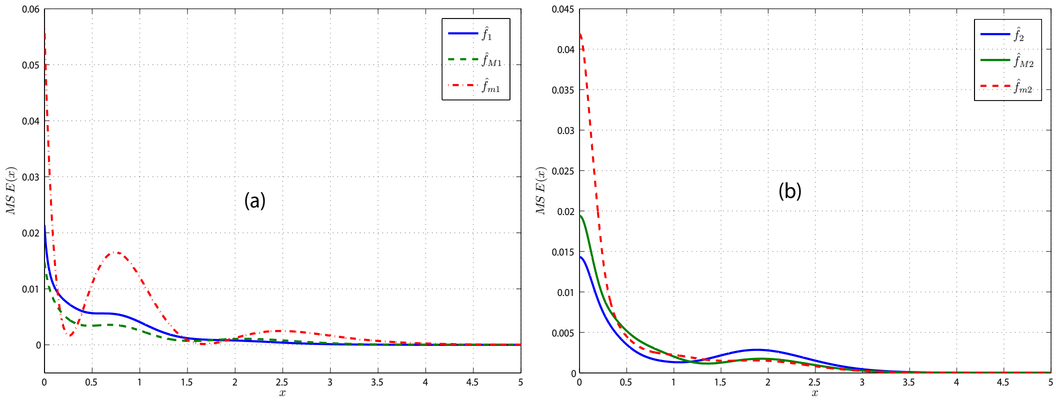

Fig. 14. MSE variation of gamma kernel estimators, case PDF_2

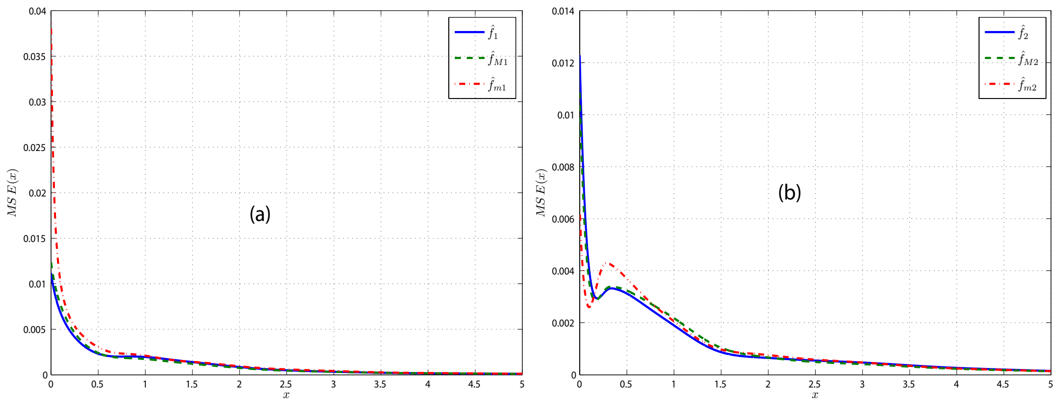

Fig. 15. MSE variation of gamma kernel estimators, case PDF_3

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