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A CHARACTERIZATION OF CONVEX FUNCTIONS AND ITS APPLICATION TO OPERATOR MONOTONE FUNCTIONS

MASATOSHI FUJII[1*], YOUNG OK KIM[2] AND RITSUO NAKAMOTO[3]

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ABSTRACT. We give a characterization of convex functions in terms of difference among values of a function. As an application, we propose an estimation of operator monotone functions: If $A > B \geq 0$ and f is operator monotone on $(0, \infty)$, then $f(A) - f(B) \geq f(\|B\| + \epsilon) - f(\|B\|) > 0$, where $\epsilon = \|(A-B)^{-1}\|^{-1}$. Moreover it gives a simple proof to Furuta's theorem: If $\log A > \log B$ for $A,\ B > 0$ and f is operator monotone on $(0, \infty)$, then there exists a $\beta > 0$ such that $f(A^\alpha) > f(B^\alpha)$ for all $0 < \alpha \leq \beta$.

1. INTRODUCTION

For a twice differentiable real-valued function f, its convexity is characterized by $f'' \geq 0$. Since there are many non-differentiable convex functions, we consider a characterization of general convex functions. We cannot use the differentiation, but the average rate of change is available. Roughly speaking, we claim that the convexity of a function is characterized by the non-decreasingness of average rate of change. It seems to be natural as a generalization of the condition $f'' \geq 0$. Actually it will be formulated as Lemma 1 in the next section.

To explain operator monotone functions, we introduce the operator order $A \geq B$ among selfadjoint operators A, B on a Hilbert space H by $(Ax, x) \geq (Bx, x)$ for all $x \in H$. In particular, A is positive if $A \geq 0$, i.e., $(Ax, x) \geq 0$ for all $x \in H$.

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* Corresponding author.

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Next, a positive operator A is said to be strictly positive, denoted by $A > 0$, if $A \geq c$ for some constant $c > 0$. So $A > B$ means that $A - B > 0$.

A real-valued continuous function f defined on $[0, \infty)$ is called operator monotone if it preserves the operator order, i.e., $f(A) \geq f(B)$ for $A \geq B \geq 0$. One of the most important examples is the power function $t \mapsto t^p$ for $0 \leq p \leq 1$ (Löwner–Heinz inequality). In general, f is called operator monotone on an interval J if $f(A) \geq f(B)$ for $A \geq B$ whose spectra contained in J. For this, we pose $\log t$ as a fundamental example of an operator monotone function on $(0, \infty)$.

Very recently, Moslehian and Najafi [9] proposed an excellent extension of the Löwner–Heinz inequality as follows:

Theorem MN. *If* $A > B \geq 0$ *and* $0 < r \leq 1$*, then* $A^r - B^r \geq \|A\|^r - (\|A\| - \epsilon)^r > 0$*, and* $\log A - \log B \geq \log \|A\| - \log(\|A\| - \epsilon) > 0$*, where* $\epsilon = \|(A-B)^{-1}\|^{-1}$*.*

In this note, we apply our characterization of concave functions and give an improvement and a generalization of Theorem MN (Theorem 5). As another application, we can give a short proof to a recent result due to Furuta [6, Theorem 2.1], which is an operator inequality related to operator monotone functions and chaotic order, i.e., the order defined by $\log A \geq \log B$ among positive invertible operators.

2. A characterization of convex functions

In this section, we propose an elementary characterization of convex functions. We essentially use average rate of change.

Lemma 2.1. *A real valued continuous function* f *on an interval* $J = [a, b)$ *with* $b \in (-\infty, +\infty]$ *is convex (resp. concave) if and only if, for each* $0 < \epsilon < b - a$*,* $D_\epsilon(t) = f(t+\epsilon) - f(t)$ *is non-decreasing (resp. non-increasing) on* $[a, b - \epsilon)$*.*

Proof. Suppose that f is convex on J. Take $s, t \in J$ with $s < t$ and $t + \epsilon \in J$. We may assume that $t - s < \epsilon$. Let $y = L(t)$ be the linear function through $(s, f(s))$ and $(s + \epsilon, f(s + \epsilon))$. Then we have

$$L(t) \geq f(t) \text{ and } L(t + \epsilon) \leq f(t + \epsilon)$$

by the convexity of f. Hence it implies that

$$\begin{aligned} D_\epsilon(t) &= f(t+\epsilon) - f(t) \\ &\geq L(t+\epsilon) - L(t) \\ &= L(s+\epsilon) - L(s) \quad \text{by the linearity of L} \\ &= f(s+\epsilon) - f(s) \\ &= D_\epsilon(s), \end{aligned}$$

as desired.

Conversely suppose that $D_\epsilon(t)$ is non-decreasing. Take $t,\ s \in J$ with $s < t = s + 2\epsilon$. Since $D_\epsilon(s) \leq D_\epsilon(s+\epsilon)$, we have

$$2f(\frac{s+t}{2}) = 2f(s+\epsilon) \leq f(s+2\epsilon) + f(s) = f(t) + f(s).$$

So f is convex. □

Corollary 2.2. *If f is strictly increasing and concave on an interval $[a, b+\delta]$ in $\mathbb{R}$ for some $\delta > 0$, then for each $0 < \epsilon \leq \delta$, $D_\epsilon(t) \geq D_\epsilon(b) > 0$ for all $t \in [a, b]$.*

Remark 2.3. Analogous argument on convexity of functions as above has been done in [8, page 2].

3. Applications to Operator monotone functions

As an application of Corollary 2.2, we give an estimation of operator monotone functions.

Lemma 3.1. *If f is non-constant and operator monotone on the interval $\mathbb{R}_+ = [0, \infty)$, then f is strictly increasing.*

Proof. First of all, we note that f is non-decreasing. Next we suppose that $f'(c) = 0$ for some $c > 0$. Noting that the Löwner matrix

$$\begin{pmatrix} f'(c) & f^{[1]}(c,d) \\ f^{[1]}(d,c) & f'(d) \end{pmatrix}$$

is positive semidefinite for any $d > 0$ by the operator monotonicity of f, where $f^{[1]}(c,d) = \frac{f(c)-f(d)}{c-d}$ is the devided difference.

Therefore its determinant is nonnegative, so that $f^{[1]}(c,d) = 0$ for any $d > 0$. This means that f is constant, which is a contradiction. Consequently we have $f' > 0$. □

Lemma 3.2. *If $C \geq 0$ and f is a concave and strictly increasing function on an interval $[a, d)$ containing the spectrum of C, then for each $0 < \epsilon < d - \|C\|$, $f(C+\epsilon) \geq f(C) + D_\epsilon(\|C\|)$.*

Proof. We first note that for a given $0 < \epsilon < d - \|C\|$, we can take $c > 0$ satisfying $0 < c < d$ and $\epsilon < c - \|C\|$. Applying Corollary 2.2 to $b = \|C\|$ and $\delta = c - \|C\|$, it follows that

$$f(C+\epsilon) - f(C) \geq D_\epsilon(\|C\|).$$

□

We here give a precise estimation of [6, Theorem 2.1] and [8, Proposition 2.2], cf. [9].

Theorem 3.3. *If $A > B \geq 0$ and f is non-constant operator monotone on $[0, \infty)$, then $f(A) - f(B) \geq f(\|B\| + \epsilon) - f(\|B\|) > 0$, where $\epsilon = \|(A-B)^{-1}\|^{-1}$.*

Proof. Since $A \geq B + \epsilon$ for $\epsilon = \|(A-B)^{-1}\|^{-1} > 0$, we have

$$f(A) \geq f(B+\epsilon).$$

Furthermore Lemmas 3.1 and 3.2 imply that

$$f(B+\epsilon) \geq f(B) + D_\epsilon(\|B\|).$$

Hence we have

$$f(A) - f(B) \geq D_\epsilon(\|B\|) = f(\|B\| + \epsilon) - f(\|B\|) > 0.$$

□

As a consequence, we have an improvement of the estimation due to Moslehian and Najafi [9]:

Corollary 3.4. *If $A > B \geq 0$ and $0 < r \leq 1$, then $A^r - B^r \geq (\|B\| + \epsilon)^r - (\|B\|)^r > 0$, and $\log A - \log B \geq \log(\|B\| + \epsilon) - \log\|B\| > 0$, where $\epsilon = \|(A - B)^{-1}\|^{-1}$.*

Remark 3.5. We note that Corollary 3.4 actually improves Theorem MN. Since $\|A\| - (\|A\| - \epsilon) = \epsilon = (\|B\| + \epsilon) - \|B\|$ and the function $t \mapsto t^r$ is strictly concave, it follows that

$$\|A\|^r - (\|A\| - \epsilon)^r \leq (\|B\| + \epsilon)^r - \|B\|^r.$$

We here pose an example:

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $A - B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \geq 1$ and so $\epsilon = 1$. Hence we have

$$\|A\|^r - (\|A\| - \epsilon)^r = 4^r - 3^r < (\|B\| + \epsilon)^r - \|B\|^r = 3^r - 2^r.$$

Now Theorem 3.3 can be regarded as a difference version. So we give a ratio version of it. It is obtained by Theorem 3.3 itself:

Corollary 3.6. *If $A > B > 0$ and f is non-constant operator monotone on $(0, \infty)$, then*

$$f(B)^{-\frac{1}{2}} f(A) f(B)^{-\frac{1}{2}} \geq 1 + (f(\|B\| + \epsilon) - f(\|B\|))\|f(B)\|^{-1},$$

where $\epsilon = \|(A - B)^{-1}\|^{-1}$.

Proof. Put $\delta = f(\|B\| + \epsilon) - f(\|B\|)$. It follows from Theorem 3.3 that

$$\begin{aligned} f(B)^{-\frac{1}{2}} f(A) f(B)^{-\frac{1}{2}} &\geq f(B)^{-\frac{1}{2}} f(B + \delta) f(B)^{-\frac{1}{2}} \\ &= 1 + \delta f(B)^{-1} \geq 1 + \delta \|f(B)\|^{-1}. \end{aligned}$$

□

As another application of Theorem 3.3, we need the chaotic order: For $A > 0$, we can define the selfadjoint operator $\log A$. So a weaker order than the operator order appears by $\log A \geq \log B$ for $A, B > 0$. We call it the chaotic order. The chaotic order plays an substantial role in operator inequalities. Among others, it brightens the Furuta inequality [5], [2], [3], [1], [4], [7] and recent development of Karcher mean theory [11].

Now we give a simple and elementary proof to the following recent theorem [6, Theorem 2.1] due to Furuta, in which we don't use any integral representation of operator monotone functions.

Theorem 3.7. *If $\log A > \log B$ for $A,\ B > 0$ and f is operator monotone on $(0, \infty)$, then there exists $\beta > 0$ such that*

$$f(A^\alpha) > f(B^\alpha) \quad \text{for all } 0 < \alpha \leq \beta.$$

Proof. Since $\log A > \log B$, it is known that there exists $\beta > 0$ such that

$$A^\alpha > B^\alpha \quad \text{for all } 0 < \alpha \le \beta.$$

Therefore it follows from Theorem 3.3 that, for each fixed $\alpha \in (0, \beta]$,

$$f(A^\alpha) > f(B^\alpha),$$

as desired. □

4. A concluding remark.

Finally we discuss an operator extension of Lemma 2.1. Namely we may expect the following conjecture:
A real valued function f on an interval $J = (a, b)$ with $b \in (-\infty, +\infty]$ is operator convex if and only if, for each $0 < \epsilon < b - a$, $D_\epsilon(t)$ is operator monotone on $(a, b - \epsilon)$. Unfortunately we have a negative answer as follows: We choose the function $f(t) = \frac{1}{t}$ on $(0, \infty)$. It is a typical example of operator convex functions. Nevertheless, $D_1(t) = -\frac{1}{t(t+1)}$ is not operator monotone. As a matter of fact, we take two 2×2 matrices A and B:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $D_1(A) \ge D_1(B)$ if and only if $A(A+1) \ge B(B+1)$. Clearly $A \ge B$, but

$$A(A+1) - B(B+1) = \begin{pmatrix} 13 & 6 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 6 & 5 \end{pmatrix} \not\ge 0.$$

This is a counterexample.

Incidentally, the operator convexity of the function $\frac{1}{t}$ is easily shown as follows: It is enough to prove the inequality

$$\left(\frac{A+B}{2}\right)^{-1} \le \frac{1}{2}(A^{-1} + B^{-1}).$$

And it is simplified by putting $C = A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}$ that

$$4(1 + C^{-1})^{-1} \le 1 + C,$$

which follows from the numerical inequality $4 \le (1 + x^{-1})(1 + x)$.

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[1] Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan.
E-mail address: mfujii@cc.osaka-kyoiku.ac.jp

[2] Department of Mathematics, Suwon University, Bongdamoup, Whasungsi, Kyungkido 445-743, Korea.
E-mail address: evergreen1317@gmail.com

[3] 3-4-13, Daihara-cho, Hitachi 316-0021, Japan.
E-mail address: r-naka@net1.jway.ne.jp