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DISJOINTNESS PRESERVING LINEAR OPERATORS BETWEEN BANACH ALGEBRAS OF VECTOR-VALUED FUNCTIONS

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Abstract. We present vector-valued versions of two theorems due to A. Jimenez–Vargas, by showing that, if $B(X,E)$ and $B(Y,F)$ are certain vector-valued Banach algebras of continuous functions and $T : B(X,E) \to B(Y,F)$ is a separating linear operator, then $\widehat{T} : \widehat{B(X,E)} \to \widehat{B(Y,F)}$, defined by $\widehat{T}\hat{f} = \widehat{Tf}$, is a weighted composition operator, where $\widehat{Tf}$ is the Gelfand transform of Tf.

Furthermore, it is shown that, under some conditions, every bijective separating map $T : B(X,E) \to B(Y,F)$ is biseparating and induces a homeomorphism between the character spaces $M(B(X,E))$ and $M(B(Y,F))$. In particular, a complete description of all biseparating, or disjointness preserving linear operators between certain vector-valued Lipschitz algebras is provided. In fact, under certain conditions, if the bijections $T : Lip^{\alpha}(X,E) \to Lip^{\alpha}(Y,F)$ and T^{-1} are both disjointness preserving, then T is a weighted composition operator in the form $Tf(y) = h(y)(f(\phi(y)))$, where ϕ is a homeomorphism from Y onto X and h is a map from Y into the set of all linear bijections from E onto F. Moreover, if T is multiplicative then $M(E)$ and $M(F)$ are homeomorphic.

1. Introduction and preliminaries

Let X be a compact Hausdorff space, $(E, \|\cdot\|)$ be a Banach algebra over the scalar field of complex numbers $\mathbb{C}$ and $C(X,E)$ be the space of all continuous

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maps from X into E. We define the uniform norm on $C(X,E)$ by

$$\|f\|_X = \sup_{x\in X}\|f(x)\|, \quad f \in C(X,E).$$

For $f, g \in C(X,E)$ and $\lambda \in \mathbb{C}$, the pointwise operations λf, $f+g$ and fg in $C(X,E)$ are defined as usual. It is easy to see that $(C(X,E), \|\cdot\|_X)$ is a Banach algebra. If $E = \mathbb{C}$ we get the ordinary function algebra $C(X,\mathbb{C}) = C(X)$ of all continuous complex-valued functions on X.

Definition 1.1. Let $(A, \|\cdot\|)$ be a Banach algebra and the character space $M(A)$ denote the set of all characters (nonzero complex-valued multiplicative linear functionals) on A.

(i) The Gelfand transform of $f \in A$ is the complex-valued function $\hat{f}$ defined by $\hat{f}(\varphi) = \varphi(f)$ on $M(A)$. Moreover, $\hat{A} = \{\hat{f} : f \in A\}$.

(ii) A is regular if $M(A) \neq \emptyset$ and for every closed subset $F \subseteq M(A)$ and every $\varphi \in M(A)\backslash F$, there exists $f \in A$ such that $\hat{f}(\varphi) = 1$ and $\hat{f}(F) \subseteq \{0\}$. If in addition, this f satisfies $\|\hat{f}\| \leq 1$, then A is called hyper-regular.

(iii) A is normal if $M(A) \neq \emptyset$ and for every closed subset $F \subseteq M(A)$ and every compact subset $K \subseteq M(A)$ with $F \cap K = \emptyset$, there exists $f \in A$ such that $\hat{f}(K) \subseteq \{1\}$ and $\hat{f}(F) \subseteq \{0\}$. If in addition, this f satisfies $\|\hat{f}\| \leq 1$, then A is called hyper-normal.

Remark 1.2. (i) A commutative Banach algebra is regular if and only if it is normal. See, for example, [18, Corollary 4.2.9] or [9, Proposition 4.1.18].

(ii) If A is a regular commutative Banach algebra such that $\hat{A}$ is closed under complex conjugation, then A is hyper-regular [18, Corollary 4.2.10].

(iii) Every commutative $C^* - algebra$ is regular and hence normal. See, for example, [18, Example 4.2.2]. Moreover, by (ii) every commutative $C^* - algebra$ is hyper-regular.

Let X be a compact Hausdorff space and E be a unital commutative Banach algebra. In the sequel, by $B(X,E)$ we mean a Banach algebra which is contained in $C(X,E)$. It is clear that if $B(X,E)$ contains the constant functions, then it is commutative if and only if E is commutative. We also recall that the cozero set of $f : X \to E$ is $coz(f) = \{x \in X : f(x) \neq 0\}$, and $supp(f)$, the support of f, is the closure of $coz(f)$ in X.

Definition 1.3. For compact Hausdorff spaces X and Y, and Banach algebras $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$, a linear map $T : B(X,E) \to B(Y,F)$ is called disjointness preserving if for every $f, g \in B(X,E)$ the equality $coz(f) \cap coz(g) = \emptyset$ implies the equality $coz(Tf) \cap coz(Tg) = \emptyset$.

Remark 1.4. It is easy to check that a linear map $T : B(X,E) \to B(Y,F)$ is disjointness preserving if and only if for every $f, g \in B(X,E)$ the equality $\|f(x)\|\|g(x)\| = 0$ for all $x \in X$ implies the equality $\|Tf(y)\|\|Tg(y)\| = 0$ for all $y \in Y$. If T has this latter property it is called a separating map by some authors. See, for example, [13], [17] and [10]. But in this paper, we use separating maps in the following sense. See, for example, [11] and [12].

Definition 1.5. If A and B are Banach algebras, a linear map $T : A \to B$ is called separating if for every $f, g \in A$, the equality $fg = 0$ implies the equality $TfTg = 0$. Moreover, T is called *biseparating* if it is bijective and both T and T^{-1} are separating.

Definition 1.6. Let A and B be Banach algebras and let $T : A \to B$ be a linear map. The map $\widehat{T} : \hat{A} \to \hat{B}$ is defined by $\widehat{T}\hat{f} = \widehat{Tf}$ for every $f \in A$.

If A and B are semisimple commutative Banach algebras, it is easy to check that the map $T : A \to B$ is separating if and only if $\widehat{T}$ is separating and, moreover, T is injective (surjective) if and only if $\widehat{T}$ is injective (surjective).

Order bounded disjointness preserving maps are also known as Lamperti operators [4].

The notion of disjointness preserving or separating operators seems to be used first in the 40's [22, 23]. Since then many mathematicians have developed this concept. For example, Abramovich made some contributions in the context of Banach lattices and vector lattices in [1, 2]. Separating linear maps for scalar-valued continuous functions, as well as the notion of automatic continuity, were studied in [5, 6, 7] and for scalar-valued Lipschitz algebras in [16]. Moreover, these maps have been studied in [13] for the algebra of continuous vector-valued functions, as well as the vector-valued Lipschitz algebras. Jarosz has also interesting results on the automatic continuity of separating linear isomorphisms in [15]. Disjointness preserving operators between certain Banach algebras of continuous functions have been studied in [3, 12]. One can also find interesting results on norm-preserving maps between Banach function algebras in [14]. Recently, as examples of weighted composition operators, disjointness preserving maps between vector-valued Lipschitz function spaces have been studied in [10].

In [16] Jimenez–Vargas has shown that for compact metric spaces X and Y, every disjointness preserving operator $T : \ell ip^{\alpha}(X) \to \ell ip^{\alpha}(Y)$ is essentially a weighted composition operator. He also proved that every bijective disjointness preserving operator $T : \ell ip^{\alpha}(X) \to \ell ip^{\alpha}(Y)$ is automatically continuous and it is, in fact, biseparating.

One of the aims of this paper is to extend the results of Jimenez–Vargas in [16] to Banach algebras of vector-valued continuous functions which are hyper-normal, semisimple, commutative and unital. First we require some definitions and notations.

Let A be a unital commutative Banach algebra. The radical of the algebra A is defined to be the intersection of all maximal ideals of A and it is denoted by $rad(A)$. The algebra A is semisimple if $rad(A) = \{0\}$.

By using a method similar to Jimenez–Vargas in [16, Theorem 2.2], we show that if $B(X, E)$ and $B(Y, F)$ are hyper-normal, semisimple, commutative and unital, and $T : B(X, E) \to B(Y, F)$ is a disjointness preserving linear map, then $\widehat{T}$ is a weighted composition operator. Furthermore, with the same conditions, we show that every bijective separating map $T : B(X, E) \to B(Y, F)$ is biseparating and induces a homeomorphism between the character spaces $M(B(X, E))$ and $M(B(Y, F))$. Then by applying the same method as in [13, Theorem 2.3], we conclude that certain disjointness preserving linear maps $T : Lip^{\alpha}(X, E) \to$

$Lip^\alpha(Y,F)$ or $T : \ell ip^\alpha(X,E) \to \ell ip^\alpha(Y,F)$ are weighted composition operators, and moreover, they induce a homeomorphism between X and Y.

Weighted composition operators between certain classes of weighted Frechet spaces and on some spaces of analytic functions, have been studied in [19].

2. Hyper-normality of vector-valued Lipschitz algebras

In this section we show that, for a compact metric space X and a commutative unital Banach algebra E, $Lip^\alpha(X,E)$ ($\ell ip^\alpha(X,E)$) is hyper-normal, or (hyper) regular if and only if E is hyper-normal, or (hyper) regular, respectively. We also show that E-valued Lipschitz algebras are semisimple if and only if E is semisimple.

Definition 2.1. Let (X,d) be a compact metric space and E be a unital commutative Banach algebra. For a constant α $(0 < \alpha \le 1)$ and a function $f : X \to E$, the Lipschitz constant of f is defined by

$$p_\alpha(f) := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\|f(x) - f(y)\|}{d(x,y)^\alpha},$$

and the vector-valued big Lipschitz algebra (of order α), or simply, the vector-valued Lipschitz algebra is defined by

$$Lip^\alpha(X,E) = \{f : X \to E : p_\alpha(f) < \infty\}.$$

Similarly, for α $(0 < \alpha < 1)$ the vector-valued little Lipschitz algebra (of order α) is defined by

$$\ell ip^\alpha(X,E) = \left\{ f \in Lip^\alpha(X,E) : \frac{\|f(x) - f(y)\|}{d(x,y)^\alpha} \to 0 \quad as \quad d(x,y) \to 0 \right\}.$$

For each $f \in Lip^\alpha(X,E)$ we define the norm by

$$\|f\|_\alpha = \|f\|_X + p_\alpha(f).$$

If $E = \mathbb{C}$ we get the ordinary complex-valued Lipschitz algebras $Lip^\alpha(X)$ and $\ell ip^\alpha(X)$. In [8] it has been shown that $(Lip^\alpha(X,E), \|\cdot\|_\alpha)$ is complete and it is, in fact, a Banach subalgebra of $C(X,E)$, and moreover, $\ell ip^\alpha(X,E)$ is a closed subalgebra of $(Lip^\alpha(X,E), \|\cdot\|_\alpha)$.

Remark 2.2. For a compact metric space X and a unital commutative Banach algebra E, we can deduce from [20, Examples 2.1(ii)] and [20, Corollary 2.2], that the maximal ideal space of $Lip^\alpha(X,E)$ is homeomorphic to the cartesian product $X \times M(E)$ in the product topology, that is,

$$M(Lip^\alpha(X,E)) \cong X \times M(E).$$

Moreover, every character ϕ on $Lip^\alpha(X,E)$ is of the form $\varphi\circ\delta_x$ for some $\varphi \in M(E)$ and for some $x \in X$ [20].

We now bring an elementary result for scalar-valued Lipschitz algebras and then extend it to the vector-valued case.

Lemma 2.3. *If X is a compact metric space then $Lip^\alpha(X)$ for $0 < \alpha \le 1$ and $\ell ip^\alpha(X)$ for $0 < \alpha < 1$ are both hyper-normal.*

Proof. Since $Lip^1(X)$ is contained in $\ell ip^\alpha(X)$ for all $0 < \alpha < 1$, it is enough to show that for any pair of disjoint compact sets C and K the function

$$f(x) = \frac{d(x,C)}{d(x,C) + d(x,K)}$$

is an element of $Lip^1(X)$, which is easy to see. $\square$

Theorem 2.4. *Let X be a compact metric space and E be a commutative unital Banach algebra. Then $Lip^\alpha(X,E)$ is hyper-normal if and only if E is hyper-normal.*

Proof. We first suppose that E is hyper-normal. Let K and F be compact subsets of $M(Lip^\alpha(X,E))$ such that $K \cap F = \emptyset$. For every $\phi \in K$, there exists a neighbourhood U_ϕ such that $U_\phi \cap F = \emptyset$. By Remark 2.2, there exist $x \in X$ and $\psi \in M(E)$ such that $\phi = \psi \circ \delta_x$. Hence there exist neighbourhoods U_x and U_ψ of x and ψ, respectively, such that $\phi \in U_x \times U_\psi \subseteq U_\phi$. Since X and $M(E)$ are compact and Hausdorff, there exist neighbourhoods V_x of x and V_ψ of ψ such that $x \in V_x \subseteq \overline{V_x} \subseteq U_x$ and $\psi \in V_\psi \subseteq \overline{V_\psi} \subseteq U_\psi$. By Lemma 2.3, $\ell ip^\alpha(X)$ is hyper-normal and hence there exists $f \in \ell ip^\alpha(X)$ such that $0 \leq f(t) \leq 1$ for all $t \in X$, $f|_{\overline{V_x}} = 0$ and $f|_{{U_x}^c} = 1$. Since E is hyper-normal, there exists $b \in E$ such that $\|\hat{b}\| \leq 1, \hat{b}|_{\overline{V_\psi}} = 0$ and $\hat{b}|_{{U_\psi}^c} = 1$. If we take $g := b + fe - fb$, where e is the unit element of E, then clearly $g \in Lip^\alpha(X,E)$ and $\hat{g}|_{V_x \times V_\psi} = 0$. To show that $\|\hat{g}\| \leq 1$ and $\hat{g}|_F = 1$ let $\varphi \in M(Lip^\alpha(X,E))$. By Remark 2.2 there exist $\gamma \in M(E)$ and $t \in X$ such that $\varphi = \gamma \circ \delta_t$. Thus we have $|\hat{g}(\varphi)| = |\varphi(g)| = |\gamma(g(t))| = |\gamma(b) + f(t) - f(t)\gamma(b)| = |1 + (1 - f(t))(\gamma(b) - 1)|$. If we take $\zeta := 1 - f(t)$ and $\beta = \gamma(b) - 1$, then $0 \leq \zeta \leq 1$ and $|1 + \beta| \leq 1$ and hence $|1 + \zeta\beta| \leq 1$. This implies that

$$|\hat{g}(\varphi)| = |\gamma(b) + f(t) - f(t)\gamma(b)| = |1 + (1 - f(t))(\gamma(b) - 1)| = |1 + \zeta\beta| \leq 1.$$

Now let $\varphi \in F$. Since $U_\phi \cap F = \emptyset$ there exist only five cases as follows:

Case 1: $t \in {U_x}^c$ and $\gamma \in U_\psi^c$. Then $f(t) = 1$ and $\gamma(b) = 1$ and hence $\hat{g}(\varphi) = 1$.

Case 2: $t \in {U_x}^c$ and $\gamma \in V_\psi$. Then $f(t) = 1$ and $\gamma(b) = 0$ and hence $\hat{g}(\varphi) = 1$.

Case 3: $t \in {U_x}^c$ and $\gamma \in U_\psi \setminus V_\psi$. Then $f(t) = 1$ and hence

$$\hat{g}(\varphi) = \gamma(b) + 1 - (\gamma(b) \cdot 1) = 1.$$

Case 4: $t \in V_x$ and $\gamma \in {U_\psi}^c$. Then $f(t) = 0$ and $\gamma(b) = 1$ and hence $\hat{g}(\varphi) = 1$.

Case 5: $t \in U_x \setminus V_x$ and $\gamma \in {U_\psi}^c$. Then $\gamma(b) = 1$ and hence

$$\hat{g}(\varphi) = 1 + f(t) - (1 \cdot f(t)) = 1.$$

If $W_\phi := V_x \times V_\psi$, then for every $\phi \in K$, there exist a neighbourhood W_ϕ in $M(Lip^\alpha(X,E))$ and a function $g \in Lip^\alpha(X,E)$ such that $\hat{g}|_{W_\phi} = 0$ and $\hat{g}|_F = 1$. Since K is compact, there exist $g_1, \cdots, g_n \in Lip^\alpha(X,E)$ such that $K \subseteq \cup_{i=1}^n W_{\phi_i}$, $\hat{g}_i|_{W_{\phi_i}} = 0$ and $\hat{g}_i|_F = 1$ for $i = 1, \cdots, n$. If we take $h = g_1 \cdots g_n$, then $h \in Lip^\alpha(X,E)$, $\|\hat{h}\| \leq 1, \hat{h}|_K = 0$ and $\hat{h}|_F = 1$. From this we now conclude that $Lip^\alpha(X,E)$ is hyper-normal.

Conversely, let $Lip^\alpha(X,E)$ be hyper-normal. Let K and F be compact subsets of $M(E)$ such that $K \cap F = \emptyset$. For a fixed element x in X, we define $K' :=$

$\{\psi \circ \delta_x : \psi \in K\}$ and $F' := \{\phi \circ \delta_x : \phi \in F\}$. It is clear that K' and F' are compact subsets of $M(Lip^\alpha(X, E))$ and $K' \cap F' = \emptyset$. Since $Lip^\alpha(X, E)$ is hyper-normal, there exists $f \in Lip^\alpha(X, E)$ such that $\|\hat{f}\| \leq 1, \hat{f}|_{K'} = 1$ and $\hat{f}|_{F'} = 0$. If $b := f(x)$, then $b \in E$, implying that $\hat{b}(\psi) = \psi(f(x)) = \hat{f}(\psi \circ \delta_x) = 1$ for every $\psi \in K$. Similarly,

$$\hat{b}(\phi) = \phi(f(x)) = \hat{f}(\phi \circ \delta_x) = 0,$$

for every $\phi \in F$. Since $\|\hat{f}\| \leq 1$, we conclude that $\|\hat{b}\| \leq 1$. Therefore, E is hyper-normal.

□

By modifying the proof of the theorem above, we also obtain the following result:

Theorem 2.5. *Let X be a compact metric space and E be a commutative unital Banach algebra. Then $Lip^\alpha(X, E)$ is (hyper) regular if and only if E is (hyper) regular.*

Theorem 2.6. *Let X be a compact Hausdorff space, E be a commutative unital Banach algebra and $B(X, E)$ contain the constant functions. Let us suppose that every character on $B(X, E)$ be of the form $\psi \circ \delta_x$ for some $\psi \in M(E)$ and $x \in X$, where δ_x is the evaluation homomorphism on $B(X, E)$. Then $B(X, E)$ is semisimple if and only if E is semisimple.*

Proof. Since every character φ on $B(X, E)$ is of the form $\psi \circ \delta_x$ for some $\psi \in M(E)$ and $x \in X$, we have

$$rad(B(X, E)) = \{f \in B(X, E) : \psi(f(x)) = 0, \psi \in M(E), x \in X\}.$$

Let E be semisimple and $f \in rad(B(X, E))$. Then for every character φ on $B(X, E)$, we have $\varphi(f) = 0$. It follows that $(\psi \circ \delta_x)(f) = \psi(f(x)) = 0$ for all $x \in X$ and all $\psi \in M(E)$ and hence $f = 0$. This implies that $B(X, E)$ is semisimple.

Conversely, let $B(X, E)$ be semisimple and $b \in rad(E)$. Let f be the constant element of $B(X, E)$, defined by $f(x) = b$ for all $x \in X$. Then for every character φ on $B(X, E)$, we have

$$\varphi(f) = (\psi \circ \delta_x)(f) = \psi(f(x)) = \psi(b) = 0,$$

for some $\psi \in M(E)$ and for some $x \in X$. Therefore, $f \in rad(B(X, E))$ and hence $f = 0$. This implies that E is semisimple. □

Remark 2.7. Since every character φ on $Lip^\alpha(X, E)$ ($\ell ip^\alpha(X, E)$) is of the form $\psi \circ \delta_x$ for some $\psi \in M(E)$ and for some $x \in X$ (see Remark 2.2), by the theorem above the algebra $Lip^\alpha(X, E)(\ell ip^\alpha(X, E))$ is semisimple if and only if E is semisimple. These results are also valid for the Banach algebra $C(X, E)$. Moreover, it was shown by Sherbert in [21, Proposition 2.1] that the scalar-valued Lipschitz algebras $Lip^\alpha(X)$ and $\ell ip^\alpha(X)$ are regular Banach function algebras. Therefore, they are normal and semisimple. See, for example, [9, Theorem 4.4.24].

3. Separating and Disjointness Preserving Linear Operators

In [16] Jimenez–Vargas proved that every disjointness preserving linear map between scalar-valued little Lipschitz algebras is a weighted composition operator. We now extend the results of Jimenez–Vargas as follows:

Theorem 3.1. *Let X, Y be compact Hausdorff spaces, E, F be unital commutative Banach algebras, and $B(X,E), B(Y,F)$ be hyper-normal semisimple commutative unital Banach algebras.*

If $T : B(X,E) \to B(Y,F)$ is a separating linear map, then

- (i) *there exists a disjoint union $M(B(Y,F)) = Y_c \cup Y_0 \cup Y_d$, where Y_0 is closed and Y_d is open in $M(B(Y,F))$.*
- (ii) *there exists a continuous map $h : Y_c \cup Y_d \to M(B(X,E))$ such that $h(\psi) \notin supp(\hat{f})$ implies $\widehat{T}\hat{f}(\psi) = 0$ for all $f \in B(X,E)$.*
- (iii) *there exists a nonvanishing function $k : Y_c \to \mathbb{C}$ such that $\widehat{T}\hat{f}(\psi) = k(\psi)\hat{f}(h(\psi))$ for every $f \in B(X,E)$ and for all $\psi \in Y_c$.*
- (iv) *$\widehat{T}\hat{f}(\psi) = 0$ for every $f \in B(X,E)$ and for all $\psi \in Y_0$.*
- (v) *$h(Y_d)$ is a finite set of nonisolated points of $M(B(X,E))$.*
- (vi) *the functional $\delta_\psi \circ \widehat{T}$ is discontinuous on $\widehat{B(X,E)}$ for each $\psi \in Y_d$.*

Proof. We divide the set $M(B(Y,F))$ into three disjoint parts: Its null part

$$Y_0 := \{\psi \in M(B(Y,F)) : \delta_\psi \circ \widehat{T} = 0\},$$

its nonnull continuous part

$Y_c := \{\psi \in M(B(Y,F)) : \delta_\psi \circ \widehat{T} : \widehat{B(X,E)} \to \mathbb{C}$ is continuous and nonzero$\}$,

and its discontinuous part

$Y_d := \{\psi \in M(B(Y,F)) : \delta_\psi \circ \widehat{T} : \widehat{B(X,E)} \to \mathbb{C}$ is discontinuous$\}$.

The proof of the theorem is set out, step by step. For steps 2, 3, 5 and 6, we follow the same method as in the proof of [16, Theorem 2.2] for $\widehat{T}$, instead of T, while presenting a different method for the proof of the other steps. We provide all the details for the sake of completeness.

Step 1. For each $\psi \in Y_c \cup Y_d$, $supp(\delta_\psi \circ \widehat{T}) \neq \emptyset$ and, in fact, it contains exactly one point.

Proof. Since $B(X,E)$ and $B(Y,F)$ are hyper-normal and semisimple commutative unital Banach algebras, by [11, Lemma 1], for every $\psi \in M(B(Y,F))$, there exists $f_\psi \in B(X,E)$ with $\widehat{T}(\hat{f}_\psi)(\psi) \neq 0$. Hence $supp(\delta_\psi \circ \widehat{T})$ contains exactly one point for every $\psi \in Y_c \cup Y_d$. □

The map $h : Y_c \cup Y_d \to M(B(X,E))$, defined by $h(\psi) = supp(\delta_\psi \circ \widehat{T})$, is called the **support map** of $\widehat{T}$.

Step 2. If $\psi \in Y_c \cup Y_d$, $f \in B(X,E)$ and $h(\psi) \notin supp(\hat{f})$, then $\widehat{T}\hat{f}(\psi) = 0$.

Proof. If $h(\psi) \notin supp(\hat{f})$, then there exists $U_{h(\psi)}$ such that $\hat{f}(\phi) = 0$ if $\phi \in U_{h(\psi)}$. Since $h(\psi) = supp(\delta_\psi \circ \widehat{T})$, there is a function $g \in B(X,E)$ such that $\widehat{T}\hat{g}(\psi) \neq 0$ and $\hat{g}(\phi) = 0$ if $\phi \notin U_{h(\psi)}$, implying that $\hat{f}(\phi)\hat{g}(\phi) = 0$ for all $\phi \in M(B(X,E))$.

Since T is separating, $\widehat{T}$ is also separating. This implies that $\widehat{T}\hat{f}(\psi)\widehat{T}\hat{g}(\psi) = 0$ and hence $\widehat{T}\hat{f}(\psi) = 0$. □

Step 3. The map $h : Y_c \cup Y_d \to M(B(X,E))$ is continuous in the weak*-topology.

Proof. Let ψ be in $Y_c \cup Y_d$ and $\{\psi_\gamma\}_{\gamma\in I}$ be a net in $Y_c \cup Y_d$ converging to ψ. Towards a contradiction, suppose that $\{h(\psi_\gamma)\}_{\gamma\in I}$ does not converge to $h(\psi)$. Then there exists a neighbourhood $N_{h(\psi)}$ and a subnet $\{h(\psi_\lambda)\}_{\lambda\in J}$ of $\{h(\psi_\gamma)\}_{\gamma\in I}$ such that $\{h(\psi_\lambda)\} \notin N_{h(\psi)}$ for each $\lambda \in J$.

By the compactness of $M(B(X,E))$ there is a subnet $\{h(\psi_\beta)\}_{\beta\in K}$ of $\{h(\psi_\lambda)\}_{\lambda\in J}$ which is convergent to an element $\phi \in M(B(X,E))$. If $\phi \neq h(\psi)$, then there exist neighbourhoods V, W of $h(\psi)$ and ϕ, respectively, such that $V \cap W = \emptyset$. Since $\{h(\psi_\beta)\}_{\beta\in K}$ converges to ϕ, there exists $\beta_0 \in K$ such that $h(\psi_\beta) \in W$ if $\beta \geq \beta_0$. Since $h(\psi) = supp(\delta_\psi \circ \widehat{T})$, there exists a function $f \in B(X,E)$ such that $\hat{f}(\lambda) = 0$ for all $\lambda \notin V$ and $\widehat{T}\hat{f}(\psi) \neq 0$. Thus $\hat{f}(\lambda) = 0$ for every $\lambda \in W$. In particular, $h(\psi_\beta) \notin supp(\hat{f})$ and hence $\widehat{T}\hat{f}(\psi_\beta) = 0$ for all $\beta \geq \beta_0$, by Step 2. Thus $\widehat{T}\hat{f}(\psi) = 0$, which is a contradiction. Consequently, $h(\psi_\beta) \to_{\beta\in K} h(\psi)$. Since $\{h(\psi)\}_{\beta\in K}$ is a subnet of $\{h(\psi_\lambda)\}_{\lambda\in J}$, it follows that $h(\psi_\beta) \notin N_{h(\psi)}$ for all $\beta \in K$, which is impossible. Therefore, $\{h(\psi_\gamma)\}_{\gamma\in I}$ converges to $h(\psi)$, implying that h is continuous. □

Step 4. For $\psi \in Y_c \cup Y_d$, let

$$M_\psi := \left\{\hat{f} \in \widehat{B(X,E)} : \hat{f}(h(\psi)) = 0\right\}, \quad J_\psi := \left\{\hat{f} \in \widehat{B(X,E)} : h(\psi) \notin supp(\hat{f})\right\}.$$

Then J_ψ is a dense subspace of M_ψ.

Proof. Note that J_ψ is, in fact, the set all functions in $\widehat{B(X,E)}$ vanishing on a neighbourhood of $h(\psi)$. Clearly J_ψ and M_ψ are vector subspaces of $\widehat{B(X,E)}$ and $J_\psi \subseteq M_\psi$. To show that J_ψ is dense in M_ψ, let $\psi \in Y_c \cup Y_d$, $\hat{f} \in M_\psi$ and $\epsilon > 0$. Define

$$\Gamma_1 := \left\{\phi \in M(B(X,E)) : |\hat{f}(\phi)| \leq \frac{\epsilon}{2}\right\}, \quad \Gamma_2 := \left\{\phi \in M(B(X,E)) : |\hat{f}(\phi)| \geq \epsilon\right\}.$$

Since $B(X,E)$ is hyper-normal, there exists $g \in B(X,E)$ such that $\|\hat{g}\| \leq 1$, $\hat{g}|_{\Gamma_1} = 0$ and $\hat{g}|_{\Gamma_2} = 1$. Since the interior of Γ_1 is a neighbourhood of $h(\psi)$ and $\hat{g}$ is zero on this neighbourhood, it follows that $\hat{g} \in J_\psi$ and hence $\hat{f}\hat{g} \in J_\psi$.

We now consider the following three cases:

Case 1: If $\phi \in \Gamma_1$, then $|\hat{f}(\phi)(1 - \hat{g}(\phi))| \leq \frac{\epsilon}{2}(1 + \|\hat{g}\|) < \epsilon$.

Case 2: If $\phi \in {\Gamma_2}^c \setminus \Gamma_1$, then $|\hat{f}(\phi)(1 - \hat{g}(\phi))| \leq \epsilon(1 + \|\hat{g}\|) < 2\epsilon$.

Case 3: If $\phi \in \Gamma_2$, then $|\hat{f}(\phi)(1 - \hat{g}(\phi))| = 0$.

Therefore, $\|\hat{f} - \hat{f}\hat{g}\| < 2\epsilon$, implying that J_ψ is dense in M_ψ. □

Step 5. There exists a nonvanishing function $k : Y_c \to \mathbb{C}$ such that

$$\widehat{T}\hat{f}(\psi) = k(\psi)\hat{f}(h(\psi)),$$

for all $f \in B(X,E)$ and all $\psi \in Y_c$.

Proof. Let $\psi \in Y_c$. Since $\delta_\psi \circ \widehat{T}$ is a nonzero continuous linear functional on $\widehat{B(X,E)}$, it follows that $ker(\delta_\psi \circ \widehat{T})$ is a proper closed subspace of $\widehat{B(X,E)}$ and moreover, $J_\psi \subset ker(\delta_\psi \circ \widehat{T})$ by Step 2. Therefore, $ker\delta_{h(\psi)} = M_\psi \subset ker(\delta_\psi \circ \widehat{T})$ by Step 4. Hence there exists a nonzero scalar $k(\psi)$ such that $\delta_\psi \circ \widehat{T} = k(\psi)\delta_{h(\psi)}$, implying that $\widehat{T}\hat{f}(\psi) = k(\psi)\hat{f}(h(\psi))$ for all $f \in B(X,E)$. □

Step 6. The set Y_0 is closed in $M(B(Y,F))$ and the set Y_d is open in $M(B(Y,F))$.

Proof. Since $Y_0 = \cap_{f\in B(X,E)} ker(\widehat{T}\hat{f})$, it follows that Y_0 is closed in $M(B(Y,F))$. To show that Y_d is open in $M(B(Y,F))$, let $\{\psi_\gamma\}_{\gamma\in I}$ be a net in $M(B(Y,F))\backslash Y_d$, which converges to a point $\psi \in M(B(Y,F))$. By Step 5, there exists a nonvanishing bounded function $k : Y_c \to \mathbb{C}$ such that

$$\begin{aligned} |\widehat{T}\hat{f}(\psi_\gamma)| &\leq sup\{|\widehat{T}\hat{f}(\psi)| : \psi \in Y_0 \cup Y_c\} \leq sup\{|\widehat{T}\hat{f}(\psi)| : \psi \in Y_c\} \\ &\leq sup\{|k(\psi)\hat{f}(h(\psi))| : \psi \in Y_c\} \leq \|k\|\|\hat{f}\|, \end{aligned}$$

for all $f \in B(X,E)$ and $\gamma \in I$, where $\|k\|$ is the supremum norm of k. However, for the boundedness of k we may take $f = 1_E$, the unit element of $B(X,E)$, in $\widehat{T}\hat{f}(\psi) = k(\psi)\hat{f}(h(\psi))$ and conclude that k is bounded. By the continuity of $\widehat{T}\hat{f}$ on $M(B(Y,F))$, we have $|\widehat{T}\hat{f}(\psi)| \leq \|k\|\|\hat{f}\|$, that is, $|\delta_\psi \circ \widehat{T}(\hat{f})| \leq \|k\|\|\hat{f}\|$. Thus the linear functional $\delta_\psi \circ \widehat{T}$ is continuous on $\widehat{B(X,E)}$ and hence $\psi \in M(B(Y,F))\backslash Y_d$. This shows that $M(B(Y,F))\backslash Y_d$ is closed and hence Y_d is open in $M(B(Y,F))$. □

Step 7. $h(Y_d)$ is a finite set of nonisolated points of $M(B(X,E))$.

Proof. For the finiteness of $h(Y_d)$, let $(h(\psi_n))_{n\in\mathbb{N}}$ be a sequence of distinct elements of $M(B(X,E))$ such that $\psi_n \in Y_d$ for all $n \in \mathbb{N}$. Moreover, suppose that there exist sequences $(V_n)_{n\in\mathbb{N}}$ and $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint neighbourhoods of $h(\psi_n)$ such that $U_n \subseteq \overline{U_n} \subseteq V_n$ for all $n \in \mathbb{N}$. Since $B(X,E)$ is hyper-normal, for each n, there exists $g_n \in B(X,E)$ such that $\hat{g}_n = 1$ on U_n and $supp(\hat{g}_n) \subseteq V_n$. On the other hand, since the linear functional $\delta_{\psi_n} \circ \widehat{T}$ is discontinuous on $\widehat{B(X,E)}$, there exists a function $h_n \in B(X,E)$ with $\|h_n\| \leq 1$ such that $|\widehat{T}\hat{h}_n(\psi_n)| \geq n^3\|g_n\|$ for all $n \in \mathbb{N}$. If $f_n := \frac{g_n h_n}{n^2\|g_n\|}$ for $n \in \mathbb{N}$, then $\hat{f}_n - \frac{\hat{h}_n}{n^2\|g_n\|} = 0$ on U_n, implying that $h(\psi_n) \notin supp(\hat{f}_n - \frac{\hat{h}_n}{n^2\|g_n\|})$. Hence $|\widehat{T}\hat{f}_n(\psi_n)| = \frac{1}{n^2\|g_n\|}|\widehat{T}\hat{h}_n(\psi_n)|$ by Step 2, so that $|\widehat{T}\hat{f}_n(\psi_n)| \geq n$. Since $B(X,E)$ is complete and $\|f_n\| < \frac{1}{n^2}$ for all $n \in \mathbb{N}$, we can define the function $f = \sum_{n=1}^{\infty} f_n \in B(X,E)$. From the fact that the Gelfand transform is a linear continuous mapping, we deduce $\hat{f} = \sum_{n=1}^{\infty} \hat{f}_n$. Since the sequence $(V_n)_{n\in\mathbb{N}}$ is pairwise disjoint and $coz(\hat{f}_n) \subseteq V_n$ for all $n \in \mathbb{N}$, it follows that $h(\psi_m) \notin supp(\hat{f}_n)$ for all $n \neq m$.

We now show that $h(\psi_m) \notin supp(\sum_{n=1,n\neq m}^{\infty} \hat{f}_n)$. If $h(\psi_m) \in supp(\sum_{n=1,n\neq m}^{\infty} \hat{f}_n)$ then

$$h(\psi_m) \in \overline{coz(\sum_{n=1,n\neq m}^{\infty} \hat{f}_n)} \subseteq \overline{\cup_{n=1,n\neq m}^{\infty} coz(\hat{f}_n)}.$$

Since V_m is an open neighbourhood of $h(\psi_m)$, there exists an element $\varphi_m \in \cup_{n=1,n\neq m}^{\infty} coz(\hat{f}_n)$ such that $\varphi_m \in V_m$. On the other hand, there exists $n \neq m$ such that $\varphi_m \in coz(\hat{f}_n) \subseteq V_n$, which is a contradiction, since $V_n \cap V_m = \emptyset$.

By Step 2 we conclude that $\delta_{\psi_m} \circ \hat{T}(\sum_{n=1,n\neq m}^{\infty} \hat{f}_n) = 0$. Since

$$\hat{f} = \hat{f}_m + \sum_{n=1,n\neq m}^{\infty} \hat{f}_n,$$

it follows that $\delta_{\psi_m} \circ \hat{T}(\hat{f}) = \delta_{\psi_m} \circ \hat{T}(\hat{f}_m)$. Therefore,

$$|\widehat{T}\hat{f}(\psi_m)| = |\widehat{T}\hat{f}_m(\psi_m)| \geq m,$$

for all $m \in \mathbb{N}$, which is a contradiction, since $\widehat{T}\hat{f} \in \widehat{B(Y,F)}$ is bounded. This proves that $h(Y_d)$ is finite.

We now show that each point of $h(Y_d)$ is a nonisolated point of $M(B(X,E))$. Let $h(\psi)$ be an isolated point of $M(B(X,E))$ for some $\psi \in Y_d$. Then there exists a neighbourhood $U_{h(\psi)}$ such that $U_{h(\psi)} = \{h(\psi)\}$. If $\hat{f}(h(\psi)) = 0$, then $h(\psi) \notin supp(\hat{f})$ and hence $\widehat{T}\hat{f}(\psi) = 0$, by Step 2. In other words, $ker(\delta_{h(\psi)}) \subseteq ker(\delta_\psi \circ \widehat{T})$ and therefore, $\delta_\psi \circ \widehat{T} = \beta_\psi \delta_{h(\psi)}$ for some nonzero scalar β_ψ. Consequently, the nonzero linear functional $\delta_\psi \circ \widehat{T}$ is continuous on $\widehat{B(X,E)}$ and hence $\psi \in Y_c$, which is a contradiction. $\square$

The proof of the theorem is now complete. $\square$

Note that the method of Jimenez–Vargas in [16] is only valid for the Lipschitz algebras, whereas by our method, the same results are valid for more general classes of vector-valued Banach algebras. We are now ready to prove that, under the same conditions as in the theorem above, every separating linear bijection between certain Banach algebras of vector-valued functions is biseparating. This is the most important part of the following theorem:

Theorem 3.2. *Let X, Y be compact Hausdorff spaces, E, F be unital commutative Banach algebras and $B(X,E)$, $B(Y,F)$ be hyper-normal semisimple commutative unital Banach algebras. Let T be a separating linear bijection from $B(X,E)$ onto $B(Y,F)$. Then $\widehat{T}$ is a weighted composition operator in the form $\widehat{T}\hat{f}(\psi) = k(\psi)\hat{f}(h(\psi))$ for all $f \in B(X,E)$ and for all $\psi \in M(B(Y,F))$, where $k \in \widehat{B(Y,F)}$ is a nonvanishing function and h is a homeomorphism from $M(B(Y,F))$ onto $M(B(X,E))$. In particular, T is biseparating.*

Proof. We adopt the same notations as in the previous theorem and divide the proof into several parts.

Part 1. $Y_0 = \emptyset$ and Y_c is compact.

Proof. Let $\psi \in Y_0$. Then $\delta_\psi \circ \widehat{T} = 0$ and $\delta_\psi(\widehat{T}\hat{f}) = 0$ for every $f \in B(X,E)$. Since T is surjective, $\widehat{T}$ is also surjective and hence for every $g \in B$, there exists $f \in B(X,E)$ such that $\hat{g} = \widehat{T}\hat{f}$. Thus $\delta_\psi(\hat{g}) = \psi(g) = 0$ for all $g \in B$ and hence $\psi = 0$, which is impossible. Therefore, $Y_0 = \emptyset$.

Since by Theorem 3.1, the set Y_d is open in $M(B(Y,F))$, it follows that $Y_c = M(B(Y,F))\backslash Y_d$ is closed and hence it is compact in $M(B)$. $\square$

Part 2. The set $h(Y_c)$ is dense in $M(B(X,E))$.

Proof. We first prove that $h(Y_c \cup Y_d)$ is dense in $M(B(X,E))$. Suppose, on the contrary, that there exists a point $\phi \in M(B(X,E))$ such that $V_\phi \cap h(Y_c \cup Y_d) = \emptyset$, where V_ϕ is a neighbourhood of ϕ. Let U_ϕ be a neighbourhood of ϕ such that $U_\phi \subseteq \overline{U_\phi} \subseteq V_\phi$ and h_ϕ be a nonzero function in $B(X,E)$ such that $supp(\hat{h}_\phi) \subseteq \overline{U_\phi}$. Hence $h(\psi) \notin supp(\hat{h}_\phi)$ for all $\psi \in Y_c \cup Y_d$, implying that $\widehat{T}\hat{h}_\phi(\psi) = 0$ for all $\psi \in Y_c \cup Y_d$, by Theorem 3.1. Since Y_0 is empty, $\widehat{T}\hat{h}_\phi = 0$. By the linearity and injectivity of $\widehat{T}$, it follows that $\hat{h}_\phi = 0$. Since $B(X,E)$ is semisimple, $h_\phi = 0$, which is impossible.

We now show that $\overline{h(Y_c \cup Y_d)} = \overline{h(Y_c)}$. It suffices to prove that $h(Y_d) \subseteq \overline{h(Y_c)}$. Let $\phi \in h(Y_d)$ and there exist a neighbourhood U_ϕ of ϕ such that $U_\phi \cap h(Y_c) = \emptyset$. Since $h(Y_d)$ is finite, there exists a neighbourhood V_ϕ of ϕ such that

$$V_\phi \backslash \{\phi\} \cap h(Y_c \cup Y_d) = \emptyset.$$

Since by Theorem 3.1, ϕ is a nonisolated point of $M(B(X,E))$, there is a point ψ in $V_\phi \backslash \{\phi\}$. Let V_ψ be a neighbourhood of ψ such that $V_\psi \subseteq V_\phi \backslash \{\phi\}$. Then $V_\psi \cap h(Y_c \cup Y_d) = \emptyset$ and this contradicts the density of $h(Y_c \cup Y_d)$ in $M(B(X,E))$. Hence $\overline{h(Y_c \cup Y_d)} = \overline{h(Y_c)}$. $\square$

Part 3. Y_d is empty.

Proof. Let $\phi \in Y_d$. Since Y_c is closed, there exists a neighbourhood V_ϕ such that $V_\phi \cap Y_c = \emptyset$. By the normality of $B(X,E)$, there exists a function h_ϕ in $B(Y,F)$ such that $\hat{h}_\phi(\phi) = 1$ and $coz(\hat{h}_\phi) \subseteq V_\phi$. By the surjectivity of $\widehat{T}$, there exists some f in $B(X,E)$ such that $\widehat{T}\hat{f} = \hat{h}_\phi$. Then $\widehat{T}\hat{f}(\psi) = \hat{h}_\phi(\psi) = 0$ for all $\psi \in Y_c$. By Theorem 3.1, $\widehat{T}\hat{f}(\psi) = k(\psi)\hat{f}(h(\psi))$ for all $\psi \in Y_c$. Since T is surjective, $k(\psi) \neq 0$ for all $\psi \in Y_c$. Hence $\hat{f}(\lambda) = 0$ for all $\lambda \in M(B(X,E))$, since $\overline{h(Y_c)} = M(B(X,E))$ by Part 2. Therefore, $\hat{f} = 0$ and $\widehat{T}\hat{f} = 0$, but $\widehat{T}\hat{f}(\phi) = \hat{h}_\phi(\phi) = 1$, which is a contradiction. $\square$

Part 4. $M(B(Y,F)) = Y_c$ and $\widehat{T}\hat{f}(\psi) = k(\psi)\hat{f}(h(\psi))$ for all $f \in B(X,E)$ and $\psi \in M(B(Y,F))$.

Proof. By Parts 1 and 3, $Y_0 = Y_d = \emptyset$. Hence $M(B(Y,F)) = Y_c$ and the result follows from Theorem 3.1. $\square$

Part 5. T^{-1} is separating and hence T is biseparating.

Proof. Let $g_1, g_2 \in B(Y,F)$ such that $\hat{g}_1\hat{g}_2 = 0$. Then there exist $f_1, f_2 \in B(X,E)$ such that $\hat{g}_1 = \hat{T}\hat{f}_1 = k(\hat{f}_1 \circ h)$, $\hat{g}_2 = \hat{T}\hat{f}_2 = k(\hat{f}_2 \circ h)$ and hence $k^2(\hat{f}_1 \circ h)(\hat{f}_2 \circ h) = 0$. Since $k(\psi) \neq 0$ for every $\psi \in M(B(Y,F))$, we have $(\hat{f}_1\hat{f}_2) \circ h = 0$. By Parts 2, 4 and the density of $h(M(B(Y,F)))$ in $M(B(X,E))$, it follows that $\hat{f}_1\hat{f}_2 = 0$ and hence $\widehat{T^{-1}}$ is separating. Since $B(X,E)$ and $B(Y,F)$ are semisimple commutative

Banach algebras, $T^{-1} : B(Y,F) \to B(X,E)$ is separating if and only if $\widehat{T^{-1}} : \widehat{B(Y,F)} \to \widehat{B(X,E)}$ is separating. Consequently, T^{-1} is separating. $\square$

Part 6. The map h is a homeomorphism from $M(B(Y,F))$ onto $M(B(X,E))$.

Proof. For the injectivity of $h : M(B(Y,F)) \to M(B(X,E))$, let ϕ, ψ be elements of $M(B(Y,F))$ with $\phi \neq \psi$ and $h(\phi) = h(\psi)$. Let U_ψ be a neighbourhood of ψ such that $\phi \notin U_\psi$. Consider $h_\psi \in B(Y,F)$ such that $\hat{h}_\psi(\psi) = 1$ and $coz(\hat{h}_\psi) \subseteq U_\psi$. Since $h_\psi \in B(Y,F)$ and $\widehat{T}$ is surjective, $\widehat{T}\hat{f} = \hat{h}_\psi$ for some $f \in B(X,E)$. By Part 4 we have

$$\hat{h}_\psi(\lambda) = \widehat{T}\hat{f}(\lambda) = k(\lambda)\hat{f}(h(\lambda)) \quad (\lambda \in M(B(Y,F))).$$

In particular, $1 = \hat{h}_\psi(\psi) = k(\psi)\hat{f}(h(\psi))$ and $0 = \hat{h}_\psi(\phi) = k(\phi)\hat{f}(h(\phi))$. Hence $\hat{f}(h(\psi)) = 1/k(\psi)$ and $\hat{f}(h(\phi)) = 0$. Since $h(\psi) = h(\phi)$, we get a contradiction. On the other hand, by Parts 2 and 4, $\overline{h(M(B(Y,F)))} = M(B(X,E))$ and hence h is continuous by Theorem 3.1. Since $M(B(Y,F))$ is compact, it follows that $h(M(B(Y,F))) = M(B(X,E))$. Therefore, $h : M(B(Y,F)) \to M(B(X,E))$ is surjective. $\square$

The proof of the theorem is now complete. $\square$

We should mention here that the proof of the theorem above follows closely [16, Theorem 3.1], except for the continuity of T. One may also compare this theorem with [13, Theorem 2.3].

Remark 3.3. By applying the results of Section 2, we conclude that if E and F are semisimple hyper-normal unital commutative Banach algebras, then the Lipschitz algebras $Lip^\alpha(X,E)$ and $\ell ip^\alpha(X,E)$ possess the same properties. Hence in this case big and little Lipschitz algebras are interesting examples satisfying the hypotheses of Theorems 3.1 and 3.2.

In [10] Esmaeili and Mahyar characterized disjointness preserving bounded linear operators between spaces of vector-valued little Lipschitz functions on compact metric spaces. In fact, they have shown that every disjointness preserving bounded linear operator between spaces of vector-valued little Lipschitz functions is a weighted composition operator.

In the following, disjointness preserving linear operators between big and little vector-valued Lipschitz algebras are characterized, without the boundedness condition.

Theorem 3.4. [17, Theorems 3.1 and 4.1] *Let X, Y be compact metric spaces and E, F be Banach algebras. Let T be a bijection from $Lip^\alpha(X,E)$ ($\ell ip^\alpha(X,E)$) onto $Lip^\alpha(Y,F)$ ($\ell ip^\alpha(Y,F)$) such that both T and T^{-1} are disjointness preserving maps. Then T is a weighted composition operator in the form*

$$Tf(y) = h(y)(f(\phi(y))), \quad (y \in Y, f \in Lip^\alpha(X,E)(\ell ip^\alpha(X,E))),$$

where ϕ is a homeomorphism from Y onto X and $h(y)$ is an invertible linear map from E onto F for each $y \in Y$. Moreover, T is bounded if and only if $h(y)$ is bounded for all $y \in Y$.

Corollary 3.5. *Let X, Y be compact metric spaces and E, F be Banach algebras. If T is a bijection from $Lip^\alpha(X, E)$ ($\ell ip^\alpha(X, E)$) onto $Lip^\alpha(Y, F)$ ($\ell ip^\alpha(Y, F)$), such that both T and T^{-1} are disjointness preserving, then X is homeomorphic to Y. In particular, if T is multiplicative then $M(E)$ is homeomorphic to $M(F)$.*

Proof. By Theorem 3.4, X is homeomorphic to Y. In the case that T is multiplicative we actually show that $M(E)$ is homeomorphic to $M(F)$. For this purpose, let y_0 be a fixed element of Y and define $\lambda : M(F) \to M(E)$ by $\lambda(\psi) = \psi \circ h(y_0)$. We first show that λ is well-defined.

By Theorem 3.4, T is a weighted composition operator in the form

$$Tf(y_0) = h(y_0)(f(\phi(y_0))), \quad (f \in Lip^\alpha(X, E)(\ell ip^\alpha(X, E))),$$

where ϕ is a homeomorphism from Y onto X and $h(y)$ is an invertible linear map from E onto F for each $y \in Y$. First we show that $h(y_0)$ is, in fact, a homomorphism. To this end, let $a, b \in E$ and take the constants functions $f = a, g = b$ in $Lip^\alpha(X, E)$. Since T is multiplicative, we have

$$\begin{aligned} h(y_0)(ab) &= h(y_0)(f(\phi(y_0))g(\phi(y_0))) = h(y_0)(fg(\phi(y_0))) = Tfg(y_0) \\ &= Tf(y_0)Tg(y_0) = h(y_0)(f(\phi(y_0)))h(y_0)(g(\phi(y_0))) \\ &= h(y_0)(a)h(y_0)(b). \end{aligned}$$

Therefore, $h(y_0)$ is a homomorphism and since $h(y_0)$ is a linear map, it follows that $\psi \circ h(y_0)$ is a homomorphism for all $\psi \in M(F)$. Since ψ is a character, there exists $b \in F$ such that $\psi(b) \neq 0$ and since $h(y_0)$ is onto, we have $h(y_0)(a) = b$, for some $a \in E$. Therefore, $\psi \circ h(y_0)(a) \neq 0$ and hence $\psi \circ h(y_0) \in M(E)$, which implies that λ is well defined.

For the injectivity of λ, let $\lambda(\psi_1) = \lambda(\psi_2)$. Then $\psi_1 \circ h(y_0) = \psi_2 \circ h(y_0)$ and hence for every $a \in E, \psi_1(h(y_0)(a)) = \psi_2(h(y_0)(a))$. Thus for every $b \in F, \psi_1(b) = \psi_2(b)$, since $h(y_0)$ is onto. Therefore, $\psi_1 = \psi_2$. For the surjectivity of λ, let $\varphi \in M(E)$ and note that $\varphi \circ h(y_0)^{-1} \in M(F)$. Then

$$\lambda(\varphi \circ h(y_0)^{-1}) = \varphi \circ h(y_0)^{-1} \circ h(y_0) = \varphi.$$

Hence λ is onto and moreover, since $M(E)$ and $M(F)$ are compact Hausdorff spaces, λ^{-1} is continuous. Therefore, λ is a homeomorphism and hence $M(E)$ is homeomorphic to $M(F)$. $\square$

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