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SKEW SYMMETRY OF A CLASS OF OPERATORS

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Abstract. An operator T on a complex Hilbert space $\mathcal{H}$ is said to be skew symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \to \mathcal{H}$ such that $CTC = -T^*$. In this paper, using an interpolation theorem related to conjugations, we give a geometric characterization for a class of operators to be skew symmetric. As an application, we get a description of skew symmetric partial isometries.

1. Introduction

The main aim of this paper is to give a geometric characterization of a class of operators being skew symmetric. This work is a continuation of [13] in which the first author and Zhu characterize skew symmetric normal operators. Let us first recall a few definitions.

Throughout this paper, we always denote by $\mathcal{H}$ a complex separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and by $\mathcal{K}(\mathcal{H})$ the ideal consisting of all compact operators on $\mathcal{H}$.

Definition 1.1. A conjugation on $\mathcal{H}$ is a conjugate-linear map $C : \mathcal{H} \longrightarrow \mathcal{H}$ satisfying $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$.

Definition 1.2. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric, if there exists a conjugation C on $\mathcal{H}$ so that $CTC = T^*$. An operator $T \in \mathcal{B}(\mathcal{H})$ is

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said to be skew symmetric, if there exists a conjugation C on $\mathcal{H}$ so that $CTC = -T^*$.

In matrix analysis, there is a lot of work on the theory of symmetric matrices and skew symmetric matrices, which has many motivations in function theory, complex analysis, moment problems and other mathematical disciplines. As a generalization of complex symmetric matrices, Garcia and Putinar [4] initiated the study for complex symmetric operators, which have many motivations in function theory, matrix analysis and other areas. Some important results concerning the internal structure of complex symmetric operators have been obtained (see [4, 5, 10, 6, 3, 8, 2, 9, 1] for references). An effective way to investigate the structure of complex symmetric operators is to characterize which special operators are complex symmetric. In fact, a lot of work concerning complex symmetric operators focuses on this basic question. Many important classes of operators such as compact operators, weighted shifts and partial isometries are studied [4, 8, 20]. However, less attention has been paid to skew symmetric operators. Using [4, Lemma 1], we conclude that an operator $T \in \mathcal{B}(\mathcal{H})$ is skew symmetric if and only if T admits a skew symmetric matrix representation with respect to some ONB $\{e_n\}$ of $\mathcal{H}$, that is $\langle Te_n, e_m\rangle = -\langle Te_m, e_n\rangle$ for all m, n. In [17], Zagorodnyuk studied the polar decomposition of skew symmetric operators and obtained some basic properties of skew symmetric operators. As the study of complex symmetric operators, an important way to investigate the structure of skew symmetric operators is to characterize the skew symmetry of concrete class of operators. In [18], Zagorodnyuk studied the skew symmetry of cyclic operators. In particular, Zhu and the first author [13] study the skew symmetry of normal operators and give two structure theorems of skew symmetric normal operators.

For non-zero vectors $u, v \in \mathcal{H}$, the rank-one operator $u \otimes v \in \mathcal{B}(\mathcal{H})$ is defined by $(u \otimes v)x = \langle x, v\rangle u$, $\forall x \in \mathcal{H}$.

The main aim of this paper is to give a geometric characterization of the following operators $T \in \mathcal{B}(\mathcal{H})$ to be skew symmetric.

$$T = \sum_{i\in\Lambda} a_i(e_i^{(1)} \otimes f_i^{(1)} - e_i^{(2)} \otimes f_i^{(2)}) \tag{1.1}$$

where $\Lambda \subset \mathbb{N}$, $a_i > 0$ for all $i \in \Lambda$, $\{e_i^{(1)}, e_i^{(2)}\}_{i\in\Lambda}$, $\{f_i^{(1)}, f_i^{(2)}\}_{i\in\Lambda}$ are two orthonormal subsets of $\mathcal{H}$. In particular, when $a_i \neq a_j$ for all $i, j \in \Lambda$ and $i \neq j$, the characterization is more explicit.

For $p, q \in \{1, 2\}$, we denote

$$(p', q') = \begin{cases} (1,2), \text{when } (p,q) = (1,2); \\ (2,2), \text{when } (p,q) = (1,1); \\ (2,1), \text{when } (p,q) = (2,1); \\ (1,1), \text{when } (p,q) = (2,2). \end{cases}$$

Theorem 1.3 (Main Theorem). *Assume $\{e_i^{(1)}, e_i^{(2)}\}_{i\in\Lambda}$, $\{f_i^{(1)}, f_i^{(2)}\}_{i\in\Lambda}$ are two orthonormal subsets of $\mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ can be written as*

$$T = \sum_{i\in\Lambda} a_i (e_i^{(1)} \otimes f_i^{(1)} - e_i^{(2)} \otimes f_i^{(2)})$$

where $\Lambda \subset \mathbb{N}$, $a_i > 0$ for all $i \in \Lambda$ and $a_i \neq a_j$ for all $i, j \in \Lambda$ with $i \neq j$. Then the following are equivalent.

(1) *T is skew symmetric;*
(2) *There exist $\{\lambda_i\}_{i\in\Lambda} \subset \mathbb{C}$ with $|\lambda_i| = 1$ for all $i \in \Lambda$ such that $\lambda_i \langle e_i^{(p)}, f_j^{(q)}\rangle = \lambda_j \langle e_j^{(p')}, f_i^{(q')}\rangle$ for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$.*
(3) *$|\langle e_i^{(p)}, f_j^{(q)}\rangle| = |\langle e_j^{(p')}, f_i^{(q')}\rangle|$ for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$, and*

$$\Big[\prod_{k=1}^{n-1} \langle e_{i_k}^{(p_k)}, f_{i_{k+1}}^{(q_k)}\rangle\Big] \langle e_{i_n}^{(p_n)}, f_{i_1}^{(q_n)}\rangle = \langle e_{i_1}^{(p'_n)}, f_{i_n}^{(q'_n)}\rangle \Big[\prod_{k=1}^{n-1} \langle e_{i_{k+1}}^{(p'_k)}, f_{i_k}^{(q'_k)}\rangle\Big]$$

for any $n \in \mathbb{N}$ and any tuple $\{i_1, i_2, \cdots, i_n\}$ in Λ^n, and any $p_k, q_k \in \{1, 2\}$ for $1 \le k \le n$.

The rest of this paper is organized as follows. In Sect.2, we give a polar decomposition theorem for skew symmetric operators and recall an interpolation theorem related to conjugations on Hilbert spaces. Further, we give a geometric characterization of the skew symmetry of the operators with form (1.1)(See Theorem 2.5). In Sect. 3, we prove Theorem 1.3. In Sect. 4, we shall characterize skew symmetric compact operators. Our result shows that skew symmetric compact operators admit the form (1.1). In Sect. 5, using Theorem 2.5, we characterize skew symmetric partial isometries.

2. Preliminaries

First, we introduce some notions.

Definition 2.1. An anticonjugation on $\mathcal{H}$ is a conjugate linear map $K : \mathcal{H} \to \mathcal{H}$ satisfying that $K^2 = -I$ and $\langle Kx, Ky\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$.

The above concept is defined in [14]. Garcia and Tener [7] prove that there is no anticonjugation on an odd dimensional Hilbert space. In particular, they show that a conjugate linear map K on $\mathbb{C}^{2n}$ is an anticonjugation if and only if there exists an ONB $\{e_1, e_2, \cdots, e_{2n}\}$ of $\mathcal{H}$ such that $Ke_i = e_{n+i}$ and $Ke_{n+i} = -e_i$ for all $1 \le i \le n$. Using a similar argument in the proof of [7, Lemma 4.4] and Zorn's Lemma, we can get the following result.

Lemma 2.2. *Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space and K be an isometric conjugate linear map on $\mathcal{H}$. Then K is an anticonjugation if and only if there exists an ONB $\{e_n, f_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ such that $Ke_n = f_n$ and $Kf_n = -e_n$ for each $n \ge 1$.*

We say that a conjugate-linear map $K : \mathcal{H} \to \mathcal{H}$ is a partial anticonjugation supported on $(\ker K)^\perp$ if $\ker K$ reduces K and $K|_{(\ker K)^\perp}$ is an anticonjugation.

The following result is based on a technique of Garcia and Putinar [5, Theorem 2].

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be skew symmetric, that is, $CTC = -T^*$ for some conjugation C on $\mathcal{H}$, and $T = U|T|$ be the polar decomposition of T. Then $T = CK|T|$, where K is a partial anticonjugation supported on $\overline{\text{ran}\,(|T|)}$, which commutes with $|T|$. In particular, $CUC = -U^*$.*

Proof. Since $T = -CT^*C$ and $T = U|T|$ is the polar decomposition of T, we have

$$T = -C|T|U^*C = -CU^*U|T|U^*C = (-CU^*C)(CU|T|U^*C).$$

It is easy to check that $-CU^*C$ is a partial isometry and $CU|T|U^*C$ is positive with $\ker(-CU^*C) = \ker(CU|T|U^*C)$. By the uniqueness of the polar decomposition of T, we have $U = -CU^*C$ and $|T| = CU|T|U^*C$.

We denote $K = CU$, it follows that $K = CU = -U^*C$. It is easy to check that K is conjugate linear and $-K^2 = U^*U$, the orthogonal projection on $\overline{\text{ran}\,(|T|)}$. Also, we have $\langle Kx, Ky\rangle = \langle CUx, CUy\rangle = \langle Uy, Ux\rangle = \langle U^*Uy, x\rangle = \langle y, x\rangle$ for all $x, y \in \overline{\text{ran}\,(|T|)}$. Since $|T| = CU|T|U^*C$, we have $|T| = -K|T|K$, and hence $K|T| = |T|K$. ☐

The following interpolation theorem related to conjugations is very useful to study the complex symmetry of operators. Using it Zhu and the first author give a geometric characterization for a norm-dense class of operators to be complex symmetric, see [19].

Lemma 2.4 ([19], Theorem 2.1)**.** *Let $\{e_i\}_{i\in\Lambda}$ and $\{f_i\}_{i\in\Lambda}$ be two orthonormal subsets of $\mathcal{H}$, where $\Lambda \subset \mathbb{N}$. Then there exists a conjugation C on $\mathcal{H}$ such that $Ce_i = f_i$ for all $i \in \Lambda$ if and only if $\langle e_i, f_j\rangle = \langle e_j, f_i\rangle$ for all $i, j \in \Lambda$.*

Let $T \in \mathcal{B}(\mathcal{H})$ and assume that T admits the following representation

$$T = \sum_{i\in\Lambda} a_i(e_i^{(1)} \otimes f_i^{(1)} - e_i^{(2)} \otimes f_i^{(2)}),$$

where $\Lambda \subset \mathbb{N}$, $a_i > 0$ for all $i \in \Lambda$, $\{e_i^{(1)}, e_i^{(2)}\}_{i\in\Lambda}$, $\{f_i^{(1)}, f_i^{(2)}\}_{i\in\Lambda}$ are two orthonormal subsets of $\mathcal{H}$. Obviously, there exists a partition $\Lambda = \cup_{k\in\Gamma}(\Lambda_k)$ of Λ such that $a_i = a_j$ if and only if $i, j \in \Lambda_k$ for some $k \in \Gamma$.

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be as above. Then T is skew symmetric if and only if there exists an* ONB *$\{g_i^{(1)}, g_i^{(2)}\}_{i\in\Lambda}$ of $\bigvee\{f_i^{(1)}, f_i^{(2)} : i \in \Lambda\}$ such that*

(1) *$\bigvee\{g_i^{(1)}, g_i^{(2)} : i \in \Lambda_k\} = \bigvee\{f_i^{(1)}, f_i^{(2)} : i \in \Lambda_k\}$ for all $k \in \Gamma$.*
(2) *$\langle g_i^{(p)}, f_j^{(q)}\rangle = (-1)^{(p+q)}\langle g_j^{(p')}, f_i^{(q')}\rangle$ for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$.*
(3) *$\langle g_i^{(p)}, e_j^{(q)}\rangle = \langle g_j^{(p')}, e_i^{(q')}\rangle$ for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$.*

Proof. "$\Longrightarrow$." Assume that T is C-skew symmetric, that is, $CTC = -T^*$. Let $g_i^{(1)} = Ce_i^{(2)}$ and $g_i^{(2)} = Ce_i^{(1)}$ for all $i \in \Lambda$. Then $\{g_i^{(1)}, g_i^{(2)} : i \in \Lambda\}$ is an orthonormal subset of $\mathcal{H}$.

Assume that $T = U|T|$ is the polar decomposition of T. One can easily deduce that $|T| = \sum_{i\in\Lambda} a_i(f_i^{(1)} \otimes f_i^{(1)} + f_i^{(2)} \otimes f_i^{(2)})$ and $U = \sum_{i\in\Lambda}(e_i^{(1)} \otimes f_i^{(1)} - e_i^{(2)} \otimes f_i^{(2)})$.

By Lemma 2.3, we have $U = CK$, where K is a partial anticonjugation supported on ran $|T|$ and $K|T| = |T|K$. It is easy to see that K is an anticonjugaton on each eigenspace of $|T|$. It follows that $K(\bigvee_{i\in\Lambda_k}\{f_i^{(1)}, f_i^{(2)}\}) \subset \bigvee_{i\in\Lambda_k}\{f_i^{(1)}, f_i^{(2)}\}$. Noting that $-K^2 = U^*U$, the orthogonal projection with the range $\bigvee_{i\in\Lambda}\{f_i^{(1)}, f_i^{(2)}\}$, we have $K(\bigvee_{i\in\Lambda_k}\{f_i^{(1)}, f_i^{(2)}\}) = \bigvee_{i\in\Lambda_k}\{f_i^{(1)}, f_i^{(2)}\}$. Since $U(\bigvee_{i\in\Lambda_k}\{f_i^{(1)}, f_i^{(2)}\}) = \bigvee_{i\in\Lambda_k}\{e_i^{(1)}, e_i^{(2)}\}$ and $CU = K$, we have $C[\bigvee_{i\in\Lambda_k}\{e_i^{(1)}, e_i^{(2)}\}] = \bigvee_{i\in\Lambda_k}\{f_i^{(1)}, f_i^{(2)}\}$ for each $k \in \Gamma$. Moreover, we have $\bigvee_{i\in\Lambda_k}\{g_i^{(1)}, g_i^{(2)}\} = C[\bigvee_{i\in\Lambda_k}\{e_i^{(1)}, e_i^{(2)}\}] = \bigvee_{i\in\Lambda_k}\{f_i^{(1)}, f_i^{(2)}\}$. Hence $a_i \neq a_j$ implies that $\langle g_i^{(p)}, f_j^{(q)}\rangle = 0 = \langle g_j^{(p')}, f_i^{(q')}\rangle$ for all $p, q \in \{1, 2\}$.

On the other hand, if $i, j \in \Lambda$ with $a_i = a_j$, we have

$$\begin{aligned} a_j\langle g_i^{(1)}, f_j^{(1)}\rangle &= \langle g_i^{(1)}, T^*e_j^{(1)}\rangle = \langle CT^*e_j^{(1)}, Cg_i^{(1)}\rangle \\ &= \langle -TCe_j^{(1)}, Cg_i^{(1)}\rangle = -\langle g_j^{(2)}, T^*e_i^{(2)}\rangle = a_i\langle g_j^{(2)}, f_i^{(2)}\rangle, \end{aligned}$$

and hence $\langle g_i^{(1)}, f_j^{(1)}\rangle = \langle g_j^{(2)}, f_i^{(2)}\rangle$. Also we have

$$\begin{aligned} a_j\langle g_i^{(1)}, f_j^{(2)}\rangle &= \langle g_i^{(1)}, -T^*e_j^{(2)}\rangle = \langle -CT^*e_j^{(2)}, Cg_i^{(1)}\rangle \\ &= \langle TCe_j^{(2)}, Cg_i^{(1)}\rangle = \langle g_j^{(1)}, T^*e_i^{(2)}\rangle = -a_i\langle g_j^{(1)}, f_i^{(2)}\rangle, \end{aligned}$$

and hence $\langle g_i^{(1)}, f_j^{(2)}\rangle = -\langle g_j^{(1)}, f_i^{(2)}\rangle$. Moreover, we have

$$\begin{aligned} a_j\langle g_i^{(2)}, f_j^{(1)}\rangle &= \langle g_i^{(2)}, T^*e_j^{(1)}\rangle = \langle CT^*e_j^{(1)}, Cg_i^{(2)}\rangle \\ &= \langle -TCe_j^{(1)}, Cg_i^{(2)}\rangle = \langle -g_j^{(2)}, T^*e_i^{(1)}\rangle = -a_i\langle g_j^{(2)}, f_i^{(1)}\rangle, \end{aligned}$$

and hence $\langle g_i^{(2)}, f_j^{(1)}\rangle = -\langle g_j^{(2)}, f_i^{(1)}\rangle$.

For all $i, j \in \Lambda$, a direct calculation shows that

$$\langle g_i^{(1)}, e_j^{(1)}\rangle = \langle Ce_i^{(2)}, e_j^{(1)}\rangle = \langle Ce_j^{(1)}, e_i^{(2)}\rangle = \langle g_j^{(2)}, e_i^{(2)}\rangle,$$

$$\langle g_i^{(1)}, e_j^{(2)}\rangle = \langle Ce_i^{(2)}, e_j^{(2)}\rangle = \langle Ce_j^{(2)}, e_i^{(2)}\rangle = \langle g_j^{(1)}, e_i^{(2)}\rangle,$$

and

$$\langle g_i^{(2)}, e_j^{(1)}\rangle = \langle Ce_i^{(1)}, e_j^{(1)}\rangle = \langle Ce_j^{(1)}, e_i^{(1)}\rangle = \langle g_j^{(2)}, e_i^{(1)}\rangle.$$

"$\Longleftarrow$." Since (3) holds for the orthonormal subsets $\{e_i^{(1)}, e_i^{(2)} : i \in \Lambda\}$ and $\{g_i^{(1)}, g_i^{(2)} : i \in \Lambda\}$, by Lemma 2.4, there exists a conjugation C on $\mathcal{H}$ such that $Ce_i^{(1)} = g_i^{(2)}$ and $Ce_i^{(2)} = g_i^{(1)}$ for all $i \in \Lambda$. Let $M = \bigvee\{e_i^{(1)}, e_i^{(2)} : i \in \Lambda\}$ and $N = \bigvee\{g_i^{(1)}, g_i^{(2)} : i \in \Lambda\}$. Note that ran $T \subset M$ and ran $T^* \subset N$, it suffices to show that $CT^*Cg_i^{(1)} = -Tg_i^{(1)}$, $CT^*Cg_i^{(2)} = -Tg_i^{(2)}$, $CT^*Ce_i^{(1)} = -Te_i^{(1)}$ and $CT^*Ce_i^{(2)} = -Te_i^{(2)}$ for all $i \in \Lambda$.

For $i, j \in \Lambda$, we claim that $a_i\langle g_j^{(1)}, f_i^{(2)}\rangle = -a_j\langle g_i^{(1)}, f_j^{(2)}\rangle$, $a_i\langle g_j^{(1)}, f_i^{(1)}\rangle = a_j\langle g_i^{(2)}, f_j^{(2)}\rangle$ and $-a_i\langle g_j^{(2)}, f_i^{(1)}\rangle = a_j\langle g_i^{(2)}, f_j^{(1)}\rangle$ for all $i, j \in \Lambda$. When $a_i = a_j$, by condition (2), this is obvious. If $a_i \neq a_j$, i, j lie in different Λ_k's, in view of condition (1), we have $\langle g_i^{(p)}, f_j^{(q)}\rangle = 0 = \langle g_j^{(p')}, f_i^{(q')}\rangle$ for all $p, q \in \{1, 2\}$. This proves the claim.

Let $i \in \Lambda$ be fixed. We have

$$\begin{aligned} CT^*Cg_i^{(1)} &= CT^*e_i^{(2)} = -a_iCf_i^{(2)} = -a_iC[\sum_{j\in\Lambda}(\langle f_i^{(2)}, g_j^{(1)}\rangle g_j^{(1)} + \langle f_i^{(2)}, g_j^{(2)}\rangle g_j^{(2)})] \\ &= -a_i[\sum_{j\in\Lambda}(\langle g_j^{(1)}, f_i^{(2)}\rangle e_j^{(2)} + \langle g_j^{(2)}, f_i^{(2)}\rangle e_j^{(1)})] \\ &= \sum_{j\in\Lambda}[a_j\langle g_i^{(1)}, f_j^{(2)}\rangle e_j^{(2)} - a_j\langle g_i^{(1)}, f_j^{(1)}\rangle e_j^{(1)}] \\ &= -Tg_i^{(1)}. \end{aligned}$$

Also we have

$$\begin{aligned} CT^*Cg_i^{(2)} &= CT^*e_i^{(1)} = a_iCf_i^{(1)} = a_iC[\sum_{j\in\Lambda}(\langle f_i^{(1)}, g_j^{(1)}\rangle g_j^{(1)} + \langle f_i^{(1)}, g_j^{(2)}\rangle g_j^{(2)})] \\ &= a_i\sum_{j\in\Lambda}[\langle g_j^{(1)}, f_i^{(1)}\rangle e_j^{(2)} + \langle g_j^{(2)}, f_i^{(1)}\rangle e_j^{(1)}] \\ &= \sum_{j\in\Lambda}[a_j\langle g_i^{(2)}, f_j^{(2)}\rangle e_j^{(2)} - a_j\langle g_i^{(2)}, f_j^{(1)}\rangle e_j^{(1)}] \\ &= -Tg_i^{(2)}. \end{aligned}$$

From the above equalities we have $Tg_i^{(1)} = a_iCf_i^{(2)}$ and $Tg_i^{(2)} = -a_iCf_i^{(1)}$.

Denote by P_N the orthogonal projection of $\mathcal{H}$ onto N. We have

$$\begin{aligned} CT^*Ce_i^{(1)} &= CT^*g_i^{(2)} = C[\sum_{j\in\Lambda}(a_j\langle g_i^{(2)}, e_j^{(1)}\rangle f_j^{(1)} - a_j\langle g_i^{(2)}, e_j^{(2)}\rangle f_j^{(2)})] \\ &= \sum_{j\in\Lambda}[a_j\langle e_j^{(1)}, g_i^{(2)}\rangle Cf_j^{(1)} - a_j\langle e_j^{(2)}, g_i^{(2)}\rangle Cf_j^{(2)}] \\ &= -\sum_{j\in\Lambda}[\langle e_j^{(1)}, g_i^{(2)}\rangle Tg_j^{(2)} + \langle e_j^{(2)}, g_i^{(2)}\rangle Tg_j^{(1)}] \\ &= -T(\sum_{j\in\Lambda}[\langle e_i^{(1)}, g_j^{(2)}\rangle g_j^{(2)} + \langle e_i^{(1)}, g_j^{(1)}\rangle g_j^{(1)}]) \\ &= -TP_Ne_i^{(1)} = -\sum_{j\in\Lambda}a_j[\langle P_Ne_i^{(1)}, f_j^{(1)}\rangle e_j^{(1)} - \langle P_Ne_i^{(1)}, f_j^{(2)}\rangle e_j^{(2)}] \\ &= -\sum_{j\in\Lambda}a_j[\langle e_i^{(1)}, P_Nf_j^{(1)}\rangle e_j^{(1)} - \langle e_i^{(1)}, P_Nf_j^{(2)}\rangle e_j^{(2)}] \\ &= -\sum_{j\in\Lambda}a_j[\langle e_i^{(1)}, f_j^{(1)}\rangle e_j^{(1)} - \langle e_i^{(1)}, f_j^{(2)}\rangle e_j^{(2)}] = -Te_i^{(1)}. \end{aligned}$$

Similarly, one can prove that $CT^*Ce_i^{(2)} = -Te_i^{(2)}$.

Now we prove that $CT^*C = -T$ and hence T is skew symmetric. □

Using above theorem, we can get the following result, which means that not every partial isometry of rank ≤ 2 is skew symmetric. However, Garcia and

Wogen [8, Corollary 1] have proved that all partial isometries of rank ≤ 2 are complex symmetric.

Corollary 2.6. *Let $\{e^{(1)}, e^{(2)}\}$, $\{f^{(1)}, f^{(2)}\}$ be two orthonormal subsets of $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ can be written as $T = e^{(1)} \otimes f^{(1)} - e^{(2)} \otimes f^{(2)}$. Then T is skew symmetric if and only if $\langle e^{(1)}, f^{(1)}\rangle = \langle e^{(2)}, f^{(2)}\rangle$.*

Proof. “$\Longleftarrow$.” Let $g^{(1)} = f^{(1)}$ and $g^{(2)} = f^{(2)}$, then it is easy to check that $\{g^{(1)}, g^{(2)}\}$ satisfies the conditions (1)(2) and (3) of Theorem 2.5. Thus T is skew symmetric.

“$\Longrightarrow$.” If T is skew symmetric, by Theorem 2.5, there exists an ONB $\{g^{(1)}, g^{(2)}\}$ of $\bigvee\{f^{(1)}, f^{(2)}\}$ which satisfies the conditions (1)(2) and (3) of this theorem. One can easily deduce that $g^{(1)} = \alpha f^{(1)}$ and $g^{(2)} = \alpha f^{(2)}$, where $|\alpha| = 1$. Since $\langle g^{(1)}, e^{(1)}\rangle = \langle g^{(2)}, e^{(2)}\rangle$, we have $\langle e^{(1)}, f^{(1)}\rangle = \langle e^{(2)}, f^{(2)}\rangle$. $\square$

3. Proof of Main Theorem

Now we can give the proof of Theorem 1.3.

Proof of Theorem 1.3. For convenience, we may directly assume that $\Lambda = \mathbb{N}$. The proof for the general case is similar.

“(1)$\Longleftrightarrow$(2).” Since $a_i \neq a_j$ for $i \neq j$, By Theorem 2.5, T is skew symmetric if and only if there exists an ONB $\{g_i^{(1)}, g_i^{(2)}\}$ of $\bigvee\{f_i^{(1)}, f_i^{(2)}\}$ for all $i \in \mathbb{N}$ such that $\langle g_i^{(1)}, f_i^{(2)}\rangle = \langle g_i^{(2)}, f_i^{(1)}\rangle = 0$, $\langle g_i^{(1)}, f_i^{(1)}\rangle = \langle g_i^{(2)}, f_i^{(2)}\rangle$ and $\langle g_i^{(p)}, e_j^{(q)}\rangle = \langle g_j^{(p')}, e_i^{(q')}\rangle$. This means that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ with $|\lambda_i| = 1$ for all $i \in \Lambda$ such that $g_i^{(1)} = \lambda_i f_i^{(1)}$ and $g_i^{(2)} = \lambda_i f_i^{(2)}$, hence $\langle \lambda_i e_i^{(p)}, f_j^{(q)}\rangle = \langle \lambda_j e_j^{(p')}, f_i^{(q')}\rangle$ for all $i, j \in \mathbb{N}$ and $p, q \in \{1, 2\}$.

“(2)$\Longrightarrow$(3).” Assume that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ with $|\lambda_i| = 1$ for all $i \in \mathbb{N}$ such that $\lambda_i\langle e_i^{(p)}, f_j^{(q)}\rangle = \lambda_j\langle e_j^{(p')}, f_i^{(q')}\rangle$ for all $i, j \in \mathbb{N}$ and $p, q \in \{1, 2\}$. Hence we have $|\langle e_i^{(p)}, f_j^{(q)}\rangle| = |\langle e_j^{(p')}, f_i^{(q')}\rangle|$ for all $i, j \in \mathbb{N}$ and $p, q \in \{1, 2\}$.

Given $n \in \mathbb{N}$, the tuple $(i_1, i_2, \cdots, i_n)$ in $\mathbb{N}^n$ and $p_k, q_k \in \{1, 2\}$ for all $1 \leq k \leq n$, we have

$$\begin{aligned}
&[\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}]\Big[\prod_{k=1}^{n-1}\langle e_{i_k}^{(p_k)}, f_{i_{k+1}}^{(q_k)}\rangle\Big]\langle e_{i_n}^{(p_n)}, f_{i_1}^{(q_n)}\rangle \\
&= \Big[\prod_{k=1}^{n-1}\lambda_{i_k}\langle e_{i_k}^{(p_k)}, f_{i_{k+1}}^{(q_k)}\rangle\Big][\lambda_{i_n}\langle e_{i_n}^{(p_n)}, f_{i_1}^{(q_n)}\rangle] \\
&= \Big[\prod_{k=1}^{n-1}\lambda_{i_{k+1}}\langle e_{i_{k+1}}^{(p_k')}, f_{i_k}^{(q_k')}\rangle\Big][\lambda_{i_1}\langle e_{i_1}^{(p_n')}, f_{i_n}^{(q_n')}\rangle] \\
&= [\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}]\langle e_{i_1}^{(p_n')}, f_{i_n}^{(q_n')}\rangle\Big[\prod_{k=1}^{n-1}\langle e_{i_{k+1}}^{(p_k')}, f_{i_k}^{(q_k')}\rangle\Big].
\end{aligned}$$

"(3)$\Longrightarrow$(2)." For $i,j\in\mathbb{N}$, we define $i\sim j$ if $i=j$ or there exist $i_1,i_2,\cdots,i_n\in\mathbb{N}$ and $p_k,q_k\in\{1,2\}$ for $0\le k\le n$ such that

$$\langle e_i^{(p_0)},f_{i_1}^{(q_0)}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_k}^{(p_k)},f_{i_{k+1}}^{(q_k)}\rangle\big]\langle e_{i_n}^{(p_n)},f_j^{(q_n)}\rangle\neq 0.$$

Since $|\langle e_i^{(p)},f_j^{(q)}\rangle|=|\langle e_j^{(p')},f_i^{(q')}\rangle|$ for all $i,j\in\mathbb{N}$ and $p,q\in\{1,2\}$, one can verify that $\sim$ is an equivalence relation on $\mathbb{N}$. Denote $\mathbb{N}/_\sim=\{\Lambda_\gamma:\gamma\in\Gamma\}$. Hence $\Lambda_\gamma\cap\Lambda_{\gamma'}=\emptyset$ for all $\gamma,\gamma'\in\Gamma$ with $\gamma\neq\gamma'$.

Let $\gamma\in\Gamma$ be fixed. Arbitrarily choose a $l_\gamma\in\Lambda_\gamma$ and set $\lambda_{l_\gamma}=1$. For $j\in\Lambda_\gamma$ with $j\neq l_\gamma$, by hypothesis, there exist $i_1,i_2,\cdots,i_n\in\Lambda_\gamma$, $p_k,q_k\in\{1,2\},0\le k\le n$ such that

$$\langle e_{l_\gamma}^{(p_0)},f_{i_1}^{(q_0)}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_k}^{(p_k)},f_{i_{k+1}}^{(q_k)}\rangle\big]\langle e_{i_n}^{(p_n)},f_j^{(q_n)}\rangle\neq 0.$$

Set

$$\lambda_j=\frac{\langle e_{l_\gamma}^{(p_0)},f_{i_1}^{(q_0)}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_k}^{(p_k)},f_{i_{k+1}}^{(q_k)}\rangle\big]\langle e_{i_n}^{(p_n)},f_j^{(q_n)}\rangle}{\langle e_{i_1}^{(p_0')},f_{l_\gamma}^{(q_0')}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_{k+1}}^{(p_k')},f_{i_k}^{(q_k')}\rangle\big]\langle e_j^{(p_n')},f_{i_n}^{(q_n')}\rangle}.$$

We are going to show that λ_j is well defined for all $j\in\mathbb{N}$. In fact, if there exist $j_1,j_2,\cdots,j_m\in\Lambda_\gamma$, $a_k,b_k\in\{1,2\}$ for $0\le k\le m$ such that

$$\langle e_{l_\gamma}^{(a_0)},f_{j_1}^{(b_0)}\rangle\big[\prod_{k=1}^{m-1}\langle e_{j_k}^{(a_k)},f_{j_{k+1}}^{(b_k)}\rangle\big]\langle e_{j_m}^{(a_m)},f_j^{(b_m)}\rangle\neq 0.$$

It suffices to show that

$$\begin{aligned}&\frac{\langle e_{l_\gamma}^{(p_0)},f_{i_1}^{(q_0)}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_k}^{(p_k)},f_{i_{k+1}}^{(q_k)}\rangle\big]\langle e_{i_n}^{(p_n)},f_j^{(q_n)}\rangle}{\langle e_{i_1}^{(p_0')},f_{l_\gamma}^{(q_0')}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_{k+1}}^{(p_k')},f_{i_k}^{(q_k')}\rangle\big]\langle e_j^{(p_n')},f_{i_n}^{(q_n')}\rangle}\\ &=\frac{\langle e_{l_\gamma}^{(a_0)},f_{j_1}^{(b_0)}\rangle\big[\prod_{k=1}^{m-1}\langle e_{j_k}^{(a_k)},f_{j_{k+1}}^{(b_k)}\rangle\big]\langle e_{j_m}^{(a_m)},f_j^{(b_m)}\rangle}{\langle e_{j_1}^{(a_0')},f_{l_\gamma}^{(b_0')}\rangle\big[\prod_{k=1}^{m-1}\langle e_{j_{k+1}}^{(a_k')},f_{j_k}^{(b_k')}\rangle\big]\langle e_j^{(a_m')},f_{j_m}^{(b_m')}\rangle}.\end{aligned}$$

On the other hand, by the condition (3), it is easy to check that the above equality holds. This shows that λ_j is well defined. Also it is easy to see that $|\lambda_j|=1$ for all $j\in\mathbb{N}$.

Arbitrarily choose $i,j\in\mathbb{N}$ and $p,q\in\{1,2\}$. We shall show that $\lambda_i\langle e_i^{(p)},f_j^{(q)}\rangle=\lambda_j\langle e_j^{(p')},f_i^{(q')}\rangle$. If $i=j$ or $\langle e_i^{(p)},f_j^{(q)}\rangle=0$, by condition (3), this is obvious. We may directly assume that $i\neq j$ and $\langle e_i^{(p)},f_j^{(q)}\rangle\neq 0$. Hence $i\sim j$. We further assume that $i,j\in\Lambda_\gamma$ for some $\gamma\in\Gamma$. We consider the following two cases.

Case 1. $i=l_\gamma$ or $j=l_\gamma$. We directly assume that $i=l_\gamma$ and $j\neq l_\gamma$. It suffices to show that $\langle e_{l_\gamma}^{(p)},f_j^{(q)}\rangle=\lambda_j\langle e_j^{(p')},f_{l_\gamma}^{(q')}\rangle$. Since $|\langle e_j^{(p')},f_{l_\gamma}^{(q')}\rangle|=|\langle e_{l_\gamma}^{(p)},f_j^{(q)}\rangle|\neq 0$, by the definition of λ_j, this is obvious.

Case 2. $i\neq l_\gamma$ and $j\neq l_\gamma$. Then λ_i can be written as

$$\lambda_i=\frac{\langle e_{l_\gamma}^{(p_0)},f_{i_1}^{(q_0)}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_k}^{(p_k)},f_{i_{k+1}}^{(q_k)}\rangle\big]\langle e_{i_n}^{(p_n)},f_i^{(q_n)}\rangle}{\langle e_{i_1}^{(p_0')},f_{l_\gamma}^{(q_0')}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_{k+1}}^{(p_k')},f_{i_k}^{(q_k')}\rangle\big]\langle e_i^{(p_n')},f_{i_n}^{(q_n')}\rangle}.$$

Further, we have

$$\begin{aligned}
\lambda_i\langle e_i^{(p)}, f_j^{(q)}\rangle &= \frac{\langle e_{l_\gamma}^{(p_0)}, f_{i_1}^{(q_0)}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_k}^{(p_k)}, f_{i_{k+1}}^{(q_k)}\rangle\big]\langle e_{i_n}^{(p_n)}, f_i^{(q_n)}\rangle}{\langle e_{i_1}^{(p_0')}, f_{l_\gamma}^{(q_0')}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_{k+1}}^{(p_k')}, f_{i_k}^{(q_k')}\rangle\big]\langle e_i^{(p_n')}, f_{i_n}^{(q_n')}\rangle}\langle e_i^{(p)}, f_j^{(q)}\rangle \\
&= \frac{\langle e_{l_\gamma}^{(p_0)}, f_{i_1}^{(q_0)}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_k}^{(p_k)}, f_{i_{k+1}}^{(q_k)}\rangle\big]\langle e_{i_n}^{(p_n)}, f_i^{(q_n)}\rangle\langle e_i^{(p)}, f_j^{(q)}\rangle}{\langle e_{i_1}^{(p_0')}, f_{l_\gamma}^{(q_0')}\rangle\big[\prod_{k=1}^{n-1}\langle e_{i_{k+1}}^{(p_k')}, f_{i_k}^{(q_k')}\rangle\big]\langle e_i^{(p_n')}, f_{i_n}^{(q_n')}\rangle\langle e_j^{(p')}, f_i^{(q')}\rangle}\langle e_j^{(p')}, f_i^{(q')}\rangle \\
&= \lambda_j\langle e_j^{(p')}, f_i^{(q')}\rangle.
\end{aligned}$$

The last equality follows from the definition of λ_j. This completes the proof. $\square$

Example 3.1. Let $T \in \mathcal{B}(\mathbb{C}^6)$ and suppose that T admits the following representation

$$T = \begin{bmatrix} 0 & 1 & \\ & 0 & 2 \\ & & 0 \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} \oplus \begin{bmatrix} 0 & -2 & \\ & 0 & -1 \\ & & 0 \end{bmatrix} \begin{matrix} f_1 \\ f_2, \\ f_3 \end{matrix}$$

where $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ is an ONB of $\mathbb{C}^6$. Then T is skew symmetric. In fact, we have $T = (e_2 \otimes e_1 - f_3 \otimes f_2) + 2(e_3 \otimes e_2 - f_2 \otimes f_1)$. By Theorem 1.3, it is easy to see that T is skew symmetric. On the other hand, using [21, Lemma 0.4], one can easily deduce that $A = \begin{bmatrix} 0 & 1 & \\ & 0 & 2 \\ & & 0 \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}$ is not complex symmetric. Also, we claim that A is not skew symmetric. Otherwise, if C is a conjugation on $\bigvee\{e_1, e_2, e_3\}$ such that $CAC = -A^*$, it is easy to see that there exist $\lambda, \mu \subset \mathbb{C}$ with $|\lambda| = |\mu| = 1$ such that $Ce_1 = \lambda e_3$ and $Ce_2 = \mu e_2$. It follows that $CACe_1 = CA(\lambda e_3) = C(2\lambda e_2) = 2\overline{\lambda}\mu e_2$ and $-T^* e_1 = -e_2$, a contradiction.

Example 3.2. Let $T \in \mathcal{B}(\mathbb{C}^4)$ and suppose that T admits the following representation

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$$

where $\{u_1, u_2, u_3, u_4\}$ is an ONB of $\mathbb{C}^4$. We claim that T is skew symmetric. In fact, a direct calculation shows that $\frac{\sqrt{10}\pm\sqrt{2}}{2}$ are eigenvalues of $|T|$. Normalized eigenvectors $\{f_1^{(1)}, f_1^{(2)}\}$ of $|T|$ and $\{e_1^{(1)}, e_1^{(2)}\}$ of $|T^*|$ corresponding to $\frac{\sqrt{10}+\sqrt{2}}{2}$ are given by

$$f_1^{(1)} = \frac{1}{\sqrt{10-4\sqrt{5}}}\begin{bmatrix} \sqrt{5}-2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, f_1^{(2)} = \frac{1}{\sqrt{10-4\sqrt{5}}}\begin{bmatrix} 0 \\ 0 \\ \sqrt{5}-2 \\ -1 \end{bmatrix},$$

$$e_1^{(1)} = \frac{1}{\sqrt{10-2\sqrt{5}}}\begin{bmatrix} \sqrt{5}-1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, e_1^{(2)} = \frac{1}{\sqrt{10-2\sqrt{5}}}\begin{bmatrix} 0 \\ 0 \\ \sqrt{5}-1 \\ -2 \end{bmatrix}.$$

Also, normalized eigenvectors $\{f_2^{(1)}, f_2^{(2)}\}$ of $|T|$ and $\{e_2^{(1)}, e_2^{(2)}\}$ of $|T^*|$ corresponding to $\frac{\sqrt{10}-\sqrt{2}}{2}$ are given by

$$f_2^{(1)} = \frac{1}{\sqrt{10+4\sqrt{5}}}\begin{bmatrix}\sqrt{5}+2\\ -1\\ 0\\ 0\end{bmatrix}, f_2^{(2)} = \frac{1}{\sqrt{10+4\sqrt{5}}}\begin{bmatrix}0\\ 0\\ \sqrt{5}+2\\ 1\end{bmatrix},$$

$$e_2^{(1)} = \frac{1}{\sqrt{10+2\sqrt{5}}}\begin{bmatrix}\sqrt{5}+1\\ -2\\ 0\\ 0\end{bmatrix}, e_2^{(2)} = \frac{1}{\sqrt{10+2\sqrt{5}}}\begin{bmatrix}0\\ 0\\ \sqrt{5}+1\\ 2\end{bmatrix}.$$

It is easy to see that $\{f_1^{(1)}, f_1^{(2)}, f_2^{(1)}, f_2^{(2)}\}$, $\{e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_2^{(2)}\}$ are two orthonormal bases of $\mathbb{C}^4$ and T can be written as

$$T = \frac{\sqrt{10}+\sqrt{2}}{2}(e_1^{(1)} \otimes f_1^{(1)} - e_1^{(2)} \otimes f_1^{(2)}) + \frac{\sqrt{10}-\sqrt{2}}{2}(e_2^{(1)} \otimes f_2^{(1)} - e_2^{(2)} \otimes f_2^{(2)}).$$

Since $\langle e_i^{(1)}, f_j^{(2)}\rangle = \langle e_j^{(1)}, f_i^{(2)}\rangle = 0$, $\langle e_i^{(2)}, f_j^{(1)}\rangle = \langle e_j^{(2)}, f_i^{(1)}\rangle = 0$ for all $1 \le i, j \le 2$, $\langle e_1^{(1)}, f_1^{(1)}\rangle = \langle e_1^{(2)}, f_1^{(2)}\rangle$, $\langle e_2^{(1)}, f_2^{(1)}\rangle = \langle e_2^{(2)}, f_2^{(2)}\rangle$ and $\langle e_1^{(1)}, f_2^{(1)}\rangle = -\langle e_2^{(2)}, f_1^{(2)}\rangle$, condition (3) of Theorem 1.3 is obviously satisfied.

4. Skew symmetric compact operators

In this section, we are going to study skew symmetric compact operators. The following result shows that skew symmetric compact operators admit the form (1.1).

Theorem 4.1. *Let $T \in \mathcal{K}(\mathcal{H})$ be skew symmetric, that is, $CTC = -T^*$ for some conjugation C on $\mathcal{H}$. Then T admits the following form*

$$T = \sum_{i\in\Lambda} a_i(Cf_i \otimes e_i - Ce_i \otimes f_i),$$

where a_i are nonzero singular values of T, repeated according to multiplicity, and $\{e_i, f_i\}$ are orthonormal eigenvectors of $|T|$ respect to a_i.

Proof. Let $\{a_i\}_{1\le i<N}$ be the nonzero singular values of T, where $N < \infty$ when T has finite rank otherwise $N = \infty$. Without loss of generality, we assume that $N = \infty$. The proof of other case is similar. Since T is compact, the eigenspaces $\mathcal{H}_n$ of $|T|$ respect to different nonzero singular values a_n are finite dimensional and mutually orthogonal. By Lemma 2.3, we have $T = CK|T|$, where K is a partial anticonjugation supported on $\overline{\text{ran}\,(|T|)}$ and $K|T| = |T|K$. It follows that $K|_{\mathcal{H}_n}$ is an anticonjugation for all n. By [7, Lemma 4.4], there is an ONB $\{e_i^{(n)}, f_i^{(n)}\}_{i=1}^{k_n}$ of $\mathcal{H}_n$ such $Ke_i^{(n)} = f_i^{(n)}$ and $Kf_i^{(n)} = -e_i^{(n)}$ for all $1 \le i \le k_n$. We have

$$Te_i^{(n)} = CK|T|e_i^{(n)} = a_nCf_i^{(n)}, Tf_i^{(n)} = CK|T|f_i^{(n)} = -a_nCe_i^{(n)}$$

for all $1 \leq i \leq k_n$ and $n \geq 1$. We conclude that

$$[T - \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} (Cf_i^{(n)} \otimes e_i^{(n)} - Ce_i^{(n)} \otimes f_i^{(n)})]|_{\text{ran } |T|} = 0.$$

Note that $\ker T = \ker |T|$, we have $T = \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} (Cf_i^{(n)} \otimes e_i^{(n)} - Ce_i^{(n)} \otimes f_i^{(n)})$. This completes the proof. $\square$

5. Skew symmetric partial isometries

In order to study skew symmetric partial isometries, we introduce the following definition.

Definition 5.1. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is Hamiltonian, if there exists an anticonjugation K on $\mathcal{H}$ such that $KTK = T^*$.

The notion of Hamiltonian operators is a generalization of Hamiltonian matrices, which have many applications in matrix Riccati equations from control theory, linear response theory and computational chemistry, see[12, 15, 16]. A matrix T on $\mathbb{C}^{2n}$ of the form

$$T = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

is Hamiltonian if $F^* = F$, $G^* = G$ and $H = -E^*$. If $\dim \mathcal{H} = \infty$ and $T \in \mathcal{B}(\mathcal{H})$ is a Hamiltonian operator, then $KTK = T^*$ for some anticongjugation K on $\mathcal{H}$. By Lemma 2.2, there exists an ONB $\{e_n, f_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ such that $Ke_n = f_n$ and $Kf_n = -e_n$ for each $n \in \mathbb{N}$. Set $\mathcal{H}_1 = \vee_{n=1}^{\infty} e_n$ and $\mathcal{H}_2 = \vee_{n=1}^{\infty} f_n$. We have $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and K_1 admits the following representation

$$K = \begin{bmatrix} 0 & -J_2 \\ J_1 & 0 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where J_1, J_2 are conjugate-linear, $J_1 e_n = f_n$ and $J_2 f_n = e_n$ for each $n \in \mathbb{N}$. A direct calculation shows that

$$T = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $E^* = -J_2 H J_1$, $F^* = J_1 F J_1$ and $G^* = J_2 G J_2$.

Goodson [11] study normal Hamiltonian operators and give a structure theorem for normal Hamiltonian operators. Garcia and Wogen [8, Theorem 2] proved that a partial isometry $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric if and only if the compression of T to its initial space is complex symmetric. Motivated by this result, we aim to characterize skew symmetric partial isometries and get the following result.

Theorem 5.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a partial isometry. Then T is skew symmetric if and only if the compression of T to its initial space is Hamiltonian. In particular,* $\dim(\ker T)^{\perp}$ *is even.*

To prove the above theorem, we give an interpolation theorem related to anticonjugations.

Theorem 5.3. *Let $\{h_i^{(1)}, h_i^{(2)}\}_{i\in\Lambda}$ and $\{g_i^{(1)}, g_i^{(2)}\}_{i\in\Lambda}$ be two orthonormal bases of $\mathcal{H}$. Then there exists an anticonjugation K on $\mathcal{H}$ such that $Kh_i^{(1)} = g_i^{(2)}$ and $Kh_i^{(2)} = g_i^{(1)}$ for all $i \in \Lambda$ if and only if $\langle h_i^{(p)}, g_j^{(q)}\rangle = -\langle h_j^{(p')}, g_i^{(q')}\rangle$ for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$.*

Proof. "$\Longrightarrow$" By the Definition 2.1, this is obvious.

"$\Longleftarrow$" For $x = \sum_{i\in\Lambda}(a_i^{(1)}h_i^{(1)} + a_i^{(2)}h_i^{(2)})$, $y = \sum_{i\in\Lambda}(b_i^{(1)}g_i^{(1)} + b_i^{(2)}g_i^{(2)})$, define $Kx = \sum_{i\in\Lambda}(\overline{a_i^{(1)}}g_i^{(2)} + \overline{a_i^{(2)}}g_i^{(1)})$, $Ky = \sum_{i\in\Lambda}(-\overline{b_i^{(1)}}h_i^{(2)} - \overline{b_i^{(2)}}h_i^{(1)})$. We are going to show that (1) K is well defined, (2) $\langle Kx, Ky\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$.

If $x \in \mathcal{H}$ can be written as

$$x = \sum_{i\in\Lambda}(a_i^{(1)}h_i^{(1)} + a_i^{(2)}h_i^{(2)}) = \sum_{i\in\Lambda}(b_i^{(1)}g_i^{(1)} + b_i^{(2)}g_i^{(2)}),$$

we shall show that

$$\sum_{i\in\Lambda}(\overline{a_i^{(1)}}g_i^{(2)} + \overline{a_i^{(2)}}g_i^{(1)}) = -\sum_{i\in\Lambda}(\overline{b_i^{(1)}}h_i^{(2)} + \overline{b_i^{(2)}}h_i^{(1)}).$$

Denote $y_1 = \sum_{i\in\Lambda}(\overline{a_i^{(1)}}g_i^{(2)} + \overline{a_i^{(2)}}g_i^{(1)})$ and $y_2 = -\sum_{i\in\Lambda}(\overline{b_i^{(1)}}h_i^{(2)} + \overline{b_i^{(2)}}h_i^{(1)})$. It suffices to show that

$$\langle y_1, h_i^{(1)}\rangle = \langle y_2, h_i^{(1)}\rangle \text{ and } \langle y_1, h_i^{(2)}\rangle = \langle y_2, h_i^{(2)}\rangle, \forall i \in \Lambda$$

For fixed $i \in \Lambda$, we have

$$\langle y_1, h_i^{(1)}\rangle = \sum_{j\in\Lambda}[\overline{a_j^{(1)}}\langle g_j^{(2)}, h_i^{(1)}\rangle + \overline{a_j^{(2)}}\langle g_j^{(1)}, h_i^{(1)}\rangle] = -\sum_{j\in\Lambda}[\overline{a_j^{(1)}}\langle g_i^{(2)}, h_j^{(1)}\rangle + \overline{a_j^{(2)}}\langle g_i^{(2)}, h_j^{(2)}\rangle],$$

$$\langle y_1, h_i^{(2)}\rangle = \sum_{j\in\Lambda}[\overline{a_j^{(1)}}\langle g_j^{(2)}, h_i^{(2)}\rangle + \overline{a_j^{(2)}}\langle g_j^{(1)}, h_i^{(2)}\rangle] = -\sum_{j\in\Lambda}[\overline{a_j^{(1)}}\langle g_i^{(1)}, h_j^{(1)}\rangle + \overline{a_j^{(2)}}\langle g_i^{(1)}, h_j^{(2)}\rangle],$$

$$\langle y_2, h_i^{(1)}\rangle = -\overline{b_i^{(2)}}, \langle y_2, h_i^{(2)}\rangle = -\overline{b_i^{(1)}}.$$

Now it suffices to prove that

$$\sum_{j\in\Lambda}[a_j^{(1)}\langle h_j^{(1)}, g_i^{(1)}\rangle + a_j^{(2)}\langle h_j^{(2)}, g_i^{(1)}\rangle] = b_i^{(1)}, \sum_{j\in\Lambda}[a_j^{(1)}\langle h_j^{(1)}, g_i^{(2)}\rangle + a_j^{(2)}\langle h_j^{(2)}, g_i^{(2)}\rangle] = b_i^{(2)},$$

for all $i \in \Lambda$.

For each $i \in \Lambda$, since $x = \sum_{j\in\Lambda}(a_j^{(1)}h_j^{(1)} + a_j^{(2)}h_j^{(2)}) = \sum_{j\in\Lambda}(b_j^{(1)}g_j^{(1)} + b_j^{(2)}g_j^{(2)})$, we have

$$b_i^{(1)} = \langle x, g_i^{(1)}\rangle = \langle\sum_{j\in\Lambda}(a_j^{(1)}h_j^{(1)} + a_j^{(2)}h_j^{(2)}), g_i^{(1)}\rangle = \sum_{j\in\Lambda}[a_j^{(1)}\langle h_j^{(1)}, g_i^{(1)}\rangle + a_j^{(2)}\langle h_j^{(2)}, g_i^{(1)}\rangle],$$

$$b_i^{(2)} = \langle x, g_i^{(2)}\rangle = \langle\sum_{j\in\Lambda}(a_j^{(1)}h_j^{(1)} + a_j^{(2)}h_j^{(2)}), g_i^{(2)}\rangle = \sum_{j\in\Lambda}[a_j^{(1)}\langle h_j^{(1)}, g_i^{(2)}\rangle + a_j^{(2)}\langle h_j^{(2)}, g_i^{(2)}\rangle].$$

This shows that K is well defined on $\mathcal{H}$.

It is easy to see that K is conjugate linear and $K^2 = -I$. We shall show that $\langle Kx, Ky\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. Assume that $x = \sum_{i\in\Lambda}(a_i^{(1)}h_i^{(1)} + a_i^{(2)}h_i^{(2)})$ and $y = \sum_{i\in\Lambda}(b_i^{(1)}g_i^{(1)} + b_i^{(2)}g_i^{(2)})$, we have

$$\begin{aligned}
\langle Kx, Ky\rangle &= \langle \sum_{i\in\Lambda}(\overline{a_i^{(1)}}g_i^{(2)} + \overline{a_i^{(2)}}g_i^{(1)}), -\sum_{j\in\Lambda}(\overline{b_j^{(1)}}h_j^{(2)} + \overline{b_j^{(2)}}h_j^{(1)})\rangle \\
&= -\sum_{i,j\in\Lambda}[\overline{a_i^{(1)}}b_j^{(1)}\langle g_i^{(2)}, h_j^{(2)}\rangle + \overline{a_i^{(1)}}b_j^{(2)}\langle g_i^{(2)}, h_j^{(1)}\rangle \\
&\quad + \overline{a_i^{(2)}}b_j^{(1)}\langle g_i^{(1)}, h_j^{(2)}\rangle + \overline{a_i^{(2)}}b_j^{(2)}\langle g_i^{(1)}, h_j^{(1)}\rangle] \\
&= \sum_{i,j\in\Lambda}[\overline{a_i^{(1)}}b_j^{(1)}\langle g_j^{(1)}, h_i^{(1)}\rangle + \overline{a_i^{(1)}}b_j^{(2)}\langle g_j^{(2)}, h_i^{(1)}\rangle \\
&\quad + \overline{a_i^{(2)}}b_j^{(1)}\langle g_j^{(1)}, h_i^{(2)}\rangle + \overline{a_i^{(2)}}b_j^{(2)}\langle g_j^{(2)}, h_i^{(2)}\rangle] \\
&= \langle \sum_{j\in\Lambda}(b_j^{(1)}g_j^{(1)} + b_j^{(2)}g_j^{(2)}), \sum_{i\in\Lambda}(a_i^{(1)}h_i^{(1)} + a_i^{(2)}h_i^{(2)})\rangle \\
&= \langle y, x\rangle.
\end{aligned}$$

Thus K is an anticonjugation on $\mathcal{H}$. It is easy to see that $Kh_i^{(1)} = g_i^{(2)}$ and $Kh_i^{(2)} = g_i^{(1)}$ for each $i \in \Lambda$. This completes the proof. $\square$

Now we are going to prove theorem 5.2.

Proof of Theorem 5.2. "$\Longrightarrow$." Assume that T is a skew symmetric partial isometry, by Lemma 2.3, there is a partial anticonjugation supported on $\overline{\operatorname{ran}(|T|)}$, which commutes with $|T|$. By [7, Lemma 4.3], it follows that $\dim \overline{\operatorname{ran}|T|}$ is even. This means that the initial space of T has even dimension. Hence we can assume that T has the following form:

$$T = \sum_{i\in\Lambda}(e_i^{(1)} \otimes f_i^{(1)} - e_i^{(2)} \otimes f_i^{(2)}),$$

where $\Lambda \subset \mathbb{N}$, $\{e_i^{(1)}, e_i^{(2)}\}_{i\in\Lambda}$ and $\{f_i^{(1)}, f_i^{(2)}\}_{i\in\Lambda}$ are two orthonormal subsets of $\mathcal{H}$. Since T is skew symmetric, by Theorem 2.5, there exists an ONB $\{g_i^{(1)}, g_i^{(2)}\}_{i\in\Lambda}$ of $\bigvee\{f_i^{(1)}, f_i^{(2)} : i \in \Lambda\}$ such that

(1) $\langle g_i^{(p)}, f_j^{(q)}\rangle = (-1)^{(p+q)}\langle g_j^{(p')}, f_i^{(q')}\rangle$ for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$.
(2) $\langle g_i^{(p)}, e_j^{(q)}\rangle = \langle g_j^{(p')}, e_i^{(q')}\rangle$ for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$.

It is easy to see that the initial space of T is $\bigvee\{f_i^{(1)}, f_i^{(2)} : i \in \Lambda\}$ and the compression of T to its initial space can be written as

$$\begin{aligned}
A = \sum_{i,j\in\Lambda}&[\langle e_i^{(1)}, g_j^{(1)}\rangle g_j^{(1)} \otimes f_i^{(1)} + \langle e_i^{(1)}, g_j^{(2)}\rangle g_j^{(2)} \otimes f_i^{(1)} \\
&- \langle e_i^{(2)}, g_j^{(1)}\rangle g_j^{(1)} \otimes f_i^{(2)} - \langle e_i^{(2)}, g_j^{(2)}\rangle g_j^{(2)} \otimes f_i^{(2)}]. \ (*)
\end{aligned}$$

Since (1) holds, by Theorem 5.3, there exists an anticonjugation K on $\bigvee\{f_i^{(1)}, f_i^{(2)} : i \in \Lambda\}$ such that $Kf_i^{(1)} = g_i^{(2)}$ and $Kf_i^{(2)} = -g_i^{(1)}$ for each $i \in \Lambda$. Now it suffices to show that $KAK = A^*$.

For fixed $i \in \Lambda$. We have

$$\begin{aligned}
KAKg_i^{(1)} &= KAf_i^{(2)} \\
&= -K[\sum_{j\in\Lambda}(\langle e_i^{(2)}, g_j^{(1)}\rangle g_j^{(1)} + \langle e_i^{(2)}, g_j^{(2)}\rangle g_j^{(2)})] \\
&= -\sum_{j\in\Lambda}(\langle g_j^{(1)}, e_i^{(2)}\rangle f_j^{(2)} - \langle g_j^{(2)}, e_i^{(2)}\rangle f_j^{(1)})(\text{by condition (2)}) \\
&= -\sum_{j\in\Lambda}(\langle g_i^{(1)}, e_j^{(2)}\rangle f_j^{(2)} - \langle g_i^{(1)}, e_j^{(1)}\rangle f_j^{(1)}) \\
&= A^* g_i^{(1)},
\end{aligned}$$

and

$$\begin{aligned}
KAKg_i^{(2)} &= -KAf_i^{(1)} \\
&= -K[\sum_{j\in\Lambda}(\langle e_i^{(1)}, g_j^{(1)}\rangle g_j^{(1)} + \langle e_i^{(1)}, g_j^{(2)}\rangle g_j^{(2)})] \\
&= -\sum_{j\in\Lambda}(\langle g_j^{(1)}, e_i^{(1)}\rangle f_j^{(2)} - \langle g_j^{(2)}, e_i^{(1)}\rangle f_j^{(1)})(\text{by condition (2)}) \\
&= -\sum_{j\in\Lambda}(\langle g_i^{(2)}, e_j^{(2)}\rangle f_j^{(2)} - \langle g_i^{(2)}, e_j^{(1)}\rangle f_j^{(1)}) \\
&= A^* g_i^{(2)},
\end{aligned}$$

Thus we have $KAK = A^*$ and A is Hamiltonian.

"$\Longleftarrow$." Assume that A is Hamiltonian and K is an anticonjugation on $\bigvee\{f_i^{(1)}, f_i^{(2)} : i \in \Lambda\}$ such that $KAK = A^*$. Let $g_i^{(1)} = -Kf_i^{(2)}$ and $g_i^{(2)} = Kf_i^{(1)}$ for each $i \in \Lambda$. By Definition 2.1, $\{g_i^{(1)}, g_i^{(2)}\}_{i\in\Lambda}$ is an ONB of $\bigvee\{f_i^{(1)}, f_i^{(2)} : i \in \Lambda\}$. It is easy to verify that

$$\langle g_i^{(p)}, f_j^{(q)}\rangle = (-1)^{(p+q)}\langle g_j^{(p')}, f_i^{(q')}\rangle \tag{1}$$

for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$. Now we can directly assume that the compression of T to its initial space has the form $(*)$. For fixed $i \in \Lambda$, we have

$$\begin{aligned}
KAKg_i^{(1)} &= KAf_i^{(2)} \\
&= -K[\sum_{j\in\lambda}(\langle e_i^{(2)}, g_j^{(1)}\rangle g_j^{(1)} + \langle e_i^{(2)}, g_j^{(2)}\rangle g_j^{(2)})] \\
&= -\sum_{j\in\Lambda}[\langle g_j^{(1)}, e_i^{(2)}\rangle f_j^{(2)} - \langle g_j^{(2)}, e_i^{(2)}\rangle f_j^{(1)}],
\end{aligned}$$

$$A^* g_i^{(1)} = \sum_{j\in\Lambda}[\langle g_i^{(1)}, e_j^{(1)}\rangle f_j^{(1)} - \langle g_i^{(1)}, e_j^{(2)}\rangle f_j^{(2)}],$$

$$\begin{aligned} KAKg_i^{(2)} &= -KAf_i^{(1)} \\ &= -K[\sum_{j\in\lambda}(\langle e_i^{(1)}, g_j^{(1)}\rangle g_j^{(1)} + \langle e_i^{(1)}, g_j^{(2)}\rangle g_j^{(2)})] \\ &= -\sum_{j\in\Lambda}[\langle g_j^{(1)}, e_i^{(1)}\rangle f_j^{(2)} - \langle g_j^{(2)}, e_i^{(1)}\rangle f_j^{(1)}], \end{aligned}$$

and

$$A^*g_i^{(2)} = \sum_{j\in\Lambda}[\langle g_i^{(2)}, e_j^{(1)}\rangle f_j^{(1)} - \langle g_i^{(2)}, e_j^{(2)}\rangle f_j^{(2)}].$$

Since $KAK = A^*$, it follows that

$$\langle g_i^{(p)}, e_j^{(q)}\rangle = \langle g_j^{(p')}, e_i^{(q')}\rangle, \tag{2}$$

for all $i, j \in \Lambda$ and $p, q \in \{1, 2\}$.

Since conditions (1) and (2) hold, by Theorem 2.5, we conclude that T is skew symmetric. $\square$

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